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On the Robust Estimation of Spatial Autoregressive Models

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The spatial autoregressive (SAR) model is commonly used for cross-sectional data with spatial dependence in the response variable. A robust M-estimator for the SAR model with independent normal random errors is proposed, which is resistant to extreme contamination or outliers in the data. The proposed estimator is defined by a set of robust estimating functions derived by modifying the estimating functions for the standard maximum likelihood (ML) estimator using the Huber function. An iterative algorithm based on iteratively reweighted least squares and a fixed point method is provided to compute the robust M-estimator. The proposed estimator is shown to be Fisher-consistent under the core model, has a bounded influence function, and is asymptotically normally distributed. A closed-form expression for the asymptotic covariance matrix estimator under the core model is derived, and used to study the asymptotic relative efficiency of the proposed estimator. Simulation studies demonstrate that the proposed robust M-estimator greatly outperforms the ML estimator as well as several other non-robust estimators of the SAR model, in terms of both estimation and inference under different contamination scenarios. Additionally, the proposed estimator offers comparable performance to some robust estimators of the SAR model under contamination. Applying the robust M-estimator for the SAR model to the U.S. federal grants and U.S. Nielsen Local Television View datasets reveals strong positive spatial dependence in the grant value among counties, insignificant positive effect of news watching on grant distribution, and evidence of outliers in the data. This results in different conclusions from those obtained using the ML estimator.

1. Introduction

The development and application of spatial econometric models has attracted much attention in recent years, with uses in fields such as economics (Ertur and Koch, 2007; Eleftheriou et al., 2019), social and environmental science (Haining, 1993), and real estate (Pace et al., 1998), among others. One of the most frequently applied spatial econometric models is the spatial autoregressive (SAR) model (Cliff and Ord, 1973; Ord, 1975; Anselin, 1988), which contains a spatial lag of the response to induce spatial dependence among nearby regions. The strength of spatial dependence in the SAR model is characterized by a single spatial dependence parameter ρ, which makes the interpretation of the SAR model relatively straightforward. Combining this with its ability to preserve the dependence structure in spatial data and spatial panel data, the SAR model has been widely applied in empirical studies; see, for instance, Caldeira (2012), Yu et al. (2016), and Hoshino (2018), among others. As a motivating application of the SAR model, we study the association between United States (U.S.) federal grants distribution and television news watching, where the grants data is subject to potential outliers. This empirical

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study is motivated by the work of Strömberg (2004), who investigated the impact of radio on the distribution of relief funds under the New Deal relief program to U.S. counties, without taking into account the possibility of spatial dependence and outliers in the data.

In the literature, different estimation methods have been developed for the SAR model. Ord (1975) discussed maximum likelihood (ML) estimation, involving the use of a Newton-Raphson algorithm to compute the spatial dependence parameter under a normality assumption for the error terms. When the error terms are not normally distributed, this estimator is better referred to as a quasi-maximum likelihood estimator; its asymptotic properties under the SAR model were studied by Lee (2004). Alternatively, Kelejian and Prucha (1998) proposed a two-stage least squares or instrumental variable estimator, which is computationally simpler than the ML estimator and also does not require the assumption of normality. However, although the two-stage least squares estimator has been shown to be consistent and asymptotically normally distributed, it is not asymptotically optimal as pointed out by Lee (2003), who subsequently proposed the asymptotically optimal instrumental variable estimator. The two-stage least squares estimators suffer from two major drawbacks: they cannot be used to test for the joint significance of all the covariates, and they are not consistent when all the covariates in the model are irrelevant. To overcome this problem, Lee (2007) suggested a generalized method-of-moments (GMM) estimator. In addition, Li et al. (2007) developed an approximate profile-likelihood estimator (APLE) for the spatial dependence parameter which allows for a closed-form expression.

All of the estimators discussed above share a common problem, namely that they are (extremely) sensitive to contamination or outliers in the data. The sensitivity of ML estimator to outliers in spatial problems is well-known (see de Luna and Genton, 2002; Genton, 2003, for instance), and indeed the non-robustness of the GMM estimator and the APLE reviewed above will be demonstrated in the simulation studies later on. Regarding our motivating U.S. federal grants dataset, some counties received exceptionally large grants relative to others while some counties received zero grants, indicating the possibility of one or more outlying responses. Following the theme developed by Elvezio Ronchetti of applying robust methods in econometrics (e.g., Krishnakumar and Ronchetti, 1997; Ronchetti and Trojani, 2001) then, we are motivated to develop a robust estimator for the SAR model, which provides a more reliable analysis of cross-section dependent data subject to potential outliers and contamination.

There is a growing literature on robust estimation and inference for spatial econometric models. Anselin (1990) proposed a robust bootstrap estimator for the SAR model, while Haining (1993) considered robust estimation of the spatial dependence parameter based on the ordinary least squares (OLS) estimator, although the OLS estimator has been shown to be inconsistent (Ord, 1975; Kelejian and Prucha, 2001). Later on, a block forward search algorithm was studied by Cerioli and Riani (2002) to detect outliers and high leverage points in spatial error models, while Genton (2003) studied the asymptotic breakdown point for the least median of squares (LMS, see Rousseeuw, 1984) estimator for the SAR model. In their study of robust indirect inference, de Luna and Genton (2002) and Genton and Ronchetti (2003) estimated the SAR model parameters using the robust indirect Yule-Walker (IYWR) estimator. A robust minimum distance estimator (RMDE) has also been proposed by Genton and Koul (2008) for the first-order quadrant autoregressive model, of which the SAR model is a special case. Furthermore, Lin and Lee (2010) suggested an optimal robust generalized method-of-moments (ORGMM) estimator for the SAR model, which is consistent and asymptotically normal even when the error terms have unknown heteroskedasticity.

In this article, we propose a new, robust M-estimator for the SAR model with independent normal random errors, where the spatial units can be irregularly located on any spatial domain with a distance measure, and which is resistant to extreme outliers in the data. Our estimator builds upon the seminal approach of Huber (1973), where the proposed estimator is defined by a set of robust estimating functions derived from the estimating functions of the ML estimator using the Huber loss function. We develop an iterative algorithm based on iterative weighted least squares, in conjunction with a fixed point method, to compute our estimator. Theoretically, we establish that the proposed robust M-estimator for the SAR model is Fisher-consistent (under the core model), and has a bounded influence function. We also establish an asymptotic normality result for our proposed estimator, along with a closed-form expression for the asymptotic covariance matrix estimate.

We numerically compute the asymptotic relative efficiency of our robust M-estimator relative to the ML estimator, which offers guidance in the choice of the tuning parameters in the associated robust estimating equations. Under different spatial weight matrices and contamination in the SAR model, we demonstrate that the proposed estimator achieves robustness against possible outliers in the error terms, with only slightly lower efficiency than the ML estimator which, in contrast, is potentially very biased when there is contamination in the error terms. Furthermore, we show that the empirical level and power of a hypothesis test for the spatial dependence parameter based on the robust M-estimator is resistant to extreme contamination, in contrast to the test based on the ML estimator which gives unreliable conclusion when outliers are present in the data.

We also carry out simulation studies to compare our proposed estimator to the other existing estimators of the SAR model, namely, the APLE, the ORGMM estimator, the IYWR estimator, the RMDE and the LMS estimator reviewed above. Under the scenario where extreme contamination was added to two neighboring error terms, we find that the proposed robust M-estimator outperforms the APLE and ORGMM estimator, which are both (substantially) biased. On the other hand, the proposed estimator has comparable performance to the IYWR estimator, the RMDE and the LMS estimator in terms of robust estimation, noting that the IYWR estimator and the RMDE are used for the SAR model on a regular grid without any covariates, and the LMS estimator tends to exhibit lower efficiency.

Applying the proposed robust M-estimator for the SAR model to the motivating U.S. federal grants and U.S. Nielsen Local Television View (NLTV) datasets, we find that although counties with a higher proportion of households that watch news channels received larger grants from the U.S. federal government, this effect is not statistically significant after accounting for potential outliers in the data. The result differs from a statistically significant positive effect observed when applying the ML estimator. Both estimators also demonstrate that grants distribution exhibited strong positive spatial dependence.

The rest of this article is organized as follows. Section 2 introduces the SAR model with independent normal random errors. Section 3 develops the proposed robust M-estimator and the iterative estimation algorithm. Section 4 develops the asymptotic properties of the proposed estimator. Sections 5 and 6 present simulation studies to assess the finite sample performance of the proposed estimator and the ML estimator, along with an application to the U.S. federal grants and NLTV datasets, respectively. We offer some concluding remarks in Section 7. All theoretical proofs of the theorems and additional simulation results, along with detailed empirical comparisons to other estimators of the SAR model, are provided in the supplementary material.

2. Spatial Autoregressive Model

Consider a set of spatial locations (or areal observations) $i = 1, \dots, n$ that are assumed to be located, possibly irregularly, on a spatial domain for which a measure of distance is available, where for the i-th location we observe a response y_i and a set of covariates $\boldsymbol{x}_i = (x_{i1}, \dots, x_{ip})^\top$ with $x_{i1} = 1$ representing an intercept term. The spatial autoregressive (SAR) model with independent normal random errors is defined as

$$y_i = \rho \sum_{j=1}^{n} w_{ij} y_j + \boldsymbol{x}_i^\top \boldsymbol{\beta} + \epsilon_i, \epsilon_i \overset{i.i.d}{\sim} N(0, \sigma^2) \text{ for } i = 1, \dots, n, \tag{1}$$

where w_{ij} is a pre-specified weight that describes the relationship between locations i and j, with $w_{ii} = 0$ for $i = 1, \dots, n$, and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_p)^\top$ denotes a vector of regression coefficients. The errors are assumed to be independent and normally distributed, with mean zero and variance σ^2. Finally, the quantity ρ denotes the spatial dependence parameter measuring the strength of spatial dependence among locations that are located near each other. Model (1) can also be written in a matrix form as

$$\boldsymbol{y} = \rho \boldsymbol{W} \boldsymbol{y} + \boldsymbol{X} \boldsymbol{\beta} + \boldsymbol{\epsilon}, \tag{2}$$

where $\boldsymbol{y} = (y_1, \dots, y_n)^\top$ and $\boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_n)^\top$ are the vectors of responses and errors, respectively, $\boldsymbol{X} = (\boldsymbol{x}_1, \dots, \boldsymbol{x}_n)^\top$ is an $n \times p$ model matrix, and $\boldsymbol{W} = (w_{ij})_{n \times n}$ is the $n \times n$ spatial weight matrix representing the spatial relationship among all n locations. Throughout this article, we use the notation $(h_{ij})_{s \times v}$ to denote a general $s \times v$ matrix whose (i, j)-th element is h_{ij} for $i = 1, \dots, s$ and $j = 1, \dots, v$.

