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# Center-outward Rank- and Sign-based VARMA Portmanteau Tests: Chitturi, Hosking, and Li–McLeod revisited

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## ABSTRACT

The pseudo-Gaussian portmanteau tests of Chitturi, Hosking, and Li and McLeod for VARMA models are revisited from a Le Cam perspective, providing a precise and more rigorous description of the asymptotic behavior of the multivariate portmanteau test statistic, which depends on the dimension  $d$  of the observations, the number  $m$  of lags involved, and the length  $n$  of the observation period. Then, based on the concepts of center-outward ranks and signs recently developed (Hallin, del Barrio, Cuesta-Albertos, and Matrán, *Annals of Statistics* 49, 1139–1165, 2021), a class of multivariate rank- and sign-based portmanteau test statistics is proposed which, under the null hypothesis and under a broad family of innovation densities, can be approximated by an asymptotically chi-square variable. The asymptotic properties of these tests are derived; simulations demonstrate their advantages over their classical pseudo-Gaussian counterpart.

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## 1. Introduction

### 1.1. The Gaussian multivariate portmanteau test

The so-called *portmanteau test* certainly ranks among the most popular and most widely used testing procedures in time series analysis. It is simple, intuitive, apparently well understood, and naturally complements eye-inspection of residual correlograms.

In their univariate forms, portmanteau test statistics were first introduced by Box and Pierce (1970) as a sum of squared residual autocorrelations of orders one through  $m$  associated with the estimation (usually, least squares or Gaussian QMLE) of the model parameters. A later version by Ljung and Box (1978) is taking into account the fact that a residual autocorrelation of order  $k$ , when computed from a series of length  $n$ , is based on a sum of  $(n - k)$  terms only; while improving finite-sample performance, that modification has no asymptotic impact, though.

Chitturi (1974) for the VAR case, Hosking (1980), and Li and McLeod (1981) for the VARMA case extended the Ljung-Box-Pierce test to the multivariate context by replacing sums of squared residual autocorrelations with sums of normalized squared elements of (estimated) residual cross-covariance matrices; modified versions in the spirit of Ljung and Box (1978) also are proposed by these authors.

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A widespread opinion is that the null distribution of the multivariate portmanteau statistic, in the  $d$ -dimensional VARMA( $p, q$ ) case, is asymptotically chi-square with  $d^2(m - p - q)$  degrees of freedom under a “broad” class of innovation densities, where  $m$  is the number of lags in the test statistic. This is, for instance, the statement in Theorem 2 of Hosking (1980). In this statement, Hosking is not very precise about what is to be understood with “asymptotically chi-square.” Actually, if only the series length  $n$  goes to infinity, the claim is incorrect. In his Appendix, he recommends  $m = O(n^{1/2})$ ; but this clearly precludes an asymptotic distribution with finitely many degrees of freedom. Chitturi (1974), and Li and McLeod (1981) are more cautious, with a somewhat vague claim that this asymptotic null distribution is “approximately chi-square with  $d^2(m - p - q)$  degrees of freedom for  $m$  and  $n$  (the series length) large enough.”

As we shall see, these statements, at best, are imprecise and the asymptotic distribution of the portmanteau test statistic under the null is not chi-square with  $d^2(m - p - q)$  degrees of freedom—not even under Gaussian innovation densities. The chi-square critical values resulting from these statements, nevertheless, are routinely used in the daily implementation of the test.

In the univariate case, a similar problem was detected by Taniguchi and Amano (2010), who show that the classical univariate portmanteau test statistics of Box and Pierce (1970) and Ljung and Box (1978) are not asymptotically chi-square if the number  $m < \infty$  of lagged residuals in the test statistic is fixed. Based on this observation, they also propose a modified Whittle likelihood ratio test which is asymptotically chi-square. In a more general setting of single-output regression with ARMA errors, Akashi et al. (2018, 2021) propose a likelihood ratio-based portmanteau test incorporating the Whittle likelihood ratio test of Taniguchi and Amano (2010) as a special case and provide a sufficient condition, in terms of Fisher information, under which their test statistic is asymptotically chi-square.

Half a century after its introduction, thus, one of the most popular tests in time series analysis still relies on vague or even faulty asymptotic statements. The first objective of this paper is to fix these asymptotic results. Rather than modifying the classical test statistics (as Taniguchi and Amano (2010) and Akashi et al. (2018, 2021) are doing in the univariate case), we are providing a precise account and a rigorous derivation of the asymptotic behavior of the portmanteau test statistic for VARMA models. Taking advantage of this, we then propose, based on the measure-transportation-based concepts of center-outward ranks and signs recently developed in Hallin et al. (2021) (see Hallin (2022) for a nontechnical survey) a class of rank-based portmanteau tests and establish their asymptotic distributions under the null. Simulations demonstrate the advantages of these tests, in terms of power and resistance to outliers, over the classical Chitturi-Hosking-Li-McLeod pseudo-Gaussian ones under non-Gaussian innovation densities.

Throughout, the theoretical tools we are using are borrowed from Le Cam’s powerful asymptotic theory of statistical experiment—a theory that was not available, fifty years ago, to Chitturi, Hosking, Li, and McLeod, who instead are making intensive use of Taylor expansions.

## 1.2. Center-outward rank-based multivariate portmanteau test

Despite their popularity, the Gaussian portmanteau tests have some undesirable drawbacks. First, their validity and consistency rely on the assumption of finite fourth-order innovation moments, an assumption that may not hold in economic and financial practice, where data are usually heavy-tailed. Second, the finite-sample performance of pseudo-Gaussian tests often is quite poor under non-Gaussian innovations (see Section 1.2 in Hallin et al. (2022b) for numerical examples). Last but not least, the pointwise chi-square approximation of the null distribution of the pseudo-Gaussian portmanteau statistic under given innovation density  $f$  is far from uniform with respect to  $f$ . As a consequence, the  $\sup_f$  of the size under innovation density  $f$  of pseudo-Gaussian tests, in general, does not converge to the nominal asymptotic level  $\alpha$  (see, e.g., Section 1.1 in Hallin et al. (2022a)): as a semiparametric test, thus, the pseudo-Gaussian portmanteau test fails to satisfy the asymptotic probability level condition.

Thanks to distribution-freeness, classical (univariate) rank-based tests are escaping that asymptotic size problem: for the ARMA case, see, e.g., Hallin and Puri (1988, 1994), Hallin and Jurečková, (1999). In a multivariate or multiple-output context, however, due to the fact that no canonical ordering is available in  $\mathbb{R}^d$  for  $d \geq 2$ , a long-standing problem has been: “what are ranks and signs in dimension  $d \geq 2$ ?” Various notions of ranks and signs have been proposed in the literature, including the componentwise ranks (Puri and Sen, 1971), the spatial ranks (Oja, 2010), the depth-based ranks (Liu, 1992), and the Mahalanobis ranks and signs (Hallin and Paindaveine, 2004). None of these ranks or signs are distribution-free under the whole family of absolutely continuous distributions, though. The *center-outward ranks and signs* recently proposed by Chernozhukov et al. (2017) and Hallin et al. (2021) are not only distribution-free under absolutely continuous distributions; they also are maximal ancillary (see Hallin et al. (2021)). They have been applied quite successfully to various statistical models of daily statistical importance, including multiple-output regression and MANOVA (Hallin et al., 2022a), goodness-of-fit (Ghosal and Sen, 2022), tests of vector independence (Deb and Sen, 2021; Shi et al., 2022a; 2022b; 2023), multiple-output quantile regression (del Barrio et al., 2022), R-estimation for VARMA models (Hallin et al., 2022b), and rank-based order selection of VAR models (Hallin et al., 2023).

In this paper, we propose a class of asymptotically distribution-free portmanteau tests for VARMA models based on residual center-outward ranks and signs. These residuals are the estimated residuals based on an estimation  $\hat{\theta}^{(n)}$  of the parameter  $\theta$  of the null VARMA model: the resulting ranks and signs, thus, are *aligned* ones, failing to achieve exact finite-sample distribution-freeness but enjoying asymptotic distribution-freeness. For sufficiently large  $m$ , however, we show that

the test statistic is arbitrarily close to an asymptotically chi-square ( $d^2(m-p-q)$  degrees of freedom) oracle statistic based on the ranks and signs of the *exact residuals*—that is, based on the actual  $\theta$  value, as opposed to the *aligned* ones; the ranks and signs of these exact residuals are fully distribution-free but, of course, cannot be computed from the observations.

While the pseudo-Gaussian portmanteau test requires plugging-in the Gaussian quasi-likelihood VARMA estimator  $\hat{\theta}^{(n)}$  for the unspecified VARMA parameter  $\theta$ , our rank-based tests rely on plugging-in the center-outward R-estimators  $\tilde{\theta}^{(n)}$ —as derived in Hallin et al. (2022b)—based on the same score function as the test statistic itself. A numerical study reveals that, when based on Gaussian scores, our rank-based test outperforms the classical pseudo-Gaussian one under a broad range of innovation distributions. This is probably the sign of a general Chernoff-Savage property (Chernoff and Savage, 1958) for which, however, we have no proof in this context.

The paper is organized as follows. Section 2 introduces the VARMA model and the local asymptotic normality of the model under some mild regularity assumptions. Section 3 revisits the Gaussian multivariate portmanteau tests of Chitturi (1974), Hosking (1980) and Li and McLeod (1981) and re-establishes their asymptotic properties under precise form via Le Cam's asymptotic theory. Section 4 proposes a class of center-outward rank-based portmanteau test statistics and similarly establishes their asymptotic properties. A brief numerical analysis of the finite-sample size and power of these tests is conducted in Section 5. Section 6 concludes. All proofs are collected in the Appendix.

## 2. Notation and general setting

In this section, we introduce the VARMA( $p, q$ ) model (Section 2.1) and the local asymptotic normality property (Section 2.2) which is the essential tool in our analysis of the asymptotic behavior of the Gaussian and center-outward rank-based multivariate portmanteau test statistics.

### 2.1. VARMA models

Consider the  $d$ -dimensional VARMA( $p, q$ ) model

$$\left(\mathbf{I}_d - \sum_{i=1}^p \mathbf{A}_i L^i\right) \mathbf{X}_t = \left(\mathbf{I}_d + \sum_{j=1}^q \mathbf{B}_j L^j\right) \epsilon_t, \quad t \in \mathbb{Z}, \quad (2.1)$$

where  $\mathbf{A}_1, \dots, \mathbf{A}_p, \mathbf{B}_1, \dots, \mathbf{B}_q$  are  $d \times d$  matrices,  $L$  denotes the lag operator, and  $\{\epsilon_t; t \in \mathbb{Z}\}$  is some i.i.d. mean-zero  $d$ -dimensional white noise process with density  $f$  with respect to the Lebesgue measure  $\mu$  on  $\mathbb{R}^d$ . Denoting by

$$\theta := \left( (\text{vec}(\mathbf{A}_1))', \dots, (\text{vec}(\mathbf{A}_p))', (\text{vec}(\mathbf{B}_1))', \dots, (\text{vec}(\mathbf{B}_q))' \right)',$$

(where  $'$  indicates transposition) the  $(p+q)d^2$ -dimensional VARMA parameter, we throughout assume that  $\theta$  satisfies the following very classical conditions ensuring identifiability and the existence of a stationary and invertible solution to (2.1).

#### Assumption 1.

(i) All solutions of the determinantal equations

$$\det\left(\mathbf{I}_d - \sum_{i=1}^p \mathbf{A}_i z^i\right) = 0 \quad \text{and} \quad \det\left(\mathbf{I}_d + \sum_{j=1}^q \mathbf{B}_j z^j\right) = 0, \quad z \in \mathbb{C}$$

lie outside the unit ball in  $\mathbb{C}$ ;

(ii)  $\det(\mathbf{A}_p) \neq 0 \neq \det(\mathbf{B}_q)$ ;

(iii)  $\mathbf{I}_d - \sum_{i=1}^p \mathbf{A}_i z^i$  and  $\mathbf{I}_d + \sum_{j=1}^q \mathbf{B}_j z^j$  have no common left factors other than  $\mathbf{I}_d$ .

Denote by  $\Theta_{p,q}$  the VARMA( $p, q$ ) parameter space—namely, the set of all  $\theta$  values satisfying Assumption 1:  $\Theta_{p,q}$  is a finite collection of open connected subsets of  $\mathbb{R}^{(p+q)d^2}$ .

Let  $\mathbf{X}^{(n)} := \{\mathbf{X}_1^{(n)}, \dots, \mathbf{X}_n^{(n)}\}$  (superscript  $(n)$  omitted whenever possible) be an observed finite realization of some solution of (2.1). For any  $\theta \in \Theta_{p,q}$ , this observation  $\mathbf{X}^{(n)}$  is asymptotically (as  $n \rightarrow \infty$ ) stationary and invertible; associated with any  $(p+q)$ -tuple  $(\mathbf{X}_0^{(n)}, \dots, \mathbf{X}_{-p+1}^{(n)}, \epsilon_0, \dots, \epsilon_{-q+1})$  of initial values, it determines, for any  $\theta$ , an  $n$ -tuple of residuals

$$\mathbf{Z}_t^{(n)}(\theta) := \left(\mathbf{I}_d - \sum_{i=1}^p \mathbf{A}_i L^i\right) \mathbf{X}_t^{(n)} - \sum_{j=1}^q \mathbf{B}_j L^j \epsilon_t, \quad t = 1, \dots, n,$$

which can be computed recursively. These residuals are i.i.d. with density  $f$  and  $\mathbf{Z}_t^{(n)}(\theta)$  coincides with  $\epsilon_t$  for any  $t$  iff  $\mathbf{X}^{(n)}$  is a solution of (2.1) with parameter value  $\theta$ .

2.2. Local asymptotic normality (LAN)

Our asymptotic analysis relies on the local asymptotic normality (LAN) property of VARMA models. That property requires mild regularity conditions on the density  $f$  of  $\epsilon_t$ .