Compared to the usual linear regression model, (2) contains an additional $\rho \boldsymbol{W} \boldsymbol{y}$ term. That is, a spatial lag term which allows for the response at a location to be affected by its nearby locations' responses. This is common in spatial econometrics, among other disciplines, to accommodate either a spillover effect or local competition (say).

2.1. Spatial Weight Matrix

Since (2) is characterized by the $n \times n$ spatial weight matrix $\boldsymbol{W}$, the n spatial units can possibly be irregularly located on any spatial domain as long as the $\boldsymbol{W}$ matrix can be constructed. Depending on the data and their background knowledge of the study, researchers can specify the spatial weight matrix $\boldsymbol{W}$ as required by the aims of their analyses. Commonly, a symmetric $\boldsymbol{W}$ can be constructed as $w_{ij} = m(d_{ij})$, where d_{ij} denotes the distance between the two locations and $m(\cdot)$ denotes a monotonically decreasing function. In other words, the spatial weight w_{ij} decreases as the distance between the two locations d_{ij} increases. The spatial weight matrix can be further row-normalized as

$$w_{ij} = \frac{m(d_{ij})}{\sum_{l=1}^{n} m(d_{il})}. \tag{3}$$

For the simulation studies in Section 5, we consider spatial weight matrices of the form (3), with $m(d_{ij}) = 1/d_{ij}$ for $i \neq j$ and $m(d_{ij}) = 0$ for $i = j$. Note the distances need not be geographic (Anselin, 1988; Cliff and Ord, 1981) e.g., it is possible to use economic distances (Case et al., 1993) or a combination of several distance metrics (Yu et al., 2016). Another common choice is based on contiguity. That is, the spatial weight matrix is constructed such that $w_{ij} = 1/n_i$ if location j shares a common border with location i that has n_i neighbors, and zero otherwise.

3. Robust M-Estimation of the SAR model

To motivate the development of our proposed robust M-estimator, we first review the ML estimator of the SAR model. Let $\boldsymbol{\theta} = (\boldsymbol{\beta}^\top, \sigma, \rho)^\top$ denote the vector of parameters in the SAR model (2). Then the log-likelihood function is

$$\ell(\boldsymbol{\theta}; \boldsymbol{y}) = -\frac{n}{2} \log(2\pi) - n \log(\sigma) + \log |\det(\boldsymbol{I}_n - \rho \boldsymbol{W})| - \frac{1}{2\sigma^2} \left\{ (\boldsymbol{I}_n - \rho \boldsymbol{W}) \boldsymbol{y} - \boldsymbol{X} \boldsymbol{\beta} \right\}^\top \left\{ (\boldsymbol{I}_n - \rho \boldsymbol{W}) \boldsymbol{y} - \boldsymbol{X} \boldsymbol{\beta} \right\},$$

and the ML estimator is $\hat{\boldsymbol{\theta}}_{ML} = \arg\max_{\boldsymbol{\theta}} \ell(\boldsymbol{\theta};\boldsymbol{y})$. Moreover, it is an M-estimator with the estimating equation $\boldsymbol{\eta}_{ML}(\boldsymbol{\theta};\boldsymbol{y}) = \partial\ell(\boldsymbol{\theta};\boldsymbol{y})/\partial\boldsymbol{\theta} = \mathbf{0}_{p+2}$, where $\mathbf{0}_\nu$ is a ν-dimensional vector of zeros, and

$$\boldsymbol{\eta}_{ML}(\boldsymbol{\theta};\boldsymbol{y}) = \begin{pmatrix} \frac{1}{\sigma^2}\boldsymbol{X}^\top\left\{(\boldsymbol{I}_n-\rho\boldsymbol{W})\boldsymbol{y}-\boldsymbol{X\beta}\right\} \\ \frac{1}{\sigma^3}\left\{(\boldsymbol{I}_n-\rho\boldsymbol{W})\boldsymbol{y}-\boldsymbol{X\beta}\right\}^\top\left\{(\boldsymbol{I}_n-\rho\boldsymbol{W})\boldsymbol{y}-\boldsymbol{X\beta}\right\}-\frac{n}{\sigma} \\ \frac{1}{\sigma^2}(\boldsymbol{Wy})^\top\left\{(\boldsymbol{I}_n-\rho\boldsymbol{W})\boldsymbol{y}-\boldsymbol{X\beta}\right\}-\mathrm{tr}(\boldsymbol{W}(\boldsymbol{I}_n-\rho\boldsymbol{W})^{-1}). \end{pmatrix}.$$

The asymptotic properties of the ML estimator $\hat{\boldsymbol{\theta}}_{ML}$ for the SAR model have been studied by Lee (2004), among others. The estimating function $\boldsymbol{\eta}_{ML}(\boldsymbol{\theta};\boldsymbol{y})$ is not bounded in $\boldsymbol{y}$, and equivalently not in $\boldsymbol{\epsilon} = (\boldsymbol{I}_n-\rho\boldsymbol{W})\boldsymbol{y}-\boldsymbol{X\beta}$. This implies its influence function is also not bounded in $\boldsymbol{y}$ and $\boldsymbol{\epsilon}$, and the ML estimator is therefore sensitive to extreme outliers in the responses and the error terms.

To overcome the lack of robustness, we modify $\boldsymbol{\eta}_{ML}(\boldsymbol{\theta};\boldsymbol{y})$ to a new robust estimating function which defines a robust M-estimator (denoted as $\hat{\boldsymbol{\theta}}$) of the SAR model. Specifically, $\hat{\boldsymbol{\theta}}$ is defined as the solution to $\boldsymbol{\eta}_R(\boldsymbol{\theta};\boldsymbol{y}) = \mathbf{0}_{p+2}$, where we denote the standardized residual vector as $\boldsymbol{z} = \{(\boldsymbol{I}_n-\rho\boldsymbol{W})\boldsymbol{y}-\boldsymbol{X\beta}\}/\sigma$, and propose

$$\boldsymbol{\eta}_R(\boldsymbol{\theta};\boldsymbol{y}) = \begin{pmatrix} \boldsymbol{X}^\top\boldsymbol{\psi}_{c_1}(\boldsymbol{z}) \\ \boldsymbol{\psi}_{c_2}(\boldsymbol{z})^\top\boldsymbol{\psi}_{c_2}(\boldsymbol{z})-n\tilde{h}(c_2) \\ \frac{1}{\sigma}\{\boldsymbol{G}(\rho)\boldsymbol{X\beta}\}^\top\boldsymbol{\psi}_{c_3}(\boldsymbol{z})+\boldsymbol{\psi}_{c_3}(\boldsymbol{z})^\top\boldsymbol{G}(\rho)^\top\boldsymbol{\psi}_{c_3}(\boldsymbol{z})-\mathrm{tr}\{\boldsymbol{G}(\rho)\}\tilde{h}(c_3) \end{pmatrix}, \quad (4)$$

for a set of positive tuning parameters c_1, c_2 and c_3. The quantity $\boldsymbol{\psi}_c : \mathbb{R}^n \to \mathbb{R}^n$ is a vector-valued function that applies the element-wise Huber function with tuning parameter $c > 0$, i.e., $h_c(z_i) = z_i\min\{1, c/|z_i|\}$. Furthermore, we have $\boldsymbol{G}(\rho) = \boldsymbol{W}(\boldsymbol{I}_n-\rho\boldsymbol{W})^{-1}$, $\tilde{h}(c) = 2c^2\{1-\Phi(c)\}-2c\phi(c)-1+2\Phi(c)$, where $\phi(\cdot)$ and $\Phi(\cdot)$ are the standard normal probability density function and cumulative distribution function, respectively.

Comparing the estimating function of the ML estimator $\boldsymbol{\eta}_{ML}(\boldsymbol{\theta};\boldsymbol{y})$ to that of our proposed robust M-estimator in (4), we see that, in addition to some constant shifts, Huber functions are now applied to the elements of the standardized residual vector $\boldsymbol{z}$. Critically, the bounded nature of the Huber function ensures that $\boldsymbol{\eta}_R(\boldsymbol{\theta};\boldsymbol{y})$ is also bounded in $\boldsymbol{y}$ and $\boldsymbol{\epsilon}$, and hence guarantees the influence function of the robust M-estimator to be bounded. This facilitates reasonable estimates of the SAR model parameters even when the data contains extreme contamination. On the other hand, the addition of constant terms to (4) ensures that the robust M-estimator is Fisher-consistent. The robust M-estimator is less efficient than the ML estimator, and the tradeoff between robustness and efficiency is controlled by the tuning parameters (c_1, c_2, c_3). When the tuning parameters tend to infinity, the robust M-estimator is equivalent to the ML estimator. We discuss the choice of the tuning parameters in Section 5.

3.1. Iterative Estimation Algorithm

We develop an iterative estimation approach to solve $\boldsymbol{\eta}_R(\boldsymbol{\theta};\boldsymbol{y}) = \mathbf{0}_{p+2}$. Recall the definition of the standardized residual vector $\boldsymbol{z}(\boldsymbol{\theta};\boldsymbol{y}) = \{z_1(\boldsymbol{\theta};\boldsymbol{y}),\ldots,z_n(\boldsymbol{\theta};\boldsymbol{y})\}^\top = \{(\boldsymbol{I}_n-\rho\boldsymbol{W})\boldsymbol{y}-\boldsymbol{X\beta}\}/\sigma$, where the dependence on $\boldsymbol{\theta}$ and $\boldsymbol{y}$ is now made explicit. Also, define the three components of the robust estimation function in (4) as $\boldsymbol{\eta}_R(\boldsymbol{\theta};\boldsymbol{y}) = \{\boldsymbol{\eta}_{R,\boldsymbol{\beta}}(\boldsymbol{\theta};\boldsymbol{y})^\top, \boldsymbol{\eta}_{R,\sigma}(\boldsymbol{\theta};\boldsymbol{y})^\top, \boldsymbol{\eta}_{R,\rho}(\boldsymbol{\theta};\boldsymbol{y})^\top\}^\top$, with $\boldsymbol{\eta}_{R,\boldsymbol{\beta}}(\boldsymbol{\theta};\boldsymbol{y}) = \boldsymbol{X}^\top\boldsymbol{\psi}_{c_1}\{\boldsymbol{z}(\boldsymbol{\theta};\boldsymbol{y})\}$, $\boldsymbol{\eta}_{R,\sigma}(\boldsymbol{\theta};\boldsymbol{y}) = \boldsymbol{\psi}_{c_2}\{\boldsymbol{z}(\boldsymbol{\theta};\boldsymbol{y})\}^\top\boldsymbol{\psi}_{c_2}\{\boldsymbol{z}(\boldsymbol{\theta};\boldsymbol{y})\}-n\tilde{h}(c_2)$, and $\boldsymbol{\eta}_{R,\rho}(\boldsymbol{\theta};\boldsymbol{y}) = (1/\sigma)\{\boldsymbol{G}(\rho)\boldsymbol{X\beta}\}^\top\boldsymbol{\psi}_{c_3}\{\boldsymbol{z}(\boldsymbol{\theta};\boldsymbol{y})\}+\boldsymbol{\psi}_{c_3}\{\boldsymbol{z}(\boldsymbol{\theta};\boldsymbol{y})\}^\top\boldsymbol{G}(\rho)^\top\boldsymbol{\psi}_{c_3}\{\boldsymbol{z}(\boldsymbol{\theta};\boldsymbol{y})\}-\mathrm{tr}\{\boldsymbol{G}(\rho)\}\tilde{h}(c_3)$.

We iterate between the following three steps for updating the parameter estimates. First, define $\alpha_i(\boldsymbol{\theta};\boldsymbol{y}) = h_{c_1}\{z_i(\boldsymbol{\theta};\boldsymbol{y})\}z_i(\boldsymbol{\theta};\boldsymbol{y})^{-1}$ if $z_i(\boldsymbol{\theta};\boldsymbol{y}) \neq 0$, and $\alpha_i(\boldsymbol{\theta};\boldsymbol{y}) = 1$ if $z_i(\boldsymbol{\theta};\boldsymbol{y}) = 0$, and let $\boldsymbol{D}_\alpha(\boldsymbol{\theta};\boldsymbol{y}) = \mathrm{diag}\{\alpha_1(\boldsymbol{\theta};\boldsymbol{y}),\ldots,\alpha_n(\boldsymbol{\theta};\boldsymbol{y})\}$ denote an $n\times n$ diagonal matrix. Then it is straightforward to show that solving $\boldsymbol{\eta}_{R,\boldsymbol{\beta}}(\boldsymbol{\theta};\boldsymbol{y}) = \mathbf{0}_p$ leads to the update

$$\boldsymbol{\beta} \Leftarrow \left\{\boldsymbol{X}^\top\boldsymbol{D}_\alpha(\boldsymbol{\theta};\boldsymbol{y})\boldsymbol{X}\right\}^{-1}\boldsymbol{X}^\top\boldsymbol{D}_\alpha(\boldsymbol{\theta};\boldsymbol{y})(\boldsymbol{y}-\rho\boldsymbol{Wy}).$$

Next, we can employ a fixed point iterative method to solve $\boldsymbol{\eta}_{R,\sigma}(\boldsymbol{\theta};\boldsymbol{y}) = 0$, which leads to the nonnegative update

$$\sigma \Leftarrow \left[\frac{\sigma^2}{n\tilde{h}(c_2)}\boldsymbol{\psi}_{c_2}\{\boldsymbol{z}(\boldsymbol{\theta};\boldsymbol{y})\}^\top\boldsymbol{\psi}_{c_2}\{\boldsymbol{z}(\boldsymbol{\theta};\boldsymbol{y})\}\right]^{\frac{1}{2}}.$$

Finally, conditional on $\boldsymbol{\beta}$ and σ, we can solve $\boldsymbol{\eta}_{R,\rho}(\rho;\boldsymbol{\beta},\sigma,\boldsymbol{y}) = 0$ as

$\arg\min_{\rho\in(a,b)}\left\{\boldsymbol{\eta}_{R,\rho}(\rho;\boldsymbol{\beta},\sigma,\boldsymbol{y})\right\}^2$, where $(a,b) = (-|\lambda_{\min}(\boldsymbol{W})|^{-1}, 1)$ with $\lambda_{\min}(\boldsymbol{W})$ denoting the smallest eigenvalue of the row-normalized $\boldsymbol{W}$. This constraint is required to ensure the nonsingularity of the matrix $\boldsymbol{I}_n-\rho\boldsymbol{W}$. As this is a one-dimensional constrained optimization problem, it can be solved relatively straightforwardly e.g., in our simulations we use the `fminbd` function in `Matlab`.

We iterate between the above three steps until convergence e.g., when changes in parameter estimates between successive iterations are below a sufficiently small threshold. Regarding the choice of initial values, we can start with the ordinary least squares estimates $\hat{\boldsymbol{\beta}}^{[0]} = (\boldsymbol{X}^\top\boldsymbol{X})^{-1}\boldsymbol{X}^\top\boldsymbol{y}$, $\hat{\sigma}^{[0]} = \{(\boldsymbol{y}-\boldsymbol{X}\hat{\boldsymbol{\beta}}^{[0]})^\top(\boldsymbol{y}-\boldsymbol{X}\hat{\boldsymbol{\beta}}^{[0]})/(n-p)\}^{1/2}$ and $\hat{\rho}^{[0]} = 0$. A formal estimation algorithm is provided in the supplementary material.

4. Theoretical Properties

In this section, we study the theoretical properties of the proposed robust M-estimator for the SAR model. Let $\boldsymbol{\theta}_0 = (\boldsymbol{\beta}_0^\top, \sigma_0, \rho_0)^\top$ denote the true value of the model parameters in (2), $\boldsymbol{A}(\boldsymbol{\theta}_0) = n^{-1}\mathrm{E}\{\boldsymbol{\eta}_R(\boldsymbol{\theta}_0;\boldsymbol{y})\boldsymbol{\eta}_R(\boldsymbol{\theta}_0;\boldsymbol{y})^\top\}$, and $\boldsymbol{B}(\boldsymbol{\theta}_0) = -n^{-1}\mathrm{E}\{\partial\boldsymbol{\eta}_R(\boldsymbol{\theta}_0;\boldsymbol{y})/\partial\boldsymbol{\theta}^\top\}$ where $\boldsymbol{\beta}_0 = (\beta_{0,1},\ldots,\beta_{0,p})^\top \in \mathbb{R}^p$ and $\sigma_0 > 0$. For a general $s \times v$ matrix $\boldsymbol{H}$, define $|\boldsymbol{H}|_\infty = \max_{1\le i\le s, 1\le j\le v}|h_{ij}|$, $\|\boldsymbol{H}\|_\infty = \max_{1\le i\le s}\sum_{j=1}^{v}|h_{ij}|$, $\|\boldsymbol{H}\|_1 = \max_{1\le j\le v}\sum_{i=1}^{s}|h_{ij}|$, and let $\boldsymbol{H}^{-\top}$ denote the transpose of $\boldsymbol{H}^{-1}$.

We require the following regularity conditions.

Condition 1. The elements of the model matrix $\boldsymbol{X}$ are uniformly bounded constants for all $n \ge 1$; that is, $\sup_{n\ge1}|\boldsymbol{X}|_\infty < \infty$.

Condition 2. The spatial weight matrix $\boldsymbol{W}$ has the form given by (3) and the true ρ_0 satisfies $\rho_0 \in (-|\lambda_{\min}(\boldsymbol{W})|^{-1}, 1)$ for such row-normalized $\boldsymbol{W}$.

Condition 3. The row sums and column sums of matrices $\boldsymbol{W}$ and $(\boldsymbol{I}_n - \rho\boldsymbol{W})^{-1}$ are uniformly bounded for all $n \ge 1$ and $\rho \in (-|\lambda_{\min}(\boldsymbol{W})|^{-1}, 1)$; that is, $\sup_{n\ge1}\|\boldsymbol{W}\|_1 < \infty$, $\sup_{n\ge1}\|\boldsymbol{W}\|_\infty < \infty$, $\sup_{n\ge1,\rho\in(-|\lambda_{\min}(\boldsymbol{W})|^{-1},1)}\|(\boldsymbol{I}_n - \rho\boldsymbol{W})^{-1}\|_1 < \infty$, and $\sup_{n\ge1,\rho\in(-|\lambda_{\min}(\boldsymbol{W})|^{-1},1)}\|(\boldsymbol{I}_n - \rho\boldsymbol{W})^{-1}\|_\infty < \infty$.

Condition 4. The matrices $\boldsymbol{A}(\boldsymbol{\theta}_0) \to \bar{\boldsymbol{A}}(\boldsymbol{\theta}_0)$ and $\boldsymbol{B}(\boldsymbol{\theta}_0) \to \bar{\boldsymbol{B}}(\boldsymbol{\theta}_0)$ as $n \to \infty$, where $\bar{\boldsymbol{A}}(\boldsymbol{\theta}_0)$ and $\bar{\boldsymbol{B}}(\boldsymbol{\theta}_0)$ are positive definite matrices.

These conditions are adapted from and are similar to those found in Lee (2004). Condition 1 assumes that the covariates are bounded, and is a simplifying assumption in line with the fact that we only consider the impact of unbounded error terms ϵ_i on the SAR model. Condition 2 guarantees that the matrix $\boldsymbol{I}_n - \rho_0\boldsymbol{W}$ is non-singular and equilibrium exists i.e., $\boldsymbol{y}$ follows a multivariate normal distribution with mean $(\boldsymbol{I}_n - \rho_0\boldsymbol{W})^{-1}\boldsymbol{X}\boldsymbol{\beta}_0$ and variance-covariance matrix $\sigma_0^2(\boldsymbol{I}_n - \rho_0\boldsymbol{W})^{-1}(\boldsymbol{I}_n - \rho_0\boldsymbol{W})^{-\top}$ (see Ord, 1975, for more details on row-normalized $\boldsymbol{W}$). Condition 3 holds when $\rho = \rho_0$, and limits the spatial dependence. This, together with Condition 4, is needed to establish a central limit theorem for the estimators. Conditions 2 and 3 together imply that the parameter space for the spatial dependence parameter is $(-|\lambda_{\min}(\boldsymbol{W})|^{-1}, 1)$ for the assumed row-normalized form of $\boldsymbol{W}$. Note for the case of symmetric $\boldsymbol{W}$, it has been shown that $(-|\lambda_{\min}(\boldsymbol{W})|^{-1}, \lambda_{\max}(\boldsymbol{W})^{-1})$ is the corresponding parameter space (Anselin, 1988), where $\lambda_{\max}(\boldsymbol{W})$ is the largest eigenvalue of $\boldsymbol{W}$. This is consistent with Condition 2 as $\lambda_{\max}(\boldsymbol{W}) = 1$ for row-normalized $\boldsymbol{W}$.