**Assumption 2.**

- (i) The innovation density  $f$  belongs to the class of *non-vanishing* Lebesgue densities on  $\mathbb{R}^d$ , i.e.,  $f(\mathbf{x}) > 0$  for all  $\mathbf{x} \in \mathbb{R}^d$ ;
- (ii)  $\int \mathbf{x}f(\mathbf{x})d\mu = \mathbf{0}$  and  $\int \mathbf{x}\mathbf{x}'f(\mathbf{x})d\mu = \Xi$  where  $\Xi$  is finite and positive definite;
- (iii)  $f^{1/2}$  is *mean-square differentiable* with *mean-square gradient*  $\mathbf{D}f^{1/2}$ , that is, there exists a square-integrable vector  $\mathbf{D}f^{1/2}$  such that, for any sequence  $\mathbf{h} \in \mathbb{R}^d$  such that  $\mathbf{0} \neq \mathbf{h} \rightarrow \mathbf{0}$ ,

$$(\mathbf{h}'\mathbf{h})^{-1} \int [f^{1/2}(\mathbf{x} + \mathbf{h}) - f^{1/2}(\mathbf{x}) - \mathbf{h}'\mathbf{D}f^{1/2}(\mathbf{x})]^2 d\mu \rightarrow 0;$$

- (iv) letting  $\varphi_f(\mathbf{x}) := (\varphi_{f_1}(\mathbf{x}), \dots, \varphi_{f_d}(\mathbf{x}))' := -2\mathbf{D}f^{1/2}(\mathbf{x})/f^{1/2}(\mathbf{x})$  (the location score function),  $\int [\varphi_i(\mathbf{x})]^4 f(\mathbf{x})d\mu < \infty$ ,  $i = 1, \dots, d$ ;
- (v) the function  $\mathbf{x} \mapsto \varphi_f(\mathbf{x})$  is piecewise Lipschitz, i.e., there exists  $K \in \mathbb{R}$  and a finite measurable partition of  $\mathbb{R}^d$  into  $J$  non-overlapping subsets  $I_1, \dots, I_J$  such that, for all  $\mathbf{x}, \mathbf{y}$  in  $I_j$ ,  $j = 1, \dots, J$ ,

$$\|\varphi_f(\mathbf{x}) - \varphi_f(\mathbf{y})\| \leq K\|\mathbf{x} - \mathbf{y}\|.$$

Conditions (i) and (iii) are standard in the context of LAN; (ii) and (iv) guarantee the existence of a full-rank information matrix for  $\theta$ . Under (v), the impact of initial values is asymptotically negligible in mean square norm; without loss of generality, thus, we henceforth assume that

$$\mathbf{X}_0^{(n)} = \dots = \mathbf{X}_{-p+1}^{(n)} = \mathbf{0} = \epsilon_0 = \dots = \epsilon_{-q+1}.$$

Denote by  $\mathcal{F}_d$  the class of densities  $f$  satisfying **Assumption 2**.

Letting  $\mathbb{P}_{\theta, f}^{(n)}$  denote the distribution of  $\mathbf{X}^{(n)}$  under parameter value  $\theta$  and innovation density  $f$ , write

$$L_{\theta+n^{-1/2}\tau^{(n)}/\theta, f}^{(n)} := \log \frac{d\mathbb{P}_{\theta+n^{-1/2}\tau^{(n)}/\theta, f}^{(n)}}{d\mathbb{P}_{\theta, f}^{(n)}},$$

where  $\tau^{(n)}$  is a bounded sequence of  $\mathbb{R}^{(p+q)d^2}$ , for the log-likelihood ratio of  $\mathbb{P}_{\theta+n^{-1/2}\tau^{(n)}/\theta, f}^{(n)}$  with respect to  $\mathbb{P}_{\theta, f}^{(n)}$  computed at  $\mathbf{X}^{(n)}$ . Define

$$\Gamma_f^{(n)}(\theta) := n^{-1/2}((n-1)^{1/2}(\text{vec}(\Gamma_{1, f}^{(n)}(\theta)))', \dots, (n-i)^{1/2}(\text{vec}(\Gamma_{i, f}^{(n)}(\theta)))', \dots, (\text{vec}(\Gamma_{n-1, f}^{(n)}(\theta)))'), \quad (2.2)$$

with the so-called *f-cross-covariance matrices*

$$\Gamma_{i, f}^{(n)}(\theta) := (n-i)^{-1} \sum_{t=i+1}^n \varphi_f(\mathbf{Z}_t^{(n)}(\theta))\mathbf{Z}_{t-i}^{(n)'}(\theta) \quad i = 1, \dots, n-1. \quad (2.3)$$

Let  $\mathbf{C}_\theta^{(n)} := (\mathbf{c}_{1, \theta}, \dots, \mathbf{c}_{n-1, \theta})$  with

$$\mathbf{c}_{i, \theta} := \begin{pmatrix} \sum_{j=0}^{i-1} \sum_{k=0}^{\min(q, i-j-1)} (\mathbf{G}_{i-j-k-1} \mathbf{B}_k) \otimes \mathbf{H}_j' \\ \vdots \\ \sum_{j=0}^{i-p} \sum_{k=0}^{\min(q, i-j-p)} (\mathbf{G}_{i-j-k-p} \mathbf{B}_k) \otimes \mathbf{H}_j' \\ \mathbf{I}_d \otimes \mathbf{H}_{i-1}' \\ \vdots \\ \mathbf{I}_d \otimes \mathbf{H}_{i-q}' \end{pmatrix}, \quad i = 1, \dots, n-1, \quad (2.4)$$

where  $\mathbf{G}_u$  and  $\mathbf{H}_u$ ,  $u \in \mathbb{Z}$  are the *Green matrices* associated with the autoregressive and moving average operators, respectively, in (2.1)—namely, the matrix coefficients of the inverted linear difference operators  $(\mathbf{A}(L))^{-1}$  and  $(\mathbf{B}(L))^{-1}$ :

$$\sum_{u=0}^{\infty} \mathbf{G}_u z^u = \left( \mathbf{I}_d - \sum_{i=1}^p \mathbf{A}_i z^i \right)^{-1} \quad \text{and} \quad \sum_{u=0}^{\infty} \mathbf{H}_u z^u = \left( \mathbf{I}_d + \sum_{j=1}^q \mathbf{B}_j z^j \right)^{-1}, \quad z \in \mathbb{C}, |z| < 1.$$

More constructive (recursive) definitions of Green's matrices can be found in [Section 1.1](#) of [Hallin \(1986\)](#) and [Section 3](#) of [Garel and Hallin \(1995\)](#) but are not needed here, and we skip them for the sake of space: the only thing we need to recall is the fact that, under [Assumption 1](#),  $\|\mathbf{G}_u\|$  and  $\|\mathbf{H}_u\|$ , just as the moduli of all solutions of the homogeneous difference equations associated with  $\mathbf{A}(L)$  and  $\mathbf{B}(L)$ , are exponentially decreasing as  $u \rightarrow \infty$ , which implies that  $\|\mathbf{c}_{i,\theta}\|$  exponentially decreases as  $i \rightarrow \infty$ . Defining

$$\Delta_f^{(n)}(\boldsymbol{\theta}) := \sum_{i=1}^{n-1} \mathbf{c}_{i,\theta} (n-i)^{1/2} \text{vec}(\Gamma_{i,f}^{(n)}(\boldsymbol{\theta})) = n^{1/2} \mathbf{C}_\theta^{(n)} \Gamma_f^{(n)}(\boldsymbol{\theta}), \quad (2.5)$$

(the central sequences), the following LAN result is established in [Garel and Hallin \(1995\)](#), [Proposition 3.1](#).

**Proposition 2.1** (Garel and Hallin (1995)). *Let [Assumptions 1](#) and [2](#) hold. Then, for any bounded sequence  $\tau^{(n)}$  in  $\mathbb{R}^{(p+q)d^2}$ , under  $\mathbb{P}_{\theta,f}^{(n)}$ , as  $n \rightarrow \infty$ ,*

$$L_{\theta+n^{-1/2}\tau^{(n)}/\theta,f}^{(n)} = \tau^{(n)'} \Delta_f^{(n)}(\boldsymbol{\theta}) - \frac{1}{2} \tau^{(n)'} \Lambda_f(\boldsymbol{\theta}) \tau^{(n)} + o_p(1), \quad (2.6)$$

with  $\Delta_f^{(n)}(\boldsymbol{\theta})$  (the central sequence) defined in (2.5) and  $(p+q)d^2 \times (p+q)d^2$  symmetric and positive definite  $\Lambda_f(\boldsymbol{\theta})$  (the information matrix—see equation (3.16) in [Garel and Hallin \(1995\)](#) for an explicit form). Still under  $\mathbb{P}_{\theta,f}^{(n)}$ ,  $\Delta_f^{(n)}(\boldsymbol{\theta})$  is asymptotically normal with mean  $\mathbf{0}$  and covariance  $\Lambda_f(\boldsymbol{\theta})$ .

### 3. The Chitturi-Hosking-Li-McLeod multivariate portmanteau test

In this section, we are revisiting the pseudo-Gaussian multivariate portmanteau test of Chitturi-Hosking-Li-McLeod and, adopting a Le Cam approach, clarify their asymptotic results.

#### 3.1. Residual cross-covariance matrices

For a Gaussian density  $f$ , the score function is  $\varphi_f(\mathbf{z}) = -\boldsymbol{\Sigma}^{-1}\mathbf{z}$ , where  $\boldsymbol{\Sigma}$  is the covariance matrix of  $\epsilon_t$ , and the  $f$ -cross-covariance matrices (2.3) reduce to

$$\Gamma_{i,\mathcal{N}}^{(n)}(\boldsymbol{\theta}) := -(n-i)^{-1} \boldsymbol{\Sigma}^{-1} \sum_{t=i+1}^n \mathbf{Z}_t(\boldsymbol{\theta}) \mathbf{Z}'_{t-i}(\boldsymbol{\theta}), \quad i = 1, \dots, n-1. \quad (3.1)$$

Note that  $\Gamma_{i,\mathcal{N}}^{(n)}(\boldsymbol{\theta})$  differs from the traditional lag- $i$  cross-covariance matrix  $(n-i)^{-1} \sum_{t=i+1}^n \mathbf{Z}_t(\boldsymbol{\theta}) \mathbf{Z}'_{t-i}(\boldsymbol{\theta})$  by a left factor  $-\boldsymbol{\Sigma}$ .

[Proposition 3.1](#) provides the asymptotic distribution of  $\Gamma_{i,\mathcal{N}}^{(n)}(\boldsymbol{\theta})$  under  $\mathbb{P}_{\theta,f}^{(n)}$  and contiguous sequences of alternatives  $\mathbb{P}_{\theta+n^{-1/2}\tau,f}^{(n)}$ . See the Appendix for a proof. In practice, of course,  $\boldsymbol{\Sigma}$  remains unspecified and has to be estimated by some  $\boldsymbol{\Sigma}^{(n)}$ —typically, the residual empirical covariance matrix. As long as  $\boldsymbol{\Sigma}^{(n)}$  is consistent, substituting it for  $\boldsymbol{\Sigma}$  in (3.1), in view of Slutsky's Lemma, has no impact on asymptotic distributions. Rather than introducing cumbersome additional notation, thus, we pursue with the  $\boldsymbol{\Sigma}$ -based definition of  $\Gamma_{i,\mathcal{N}}^{(n)}(\boldsymbol{\theta})$ . As for its rank-based counterparts defined in [Section 4.3](#), they do not involve any  $\boldsymbol{\Sigma}$  nor any of the parameters of the actual innovation distribution.

**Proposition 3.1.** *Let [Assumptions 1](#) and [2](#) hold. Then, for any positive integers  $i \neq j$ , the vectors*

$$(n-i)^{1/2} \text{vec}(\Gamma_{i,\mathcal{N}}^{(n)}(\boldsymbol{\theta})) \quad \text{and} \quad (n-j)^{1/2} \text{vec}(\Gamma_{j,\mathcal{N}}^{(n)}(\boldsymbol{\theta}))$$

are jointly asymptotically normal, with mean  $(\mathbf{0}', \mathbf{0}')'$  under  $\mathbb{P}_{\theta,f}^{(n)}$ , mean

$$\left( (\boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma}^{-1}) \mathbf{c}'_{i,\theta} \tau \right)', \left( (\boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma}^{-1}) \mathbf{c}'_{j,\theta} \tau \right)' \quad (3.2)$$

under  $\mathbb{P}_{\theta+n^{-1/2}\tau,f}^{(n)}$  and covariance  $\begin{pmatrix} \boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma}^{-1} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma}^{-1} \end{pmatrix}$  under both.

To determine the asymptotic behavior of the Gaussian portmanteau test, the asymptotic linearity of  $(n-i)^{1/2} \text{vec}(\Gamma_{i,\mathcal{N}}^{(n)}(\boldsymbol{\theta}))$  is required. That result is established in Lemma 4 of [Hallin and Paindaveine \(2005\)](#) under the quite restrictive assumption that the distribution of  $\epsilon_t$  is elliptically symmetric. Here, we make the assumption that asymptotic linearity holds for general  $f$  satisfying [Assumption 2](#); the form of the linear term on the right-hand side of (3.3) below follows from the form of the shift matrices in (3.2)—itself a consequence of Le Cam's third Lemma.

**Assumption 3.** For any positive integer  $i$ ,

$$(n-i)^{1/2} [\text{vec}(\Gamma_{i,\mathcal{N}}^{(n)}(\boldsymbol{\theta} + n^{-1/2}\boldsymbol{\tau})) - \text{vec}(\Gamma_{i,\mathcal{N}}^{(n)}(\boldsymbol{\theta}))] = (\boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma}^{-1})\mathbf{c}'_{i,\boldsymbol{\theta}}\boldsymbol{\tau} + o_p(1) \quad (3.3)$$

under  $P_{\boldsymbol{\theta},f}^{(n)}$ , as  $n \rightarrow \infty$ .

We refer to Section 4 of van den Akker et al. (2015) for primitive conditions.

The Gaussian portmanteau test is based on the plug-in of  $\hat{\boldsymbol{\theta}}_{\mathcal{N}}^{(n)}$ , the Gaussian quasi-maximum likelihood estimator (QMLE) of  $\boldsymbol{\theta}$ , in the test statistic to be defined. Recall that  $\hat{\boldsymbol{\theta}}_{\mathcal{N}}^{(n)}$  is defined as the solution of  $\Delta_{\mathcal{N}}^{(n)}(\boldsymbol{\theta}) = \mathbf{0}$  where

$$\Delta_{\mathcal{N}}^{(n)}(\boldsymbol{\theta}) := \sum_{i=1}^{n-1} \mathbf{c}_{i,\boldsymbol{\theta}}(n-i)^{1/2} \text{vec}(\Gamma_{i,\mathcal{N}}^{(n)}(\boldsymbol{\theta}))$$

is the Gaussian central sequence. A finite fourth-order moment assumption on  $\epsilon_t$  is required in order for  $\hat{\boldsymbol{\theta}}_{\mathcal{N}}^{(n)}$  to be root- $n$ -consistent; see, e.g., Mélard (2022) for a recent proof.

**Assumption 4.**  $E\left[\text{vec}(\epsilon_t \epsilon_t') (\text{vec}(\epsilon_t \epsilon_t'))'\right] < \infty$ .

In view of (3.3), the plug-in impact of substituting  $\hat{\boldsymbol{\theta}}_{\mathcal{N}}^{(n)}$  for the actual  $\boldsymbol{\theta}$  in a quadratic form of  $(n-i)^{1/2} \text{vec}(\Gamma_{i,\mathcal{N}}^{(n)}(\boldsymbol{\theta}))$ ,  $i = 1, \dots, m$  is asymptotically non-negligible. Taking into account the fact that  $\Delta_{\mathcal{N}}^{(n)}(\hat{\boldsymbol{\theta}}_{\mathcal{N}}^{(n)}) = \mathbf{0}$ , however, that impact can be neutralized in view of Lemma 3.2 below.