In robust statistics, a model is regarded as the core model when the data generating model differs from the core model by a small perturbation, e.g., outliers. In this article, the core model is the SAR model with independent normal errors (2). With this in mind, we first establish the robustness property of the robust M-estimator $\hat{\boldsymbol{\theta}}$.

Theorem 1. *The robust M-estimator $\hat{\boldsymbol{\theta}}$ is Fisher-consistent under the core model (2). That is, $\mathrm{E}\{\boldsymbol{\eta}_R(\boldsymbol{\theta}_0;\boldsymbol{y})\} = \boldsymbol{0}_{p+2}$. Furthermore, the influence function of the robust M-estimator is bounded and continuous in $\boldsymbol{y}$.*

The above result implies that the proposed robust M-estimator $\hat{\boldsymbol{\theta}}$ will estimate the true parameter $\boldsymbol{\theta}_0$ of the core model when there is no contamination. Moreover, Theorem 1 implies that $\hat{\boldsymbol{\theta}}$ is robust to extreme contamination or outliers, and thus overcomes the sensitivity of the ML estimator $\hat{\boldsymbol{\theta}}_{ML}$. We will empirically demonstrate this in Section 5.

Next, we present a result on the asymptotic distribution of the robust M-estimator.

Theorem 2. *Assume Conditions 1 - 4 are satisfied. Then under the core model (2) with true parameter $\boldsymbol{\theta}_0$, we have*

$$\sqrt{n}(\hat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) \xrightarrow{d} \boldsymbol{N}(\boldsymbol{0}_{p+2}, \bar{\boldsymbol{B}}(\boldsymbol{\theta}_0)^{-1}\bar{\boldsymbol{A}}(\boldsymbol{\theta}_0)\bar{\boldsymbol{B}}(\boldsymbol{\theta}_0)^{-\top}).$$

The above result enables us to perform statistical inference (e.g. construct confidence intervals for the parameters) using the proposed robust M-estimator. The two matrices $\bar{\boldsymbol{A}}(\boldsymbol{\theta}_0)$ and $\bar{\boldsymbol{B}}(\boldsymbol{\theta}_0)$ in Theorem 2 are usually unknown. To use Theorem 2 in practice, we develop closed-form expressions for the matrices $\boldsymbol{B}(\boldsymbol{\theta}_0)$ and $\boldsymbol{A}(\boldsymbol{\theta}_0)$. We provide the details in the supplementary material, but briefly the former is obtained by finding the partial derivatives of the scaled robust estimating function $n^{-1}\boldsymbol{\eta}_R(\boldsymbol{\theta};\boldsymbol{y})$ with respect to $\boldsymbol{\theta}$, evaluating at $\boldsymbol{\theta}_0$, and then obtaining their negative expectation under the core model. For the matrix $\boldsymbol{A}(\boldsymbol{\theta}_0)$, we find the expectation under the core model of the matrix $n^{-1}\boldsymbol{\eta}_R(\boldsymbol{\theta}_0;\boldsymbol{y})\boldsymbol{\eta}_R(\boldsymbol{\theta}_0;\boldsymbol{y})^\top$. Using the fact that $\boldsymbol{A}(\boldsymbol{\theta}_0) \to \bar{\boldsymbol{A}}(\boldsymbol{\theta}_0)$ and $\boldsymbol{B}(\boldsymbol{\theta}_0) \to \bar{\boldsymbol{B}}(\boldsymbol{\theta}_0)$, we can then estimate the asymptotic covariance matrix in Theorem 2 as

$$\boldsymbol{V}(\hat{\boldsymbol{\theta}}) = \boldsymbol{B}(\hat{\boldsymbol{\theta}})^{-1}\boldsymbol{A}(\hat{\boldsymbol{\theta}})\boldsymbol{B}(\hat{\boldsymbol{\theta}})^{-\top}.$$

Based on $\boldsymbol{V}(\hat{\boldsymbol{\theta}})$ and Theorem 2, we can then construct $100(1-\alpha)\%$ approximate confidence intervals, say, for the q-th element of $\boldsymbol{\theta}_0$ as

$$\left[\hat{\theta}_q - \Phi^{-1}\left(1 - \frac{\alpha}{2}\right)\frac{V_{qq}(\hat{\boldsymbol{\theta}})^{\frac{1}{2}}}{\sqrt{n}}, \hat{\theta}_q + \Phi^{-1}\left(1 - \frac{\alpha}{2}\right)\frac{V_{qq}(\hat{\boldsymbol{\theta}})^{\frac{1}{2}}}{\sqrt{n}})\right], \tag{5}$$

where $\hat{\theta}_q$ is the q-th element of $\hat{\boldsymbol{\theta}}$, and $V_{qq}(\hat{\boldsymbol{\theta}})$ is the (q,q)-th element of $\boldsymbol{V}(\hat{\boldsymbol{\theta}})$. We examine the empirical coverage of these approximate confidence intervals in the next section.

5. Numerical Study

We conducted numerical studies to evaluate the finite sample performance of the proposed robust M-estimator, relative to the ML estimator of the SAR model. The response vector $\boldsymbol{y}$ was simulated from $\boldsymbol{y} = (\boldsymbol{I}_n - \rho_0 \boldsymbol{W})^{-1}\boldsymbol{X}\beta_0 + (\boldsymbol{I}_n - \rho_0 \boldsymbol{W})^{-1}\boldsymbol{\epsilon}$ i.e., the core SAR model is in (2) with true parameter vector $\boldsymbol{\theta}_0 = (\beta_0, \sigma_0, \rho_0)^\top = (1, 1, 0.5)^\top$ (following Lee, 2004). That is, $\boldsymbol{\beta}$ consists of only $p = 1$ coefficient with no intercept. The elements of the $n \times 1$ model matrix $\boldsymbol{X}$ were generated independently from the standard normal distribution, noting the same model matrix was used across different simulated datasets. We set the spatial weight matrix to be a row-normalized distance-based matrix based on a one-dimensional regular grid $\boldsymbol{W}_d = (w_{d,ij})_{n\times n}$, where $w_{d,ij} = (1/d_{ij})/(\sum_{j=1}^n 1/d_{ij})$ for $i \neq j$ and zero otherwise, and $d_{ij} = |i - j|$. Additional simulations to compare the proposed estimator and the ML estimator, where we set the spatial weight matrix as $\boldsymbol{W}_r = (w_{r,ij})_{n\times n}$ with elements $w_{r,ij} = (1/r_{ij})/(\sum_{j=1}^n 1/r_{ij})$ for $i \neq j$ and zero otherwise and $r_{ij} = r_{ji}$ are generated independently from a uniform distribution $U(n^{-1/2}, n^{1/2})$ (similar to the weight matrix considered in Zou et al., 2017), are presented in Section S.5 of the supplementary material.

We generated the errors ϵ_i under four different contamination scenarios:

1. *No contamination*: $\epsilon_i \sim N(0, \sigma_0^2); i = 1, \dots, n$;
2. *Mixture contamination*: $\epsilon_i \sim \omega N(0, \sigma_0^2) + (1 - \omega)N(d_1, d_2); i = 1, \dots, n$, where $\omega \in \{0.01, 0.02, 0.05, 0.10\}$, $d_1 \in \{0, 3, 5\}$, $d_2 \in \{1, 9\}$;
3. *Extreme contamination of a single error term*: $\epsilon_i \sim N(0, \sigma_0^2); i = 2, \dots, n$, and $\epsilon_1 \sim N(C, \sigma_0^2)$ where $C \in \{-150, -100, -50, 50, 100, 150\}$;
4. *Extreme contamination of two error terms*: $\epsilon_i \sim N(0, \sigma_0^2); i = 3, \dots, n$, and $\epsilon_{i'} \sim N(C, \sigma_0^2)$ for $i' = 1, 2$ where $C \in \{-100, -75, -50, 50, 75, 100\}$.

For the first case of no contamination, we considered a range of sample sizes $n = \{20, 60, 100, 140, 180, 200, 400, 600\}$ to assess the asymptotic properties of the proposed estimator. For the remaining three scenarios, we fixed $n = 200$ and focused on assessing the robustness of the proposed estimator relative to the ML estimator. For each combination of n and spatial weight matrix, we constructed 1000 simulated datasets.

We also studied robust hypothesis tests of $H_0 : \rho_0 = 0$ vs $H_1 : \rho_0 \neq 0$ using the proposed estimator and the ML estimator, along with their corresponding estimated standard errors. The detailed results of these in terms of empirical level and power are presented in Section S.6 of the supplementary material. Furthermore, we also performed several simulation studies to compare the robust M-estimator to the other estimators of the SAR model in terms of robust estimation. More specifically, we considered adding contamination $C \in \{-100, -75, -50, 0, 50, 75, 100\}$ to two neighboring error terms (similar to contamination scenarios 1 and 4 when $C = 0$ and $C \neq 0$, respectively) located on a two-dimensional regular grid, and then compared our proposed estimator to the IYWR estimator of de Luna and Genton (2002) and Genton and Ronchetti (2003), the RMDE estimator of Genton and Koul (2008), the LMS estimator discussed in Genton (2003), the APLE of Li et al. (2007), and the ORGMM estimator of Lin and Lee (2010). The details of this simulation design and full results are provided in Section S.7 of the supplementary material.

5.1. Choosing Tuning Parameters c_1, c_2 and c_3

Let $\boldsymbol{V}_{ML}(\boldsymbol{\theta}_0) = [-n^{-1}\mathrm{E}\{\partial \boldsymbol{\eta}_{ML}(\boldsymbol{\theta}_0)/\partial \boldsymbol{\theta}^\top\}]^{-1}$ denote the estimate of the asymptotic covariance matrix of the ML estimator obtained by transforming the expression derived by Lee (2004), who considered the estimation of $(\boldsymbol{\beta}^\top, \sigma^2, \rho)$ instead of $(\boldsymbol{\beta}^\top, \sigma, \rho)$. Also, let $V_{ML,qq}(\boldsymbol{\theta}_0)$ denote the (q, q)-th element of $\boldsymbol{V}_{ML}(\boldsymbol{\theta}_0)$, and $\hat{\theta}_{ML,q}$ denote the q-th element of $\hat{\boldsymbol{\theta}}_{ML}$. Using the true parameters $\boldsymbol{\theta}_0$ and setting $n = 200$ to be the same sample size considered for all four contamination scenarios above, we computed the asymptotic relative efficiency (ARE) of $\hat{\theta}_q$ relative to $\hat{\theta}_{ML,q}$ under the core model (2) as $V_{ML,qq}(\boldsymbol{\theta}_0)/V_{qq}(\boldsymbol{\theta}_0)$ for $q = 1, 2, 3$, where the tuning parameters (c_1, c_2, c_3) were varied across the grid of $[0.01, 8]^3$ to guide the choice of the tuning parameters.

Figure 1 presents the AREs for the spatial weight matrix set to $\boldsymbol{W}_d$; results for $\boldsymbol{W}_r$ are presented in the supplementary material. It can be seen that the AREs of $\hat{\beta}$ and $\hat{\sigma}$ were most strongly affected by the choice of c_1 and c_2, respectively, while the ARE of $\hat{\rho}$ was mainly affected by both the choices of c_2 and c_3. The AREs for the proposed robust M-estimator tended to one when all three tuning parameters got closer to eight, which is consistent with our expectation that the robust M-estimator is equivalent to the ML estimator when all the tuning parameters tend to infinity.

Based on the above results, we found that the choice of $(c_1, c_2, c_3) = (1.4, 2.4, 1.65)$ produced AREs of approximately 0.95 for all components of $\hat{\boldsymbol{\theta}}$. More precisely, when $n = 200$ and using the model matrix $\boldsymbol{X}$ as set up above, this set of tuning parameters resulted in AREs for $\hat{\beta}$, $\hat{\sigma}$ and $\hat{\rho}$ equal to 0.9554, 0.9531 and 0.9545 respectively when the spatial weight matrix was set to $\boldsymbol{W}_d$, and AREs equal to 0.9554, 0.9531 and 0.9543 respectively when the spatial weight matrix was set to $\boldsymbol{W}_r$. In light of these results, we used this set of tuning parameters throughout the simulation studies. It is also worth noting that the same set of tuning parameters was used for the numerical studies involving locations on a two-dimensional spatial domain in Section S.7 of the supplementary material, where it further demonstrated reasonable performance of the robust M-estimator. Also, the proposed estimator using the same set of tuning parameters exhibited satisfactory performance in the simulation studies involving the spatial weight matrix $\boldsymbol{W}_r$ in Section S.5 of the supplementary material, where the

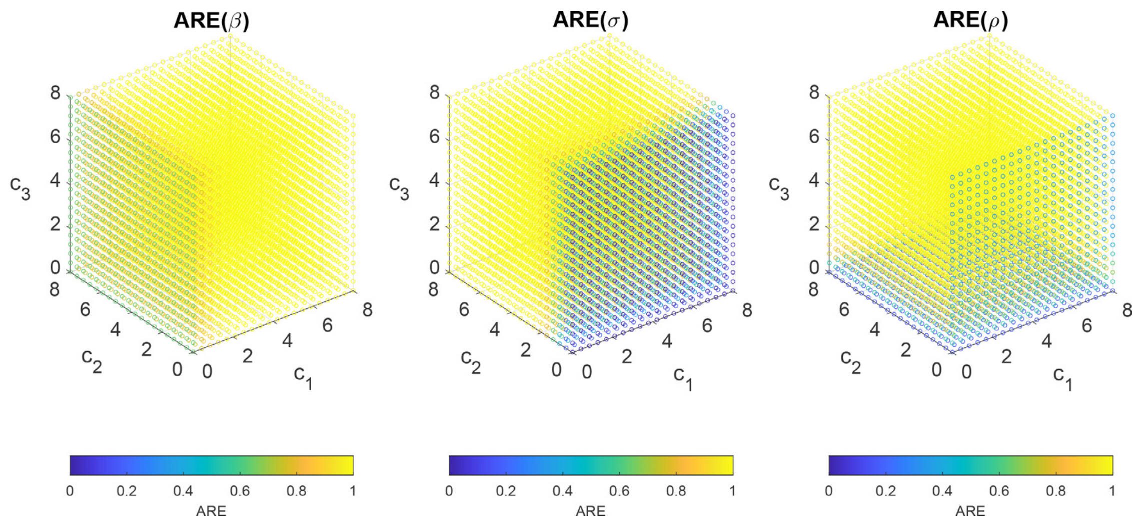

Fig. 1. Asymptotic relative efficiencies (ARE) of the components of $\hat{\boldsymbol{\theta}}$ relative to $\hat{\boldsymbol{\theta}}_{ML}$, under the core model using various choices of tuning parameters $(c_1, c_2, c_3) \in [0.01, 8]^3$, where $\boldsymbol{\theta}_0 = (\beta_0, \sigma_0, \rho_0)^\top = (1, 1, 0.5)^\top$, $\boldsymbol{W} = \boldsymbol{W}_d$ and $n = 200$.

r_{ij}'s used in the construction of $\boldsymbol{W}_r$ above can be thought as pairwise distances of locations on a two-dimensional spatial domain. Therefore, this provided support for the use of the same set of tuning parameters in the two-dimensional spatial U.S. federal grants dataset in Section 6.

5.2. Results for Robust Estimation and Inference

We employed the proposed algorithm in Section 3.1 to calculate robust M-estimates for the SAR model, with the tuning parameters set to $(c_1, c_2, c_3) = (1.4, 2.4, 1.65)$ as discussed above. We compared these with ML estimates, which were obtained using the `sar` function in `Matlab` that makes use of a Monte Carlo approximation to compute the log-determinant term $\log|\det(\boldsymbol{I}_n - \rho\boldsymbol{W})|$ in the log-likelihood (these have been shown to produce similar estimation results compared to the ML estimator based on full computation of the log determinant term, LeSage and Pace, 2009). For both estimators, we assessed performance as follows:

- Empirical bias $\text{Bias}(\hat{\theta}_q) = (1/1000)\sum_{b=1}^{1000}\hat{\theta}_q^{(b)} - \theta_{0,q}$, where $\theta_{0,q}$ denotes the q-th element of the true parameter vector $\boldsymbol{\theta}_0$ and $\hat{\boldsymbol{\theta}}^{(b)} = (\hat{\beta}^{(b)}, \hat{\sigma}^{(b)}, \hat{\rho}^{(b)})^\top$ denotes the estimates from the b-th simulated dataset.
- Root mean squared error, defined as $\text{RMSE}(\hat{\theta}_q) = \sqrt{\text{Bias}(\hat{\theta}_q)^2 + \text{ESD}(\hat{\theta}_q)^2}$, where $\text{ESD}(\hat{\theta}_q) = \left(\sum_{b=1}^{1000}\{\hat{\theta}_q^{(b)} - (\sum_{b=1}^{1000}\hat{\theta}_q^{(b)}/1000)\}^2/999\right)^{1/2}$ denotes the empirical standard deviation.
- We also computed average estimated standard error $\text{ASE}(\hat{\theta}_q) = \sum_{b=1}^{1000} V_{qq}(\hat{\boldsymbol{\theta}}^{(b)})^{1/2}/1000$, and compared it to the empirical standard deviation based on the ratio $\text{ASE}(\hat{\theta}_q)/\text{ESD}(\hat{\theta}_q)$.
- Empirical coverage probability (CP) of the 95% confidence intervals for $\theta_{0,q}$ e.g., based on (5) for the robust M-estimator.

Table 1 presents the simulation results under the core model (2) with no contamination. As the sample size n increased, the biases and RMSE of $\hat{\beta}, \hat{\sigma}$ and $\hat{\rho}$ all converged to zero, which is consistent with the proposed robust M-estimator being Fisher-consistent. Also, their ratios ASE/ESD were close to one, and their CP converged to 95% as n increased, which provides some empirical confirmation of the derived closed-form expression of the asymptotic covariance matrix estimate and the asymptotic normality of the robust M-estimator, respectively. Both the robust M-estimator and the ML estimator had similar performance in terms of average biases, while the ML estimator always had slightly smaller RMSE.

Table 2 shows that the robust M-estimator outperformed the ML estimator under mixture contamination. The RMSEs of $\hat{\beta}$ and $\hat{\sigma}$ were always smaller than those of $\hat{\beta}_{ML}$ and $\hat{\sigma}_{ML}$, respectively, especially when the proportion of contamination $\omega = 0.05$. When the contaminations considered were $0.05N(5, 1)$ and $0.05N(5, 9)$, $\hat{\rho}$ had significantly smaller bias and RMSE compared to $\hat{\rho}_{ML}$. The ratios ASE/ESD for the robust M-estimator were close to one, except for some underestimation and overestimation of $\hat{\sigma}$ and $\hat{\rho}$, respectively, under the $0.05N(5, 1)$ and $0.05N(5, 9)$ contamination settings. Similar (or even worse in certain cases) underestimation and overestimation could be seen for $\hat{\sigma}_{ML}$ and $\hat{\rho}_{ML}$, respectively. As for the empirical coverage probability, the CPs of the proposed robust M-estimator were also closer to 95% than the ML estimator, particularly for the variance parameter σ_0. Simulation results for other mixture contamination scenarios, e.g., $d_1 = 3$ or $\omega \in \{0.02, 0.10\}$, are provided in Section S.6 of the supplementary material, and they were very similar to the results in Table 2.

Table 1
Simulations results using the SAR model (2) with $\boldsymbol{\theta}_0 = (\beta_0, \sigma_0, \rho_0)^\top = (1, 1, 0.5)^\top$, $\boldsymbol{W}_d$ as the spatial weight matrix, $\epsilon_i \sim N(0, \sigma_0^2)$ for $i = 1, \ldots, n$, and $n \in (20, 60, 100, 140, 180, 200, 400, 600)$.