Denote by  $\Gamma_{i,\mathcal{N}}^{(n)*}(\boldsymbol{\theta})$  the matrix of residuals in the regression of  $\Gamma_{i,\mathcal{N}}^{(n)}(\boldsymbol{\theta})$ 's entries with respect to  $\Delta_{\mathcal{N}}^{(n)}(\boldsymbol{\theta})$  in their joint asymptotic (under  $P_{\boldsymbol{\theta},f}^{(n)}$ ) distribution; namely, let

$$(n-i)^{1/2} \text{vec}(\Gamma_{i,\mathcal{N}}^{(n)*}(\boldsymbol{\theta})) := (n-i)^{1/2} \text{vec}(\Gamma_{i,\mathcal{N}}^{(n)}(\boldsymbol{\theta})) - (\boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma}^{-1})\mathbf{c}'_{i,\boldsymbol{\theta}} \left( \sum_{i=1}^{n-1} \mathbf{c}_{i,\boldsymbol{\theta}} (\boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma}^{-1})\mathbf{c}'_{i,\boldsymbol{\theta}} \right)^{-1} \Delta_{\mathcal{N}}^{(n)}(\boldsymbol{\theta}) \quad (3.4)$$

$$= (n-i)^{1/2} \text{vec}(\Gamma_{i,\mathcal{N}}^{(n)}(\boldsymbol{\theta})) - (\boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma}^{-1})\mathbf{c}'_{i,\boldsymbol{\theta}} \left( \sum_{i=1}^{n-1} \mathbf{c}_{i,\boldsymbol{\theta}} (\boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma}^{-1})\mathbf{c}'_{i,\boldsymbol{\theta}} \right)^{-1} \times \left( \sum_{i=1}^{n-1} \mathbf{c}_{i,\boldsymbol{\theta}} (n-i)^{1/2} \text{vec}(\Gamma_{i,\mathcal{N}}^{(n)}(\boldsymbol{\theta})) \right). \quad (3.5)$$

Lemma 3.2 below shows that  $\Gamma_{i,\mathcal{N}}^{(n)}(\hat{\boldsymbol{\theta}}_{\mathcal{N}}^{(n)})$  and  $\Gamma_{i,\mathcal{N}}^{(n)*}(\boldsymbol{\theta})$  under  $P_{\boldsymbol{\theta},f}^{(n)}$  only differ by a  $o_p(n^{-1/2})$  quantity: under  $P_{\boldsymbol{\theta},f}^{(n)}$  and contiguous alternatives, thus, using  $\Gamma_{i,\mathcal{N}}^{(n)}(\hat{\boldsymbol{\theta}}_{\mathcal{N}}^{(n)})$  in the portmanteau test statistic is asymptotically equivalent to using  $\Gamma_{i,\mathcal{N}}^{(n)*}(\boldsymbol{\theta})$ .

The advantage of the latter (an oracle statistic) is that it no longer involves  $\hat{\boldsymbol{\theta}}_{\mathcal{N}}^{(n)}$ , which simplifies the derivation of asymptotic results. See the Appendix for a proof.

**Lemma 3.2.** Let Assumptions 1, 2, 3, and 4 hold. Then

$$(n-i)^{1/2} \text{vec}(\Gamma_{i,\mathcal{N}}^{(n)}(\hat{\boldsymbol{\theta}}_{\mathcal{N}}^{(n)}) - \Gamma_{i,\mathcal{N}}^{(n)*}(\boldsymbol{\theta})) = o_p(1), \quad (3.6)$$

for any fixed  $i \geq 1$ , under any  $P_{\boldsymbol{\theta},f}^{(n)}$  and any contiguous  $P_{\boldsymbol{\theta}+n^{-1/2}\boldsymbol{\tau},f}^{(n)}$  as  $n \rightarrow \infty$ .

Note, however, that the asymptotic equivalence (3.6) results from the fact that the estimator  $\hat{\boldsymbol{\theta}}_{\mathcal{N}}^{(n)}$  and the cross-covariances  $\Gamma_{i,\mathcal{N}}^{(n)}$  both are associated with the Gaussian distribution (although  $f$  needs not to be Gaussian): the same result would not hold with non-Gaussian quasi-maximum likelihood estimators (for instance, the estimator  $\hat{\boldsymbol{\theta}}_f^{(n)}$  obtained from the actual likelihood equations  $\Delta_f^{(n)}(\boldsymbol{\theta}) = \mathbf{0}$ , with  $f$  non-Gaussian. Nor does it hold, e.g., with robust or R-estimators (see Hallin et al. (2022b)) of  $\boldsymbol{\theta}$  plugged into  $\Gamma_{i,\mathcal{N}}^{(n)}$ .

### 3.2. The (pseudo-)Gaussian test

The statistic of the Gaussian portmanteau test of Hosking (1980) (to be performed as a pseudo-Gaussian test) takes the form

$$Q_{m,\mathcal{N}}^{(n)}(\hat{\boldsymbol{\theta}}_{\mathcal{N}}^{(n)}) := \sum_{i=1}^m (n-i) \text{vec}(\Gamma_{i,\mathcal{N}}^{(n)}(\hat{\boldsymbol{\theta}}_{\mathcal{N}}^{(n)}))' (\widehat{\boldsymbol{\Sigma}} \otimes \widehat{\boldsymbol{\Sigma}})^{-1} \text{vec}(\Gamma_{i,\mathcal{N}}^{(n)}(\hat{\boldsymbol{\theta}}_{\mathcal{N}}^{(n)})) = n \Gamma_{\mathcal{N}}^{(m,n)'}(\hat{\boldsymbol{\theta}}_{\mathcal{N}}^{(n)}) (\mathbf{I}_m \otimes \widehat{\boldsymbol{\Sigma}} \otimes \widehat{\boldsymbol{\Sigma}})^{-1} \Gamma_{\mathcal{N}}^{(m,n)}(\hat{\boldsymbol{\theta}}_{\mathcal{N}}^{(n)}),$$

where

$$\Gamma_{\mathcal{N}}^{(m,n)}(\boldsymbol{\theta}) := n^{-1/2} \left( (n-1)^{1/2} \left( \text{vec}(\Gamma_{1,\mathcal{N}}^{(n)}(\boldsymbol{\theta})) \right)', \dots, (n-m)^{1/2} \left( \text{vec}(\Gamma_{m,\mathcal{N}}^{(n)}(\boldsymbol{\theta})) \right)' \right)'$$

$$\text{and } \widehat{\boldsymbol{\Sigma}} := n^{-1} \sum_{t=1}^n \mathbf{Z}_t(\widehat{\boldsymbol{\theta}}_{\mathcal{N}}^{(n)}) \mathbf{Z}_t'(\widehat{\boldsymbol{\theta}}_{\mathcal{N}}^{(n)}).$$

The asymptotic behavior of  $Q_{m,\mathcal{N}}^{(n)}(\widehat{\boldsymbol{\theta}}_{\mathcal{N}}^{(n)})$  is obtained by approximating it (for sufficiently large  $m$ ) by a statistic that is asymptotically chi-square with  $d^2(m-p-q)$  degrees of freedom as  $n \rightarrow \infty$ . Specifically, it follows from [Lemma 3.2](#) and the exponential decrease (as  $i \rightarrow \infty$ ) of the  $\|\mathbf{c}_{i,\theta}\|$ 's that, for  $m$  large enough,  $(n-i)^{1/2} \text{vec}(\Gamma_{i,\mathcal{N}}^{(n)}(\widehat{\boldsymbol{\theta}}_{\mathcal{N}}^{(n)}))$  is arbitrarily close to

$$\begin{aligned} & (n-i)^{1/2} \text{vec}(\Gamma_{i,\mathcal{N}}^{(n)**}(\boldsymbol{\theta})) \\ & := (n-i)^{1/2} \text{vec}(\Gamma_{i,\mathcal{N}}^{(n)}(\boldsymbol{\theta})) - (\boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma}^{-1}) \mathbf{c}'_{i,\theta} \left( \sum_{i=1}^m \mathbf{c}_{i,\theta} (\boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma}^{-1}) \mathbf{c}'_{i,\theta} \right)^{-1} \left( \sum_{i=1}^m \mathbf{c}_{i,\theta} (n-i)^{1/2} \text{vec}(\Gamma_{i,\mathcal{N}}^{(n)}(\boldsymbol{\theta})) \right), \end{aligned}$$

a quantity that results from truncating at  $m$  terms the sums in the right-hand side of [\(3.5\)](#).

Let

$$Q_{m,\mathcal{N}}^{(n)**}(\boldsymbol{\theta}) := \sum_{i=1}^m (n-i) \left( \text{vec}(\Gamma_{i,\mathcal{N}}^{(n)**}(\boldsymbol{\theta})) \right)' (\boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma})^{-1} \text{vec}(\Gamma_{i,\mathcal{N}}^{(n)**}(\boldsymbol{\theta})).$$

The following result states that  $Q_{m,\mathcal{N}}^{(n)}(\widehat{\boldsymbol{\theta}}_{\mathcal{N}}^{(n)})$  can be approximated by  $Q_{m,\mathcal{N}}^{(n)**}(\boldsymbol{\theta})$ , where  $Q_{m,\mathcal{N}}^{(n)**}(\boldsymbol{\theta})$  is asymptotically chi-square under  $P_{\boldsymbol{\theta},f}^{(n)}$  as  $n \rightarrow \infty$ .

**Proposition 3.3.** *Let [Assumptions 1, 2, 3, and 4](#) hold. Then, under  $P_{\boldsymbol{\theta},f}^{(n)}$  and contiguous alternatives,*

(i) *for all  $\delta, \varepsilon > 0$ , there exist integers  $M_{\delta,\varepsilon}$ , and  $N_{\delta,\varepsilon}$  such that*

$$P\left(|Q_{m,\mathcal{N}}^{(n)}(\widehat{\boldsymbol{\theta}}_{\mathcal{N}}^{(n)}) - Q_{m,\mathcal{N}}^{(n)**}(\boldsymbol{\theta})| < \delta\right) > 1 - \varepsilon$$

*for all  $n \geq N_{\delta,\varepsilon}$  and fixed  $m \geq M_{\delta,\varepsilon}$ ;*

(ii)  *$Q_{m,\mathcal{N}}^{(n)**}(\boldsymbol{\theta})$  is asymptotically chi-square with  $d^2(m-p-q)$  degrees of freedom as  $n \rightarrow \infty$ .*

See the Appendix for a proof.

## 4. A portmanteau test based on multivariate center-outward ranks and signs

### 4.1. Pseudo-Gaussian versus rank-based tests

Pseudo-Gaussian tests are generally considered asymptotically valid over some class  $\mathcal{P}$  of distributions (containing the Gaussian). And the critical value obtained from the common asymptotic distribution of a pseudo-Gaussian test statistic indeed yields, for each  $P \in \mathcal{P}$ , a test with correct asymptotic size  $\alpha$ , say. As mentioned in the introduction, the convergence to  $\alpha$  of the finite-sample sizes of the resulting tests, however, in general is highly non-uniform: while converging to  $\alpha$  pointwise in  $P \in \mathcal{P}$ , the  $\sup_{P \in \mathcal{P}}$  of these sizes typically fails to do so. If  $P$  remains unspecified, which is the case when pseudo-Gaussian tests are performed, the  $\alpha$ -level constraint should apply to the  $\sup_{P \in \mathcal{P}}$  of the size under  $P$  which, typically, does not converge to  $\alpha$ . This is to be kept in mind when performing pseudo-Gaussian tests.

Distribution-free tests do not suffer from the same drawback; since their finite-sample size (hence also their asymptotic size) does not depend on  $P \in \mathcal{P}$ , pointwise and uniform convergence are equivalent under distribution-freeness. The typical example of distribution-free tests is that of rank-based tests. The advantages in terms of size and validity of rank tests over the pseudo-Gaussian ones, moreover, are not obtained at the cost of power and efficiency, as shown, for location and simple-output regression, by the celebrated Chernoff-Savage and Hodges-Lehmann results ([Chernoff and Savage \(1958\)](#); [Hodges and Lehmann \(1956\)](#)) and, for univariate ARMA time series, by [Hallin \(1994\)](#) and [Hallin and Tribel \(2000\)](#). Although no fully general versions of the Chernoff-Savage and Hodges-Lehmann inequalities have been established so far, in the multiple-output case, for center-outward ranks and signs, partial results (restricted to elliptical densities  $f$ ) have been obtained by [Hallin and Paindaveine \(2002a\)](#) and [Deb et al. \(2021\)](#), and, for elliptical VARMA models, by [Hallin and Paindaveine \(2002b, 2005\)](#). These inequalities are quite likely to hold beyond the class of elliptical densities, though.

The basic tools for constructing our rank-based multivariate portmanteau test are the center-outward ranks and signs proposed by [Hallin et al. \(2021\)](#), based on ideas and results from measure transportation theory. In [Section 4.2](#), we introduce the center-outward distribution function and its empirical version, from which the center-outward ranks and signs can be defined.

4.2. Center-outward ranks and signs

Denote by  $\mathbb{S}_d$  and  $\bar{\mathbb{S}}_d$  the open and closed unit ball, respectively, and by  $\mathcal{S}_{d-1}$  the unit hypersphere in  $\mathbb{R}^d$ . Let  $\mathcal{P}_d^\pm$  denote the family of all distributions  $P$  with densities in  $\mathcal{F}_d$  such that, for all positive  $r \in \mathbb{R}$ , there exist constants  $L_r^- > 0$  and  $L_r^+ < \infty$  for which

$$L_r^- \leq f(\mathbf{x}) \leq L_r^+ \quad \text{for all } \mathbf{x} \in r\bar{\mathbb{S}}_d.$$

For  $P$  in this family, the center-outward distribution functions defined below are continuous: see Figalli (2018). More general cases are studied in del Barrio et al. (2020), but require more cautious and less intuitive definitions of these center-outward functions which, for the sake of simplicity, we do not consider here. Denote by  $U_d$  the spherical uniform distribution over  $\mathbb{S}_d$ , that is, the product of a uniform measure over the hypersphere  $\mathcal{S}_{d-1}$  and a uniform over the unit interval of distances to the origin.

The center-outward distribution function  $\mathbf{F}_\pm$  of  $P$  is defined as the a.e. unique gradient of convex function mapping  $\mathbb{R}^d$  to  $\mathbb{S}_d$  and pushing  $P$  forward to  $U_d$  (that is, such that  $\mathbf{F}_\pm(\mathbf{X}) \sim U_d$  if  $\mathbf{X} \sim P$ ). For  $P \in \mathcal{P}_d^\pm$ , such mapping is a homeomorphism between  $\mathbb{S}_d \setminus \{\mathbf{0}\}$  and  $\mathbb{R}^d \setminus \mathbf{F}_\pm^{-1}(\{\mathbf{0}\})$  (Figalli (2018)) and the corresponding center-outward quantile function is defined as  $\mathbf{Q}_\pm := \mathbf{F}_\pm^{-1}$  (letting, with a small abuse of notation,  $\mathbf{Q}_\pm(\mathbf{0}) := \mathbf{F}_\pm^{-1}(\{\mathbf{0}\})$ ). For any given distribution  $P$ , the quantile function  $\mathbf{Q}_\pm$  induces a collection of continuous, connected, and nested quantile contours  $\mathbf{Q}_\pm(r\mathcal{S}_{d-1})$  and regions  $\mathbf{Q}_\pm(r\bar{\mathbb{S}}_d)$  of order  $r \in [0, 1]$ ; the center-outward median  $\mathbf{Q}_\pm(\mathbf{0})$  is a uniquely defined compact set of Lebesgue measure zero. We refer to Hallin et al. (2021) for details.