		Robust M-Estimator				ML Estimator			
n	$\boldsymbol{\theta}$	Bias	RMSE	ASE/ESD	CP	Bias	RMSE	ASE/ESD	CI
20	β	-0.0242	0.2972	0.9797	0.9280	-0.0209	0.2941	0.9645	0.9310
	σ	-0.0529	0.1717	0.9452	0.8710	-0.0555	0.1681	0.9474	0.8680
	ρ	-0.2370	0.4297	0.9079	0.9490	-0.2305	0.4194	0.9029	0.9400
60	β	0.0061	0.1272	0.9918	0.9430	0.0059	0.1250	0.9848	0.9430
	σ	-0.0149	0.0947	0.9889	0.9370	-0.0167	0.0935	0.9795	0.9360
	ρ	-0.0940	0.2506	0.9750	0.9520	-0.0943	0.2468	0.9726	0.9530
100	β	0.0015	0.0929	0.9960	0.9490	0.0024	0.0914	0.9895	0.9480
	σ	-0.0102	0.0725	1.0016	0.9400	-0.0110	0.0710	1.0005	0.9380
	ρ	-0.0606	0.2179	0.9671	0.9450	-0.0611	0.2136	0.9678	0.9480
140	β	-0.0029	0.0807	1.0102	0.9450	-0.0034	0.0792	1.0074	0.9510
	σ	-0.0053	0.0632	0.9700	0.9360	-0.0052	0.0618	0.9685	0.9350
	ρ	-0.0587	0.2124	0.9620	0.9390	-0.0581	0.2102	0.9503	0.9410
180	β	0.0022	0.0735	1.0060	0.9520	0.0017	0.0717	1.0082	0.9520
	σ	-0.0063	0.0550	0.9849	0.9290	-0.0064	0.0534	0.9903	0.9320
	ρ	-0.0284	0.1837	0.9929	0.9380	-0.0276	0.1809	0.9847	0.9370
200	β	-0.0008	0.0716	0.9820	0.9450	-0.0002	0.0696	0.9880	0.9440
	σ	-0.0043	0.0527	0.9745	0.9320	-0.0046	0.0512	0.9787	0.9350
	ρ	-0.0291	0.1790	0.9881	0.9480	-0.0290	0.1745	0.9918	0.9490
400	β	-0.0004	0.0494	0.9868	0.9460	0.0000	0.0483	0.9879	0.9440
	σ	-0.0036	0.0374	0.9704	0.9390	-0.0039	0.0364	0.9758	0.9440
	ρ	-0.0124	0.1441	0.9936	0.9490	-0.0130	0.1395	1.0049	0.9530
600	β	-0.0011	0.0418	0.9754	0.9520	-0.0010	0.0408	0.9763	0.9430
	σ	-0.0006	0.0297	0.9966	0.9440	-0.0010	0.0291	0.9946	0.9420
	ρ	-0.0160	0.1262	0.9982	0.9460	-0.0158	0.1230	1.0015	0.9500

Table 2
Simulation results using the SAR model (2) with $\boldsymbol{\theta}_0 = (\beta_0, \sigma_0, \rho_0)^\top = (1, 1, 0.5)^\top$, $\boldsymbol{W}_d$ as the spatial weight matrix, $\epsilon_i \sim \omega N(0, \sigma_0^2) + (1-\omega)N(d_1, d_2)$ for $i = 1, \ldots, n$, $\omega \in \{0.01, 0.05\}$, $d_1 \in \{0, 5\}$, $d_2 \in \{1, 9\}$, and $n = 200$.

		Robust M-Estimator				ML Estimator			
$\omega N(d_1, d_2)$	$\boldsymbol{\theta}$	Bias	RMSE	ASE/ESD	CP	Bias	RMSE	ASE/ESD	CP
$0.01N(0,9)$	β	-0.0028	0.0714	0.9992	0.9530	-0.0032	0.0725	0.9850	0.9530
	σ	0.0093	0.0529	0.9951	0.9510	0.0338	0.0784	0.7328	0.8530
	ρ	-0.0286	0.1711	1.0408	0.9590	-0.0286	0.1693	1.0341	0.9610
$0.01N(5,1)$	β	-0.0034	0.0709	1.0196	0.9590	-0.0021	0.0745	1.0261	0.9540
	σ	0.0225	0.0627	0.8975	0.9050	0.1065	0.1423	0.5883	0.5230
	ρ	-0.0268	0.1778	0.9979	0.9450	-0.0118	0.1811	0.9485	0.9370
$0.01N(5,9)$	β	-0.0001	0.0730	0.9843	0.9560	-0.0003	0.0785	0.9963	0.9500
	σ	0.0168	0.0578	0.9441	0.9440	0.1329	0.1897	0.4197	0.4970
	ρ	-0.0302	0.1717	1.0404	0.9580	-0.0179	0.1771	0.9821	0.9450
$0.05N(0,9)$	β	-0.0026	0.0757	0.9994	0.9480	-0.0018	0.0834	0.9703	0.9370
	σ	0.0710	0.0961	0.8489	0.7500	0.1717	0.2009	0.5635	0.2820
	ρ	-0.0377	0.1833	0.9978	0.9490	-0.0389	0.1907	0.9524	0.9390
$0.05N(5,1)$	β	-0.0001	0.0787	1.0455	0.9660	0.0012	0.1024	0.9810	0.9440
	σ	0.1649	0.1910	0.6219	0.3070	0.4543	0.4743	0.5375	0.0060
	ρ	0.0155	0.1157	1.5274	0.9870	0.1425	0.1868	1.2379	0.8010
$0.05N(5,9)$	β	0.0010	0.0759	1.0539	0.9660	0.0013	0.1112	0.9650	0.9400
	σ	0.1315	0.1538	0.7289	0.4180	0.5547	0.5910	0.3835	0.0080
	ρ	0.0023	0.1245	1.4148	0.9840	0.1221	0.1774	1.2031	0.8790

Table 3 demonstrates that the robust M-estimator generally performed well in terms of RMSE, ASE/ESD and CP when extreme contamination was added to a single error term. There was a minor issue where the ratio ASE/ESD for $\hat{\rho}$ showed overestimation of its ESD and overcoverage of its CP, although this issue was much worse for $\hat{\rho}_{ML}$. The ML estimators $\hat{\beta}_{ML}$ and $\hat{\sigma}_{ML}$ were both severely biased, and their RMSEs were significantly larger than those of the proposed robust M-estimator. The ML estimators also suffered from severe overestimation of their ESDs, which led to CPs of either 0% or 100%. Finally, Table 4 shows that the robust M-estimator performed well when extreme contamination was added to ϵ_1 and ϵ_2, while the ML estimators $\hat{\sigma}_{ML}$ and $\hat{\rho}_{ML}$ exhibited much larger biases and RMSEs, overestimation of the ESDs, and poor CPs.

Comparing the estimation results of the ML estimator in Tables 3 and 4, it can be seen that the biases in $\hat{\beta}_{ML}$ when only ϵ_1 was contaminated were passed onto $\hat{\rho}_{ML}$ when both ϵ_1 and ϵ_2 were contaminated. The reason $\hat{\rho}_{ML}$ was relatively unaffected when only ϵ_1 was contaminated was that this type of contamination only involved a *single* error term but not its neighboring error terms, so the effect of the contamination was absorbed by $\hat{\beta}_{ML}$. On the other hand, based on the

Table 3
Simulation results using the SAR model (2) with $\boldsymbol{\theta}_0 = (\beta_0, \sigma_0, \rho_0)^\top = (1, 1, 0.5)^\top$, $\boldsymbol{W}_d$ as the spatial weight matrix, $\epsilon_i \sim N(0, \sigma_0^2)$ for $i = 2, \ldots, n$, $\epsilon_1 \sim N(C, \sigma_0^2)$, $C \in \{-150, -100, -50, 50, 100, 150\}$, and $n = 200$.

		Robust M-Estimator				ML Estimator			
C	$\boldsymbol{\theta}$	Bias	RMSE	ASE/ESD	CP	Bias	RMSE	ASE/ESD	CP
50	β	-0.0087	0.0719	1.0040	0.9540	-0.2718	0.2804	3.6572	0.9990
	σ	0.0145	0.0540	1.0028	0.9520	2.6631	2.6641	2.5737	0.0000
	ρ	-0.0080	0.1496	1.1570	0.9490	0.0074	0.0814	2.3653	1.0000
100	β	-0.0128	0.0709	1.0315	0.9560	-0.5509	0.5551	7.2351	1.0000
	σ	0.0183	0.0572	0.9656	0.9410	6.1161	6.1165	4.9766	0.0000
	ρ	-0.0264	0.0999	1.8498	0.9990	0.0053	0.0430	4.5431	1.0000
150	β	-0.0097	0.0721	1.0128	0.9460	-0.8212	0.8241	10.5869	1.0000
	σ	0.0237	0.0575	1.0026	0.9470	9.6159	9.6161	7.6214	0.0000
	ρ	-0.0061	0.0912	1.9249	1.0000	0.0099	0.0310	6.5781	1.0000
-50	β	0.0091	0.0733	0.9808	0.9480	0.2738	0.2828	3.5726	1.0000
	σ	0.0099	0.0512	1.0326	0.9580	2.6635	2.6643	2.8116	0.0000
	ρ	-0.0054	0.1314	1.3124	0.9650	0.0008	0.0788	2.4223	1.0000
-100	β	0.0136	0.0705	1.0381	0.9510	0.5524	0.5565	7.2841	1.0000
	σ	0.0168	0.0542	1.0137	0.9500	6.1174	6.1178	5.0914	0.0000
	ρ	-0.0032	0.0920	1.8870	0.9980	0.0046	0.0409	4.7486	1.0000
-150	β	0.0052	0.0723	1.0081	0.9560	0.8182	0.8213	10.3572	1.0000
	σ	0.0286	0.0623	0.9559	0.9210	9.6178	9.6180	7.5998	0.0000
	ρ	0.0190	0.0900	1.9380	1.0000	0.0042	0.0292	6.7046	1.0000

Table 4
Simulation results using the SAR model (2) with $\boldsymbol{\theta}_0 = (\beta_0, \sigma_0, \rho_0)^\top = (1, 1, 0.5)^\top$, $\boldsymbol{W}_d$ as the spatial weight matrix, $\epsilon_i \sim N(0, \sigma_0^2)$ for $i = 3, \ldots, n$, $\epsilon_{i'} \sim N(C, \sigma_0^2)$ for $i' = 1, 2$, $C \in \{-100, -75, -50, 50, 75, 100\}$, and $n = 200$.