Turning to the sample, the residuals  $\mathbf{Z}_1^{(n)}(\boldsymbol{\theta}), \dots, \mathbf{Z}_n^{(n)}(\boldsymbol{\theta})$  under  $P_{\boldsymbol{\theta},f}^{(n)}$  are i.i.d. with density  $f \in \mathcal{F}_d$  and center-outward distribution function  $\mathbf{F}_\pm$ . For the empirical counterpart  $\mathbf{F}_\pm^{(n)}$  of  $\mathbf{F}_\pm$ , let  $n$  factorize into  $n = n_R n_S + n_0$ , for  $n_R, n_S, n_0 \in \mathbb{N}$  and  $0 \leq n_0 < \min\{n_R, n_S\}$ , where  $n_R \rightarrow \infty$  and  $n_S \rightarrow \infty$  as  $n \rightarrow \infty$ , and consider a sequence  $\mathfrak{G}^{(n)}$  of grids, where each grid consists of the  $n_R n_S$  intersections between an  $n_S$ -tuple  $(\mathbf{u}_1, \dots, \mathbf{u}_{n_S})$  of unit vectors, and the  $n_R$  hyperspheres with radii  $1/(n_R + 1), \dots, n_R/(n_R + 1)$  centered at the origin, along with  $n_0$  copies of the origin. The only requirement is that the discrete distribution with probability masses  $1/n$  at each gridpoint and probability mass  $n_0/n$  at the origin converges weakly to the uniform  $U_d$  over the ball  $\mathbb{S}_d$ . Then, we define  $\mathbf{F}_\pm^{(n)}(\mathbf{Z}_t^{(n)})$ , for  $t = 1, \dots, n$  as the solution (optimal mapping) of a coupling problem between the residuals and the gridpoints. Specifically, the empirical center-outward distribution function is the (random) discrete mapping

$$\mathbf{F}_\pm^{(n)} : \mathbf{Z}^{(n)} := (\mathbf{Z}_1^{(n)}, \dots, \mathbf{Z}_n^{(n)}) \mapsto (\mathbf{F}_\pm^{(n)}(\mathbf{Z}_1^{(n)}), \dots, \mathbf{F}_\pm^{(n)}(\mathbf{Z}_n^{(n)}))$$

satisfying

$$\sum_{t=1}^n \|\mathbf{Z}_t^{(n)} - \mathbf{F}_\pm^{(n)}(\mathbf{Z}_t^{(n)})\|^2 = \min_{T \in \mathcal{T}} \sum_{t=1}^n \|\mathbf{Z}_t^{(n)} - T(\mathbf{Z}_t^{(n)})\|^2, \tag{4.1}$$

where  $\mathbf{Z}_t^{(n)} = \mathbf{Z}_t^{(n)}(\boldsymbol{\theta})$ , the set  $\{\mathbf{F}_\pm^{(n)}(\mathbf{Z}_t^{(n)}) | t = 1, \dots, n\}$  coincides with the  $n$  points of the grid, and  $\mathcal{T}$  denotes the set of all possible bijective mappings between  $\mathbf{Z}_1^{(n)}, \dots, \mathbf{Z}_n^{(n)}$  and the  $n$  gridpoints. Intuition for this mapping in dimension  $d = 2$  is provided in Figure 1.

Based on this empirical center-outward distribution function, the center-outward ranks are defined as

$$R_{\pm,t}^{(n)} := R_{\pm,t}^{(n)}(\boldsymbol{\theta}) := (n_R + 1) \|\mathbf{F}_\pm^{(n)}(\mathbf{Z}_t^{(n)})\|, \tag{4.2}$$

the center-outward signs as

$$\mathbf{S}_{\pm,t}^{(n)} := \mathbf{S}_{\pm,t}^{(n)}(\boldsymbol{\theta}) := \mathbf{F}_\pm^{(n)}(\mathbf{Z}_t^{(n)}) / [\|\mathbf{F}_\pm^{(n)}(\mathbf{Z}_t^{(n)}) \neq \mathbf{0}\| \|\mathbf{F}_\pm^{(n)}(\mathbf{Z}_t^{(n)})\|]. \tag{4.3}$$

It follows that  $\mathbf{F}_\pm^{(n)}(\mathbf{Z}_t^{(n)})$  factorizes into

$$\mathbf{F}_\pm^{(n)}(\mathbf{Z}_t^{(n)}) = \frac{R_{\pm,t}^{(n)}}{n_R + 1} \mathbf{S}_{\pm,t}^{(n)}, \quad \text{whence} \quad \mathbf{Z}_t^{(n)} = \mathbf{Q}_\pm^{(n)} \left( \frac{R_{\pm,t}^{(n)}}{n_R + 1} \mathbf{S}_{\pm,t}^{(n)} \right). \tag{4.4}$$

Those ranks and signs are jointly distribution-free under  $P_{\boldsymbol{\theta},f}^{(n)}$  (for  $P \in \mathcal{P}_d^\pm$ ): more precisely, under  $P_{\boldsymbol{\theta},f}^{(n)}$ , the  $n$ -tuple  $\mathbf{F}_\pm^{(n)}(\mathbf{Z}_1^{(n)}), \dots, \mathbf{F}_\pm^{(n)}(\mathbf{Z}_n^{(n)})$  is uniformly distributed over the  $n!/n_0!$  permutations with repetition of the  $n$  underlying gridpoints (the origin having multiplicity  $n_0$ ). Moreover, the center-outward distribution functions, ranks, and signs inherit, from the invariance properties of Euclidean distances, elementary but remarkable invariance and equivariance properties with respect to shift, global scale, and orthogonal transformations: see Proposition 2.2 in Hallin et al. (2022a) for details.

An intuitive choice of  $n_R$  and  $n_S$  in the factorization of  $n$  is  $n_R \approx n^{1/d}$  and  $n_S \approx n^{(d-1)/d}$ ;  $n_R$ , indeed, is the cardinality of a one-dimensional grid over  $[0,1]$ ,  $n_S$  the cardinality of a grid over the  $(d-1)$ -sphere  $\mathcal{S}_{d-1}$ . Other heuristic criteria are possible, though. Mordant (2021), for instance, since the grid is supposed to provide an approximation of the spherical uniform, suggests minimizing the Wasserstein distance between the empirical distribution over the grid and the spherical uniform: see Section 3.2 of Hallin and Mordant (2022).

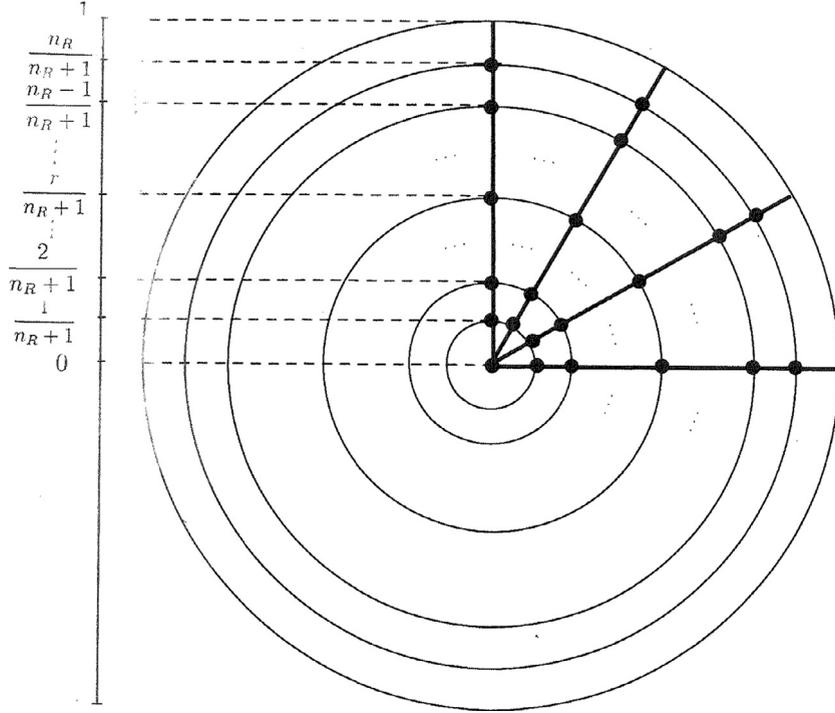


Fig. 1. A regular grid of  $n = n_R n_S$  points over  $\mathbb{S}_2$ .

#### 4.3. Center-outward rank-based cross-covariance matrices

Writing  $\mathbf{F}_{\pm,t}^{(n)}$ ,  $R_{\pm,t}^{(n)}$ , and  $\mathbf{S}_{\pm,t}^{(n)}$  for  $\mathbf{F}_{\pm}^{(n)}(\mathbf{Z}_t^{(n)}(\boldsymbol{\theta}))$ ,  $R_{\pm,t}^{(n)}(\boldsymbol{\theta})$ , and  $\mathbf{S}_{\pm,t}^{(n)}(\boldsymbol{\theta})$ , respectively, consider the center-outward rank-based counterpart of  $\Gamma_f^{(n)}(\boldsymbol{\theta})$ . Specifically, define

$$\Gamma_{\sim \mathbf{J}_1, \mathbf{J}_2}^{(n)}(\boldsymbol{\theta}) := n^{-1/2} \left( (n-1)^{1/2} (\text{vec}(\Gamma_{\sim 1, \mathbf{J}_1, \mathbf{J}_2}^{(n)}(\boldsymbol{\theta})))', \dots, (n-i)^{1/2} (\text{vec}(\Gamma_{\sim i, \mathbf{J}_1, \mathbf{J}_2}^{(n)}(\boldsymbol{\theta})))', \dots, (\text{vec}(\Gamma_{\sim n-1, \mathbf{J}_1, \mathbf{J}_2}^{(n)}(\boldsymbol{\theta})))' \right), \quad (4.5)$$

with, for  $i = 1, \dots, n-1$ ,

$$\Gamma_{\sim i, \mathbf{J}_1, \mathbf{J}_2}^{(n)}(\boldsymbol{\theta}) := (n-i)^{-1} \sum_{t=i+1}^n \mathbf{J}_1 \left( \frac{R_{\pm,t}^{(n)}}{n_R+1} \mathbf{S}_{\pm,t}^{(n)} \right) \mathbf{J}_2' \left( \frac{R_{\pm,t-i}^{(n)}}{n_R+1} \mathbf{S}_{\pm,t-i}^{(n)} \right) \quad (4.6)$$

where  $\mathbf{J}_1$  and  $\mathbf{J}_2 : \mathcal{S}_d \rightarrow \mathbb{R}$  are score functions satisfying Assumption 5 below. Call  $\Gamma_{\sim i, \mathbf{J}_1, \mathbf{J}_2}^{(n)}(\boldsymbol{\theta})$  a (residual) rank-based cross-covariance matrix with lag  $i$ : this matrix is distribution-free under  $\mathbb{P}_{\boldsymbol{\theta}, f}^{(n)}$  due to distribution-freeness of the center-outward ranks and signs. In order to establish its asymptotic distribution, we make the following assumption on  $\mathbf{J}_1$  and  $\mathbf{J}_2$ .

**Assumption 5.** The score functions  $\mathbf{J}_1$  and  $\mathbf{J}_2$

- (i) are continuous over  $\mathcal{S}_d$ ;
- (ii) are square-integrable, that is,  $\int_{\mathcal{S}_d} \|\mathbf{J}_\ell(\mathbf{u})\|^2 dU_d < \infty$  for  $\ell = 1, 2$ .

Moreover,

- (iii) for any sequence  $\mathbf{s}^{(n)} := \{\mathbf{s}_1^{(n)}, \dots, \mathbf{s}_n^{(n)}\}$  of  $n$ -tuples in  $\mathcal{S}_d$  such that the uniform discrete distribution over  $\mathbf{s}^{(n)}$  converges weakly to  $U_d$ ,

$$\lim_{n \rightarrow \infty} n^{-1} \sum_{t=1}^n \|\mathbf{J}_\ell(\mathbf{s}_t^{(n)})\|^2 = \int_{\mathcal{S}_d} \|\mathbf{J}_\ell(\mathbf{u})\|^2 dU_d, \quad \ell = 1, 2. \quad (4.7)$$

Let

$$\begin{aligned} \boldsymbol{\mu}_{\mathbf{J}_1, \mathbf{J}_2}^{(n)} &:= E(\mathbf{J}_1(\mathbf{F}_{\pm, 2}^{(n)})\mathbf{J}_2'(\mathbf{F}_{\pm, 1}^{(n)})), \\ \mathbf{D}_{\mathbf{J}_1, \mathbf{J}_2} &:= \left[ \int_{\mathbb{S}_d} \mathbf{J}_2(\mathbf{u}_2)\mathbf{J}_2'(\mathbf{u}_2)dU_d(\mathbf{u}_2) \right] \otimes \left[ \int_{\mathbb{S}_d} \mathbf{J}_1(\mathbf{u}_1)\mathbf{J}_1'(\mathbf{u}_1)dU_d(\mathbf{u}_1) \right] \\ &\quad - \left[ \int_{\mathbb{S}_d} \mathbf{J}_2(\mathbf{u})dU_d \int_{\mathbb{S}_d} \mathbf{J}_2'(\mathbf{u})dU_d \right] \otimes \left[ \int_{\mathbb{S}_d} \mathbf{J}_1(\mathbf{u})dU_d \int_{\mathbb{S}_d} \mathbf{J}_1'(\mathbf{u})dU_d \right], \end{aligned}$$

and

$$\mathbf{K}_{\mathbf{J}_1, \mathbf{J}_2, f} := E[\mathbf{J}_2(\mathbf{F}_{\pm}(\epsilon_t))\epsilon_t'] \otimes E[\mathbf{J}_1(\mathbf{F}_{\pm}(\epsilon_t))\phi_f'(\epsilon_t)].$$

We then have the following result from Hallin et al. (2022b).

**Proposition 4.1.** *Let Assumptions 1, 2, and 5 hold. Then, for any positive integers  $i \neq j$ , the vectors*

$$(n-i)^{1/2} \text{vec}(\Gamma_{\sim i, \mathbf{J}_1, \mathbf{J}_2}^{(n)}(\boldsymbol{\theta}) - \boldsymbol{\mu}_{\mathbf{J}_1, \mathbf{J}_2}^{(n)}) \quad \text{and} \quad (n-j)^{1/2} \text{vec}(\Gamma_{\sim j, \mathbf{J}_1, \mathbf{J}_2}^{(n)}(\boldsymbol{\theta}) - \boldsymbol{\mu}_{\mathbf{J}_1, \mathbf{J}_2}^{(n)})$$

are jointly asymptotically normal, with mean  $(\mathbf{0}', \mathbf{0}')$  under  $\mathbb{P}_{\boldsymbol{\theta}, f}^{(n)}$ , mean

$$\left( (\mathbf{K}_{\mathbf{J}_1, \mathbf{J}_2, f} \mathbf{c}_{i, \boldsymbol{\theta}}' \boldsymbol{\tau})', (\mathbf{K}_{\mathbf{J}_1, \mathbf{J}_2, f} \mathbf{c}_{j, \boldsymbol{\theta}}' \boldsymbol{\tau})' \right)' \tag{4.8}$$

under  $\mathbb{P}_{\boldsymbol{\theta} + n^{-1/2} \boldsymbol{\tau}, f}^{(n)}$  and covariance  $\begin{pmatrix} \mathbf{D}_{\mathbf{J}_1, \mathbf{J}_2} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_{\mathbf{J}_1, \mathbf{J}_2} \end{pmatrix}$  under both.

Without loss of generality, we will assume that  $\boldsymbol{\mu}_{\mathbf{J}_1, \mathbf{J}_2}^{(n)} = o(n^{-1/2})$ . A sufficient condition is either  $\int_{\mathbb{S}_d} \mathbf{J}_1(\mathbf{u})dU_d = \mathbf{0}$  or  $\int_{\mathbb{S}_d} \mathbf{J}_2(\mathbf{u})dU_d = \mathbf{0}$ ; indeed, following the same lines as in Lemma 1 of Hallin and La Vecchia (2020), one has

$$\boldsymbol{\mu}_{\mathbf{J}_1, \mathbf{J}_2}^{(n)} - E(\mathbf{J}_1(\mathbf{F}_{\pm, 1}))E(\mathbf{J}_2'(\mathbf{F}_{\pm, 1})) = o(n^{-1/2}).$$

Moreover, note that, for the scores in Examples 1-3 of Section 4.7,  $\boldsymbol{\mu}_{\mathbf{J}_1, \mathbf{J}_2}^{(n)} = \mathbf{0}$  holds whenever the regular grid  $\mathfrak{G}^{(n)}$  is symmetric with respect to the origin. See Hallin et al. (2023) for details.

To define the rank-based test statistic and derive its asymptotic theory, we assume the following asymptotic linearity of  $(n-i)^{1/2} \text{vec}(\Gamma_{\sim i, \mathbf{J}_1, \mathbf{J}_2}^{(n)}(\boldsymbol{\theta}))$ , where the form of the right-hand side of (4.9) again follows from the form of the shift matrix in (4.8).

**Assumption 6.** For any positive integer  $i$ , as  $n \rightarrow \infty$ ,

$$(n-i)^{1/2} \left[ \text{vec}(\Gamma_{\sim i, \mathbf{J}_1, \mathbf{J}_2}^{(n)}(\boldsymbol{\theta} + n^{-1/2} \boldsymbol{\tau})) - (\Gamma_{\sim i, \mathbf{J}_1, \mathbf{J}_2}^{(n)}(\boldsymbol{\theta})) \right] = -\mathbf{K}_{\mathbf{J}_1, \mathbf{J}_2, f} \mathbf{c}_{i, \boldsymbol{\theta}}' \boldsymbol{\tau} + o_p(1) \tag{4.9}$$

under  $\mathbb{P}_{\boldsymbol{\theta}, f}^{(n)}$  (hence also under  $\mathbb{P}_{\boldsymbol{\theta} + n^{-1/2} \boldsymbol{\tau}, f}^{(n)}$ ).

Sufficient conditions for (4.9) can be found in van den Akker et al. (2015).

Let  $\Delta_{\sim \mathbf{J}_1, \mathbf{J}_2}^{(n)}(\boldsymbol{\theta}) := \sum_{i=1}^{n-1} \mathbf{c}_{i, \boldsymbol{\theta}} (n-i)^{1/2} \text{vec}(\Gamma_{\sim i, \mathbf{J}_1, \mathbf{J}_2}^{(n)}(\boldsymbol{\theta}))$  denote the rank-based central sequence. The asymptotic linearity of  $\Delta_{\sim \mathbf{J}_1, \mathbf{J}_2}^{(n)}(\boldsymbol{\theta})$ , that is,

$$\Delta_{\sim \mathbf{J}_1, \mathbf{J}_2}^{(n)}(\boldsymbol{\theta} + n^{-1/2} \boldsymbol{\tau}) - \Delta_{\sim \mathbf{J}_1, \mathbf{J}_2}^{(n)}(\boldsymbol{\theta}) = - \sum_{i=1}^{n-1} \mathbf{c}_{i, \boldsymbol{\theta}} \mathbf{K}_{\mathbf{J}_1, \mathbf{J}_2, f} \mathbf{c}_{i, \boldsymbol{\theta}}' \boldsymbol{\tau} + o_p(1) \tag{4.10}$$

both under  $\mathbb{P}_{\boldsymbol{\theta}, f}^{(n)}$  and under  $\mathbb{P}_{\boldsymbol{\theta} + n^{-1/2} \boldsymbol{\tau}, f}^{(n)}$  as  $n \rightarrow \infty$  readily follows from Assumption 6.

#### 4.4. Center-outward rank-based portmanteau tests

Recall from Lemma 3.2 that the pseudo-Gaussian portmanteau test relies on a plug-in of the QMLE  $\hat{\boldsymbol{\theta}}_{\mathcal{N}}^{(n)}$ , which kind of neutralizes the asymptotic impact of computing cross-covariances from estimated residuals rather than exact ones. Lemma 4.2 below shows that a similar result holds for the rank-based cross-covariance matrix  $\Gamma_{\sim i, \mathbf{J}_1, \mathbf{J}_2}^{(n)}$  when computed at the R-estimator  $\hat{\boldsymbol{\theta}}_{\sim \mathbf{J}_1, \mathbf{J}_2}^{(n)}$  of Hallin et al. (2022b). That result will not hold if the R-estimator and the rank-based cross-covariance matrix are based on distinct score functions, or if the QMLE is plugged in instead of the adequate R-estimator.

Let  $\Upsilon_{\mathbf{J}_1, \mathbf{J}_2, f}^{(n)}(\boldsymbol{\theta}) := \sum_{i=1}^{n-1} \mathbf{c}_{i, \boldsymbol{\theta}} \mathbf{K}_{\mathbf{J}_1, \mathbf{J}_2, f} \mathbf{c}_{i, \boldsymbol{\theta}}'$  denote the shift matrix in (4.10) and let  $\hat{\Upsilon}_{\mathbf{J}_1, \mathbf{J}_2}^{(n)}$  be a consistent (under  $\mathbb{P}_{\boldsymbol{\theta}, f}^{(n)}$ ) estimator thereof; see Hallin et al. (2022b) for details. Further, denote by  $\bar{\boldsymbol{\theta}}^{(n)}$  a preliminary  $\sqrt{n}$ -consistent and asymptotically discrete

estimator of  $\theta$ . Recall that an estimator  $\tilde{\theta}^{(n)}$  of  $\theta$  is called *asymptotically discrete* if the number of values it takes in balls of radius  $cn^{-1/2}$  ( $c > 0$ ) is bounded as  $n \rightarrow \infty$ , which is easily obtained by discretization. This assumption, which is classical in Le Cam asymptotics, is needed in asymptotic statements but has no finite-sample implications. The *one-step R-estimator* of  $\theta$  based on scores  $\mathbf{J}_1$  and  $\mathbf{J}_2$  is defined as

$$\tilde{\theta}_{\mathbf{J}_1, \mathbf{J}_2}^{(n)} := \tilde{\theta}^{(n)} + n^{-1/2} (\widehat{\Upsilon}_{\mathbf{J}_1, \mathbf{J}_2}^{(n)})^{-1} \Delta_{\mathbf{J}_1, \mathbf{J}_2}^{(n)} (\tilde{\theta}^{(n)}). \quad (4.11)$$

Now, consider the matrix  $\Gamma_{\mathbf{J}_1, \mathbf{J}_2, f}^{(n)*}(\theta)$  of residuals in the regression of the components of  $(n-i)^{1/2} \text{vec} \Gamma_{\mathbf{J}_1, \mathbf{J}_2}^{(n)}(\theta)$  with respect to  $\Delta_{\mathbf{J}_1, \mathbf{J}_2}^{(n)}(\theta)$  in the covariance matrix (note, however, that this matrix is not the asymptotic covariance matrix of  $((n-i)^{1/2} \text{vec}' \Gamma_{\mathbf{J}_1, \mathbf{J}_2}^{(n)}(\theta), \Delta_{\mathbf{J}_1, \mathbf{J}_2}^{(n)'}(\theta))'$ )

$$\begin{pmatrix} \mathbf{D}_{\mathbf{J}_1, \mathbf{J}_2} & \mathbf{K}_{\mathbf{J}_1, \mathbf{J}_2, f} \mathbf{c}'_{i, \theta} \\ \mathbf{c}_{i, \theta} \mathbf{K}_{\mathbf{J}_1, \mathbf{J}_2, f} & \sum_{i=1}^{n-1} \mathbf{c}_{i, \theta} \mathbf{K}_{\mathbf{J}_1, \mathbf{J}_2, f} \mathbf{c}'_{i, \theta} \end{pmatrix}.$$

Namely, define the oracle (it depends on the actual  $f$ ) statistic

$$\begin{aligned} & (n-i)^{1/2} \text{vec}(\Gamma_{\mathbf{J}_1, \mathbf{J}_2, f}^{(n)*}(\theta)) \\ & := (n-i)^{1/2} \text{vec}(\Gamma_{\mathbf{J}_1, \mathbf{J}_2}^{(n)}(\theta)) - \mathbf{K}_{\mathbf{J}_1, \mathbf{J}_2, f} \mathbf{c}'_{i, \theta} \left( \sum_{i=1}^{n-1} \mathbf{c}_{i, \theta} \mathbf{K}_{\mathbf{J}_1, \mathbf{J}_2, f} \mathbf{c}'_{i, \theta} \right)^{-1} \Delta_{\mathbf{J}_1, \mathbf{J}_2}^{(n)}(\theta) \\ & = (n-i)^{1/2} \text{vec}(\Gamma_{\mathbf{J}_1, \mathbf{J}_2}^{(n)}(\theta)) - \mathbf{K}_{\mathbf{J}_1, \mathbf{J}_2, f} \mathbf{c}'_{i, \theta} \left( \sum_{i=1}^{n-1} \mathbf{c}_{i, \theta} \mathbf{K}_{\mathbf{J}_1, \mathbf{J}_2, f} \mathbf{c}'_{i, \theta} \right)^{-1} \\ & \quad \times \left( \sum_{i=1}^{n-1} \mathbf{c}_{i, \theta} (n-i)^{1/2} \text{vec}(\Gamma_{\mathbf{J}_1, \mathbf{J}_2}^{(n)}(\theta)) \right). \end{aligned} \quad (4.12)$$

**Lemma 4.2** below shows that  $(n-i)^{1/2} \Gamma_{\mathbf{J}_1, \mathbf{J}_2}^{(n)}(\theta)$  and the oracle statistic  $(n-i)^{1/2} \Gamma_{\mathbf{J}_1, \mathbf{J}_2, f}^{(n)*}(\theta)$  are asymptotically equivalent. Hence, the asymptotics of the rank-based test statistic constructed from  $(n-i)^{1/2} \Gamma_{\mathbf{J}_1, \mathbf{J}_2}^{(n)}(\theta)$  coincide with those of its counterpart built from  $(n-i)^{1/2} \Gamma_{\mathbf{J}_1, \mathbf{J}_2, f}^{(n)*}(\theta)$ . See the Appendix for a proof.

**Lemma 4.2.** *Let Assumptions 1, 2, 5, and 6 hold. Then*

- (i)  $\Delta_{\mathbf{J}_1, \mathbf{J}_2}^{(n)}(\theta) = o_p(1)$ ,
- (ii)  $(n-i)^{1/2} \text{vec} \left( \Gamma_{\mathbf{J}_1, \mathbf{J}_2, f}^{(n)*}(\theta) - \Gamma_{\mathbf{J}_1, \mathbf{J}_2}^{(n)}(\theta) \right) = o_p(1)$ , and
- (iii)  $(n-i)^{1/2} \text{vec} \left( \Gamma_{\mathbf{J}_1, \mathbf{J}_2, f}^{(n)*}(\theta) - \Gamma_{\mathbf{J}_1, \mathbf{J}_2, f}^{(n)*}(\theta) \right) = o_p(1)$

for any fixed  $i \geq 1$ , under  $P_{\theta, f}^{(n)}$  and  $P_{\theta+n^{1/2}\tau, f}^{(n)}$ ,  $\tau \in \mathbb{R}^{(p+q)d^2}$ , as  $n \rightarrow \infty$ .

Note that the asymptotic covariance matrix of  $(n-i)^{1/2} \text{vec}(\Gamma_{\mathbf{J}_1, \mathbf{J}_2}^{(n)}(\theta))$  in Proposition 4.1, which does not depend on the underlying density  $f$ , differs from the corresponding shift matrix  $\mathbf{K}_{\mathbf{J}_1, \mathbf{J}_2, f}$ , which is not even symmetric for a general  $f$ . This was not the case for the asymptotic covariance matrix of  $(n-i)^{1/2} \text{vec}(\Gamma_{i, \mathcal{N}}^{(n)}(\theta))$ . As a consequence, the construction of rank-based portmanteau tests is more intricate than in the classical pseudo-Gaussian case and requires some additional notation.

In view of Lemma 4.2, the test statistic should involve a consistent estimator of the covariance matrix of  $(n-i)^{1/2} \text{vec}(\Gamma_{\mathbf{J}_1, \mathbf{J}_2}^{(n)*}(\theta))$  rather than a consistent estimator of the covariance matrix of  $(n-i)^{1/2} \text{vec}(\Gamma_{\mathbf{J}_1, \mathbf{J}_2}^{(n)}(\theta))$ . Let

$$\Omega_{\mathbf{J}_1, \mathbf{J}_2, f}^{(m)}(\theta) := \mathbf{W}_{\mathbf{J}_1, \mathbf{J}_2, f}^{(m)}(\theta) \mathbf{W}_{\mathbf{J}_1, \mathbf{J}_2, f}^{(m)'}(\theta) \quad (4.13)$$

with

$$\mathbf{W}_{i, \mathbf{J}_1, \mathbf{J}_2, f}^{(m)}(\boldsymbol{\theta}) := (\mathbf{e}_i^{(m)'} \otimes \mathbf{D}_{\mathbf{J}_1, \mathbf{J}_2}^{1/2}) - \mathbf{K}_{\mathbf{J}_1, \mathbf{J}_2, f} \mathbf{c}_{i, \boldsymbol{\theta}}' \left( \sum_{i=1}^m \mathbf{c}_{i, \boldsymbol{\theta}} \mathbf{K}_{\mathbf{J}_1, \mathbf{J}_2, f} \mathbf{c}_{i, \boldsymbol{\theta}}' \right)^{-1} \mathbf{C}_{\boldsymbol{\theta}}^{(m+1)} (\mathbf{I}_m \otimes \mathbf{D}_{\mathbf{J}_1, \mathbf{J}_2}^{1/2}) \quad (4.14)$$

and  $\mathbf{e}_i^{(m)}$  standing for the  $i$ th vector of the canonical basis in  $\mathbb{R}^m$ .