		Robust M-Estimator				ML Estimator			
C	$\boldsymbol{\theta}$	Bias	RMSE	ASE/ESD	CP	Bias	RMSE	ASE/ESD	CP
50	β	-0.0044	0.0711	1.0268	0.9550	-0.0207	0.0723	4.6612	1.0000
	σ	0.0311	0.0600	1.0326	0.9380	3.6791	3.6797	3.5454	0.0000
	ρ	-0.0054	0.0974	1.8007	0.9930	0.4678	0.4680	1.8110	0.0000
75	β	-0.0017	0.0709	1.0355	0.9550	-0.0228	0.0721	6.9935	1.0000
	σ	0.0388	0.0665	0.9894	0.8970	5.9292	5.9296	4.9802	0.0000
	ρ	0.0119	0.0949	1.8309	1.0000	0.4703	0.4705	1.7276	0.0000
100	β	-0.0036	0.0694	1.0611	0.9600	-0.0270	0.0725	9.4325	1.0000
	σ	0.0409	0.0701	0.9400	0.8830	8.1914	8.1917	6.5333	0.0000
	ρ	-0.0062	0.0733	2.4111	1.0000	0.4710	0.4712	1.7321	0.0000
-50	β	-0.0046	0.0723	1.0085	0.9490	-0.0033	0.0699	4.6077	1.0000
	σ	0.0297	0.0604	1.0052	0.9370	3.6652	3.6659	3.4006	0.0000
	ρ	0.0197	0.0868	2.0243	0.9990	0.4704	0.4706	1.7197	0.0000
-75	β	-0.0035	0.0714	1.0339	0.9540	0.0049	0.0680	7.0282	1.0000
	σ	0.0439	0.0706	0.9696	0.8720	5.9119	5.9123	5.0997	0.0000
	ρ	0.0369	0.0955	1.9116	1.0000	0.4713	0.4715	1.6979	0.0000
-100	β	-0.0022	0.0701	1.0554	0.9580	0.0070	0.0674	9.4419	1.0000
	σ	0.0468	0.0716	0.9934	0.8800	8.1758	8.1761	6.4540	0.0000
	ρ	0.0065	0.0628	2.7907	1.0000	0.4727	0.4728	1.6332	0.0000

construction of $\boldsymbol{W}_d$, ϵ_1 and ϵ_2 were the closest neighbors to each other with their row-normalized spatial weights given as $w_{d,12} = 0.1703$ and $w_{d,21} = 0.1456$. Therefore, when extreme contamination was added to these two error terms, $\hat{\rho}_{ML}$ interpreted this as strong spatial dependence between the two spatial units, which resulted in a large positive bias for $\hat{\rho}_{ML}$. By contrast, the proposed robust M-estimator remained resistant to such extreme contamination. We also refer the reader to the simulation results using $\boldsymbol{W}_r$ in the supplementary material, which offer further support to the above explanations.

Therefore, our proposed robust M-estimator sacrificed a small amount of efficiency under the core model with no contamination, but performed substantially better than the ML estimator under the different contamination settings. The asymptotic covariance matrix estimate under the core model proposed in Section 4 offered reasonable estimates of the uncertainty in the robust M-estimator, even when there was contamination of various types.

Results from investigating the finite sample performance for testing $H_0 : \rho_0 = 0$ vs $H_1 : \rho_0 \neq 0$ are provided in Table S.9 of the supplementary material. Overall, it can be seen that the empirical level and power of the test based on the robust M-estimator was satisfactory and relatively unaffected by the contamination, while unsurprisingly the test based on the ML estimator was unreliable when $C \neq 0$ as its empirical level and power was always 100% due to the large positive biases in $\hat{\rho}_{ML}$ as seen in Table 4.

Next, we summarize the results based on comparing the proposed robust M-estimator to other estimators of the SAR model in the literature; see Section S.7 of the supplementary material for full results. When there was no contamination (i.e. $C = 0$), all estimators performed similarly in terms of biases and RMSEs. When $C \neq 0$, Table S.10 shows that the robust M-estimator of σ exhibited better estimation performance than the IYWR estimator of σ, which was biased, while the IYWR estimator of ρ performed slightly better than the robust M-estimator for very large values of C. Tables S.12 and S.13 show that our proposed robust M-estimator offered similar performances in terms of robust estimation as the RMDE and the LMS estimator, respectively, although the LMS estimator tended to show higher RMSEs due to its comparably lower efficiency. Furthermore, Tables S.14 and S.15 demonstrate that our robust M-estimator strongly outperformed the APLE and the ORGMM estimator, respectively, when $C \neq 0$. This is not surprising since the APLE is not a robust estimator, while the ORGMM estimator is only robust to heteroskedasticity but not extreme contamination in the error terms. It is worth emphasizing that these simulations considered regular locations on a two-dimensional spatial domain, and that some of the simulations did not include covariates to allow comparisons with some of the other estimators (which were designed with no $\boldsymbol{X}$). Nevertheless, our robust M-estimator still works for more general cases i.e., it can deal with irregularly located locations on any spatial domain, and include estimation of $\boldsymbol{\beta}$ when covariates are present in the data; see the application in Section 6 which exemplifies this.

Finally, as a further comparison, we fitted the SAR model using the robust M-estimator to a real dataset in Figure 9 of Genton and Ronchetti (2003) that consists of the residual values from fitting a linear trend to reflectance values extracted from an extensive aerial survey along the south coast of England. The results are provided in Table S.11 of the supplementary material, from which we see that our proposed robust M-estimator performed similarly to the IYWR estimator, while the classical (non-robust) Yule-Walker estimates of σ and ρ were more different. In summary, the comparison with other estimators of the SAR model indicated that the proposed robust M-estimator overall exhibited strong performance under a variety of contamination scenarios.

6. Application to U.S. Federal Grants Data

We applied the proposed robust M-estimator for the SAR model to investigate the association between United States federal grants distribution and television (TV) news watching at the county level. Previous literature has shown that media usage exhibited major social and economic impacts on its users (Chong and Ferrara, 2009; Olken, 2009; Keefer and Khemani, 2014; Kearney and Levine, 2015). In politics specifically, Strömberg (2004) found that radio access had a substantial influence on the distribution of relief funds under the New Deal relief program by the U.S. federal government, while Martin and Yurukoglu (2017) and Prior and Bougher (2018) among others have also examined the impact of media on political decisions. We are interested in whether the U.S. federal government issued larger grants to counties with a higher proportion of TV news viewing households, presumably with the reasoning being that this could create a better impression of the federal government through larger exposure to local residents. Moreover, local residents of those counties were expected to have higher political awareness, so it was likely that the U.S. federal government targeted those counties by distributing more grants to them in order to gain the political preferences of their residents. It is worth highlighting that most previous studies on media impact have not taken into account the possibility of spatial dependence, contamination or a long-tail in their data; we address this by employing a robust M-estimation method for the SAR model.

The spatial units in our analysis were the counties in the U.S., and the response variable was based on the dollar value (in U.S. dollars) of federal grants distributed to different counties in the 2007 financial year (1 October 2006 - 30 September 2007) by the federal government. The data were sourced from the U.S. Census Bureau-Governments Division. Briefly, federal grants are financial aid issued to recipients including local governments to perform a public purpose of support or stimulation in areas such as health, education and transportation. We defined the response variable for the i-th county as $y_i = \log(\text{Grant}_i/1000 + 0.001)$, with the transformation used since the value of grants Grant_i ranged from \$0 to \$26,966,852,000. In Figure 2 we observed clusters of similar colors, suggesting the possibility of spatial dependence in the grant values. In addition, it can be seen that some counties received exceptionally large grants compared to others while some counties received nothing, suggesting the presence of outlying units or a long-tail in the data. More detailed choropleth maps of grant values for the counties of different states, together with the counties in their neighboring states, are provided in Section S.8 of the supplementary material. These choropleth maps also indicate evidence of spatial dependence in the value of federal grants received among counties, e.g., the clusters in Figure S.2 consisting of counties in the California, Arizona and Nevada states.

The main covariate of interest in this application was the total news rating for each county in the 2005 financial year (1 October 2004 - 30 September 2005), computed using channel rating data from the Nielsen Local Television View (NLTV) database. The total news rating for a county was defined as the average estimated percentage over the 2005 financial year of all TV households in the county that tune to the list of cable news channels classified as 'News' content by Nielsen. We also included other covariates in the SAR model, such as per capita personal income and total population size, which are expected to affect the amount of federal grants issued to a county; see Table 5 for the complete list. These county-level economic data, including the federal grants data, were collected in different surveys such as Census-American Community Survey and Bureau of Labor Statistics - Current Population Survey, and we refer to them collectively as the Census County data. The NLTV dataset was merged with the Census County data based on county name, which resulted in a final dataset consisting of $n = 1385$ counties with $k = 10$ covariates. Details of the matching process and data sources for the county-

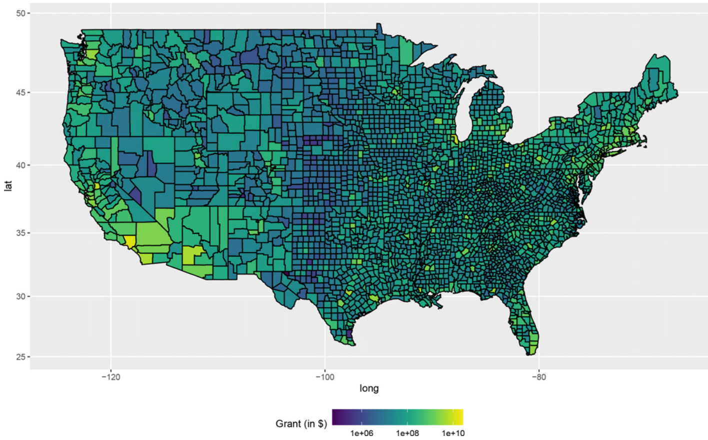

Fig. 2. Choropleth map of federal grant values (in U.S. dollars) for U.S. counties in the financial year 2007, noting that seven counties received $0 grants. Some counties were not included in the final sample due to matching issues between Census County and NLTV datasets, or their total news ratings estimated using numbers of households that were below the minimum sample size defined by Nielsen.

Table 5
Regression coefficients β and the definition of their associated covariates, for the application to the U.S. federal grants data.

β	Covariate
β_0	Intercept
β_1	Total News Rating for 2005 financial year
β_2	Total population (estimate at July 1) in 2005
β_3	Civilian labor force unemployment rate in 2005
β_4	Number of violent crimes known to police in 2005
β_5	Vote cast for president, percent for Democratic in 2004
β_6	Ratio of Hispanic or Latino households between 2005-2009
β_7	People of all ages in poverty, percent in 2005
β_8	Per capita personal income in 2005
β_9	Black population, percent (estimate at July 1) in 2005
β_{10}	Asian population, percent (estimate at July 1) in 2005

level economic data are provided in the supplementary material. The two-year lag between the response variable and the covariates was purposefully chosen since the effect of the latter on grant decisions was not expected to be immediate due to the nature of the grant lifecycle (which involves a long process of planning, developing, and reviewing).