For sufficiently large  $m$ , due to the exponential decrease of  $\|\mathbf{c}_{i, \boldsymbol{\theta}}\|$ , the matrices  $\Omega_{i, \mathbf{J}_1, \mathbf{J}_2, f}^{(m)}(\boldsymbol{\theta}^{(n)})$  are (uniformly in  $i$ ) arbitrarily close to the covariance matrices of  $(n-i)^{1/2} \text{vec}(\Gamma_{i, \mathbf{J}_1, \mathbf{J}_2, f}^{(n)*}(\boldsymbol{\theta}))$  as  $n \rightarrow \infty$ . Letting

$$\mathbf{E}_{\mathbf{J}_1, \mathbf{J}_2, f}^{(m)}(\boldsymbol{\theta}) := \mathbf{I}_{md^2} - (\mathbf{I}_m \otimes \mathbf{K}_{\mathbf{J}_1, \mathbf{J}_2, f}) \mathbf{C}_{\boldsymbol{\theta}}^{(m+1)'} \left( \sum_{i=1}^m \mathbf{c}_{i, \boldsymbol{\theta}} \mathbf{K}_{\mathbf{J}_1, \mathbf{J}_2, f} \mathbf{c}_{i, \boldsymbol{\theta}}' \right)^{-1} \mathbf{C}_{\boldsymbol{\theta}}^{(m+1)}, \quad (4.15)$$

we have

$$\text{diag}(\Omega_{i, \mathbf{J}_1, \mathbf{J}_2, f}^{(m)}(\boldsymbol{\theta}))_{1 \leq i \leq m} = \mathbf{E}_{\mathbf{J}_1, \mathbf{J}_2, f}^{(m)}(\boldsymbol{\theta}) (\mathbf{I}_m \otimes \mathbf{D}_{\mathbf{J}_1, \mathbf{J}_2}) \mathbf{E}_{\mathbf{J}_1, \mathbf{J}_2, f}^{(m)'}(\boldsymbol{\theta}). \quad (4.16)$$

Heuristically,  $\mathbf{E}_{\mathbf{J}_1, \mathbf{J}_2, f}^{(m)}(\boldsymbol{\theta})$ , which appears in the rank-based test statistic (4.18), is taking into account the difference between  $\Gamma_{i, \mathbf{J}_1, \mathbf{J}_2, f}^{(n)*}(\boldsymbol{\theta})$  and  $\Gamma_{i, \mathbf{J}_1, \mathbf{J}_2}^{(n)}(\boldsymbol{\theta})$  in (4.12).

Our rank-based test statistics involve consistent estimators of the quantities defined above. Denote by  $\widehat{\mathbf{K}}_{\mathbf{J}_1, \mathbf{J}_2}^{(n)}$  a consistent estimator of  $\mathbf{K}_{\mathbf{J}_1, \mathbf{J}_2, f}$ ; such  $\widehat{\mathbf{K}}_{\mathbf{J}_1, \mathbf{J}_2}^{(n)}$  can be obtained via (4.9)—for example, letting

$$\tau_j = -\mathbf{c}_{1, \boldsymbol{\theta}^{(n)}}'_{\mathbf{J}_1, \mathbf{J}_2} \left( \mathbf{c}'_{1, \boldsymbol{\theta}^{(n)}}_{\mathbf{J}_1, \mathbf{J}_2} \mathbf{c}_{1, \boldsymbol{\theta}^{(n)}}_{\mathbf{J}_1, \mathbf{J}_2} \right)^{-1} \mathbf{e}_j^{(d^2)},$$

where  $\mathbf{e}_j^{(d^2)}$  denotes  $j$ th vector of the canonical basis in  $\mathbb{R}^{d^2}$ ,

$$(n-1)^{1/2} \left[ \text{vec}(\Gamma_{1, \mathbf{J}_1, \mathbf{J}_2}^{(n)}(\boldsymbol{\theta}^{(n)}) + n^{-1/2} \tau_j) - \Gamma_{1, \mathbf{J}_1, \mathbf{J}_2}^{(n)}(\boldsymbol{\theta}^{(n)}) \right]$$

is a consistent estimator of the  $j$ th column of  $\mathbf{K}_{\mathbf{J}_1, \mathbf{J}_2, f}$ . For  $j = 1, \dots, d^2$ , this yields a consistent estimator of  $\mathbf{K}_{\mathbf{J}_1, \mathbf{J}_2, f}$ . Let  $\widehat{\Omega}_{i, \mathbf{J}_1, \mathbf{J}_2, f}^{(m)}(\boldsymbol{\theta})$  and  $\widehat{\mathbf{E}}_{\mathbf{J}_1, \mathbf{J}_2, f}^{(m)}(\boldsymbol{\theta})$  stand for the consistent estimators of  $\Omega_{i, \mathbf{J}_1, \mathbf{J}_2, f}^{(m)}(\boldsymbol{\theta})$  and  $\mathbf{E}_{\mathbf{J}_1, \mathbf{J}_2, f}^{(m)}(\boldsymbol{\theta})$  obtained by plugging  $\widehat{\mathbf{K}}_{\mathbf{J}_1, \mathbf{J}_2}^{(n)}$  into (4.13)–(4.14) and (4.15). Then the center-outward rank-based portmanteau test statistic we are proposing takes the form

$$\begin{aligned} Q_{\sim m; \mathbf{J}_1, \mathbf{J}_2}^{(n)}(\boldsymbol{\theta}^{(n)}) &:= \sum_{i=1}^m (n-i) \text{vec}(\Gamma_{i, \mathbf{J}_1, \mathbf{J}_2}^{(n)}(\boldsymbol{\theta}^{(n)}))' (\widehat{\Omega}_{i, \mathbf{J}_1, \mathbf{J}_2, f}^{(m)}(\boldsymbol{\theta}^{(n)}))^{-1} \\ &\quad \times \text{vec}(\Gamma_{i, \mathbf{J}_1, \mathbf{J}_2}^{(n)}(\boldsymbol{\theta}^{(n)})), \end{aligned} \quad (4.17)$$

where  $\mathbf{A}^-$  denotes the Moore–Penrose inverse of a matrix  $\mathbf{A}$ . Alternatively, in view of (4.16),  $Q_{\sim m; \mathbf{J}_1, \mathbf{J}_2}^{(n)}(\boldsymbol{\theta}^{(n)})$  can be written as

$$\begin{aligned} Q_{\sim m; \mathbf{J}_1, \mathbf{J}_2}^{(n)}(\boldsymbol{\theta}^{(n)}) &= n \Gamma_{\sim \mathbf{J}_1, \mathbf{J}_2}^{(m, n)'}(\boldsymbol{\theta}^{(n)}) \left( \widehat{\mathbf{E}}_{\mathbf{J}_1, \mathbf{J}_2, f}^{(m)}(\boldsymbol{\theta}^{(n)}) (\mathbf{I}_m \otimes \mathbf{D}_{\mathbf{J}_1, \mathbf{J}_2}) \widehat{\mathbf{E}}_{\mathbf{J}_1, \mathbf{J}_2, f}^{(m)'}(\boldsymbol{\theta}^{(n)}) \right)^{-1} \\ &\quad \times \Gamma_{\sim \mathbf{J}_1, \mathbf{J}_2}^{(m, n)}(\boldsymbol{\theta}^{(n)}) \end{aligned} \quad (4.18)$$

where

$$\Gamma_{\sim \mathbf{J}_1, \mathbf{J}_2}^{(m, n)}(\boldsymbol{\theta}) := n^{-1/2} \left( (n-1)^{1/2} (\text{vec}(\Gamma_{1, \mathbf{J}_1, \mathbf{J}_2}^{(n)}(\boldsymbol{\theta})))', \dots, (n-m)^{1/2} (\text{vec}(\Gamma_{m, \mathbf{J}_1, \mathbf{J}_2}^{(n)}(\boldsymbol{\theta})))' \right)'$$

is a truncated version of  $\Gamma_{\sim \mathbf{J}_1, \mathbf{J}_2}^{(n)}(\boldsymbol{\theta})$ .

#### 4.5. Asymptotic distribution

In view of Lemma 4.2 and the consistency of  $\boldsymbol{\theta}_{\sim \mathbf{J}_1, \mathbf{J}_2}^{(n)}$  and  $\widehat{\mathbf{K}}_{\mathbf{J}_1, \mathbf{J}_2}^{(n)}$ , we have

$$Q_{\sim m; \mathbf{J}_1, \mathbf{J}_2}^{(n)}(\boldsymbol{\theta}^{(n)}) = Q_{\sim m; \mathbf{J}_1, \mathbf{J}_2, f}^{(n)*}(\boldsymbol{\theta}) + o_p(1) \quad (4.19)$$

where

$$Q_{\sim m; \mathbf{J}_1, \mathbf{J}_2, f}^{(n)*}(\boldsymbol{\theta}) := \sum_{i=1}^m (n-i) \text{vec}(\Gamma_{i, \mathbf{J}_1, \mathbf{J}_2, f}^{(n)*}(\boldsymbol{\theta}))' (\Omega_{i, \mathbf{J}_1, \mathbf{J}_2, f}^{(m)}(\boldsymbol{\theta}))^{-1} \text{vec}(\Gamma_{i, \mathbf{J}_1, \mathbf{J}_2, f}^{(n)*}(\boldsymbol{\theta})).$$

Moreover, note that for  $m$  large enough, due again to the exponential decrease of  $\|\mathbf{c}_{i,\theta}\|$ ,  $Q_{\sim m; \mathbf{J}_1, \mathbf{J}_2, f}^{(n)**}(\boldsymbol{\theta})$  is arbitrarily close to

$$Q_{\sim m; \mathbf{J}_1, \mathbf{J}_2, f}^{(n)**}(\boldsymbol{\theta}) := \sum_{i=1}^m (n-i) \text{vec}(\Gamma_{\sim i; \mathbf{J}_1, \mathbf{J}_2, f}^{(m,n)**}(\boldsymbol{\theta}))' (\Omega_{\sim i; \mathbf{J}_1, \mathbf{J}_2, f}^{(m)}(\boldsymbol{\theta}))^{-1} \text{vec}(\Gamma_{\sim i; \mathbf{J}_1, \mathbf{J}_2, f}^{(m,n)**}(\boldsymbol{\theta})) \quad (4.20)$$

where

$$(n-i)^{1/2} \text{vec}(\Gamma_{\sim i; \mathbf{J}_1, \mathbf{J}_2, f}^{(m,n)**}(\boldsymbol{\theta})) := (n-i)^{1/2} \text{vec}(\Gamma_{\sim i; \mathbf{J}_1, \mathbf{J}_2}^{(n)}(\boldsymbol{\theta})) - \mathbf{K}_{\mathbf{J}_1, \mathbf{J}_2, f} \mathbf{c}'_{i,\theta} \left( \sum_{i=1}^m \mathbf{c}_{i,\theta} \mathbf{K}_{\mathbf{J}_1, \mathbf{J}_2, f} \mathbf{c}'_{i,\theta} \right)^{-1} \\ \times \sum_{i=1}^m \mathbf{c}_{i,\theta} (n-i)^{1/2} \text{vec}(\Gamma_{\sim i; \mathbf{J}_1, \mathbf{J}_2}^{(n)}(\boldsymbol{\theta})) \quad (4.21)$$

is an approximation of  $(n-i)^{1/2} \text{vec}(\Gamma_{\sim i; \mathbf{J}_1, \mathbf{J}_2, f}^{(n)**}(\boldsymbol{\theta}))$  resulting from truncating at  $m$  the summation in the right-hand side of (4.12); the asymptotic covariance matrix of  $(n-i)^{1/2} \text{vec}(\Gamma_{\sim i; \mathbf{J}_1, \mathbf{J}_2, f}^{(m,n)**}(\boldsymbol{\theta}))$  is  $\Omega_{\sim i; \mathbf{J}_1, \mathbf{J}_2, f}^{(m)}(\boldsymbol{\theta})$  under  $P_{\boldsymbol{\theta}, f}^{(n)}$ .

The following result is the rank-based counterpart of Proposition 3.3 and states that  $Q_{\sim m; \mathbf{J}_1, \mathbf{J}_2}^{(n)}(\boldsymbol{\theta}^{(n)})$ , for  $m$  large enough, can be approximated by  $Q_{\sim m; \mathbf{J}_1, \mathbf{J}_2, f}^{(n)**}(\boldsymbol{\theta})$  which is asymptotically chi-square under  $P_{\boldsymbol{\theta}, f}^{(n)}$  as  $n \rightarrow \infty$ . Note that  $\Gamma_{\sim i; \mathbf{J}_1, \mathbf{J}_2}^{(m,n)}(\boldsymbol{\theta})$ , which depends on the ranks of the “true” residuals, is fully distribution-free, while (due to the presence of  $\mathbf{K}_{\mathbf{J}_1, \mathbf{J}_2, f}$ )  $\Gamma_{\sim i; \mathbf{J}_1, \mathbf{J}_2, f}^{(m,n)**}(\boldsymbol{\theta})$  in (4.21), hence  $Q_{\sim m; \mathbf{J}_1, \mathbf{J}_2, f}^{(n)**}(\boldsymbol{\theta})$  in (4.20), are not. However,  $Q_{\sim m; \mathbf{J}_1, \mathbf{J}_2, f}^{(n)**}(\boldsymbol{\theta})$  is asymptotically distribution-free.

**Proposition 4.3.** *Let Assumptions 1, 2, 5, and 6 hold. Then, under  $P_{\boldsymbol{\theta}, f}^{(n)}$*

(i) *for all  $\delta, \varepsilon > 0$ , there exist  $M_{\delta, \varepsilon}$  and  $N_{\delta, \varepsilon} \in \mathbb{N}$  such that*

$$P\left(\left| Q_{\sim m; \mathbf{J}_1, \mathbf{J}_2}^{(n)}(\boldsymbol{\theta}^{(n)}) - Q_{\sim m; \mathbf{J}_1, \mathbf{J}_2, f}^{(n)**}(\boldsymbol{\theta}) \right| < \delta\right) > 1 - \varepsilon$$

*for all  $n \geq N_{\delta, \varepsilon}$  and  $M_{\delta, \varepsilon} \leq m \leq n - 1$ ;*

(ii) *irrespective of the innovation density  $f$ ,  $Q_{\sim m; \mathbf{J}_1, \mathbf{J}_2, f}^{(n)**}(\boldsymbol{\theta})$  is asymptotically chi-square with  $d^2(m - p - q)$  degrees of freedom as  $n \rightarrow \infty$ .*

See the Appendix for a proof.

#### 4.6. Local powers and AREs

Having obtained the asymptotic distributions of portmanteau test statistics under the null, natural questions are: what are their local asymptotic powers? can we derive their asymptotic relative efficiencies (AREs) with respect, for instance, to the classical procedure?

Local powers and AREs, however, depend on the type of alternative under consideration and are usually obtained via an application of Le Cam’s third Lemma. This requires an embedding of the VARMA( $p, q$ ) model that has been studied so far into a larger model enjoying LAN and containing the alternatives of interest. A simple example is the VARMA( $p_1, q_1$ ) model with  $p_1 > p$  and/or  $q_1 > q$ , for which LAN results are available (Garel and Hallin, 1995); call it the “VARMA( $p_1, q_1$ ) case”. Other alternatives of interest are, e.g., the presence of bilinear terms in the data-generating process, or the conditional heteroskedasticity of the  $\boldsymbol{\epsilon}_t$ ’s—for which LAN, in the vicinity of VARMA( $p, q$ ) models, is not available in the literature. While we have no doubt that such LAN structures hold under appropriate assumptions, establishing such results is beyond the scope of this paper, and we are restricting this section to a discussion of the VARMA( $p_1, q_1$ ) case.

Now, even in the VARMA( $p_1, q_1$ ) case, things are not as simple as it may appear at first sight. While asymptotic shifts under local VARMA( $p_1, q_1$ ) alternatives are easily obtained for the Gaussian  $(n-i)^{1/2} \text{vec}(\Gamma_{i, \mathcal{N}}^{(n)**}(\boldsymbol{\theta}))$  and the rank-based  $(n-i)^{1/2} \text{vec}(\Gamma_{\sim i; \mathbf{J}_1, \mathbf{J}_2, f}^{(n)**}(\boldsymbol{\theta}))$ , hence, in view of Lemmas 3.2 and 4.2, for  $(n-i)^{1/2} \text{vec}(\Gamma_{i, \mathcal{N}}^{(n)}(\hat{\boldsymbol{\theta}}_{\mathcal{N}}^{(n)}))$  and  $(n-i)^{1/2} \text{vec}(\Gamma_{\sim i; \mathbf{J}_1, \mathbf{J}_2}^{(n)}(\hat{\boldsymbol{\theta}}_{\mathbf{J}_2}^{(n)}))$ , the quadratic forms  $Q_{m; \mathcal{N}}^{(n)*}(\boldsymbol{\theta})$  and  $Q_{\sim m; \mathbf{J}_1, \mathbf{J}_2}^{(n)*}(\boldsymbol{\theta})$  are not idempotent (not even asymptotically so) in these statistics and are not asymptotically chi-square under the null. As a consequence, the asymptotic shifts of  $(n-i)^{1/2} \text{vec}(\Gamma_{i, \mathcal{N}}^{(n)}(\hat{\boldsymbol{\theta}}_{\mathcal{N}}^{(n)}))$  and  $(n-i)^{1/2} \text{vec}(\Gamma_{\sim i; \mathbf{J}_1, \mathbf{J}_2, f}^{(n)**}(\hat{\boldsymbol{\theta}}_{\mathbf{J}_2}^{(n)}))$  do not induce, for the portmanteau statistics  $Q_{m; \mathcal{N}}^{(n)}(\hat{\boldsymbol{\theta}}_{\mathcal{N}}^{(n)}) = Q_{m; \mathcal{N}}^{(n)*}(\boldsymbol{\theta}) + o_p(1)$  and  $Q_{\sim m; \mathbf{J}_1, \mathbf{J}_2}^{(n)}(\hat{\boldsymbol{\theta}}_{\mathbf{J}_2}^{(n)}) = Q_{\sim m; \mathbf{J}_1, \mathbf{J}_2}^{(n)*}(\boldsymbol{\theta}) + o_p(1)$ , the usual chi-square noncentrality parameters, and the classical methods expressing AREs in terms of ratios of noncentrality parameters do not apply.

These classical methods, on the other hand, do apply to the asymptotically chi-square approximations  $Q_{m,\mathcal{N}}^{(n)**}(\boldsymbol{\theta})$  and  $Q_{\sim m; \mathbf{J}_1, \mathbf{J}_2}^{(n)**}(\boldsymbol{\theta})$ . These, however, are oracle test statistics that cannot be computed from the observations and the resulting AREs cannot be considered as approximations of the actual portmanteau AREs; the approximation errors in Propositions 3.3 and 4.3, indeed, do not go to zero as  $n \rightarrow \infty$ . Such AREs are of limited practical value, thus. Accordingly, we will not proceed any further with their numerical evaluation, and rather recommend, for power comparisons, the Monte Carlo approach adopted in Section 5.

#### 4.7. Some standard score functions

The rank-based test statistic  $Q_{\sim m; \mathbf{J}_1, \mathbf{J}_2}^{(n)}(\boldsymbol{\theta}^{(n)})$  depends on the choice of the score functions  $\mathbf{J}_1$  and  $\mathbf{J}_2$ . Here we provide three examples of standard score functions, extending scores that are widely applied in the univariate (see e.g. Hallin and La Vecchia (2020)) and the elliptical setting (see Hallin and Paindaveine (2004)).

**Example 1** (Sign test scores). Setting  $\mathbf{J}_\ell \left( \frac{R_{\pm,t}^{(n)}}{n_R+1} \mathbf{S}_{\pm,t}^{(n)} \right) = \mathbf{S}_{\pm,t}^{(n)}$ ,  $\ell = 1, 2$  yields the center-outward sign-based cross-covariance matrices

$$\Gamma_{\sim i, \text{sign}}^{(n)}(\boldsymbol{\theta}) = (n-i)^{-1} \sum_{t=i+1}^n \mathbf{S}_{\pm,t}^{(n)}(\boldsymbol{\theta}) \mathbf{S}_{\pm,t-i}^{(n)\prime}(\boldsymbol{\theta}), \quad i = 1, \dots, n-1.$$

The resulting test statistic  $Q_{\sim m; \mathbf{J}_1, \mathbf{J}_2}^{(n)}(\boldsymbol{\theta}^{(n)})$  entirely relies on the center-outward signs  $\mathbf{S}_{\pm,t}^{(n)}(\boldsymbol{\theta})$ , which justifies the terminology *sign test scores*.

**Example 2** (Spearman scores). Another simple choice is  $\mathbf{J}_1(\mathbf{u}) = \mathbf{u} = \mathbf{J}_2(\mathbf{u})$ . The corresponding rank-based cross-covariance matrices are

$$\Gamma_{\sim i, \text{Sp}}^{(n)}(\boldsymbol{\theta}) = (n-i)^{-1} \sum_{t=i+1}^n \mathbf{F}_{\pm,t}^{(n)} \mathbf{F}_{\pm,t-i}^{(n)\prime}, \quad i = 1, \dots, n-1$$

reducing, for  $d = 1$ , to Spearman autocorrelations, whence the terminology *Spearman scores*.

**Example 3** (van der Waerden or spherical normal scores). Let

$$\mathbf{J}_\ell \left( \frac{R_{\pm,t}^{(n)}}{n_R+1} \mathbf{S}_{\pm,t}^{(n)} \right) = \mathbf{J}_\ell \left( \frac{R_{\pm,t}^{(n)}}{n_R+1} \right) \mathbf{S}_{\pm,t}^{(n)}, \quad \ell = 1, 2,$$

with  $J_\ell(u) = ((F_d^{\chi^2})^{-1}(u))^{1/2}$ , where  $F_d^{\chi^2}$  denotes the chi-square distribution function with  $d$  degrees of freedom. This yields the *spherical van der Waerden (vdW) rank scores*, with cross-covariance matrices

$$\Gamma_{\sim i, \text{vdW}}^{(n)}(\boldsymbol{\theta}) = (n-i)^{-1} \sum_{t=i+1}^n \left[ (F_d^{\chi^2})^{-1} \left( \frac{R_{\pm,t}^{(n)}(\boldsymbol{\theta})}{n_R+1} \right) \right]^{1/2} \left[ (F_d^{\chi^2})^{-1} \left( \frac{R_{\pm,t-i}^{(n)}(\boldsymbol{\theta})}{n_R+1} \right) \right]^{1/2} \mathbf{S}_{\pm,t}^{(n)}(\boldsymbol{\theta}) \mathbf{S}_{\pm,t-i}^{(n)\prime}(\boldsymbol{\theta}), \quad i = 1, \dots, n-1.$$

## 5. Numerical assessment of finite-sample performance and resistance to outliers

In this section, we investigate through a brief Monte Carlo experiment the finite-sample performance of the center-outward rank-based and pseudo-Gaussian tests under various innovation densities, including contaminated ones.

### 5.1. Size and power

We first compare the sizes of these tests, hence their validity, by generating  $N = 300$  replications of sample size  $n = 1000$  from the bivariate VARMA(1, 1) model with

$$\mathbf{A}_1 = \begin{pmatrix} 0.5 & 0.2 \\ -0.1 & 0.4 \end{pmatrix} \quad \text{and} \quad \mathbf{B}_1 = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.4 \end{pmatrix} \quad (5.1)$$

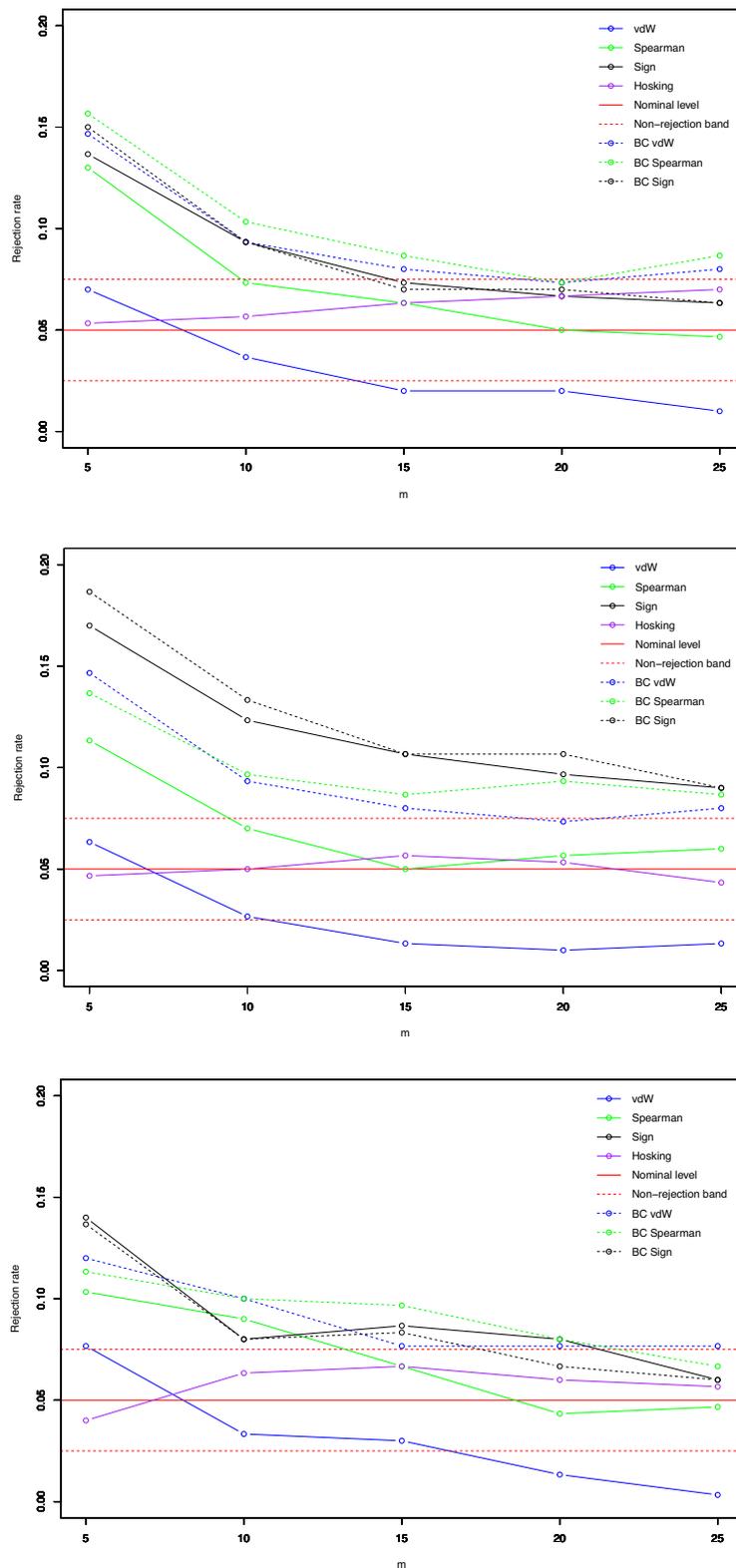
and three types of innovation densities  $f$ : spherical normal, mixture of three Gaussians, and skew- $t$  distribution with 3 degrees of freedom (denoted as skew- $t_3$ ; see Azzalini and Capitanio (2003) for a definition), respectively.

The mixture is of the form

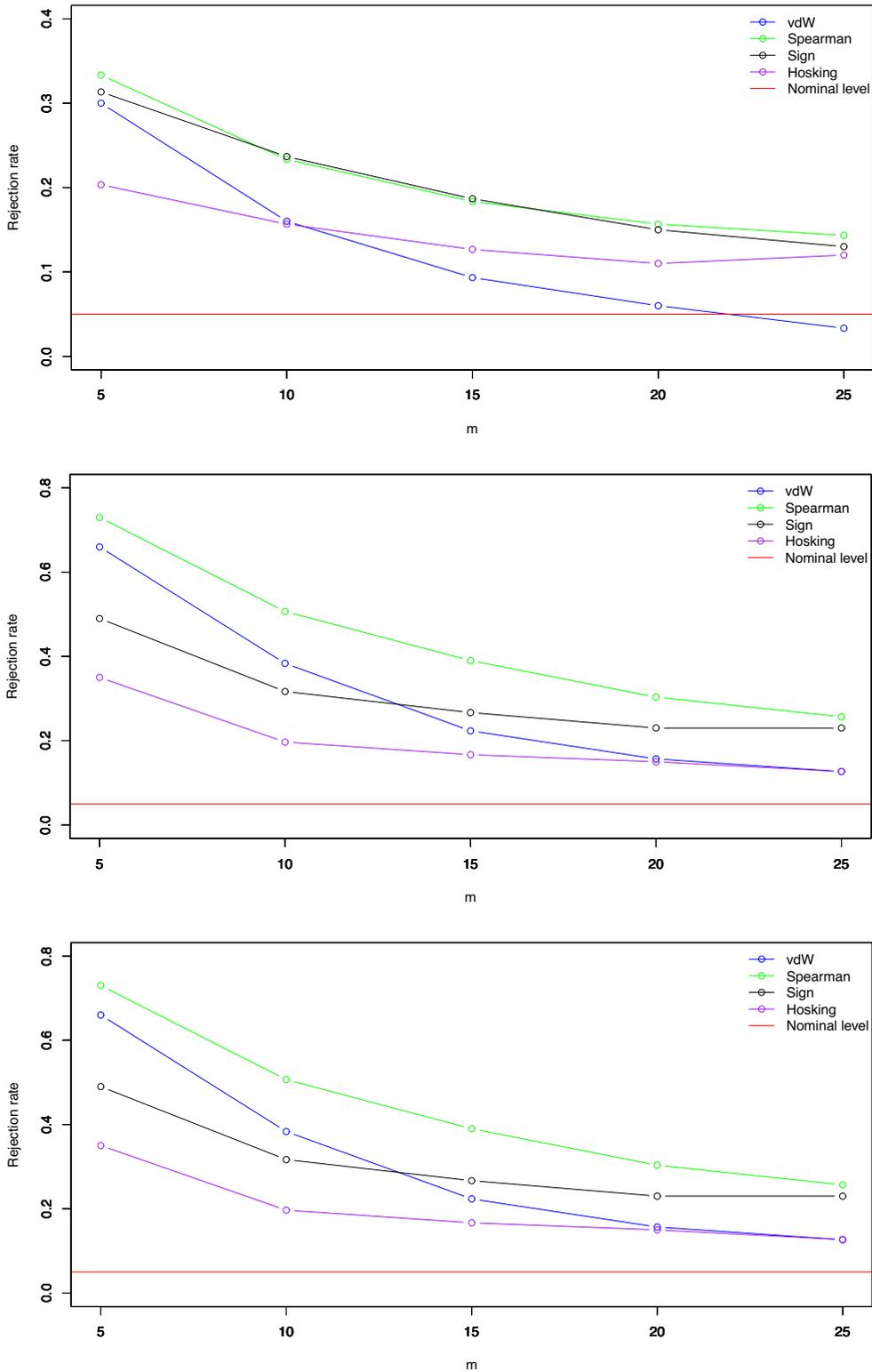
$$\frac{3}{8} \mathcal{N}(\boldsymbol{\mu}_1, \boldsymbol{\Sigma}_1) + \frac{3}{8} \mathcal{N}(\boldsymbol{\mu}_2, \boldsymbol{\Sigma}_2) + \frac{1}{4} \mathcal{N}(\boldsymbol{\mu}_3, \boldsymbol{\Sigma}_3), \quad (5.2)$$

with  $\boldsymbol{\mu}_1 = (-5, 0)'$ ,  $\boldsymbol{\mu}_2 = (5, 0)'$ ,  $\boldsymbol{\mu}_3 = (0, 0)'$ , and

$$\boldsymbol{\Sigma}_1 = \begin{pmatrix} 7 & 5 \\ 5 & 5 \end{pmatrix}, \quad \boldsymbol{\Sigma}_2 = \begin{pmatrix} 7 & -6 \\ 6 & 6 \end{pmatrix}, \quad \boldsymbol{\Sigma}_3 = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}.$$



**Fig. 2.** Rejection frequencies (nominal level 5%; asymptotic chi-square critical values and, for the van der Waerden, Spearman, and sign tests, bias-corrected (BC) permutational critical values), for  $m = 5, 10, \dots, 25$ , of the Gaussian, van der Waerden, Spearman, and sign portmanteau tests for unspecified VARMA(1,1) model, under the VARMA(1,1) model (5.1) with spherical normal (upper panel), mixture (5.2) of three Gaussians (middle panel) and skew- $t_3$  (lower panel) innovation densities. Number of observations  $n = 1000$ ;  $N = 300$  replications. The solid and dashed horizontal lines indicate the nominal level  $\alpha = 5\%$  and the rejection limits of the 5% two-sided test of the hypothesis that the actual level indeed is 5%.