To construct $\boldsymbol{W}$, we defined the spatial weight between counties i and j based on the inverse distance between the two counties. That is, $w_{ij} = (1/d_{ij})/\sum_{l=1}^{n}(1/d_{il})$ where d_{ij} is the great circle distance between counties i and j calculated using the Haversine formula on the internal points of the counties. We used the 2010 Census county distances data from the County Distance Database of the National Bureau of Economic Research, noting county distances are expected to remain relatively constant over time. To obtain the robust M-estimate of the SAR model, we used the `fsolve` function in `Matlab` with the default settings to solve the robust estimating equation $\boldsymbol{\eta}_R(\boldsymbol{\theta};\boldsymbol{y}) = \boldsymbol{0}_{13}$, with the same set of tuning parameters as in the simulation studies i.e., $(c_1, c_2, c_3) = (1.4, 2.4, 1.65)$. We did not use the proposed iterative estimation algorithm in Section 3.1 purely to reduce the computational time, as each iterative step would have involved the calculation of very large matrices given $n = 1385$.

Table 6 reports the estimated parameters for the fitted SAR model using the proposed robust M-estimation, along with fitting using standard ML estimation. Corresponding standard errors, along with p-values based on Wald tests are also provided. Focusing on the robust M-estimator, results show that although the total news rating had a positive effect on grant value ($\hat{\beta}_1 = 0.0336$), this effect was not statistically significant at the 5% level after controlling for the effect of other covariates. The estimated spatial dependence parameter $\hat{\rho} = 0.9791$ was statistically significant at the 5% level, suggesting that the value of the grant received by a county was strongly positively affected by the values of grants received by its nearby counties. This reflected the spatial dependencies seen in the initial exploration of the data, and thus supported the need to account for spatial dependence via e.g., the SAR model. Turning to the other economic variables, most of the regression slopes were in line with their expected effects on grant values. For example, we found that grant values were larger for counties with larger populations ($\hat{\beta}_2 = 0.0177$), larger numbers of violent crimes ($\hat{\beta}_4 = 0.0464$), higher percentage of votes cast for Democratic in the 2004 U.S. presidential election ($\hat{\beta}_5 = 0.0117$), higher proportion of people in poverty ($\hat{\beta}_7 = 0.0292$), and higher per capita personal income ($\hat{\beta}_8 = 0.0730$). Grant values were also significantly larger for counties with a higher proportion of Hispanic or Latino households ($\hat{\beta}_6 = 0.5049$), and a higher percentage of Asian population ($\hat{\beta}_{10} = 0.0406$), which might be due to their need for more support. This could also be due to the federal government attempting to affect polit-

Table 6
Estimation results of fitting the SAR model to U.S. federal grants data, using the robust M-estimator (left) and the standard ML estimator (right). Coefficients with asterisk denote those which are statistically significant at the 5% level.

	Robust M-Estimator			ML Estimator		
	$\hat{\theta}_q$	SE($\hat{\theta}_q$)	P − value	$\hat{\theta}_{ML,q}$	SE($\hat{\theta}_{ML,q}$)	P − value
β_0	-3.0203*	0.2993	<0.01	-1.7021*	0.3149	<0.01
β_1	0.0336	0.0395	0.3950	0.1441*	0.0603	0.0169
β_2	0.0177*	0.0010	<0.01	-0.00004	0.0015	0.9813
β_3	0.0078	0.0157	0.6211	-0.0159	0.0240	0.5066
β_4	0.0464*	0.0131	<0.01	0.1618*	0.0200	<0.01
β_5	0.0117*	0.0022	<0.01	0.0165*	0.0033	<0.01
β_6	0.5049*	0.2191	0.0212	1.0453*	0.3349	<0.01
β_7	0.0292*	0.0057	<0.01	-0.0099	0.0088	0.2600
β_8	0.0730*	0.0050	<0.01	0.0342*	0.0076	<0.01
β_9	0.0027	0.0020	0.1696	0.0097*	0.0030	<0.01
β_{10}	0.0406*	0.0092	<0.01	0.0392*	0.0141	<0.01
σ	0.7821*	0.0152	<0.01	1.2231*	0.0232	<0.01
ρ	0.9791*	0.0182	<0.01	0.9918*	0.0057	<0.01

ical preferences through the distribution of larger grants. Civilian labour force unemployment rate and percentage of Black population were found to have non-significant positive effects on the value of grants issued to counties.

The results obtained from using robust M-estimation were quite different from those obtained using ML estimation. This is exemplified by $\hat{\rho}_{ML}$ and $\hat{\sigma}_{ML}$ both being larger than $\hat{\rho}$ and $\hat{\sigma}$, respectively. We conjecture that this could be due to the presence of extreme response values in some nearby counties, e.g., Bexar and Travis counties in Texas State, which are located near each other, both received exceptionally large grants (see Figure S.5). This would be consistent with $\hat{\rho}_{ML}$ and $\hat{\sigma}_{ML}$ possessing large positive biases when the error terms of two neighboring locations were severely contaminated (see the simulation results in Section 5). Put another way, the stronger spatial dependence inferred from the ML estimate could be an overestimation due to the contamination in counties located near each other. Furthermore, the ML estimate of the effect of total news rating ($\hat{\beta}_{ML,1} = 0.1441$) was significant at the 5% level, and much larger than the corresponding robust M-estimate. The ML estimates for β_2 (total population), β_3 (unemployment rate) and β_7 (people in poverty) also had opposite signs compared to the corresponding robust M-estimates, while β_9 (Black population) was found to be statistically significant when using ML estimation but not significant when using our robust M-estimation. Conversely, β_2 (total population) and β_7 (people in poverty) were found to be not statistically significant based on the ML estimate, but were significant based on the robust M-estimate.

Again, we conjecture that these differences in the estimates of covariate effects could be due to the outliers in the data, since it was shown in the simulation studies that the ML estimate for β was biased when there was extreme contamination in one error term and its standard error could be substantially overestimated, while the robust M-estimate was able to indirectly identify the outliers and reduce their impact on the parameter estimation of the SAR model through the use of the Huber function.

To further demonstrate the above point, we constructed residual diagnostics for the SAR model fitted using robust M-estimation, which allowed us to identify three outlying counties, namely, Los Angeles County of California State which received the second largest grant, and New York County and Queens County of New York State which both received zero grants (see Section S.8 of the supplementary material for details of this). After removing these outliers, we refitted the SAR model with ML estimation, from which the results (Table S.17) became much more consistent with the robust M-estimates in Table 6, e.g., the positive effect of total news rating was no longer statistically significant at the 5% level, and the magnitude of the effect reduced substantially, which supported the implicit down-weighting of outliers by the robust M-estimation. We also constructed a hair-plot (a graphical tool for visualizing influential observations proposed by Genton and Ruiz-Gazen, 2010) in Figure S.8 of the supplementary material to assess and identify three influential counties on the spatial dependence parameter estimate, namely Kings County, Nassau County and New York County of New York State. It can be seen that a large positive contamination for the responses of these influential counties led to a decrease in the spatial dependence parameter estimate. This may be attributed to their common neighbor i.e., the Queens County of New York State that received zero grant which was also identified as one of the outliers in the residual diagnostics discussed above. To summarize, this application illustrated the importance of allowing for the possibility of data contamination or long-tails when dealing with real-world spatially-indexed data, through the use of robust estimation procedures such as our proposed robust M-estimator.

7. Conclusion

We have developed a robust M-estimator for the parameters of a SAR model, based on applying the Huber function to the standardized residual in the ML estimating function. We proposed an iterative estimation algorithm to compute the proposed estimator, and demonstrated that it is Fisher-consistent with an estimating function that is continuous and bounded in the errors. We also established the asymptotic normality of the robust M-estimator under the core model, and derived a closed-form estimate for the corresponding asymptotic covariance matrix to facilitate statistical inference. Simulation studies demonstrate that the robust M-estimator has comparable performance to the ML estimator when the data has no contamination, while it greatly outperforms the ML estimator in terms of parameter estimation and empirical coverage probability under different and potentially extreme contamination of the error terms. Furthermore, the empirical level and power of the test of the spatial dependence parameter based on the proposed estimator outperformed that of the ML estimator when two neighboring error terms were contaminated. In comparison to other estimators of the SAR model, our proposed robust M-estimator clearly outperformed other non-robust estimators, and exhibited comparable performance to some existing robust estimators while having the added benefit of allowing for irregularly located locations and covariates in the data and higher efficiency in certain instances. Fitting the SAR model to the U.S. federal grants data using our robust M-estimator revealed strong positive spatial dependence in the grant values among counties, and also identified extreme responses in the data through the differences in the results obtained using robust M-estimation versus ML estimation.

Other than the spatial economics field, the proposed robust M-estimation method can also be applied to other fields, e.g. studying the influence among firms with a network (see Zou et al., 2021), along with generalizations to the case of the spatial Durbin model (LeSage and Pace, 2009) which contains both spatial lags of the response variable and the covariates. It would also be useful to extend our estimation method to handle spatial panel data, for example, by considering a similar robust estimator for spatial lag fixed effects panel model which could be reduced to an equivalent form as the SAR model via a demeaning transformation. Another possible extension of our robust M-estimator is to account for contamination in the covariates in addition to the error terms. Genton and Lucas (2003) proposed a unifying definition of breakdown point in finite and asymptotic samples of independent or dependent observations based on the smallest fraction of outliers for which the set of possible badness values (e.g. bias) can take on only a finite number of values. This definition was then applied by Genton (2003) to study the asymptotic breakdown point of the LMS estimator for the first-order unidimensional SAR model without covariates, and he showed that the LMS estimator for ρ has only 14.4% breakdown point under the first-order unidimensional SAR model compared to the 50% breakdown point of LMS estimator for the regression coefficient in the classical linear regression model with independent observations (see Rousseeuw, 1984). It would be interesting to extend the ideas of Genton (2003) for the first-order unidimensional SAR model with a single model parameter ρ to the more general SAR model considered in this article, i.e., the SAR model with a general spatial weight matrix $\boldsymbol{W}$ and model parameters $(\boldsymbol{\beta}^\top, \sigma, \rho)^\top$. Detailed theoretical investigation of the asymptotic breakdown point for the robust M-estimator could help to further understand the large sample robustness properties of the proposed estimator and facilitate comparison of the breakdown point of the robust M-estimator that utilizes the Huber function under the SAR model with dependent observations to that of the Huber estimator (Huber, 1973) under the classical linear regression model with independent observations; such an investigation would warrant a separate substantial research project. Finally, other variants of the robust M-estimator can be considered by using alternative bounded functions such as the logistic function to bound the standardized residual. This will allow us to compare the statistical performances of different robust M-estimators, and thus provide guidance in choosing appropriate bounded functions.

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Supplementary material

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