**Fig. 3.** Rejection frequencies (nominal level 5%; asymptotic chi-square critical values), for  $m = 5, 10, \dots, 25$ , of the Gaussian, van der Waerden, Spearman, and sign portmanteau tests for unspecified VARMA(1,1) model, under the VARMA(1,2) alternative (5.3) with spherical normal (upper panel), mixture (5.2) of three Gaussians (middle panel) and skew  $t_3$  (lower panel) innovation densities. Number of observations  $n = 1000$ ;  $N = 300$  replications. The solid horizontal line indicates the nominal level  $\alpha = 5\%$ .

**Table 1**

Optimal lag numbers  $m_0$  and empirical powers (against the VARMA alternative (5.3)) of the van der Waerden, Spearman, sign and Hosking portmanteau tests, under spherical normal, mixture of three Gaussians, and skew  $t_3$  innovation densities, respectively. Boldface indicates the winner (consistently, van der Waerden with  $m = 5$  or 6) in each column.

Test	Gaussian		Mixture		Skew- $t_3$	
	$m_0$	Power	$m_0$	Power	$m_0$	Power
vdW	6	<b>0.272</b>	5	<b>0.660</b>	5	<b>0.580</b>
Spearman	14	0.193	10	0.507	10	0.400
Sign	22	0.142	–	–	15	0.217
Hosking	5	0.203	5	0.350	5	0.237

For each replication, a VARMA(1,1) model has been estimated, via Gaussian likelihood, van der Waerden, Spearman, and sign R-estimation, respectively (ranks and signs computed from a grid with  $n_R = 25$ ,  $n_S = 40$ , and  $n_0 = 0$ ). Based on these estimators, the Gaussian, van der Waerden, Spearman, and sign portmanteau tests (ranks and signs similarly computed from a grid with  $n_R = 25$ ,  $n_S = 40$ , and  $n_0 = 0$ ) were performed for each replication at 5% nominal level for  $m = 5, 10, \dots, 25$ . Note that a series length of  $n = 1000$  in dimension two—roughly corresponding to a series length of  $n \approx \sqrt{1000} \approx 32$  in dimension one—is not a very long one.

The rejection frequencies under the null hypothesis of an unspecified VARMA(1,1) model are shown in Figure 2 for the spherical Gaussian, the mixture of three Gaussians (5.2), and the skew- $t_3$  innovation densities. The dotted horizontal lines provide the 5% critical band outside which the empirical size of a test is significantly different from the nominal level  $\alpha = 5\%$ .

Irrespective of the innovation density, rejection frequencies (for fixed  $n$ ) tend to decrease as  $m$  increases. As a function of  $m$ , the decrease depends on the scores and the actual innovation. It appears, for instance, that the van der Waerden test meets the nominal  $\alpha$ -level constraint for  $m \approx 8$  under all three innovation densities, while the Spearman test requires  $m$  to be as large as 10 (under the mixture of Gaussians or the Skew- $t_3$ ) or even 14 (under the spherical Gaussian) for not exhibiting significant over-rejection. In that respect, van der Waerden is “more  $m$ -parsimonious” than Spearman, and the sign test is particularly “greedy” (never meeting the nominal 5% size requirement for  $m$  between 5 and 25 in the mixture case). As for Hosking’s traditional pseudo-Gaussian test, it is meeting the nominal level constraint for  $m$  as small as 5.

In order to compare the powers of the various portmanteau tests, we simulated the bivariate VARMA(1, 2) model with matrix coefficients

$$\mathbf{A}_1 = \begin{pmatrix} 0.5 & 0.2 \\ -0.1 & 0.4 \end{pmatrix}, \quad \mathbf{B}_1 = \begin{pmatrix} 0.3 & 0 \\ 0 & 0.4 \end{pmatrix}, \quad \text{and} \quad \mathbf{B}_2 = \begin{pmatrix} 0.07 & 0.03 \\ -0.02 & 0.1 \end{pmatrix} \quad (5.3)$$

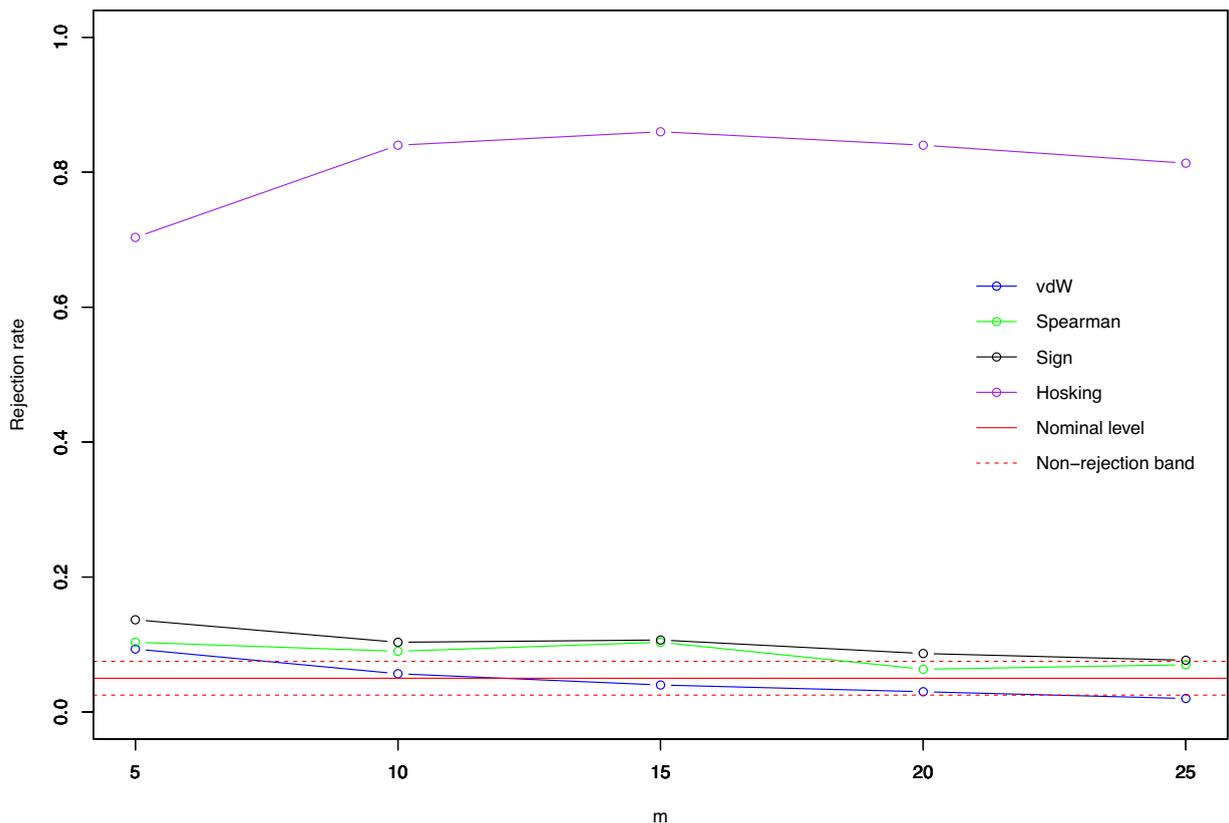
under the same innovation densities as above.

The rejection frequencies, under the VARMA(1,2) alternative (5.3) with spherical normal, mixture (5.2) of three Gaussians, and skew- $t_3$  innovation densities, of the same Gaussian, van der Waerden, Spearman, and sign portmanteau tests as in Figure 2 are shown in Figure 3 for  $m = 5, 10, \dots, 25$ . As expected, rejection frequencies tend to decrease as  $m$  increases, irrespective of the innovation density. Interestingly, for given  $m$ , Spearman has greater power than Hosking’s traditional test, even under normal innovations, for all  $m \leq 25$ . So has the sign test—but this is largely due to the fact that, as shown in Figure 2, it is over-rejecting. Now, a fair comparison should take the results of Figure 2 into account. Table 1 provides, for each test and each innovation density, the optimal lag numbers  $m_0$ —that is, the lag compatible with the 5% level condition at which the empirical power is maximal—along with that power. Inspection of the table reveals that van der Waerden with  $m = 5$  or 6 lags is the uniform winner, and more parcimonious than Spearman, which is second best, albeit with more than  $m = 10$  lags. Both are overperforming Hosking’s traditional test.

These conclusions, of course, are based on a very limited Monte Carlo experiment. More general innovation densities and more general alternatives (bilinear, heteroskedastic, etc.) clearly should be considered. However, a general phenomenon emerges: increasing the number  $m$  of lags while  $n$  is fixed helps (see Propositions 3.3 and 4.3) improve the chi-square approximation, but comes at the cost of additional degrees of freedom, hence larger critical values and lower rejection frequencies. The choice of a test statistic, thus, relies on a trade-off: once it is large enough for the level constraint to be satisfied, increasing  $m$  is not desirable unless a gain of power compensates for the loss caused by the additional degrees of freedom.

## 5.2. Robustness

Next, we investigate the robustness properties of the center-outward rank-based and pseudo-Gaussian tests. A thorough theoretical treatment of this issue is beyond the scope of this paper, and we only perform a limited Monte-Carlo study, focusing on two cases: resistance to a temporary change in the innovation density, with a short “crisis period” of heavy-tailed innovations, and resistance to the presence of a patchy outlier. In both cases, the level and power of the rank-based test remains largely unaffected while the pseudo-Gaussian severely over-rejects.



**Fig. 4.** Rejection frequencies (nominal level 5%; asymptotic chi-square critical values), for  $m = 5, 10, \dots, 25$ , of the Gaussian, van der Waerden, Spearman, and sign portmanteau tests for unspecified VARMA(1,1) model, under the VARMA(1,1) model (5.1) with spherical normal innovations contaminated by 30 points of skew- $t_2$  ones. Number of observations  $n = 1030$ ; number of replications  $N = 300$ . The solid and dashed horizontal lines indicate the nominal level  $\alpha = 5\%$  and the rejection limits of the 5% two-sided test of the hypothesis that the actual level indeed is 5%.

### 5.2.1. Resistance to changes in the innovation density

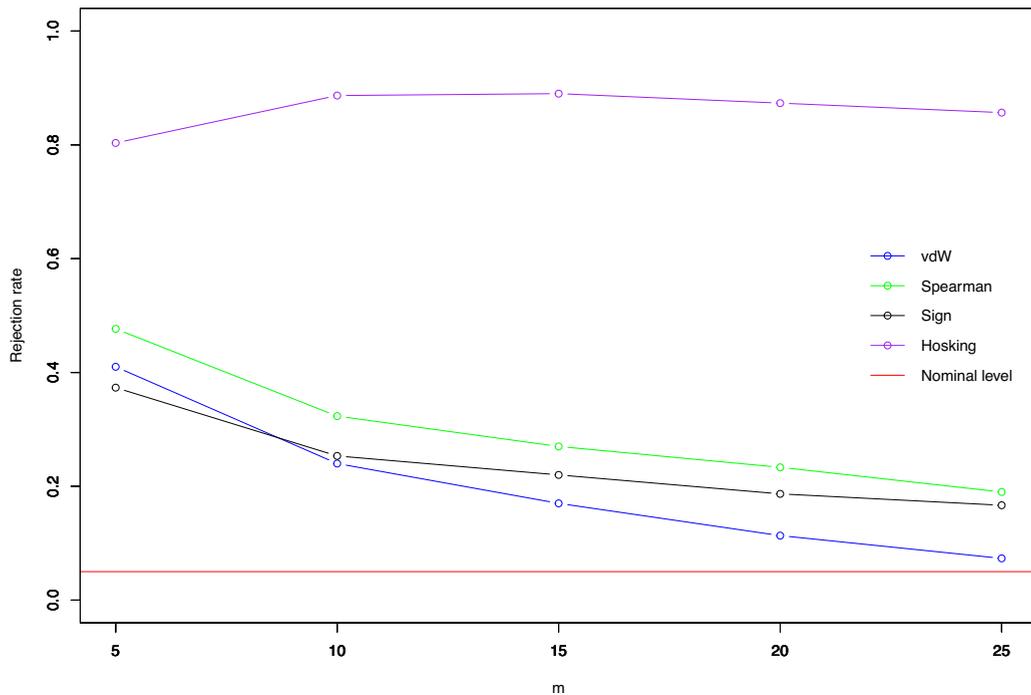
In this section, we consider, for  $-29 \leq t \leq 1000$  (hence,  $n = 1030$  observations), the same VARMA(1,1) and VARMA(1,2) data-generating processes as in Section 5.1, with spherical Gaussian innovations for  $t = 1, \dots, 1000$  but, for the initial period  $t = -29, \dots, 0$  (that is, 3% of the total observation period), innovation outliers from a centered skew- $t_2$ . This can be interpreted as a crisis-exit scenario with crisis period ending at time  $t = 0$ —an information that is not available to the analyst.

The rejection frequencies under the VARMA(1,1) null hypothesis (5.1) and the VARMA(1,2) alternative (5.3) are shown in Figures 4 and 5, respectively. Comparing Figure 4 and the top panel of Figure 2 reveals that the size of the rank-based tests under the null are remarkably robust (the van der Waerden satisfies the nominal 5% constraint for  $m \geq 5$ , as it did in the upper panel of Figure 2) while the size of Hosking's pseudo-Gaussian test, with a rejection frequency under the null of about 0.80, is very severely affected: the impact on Hosking of the initial crisis (3% of the observation period), thus, is quite persistent. The high rejection frequency of the classical portmanteau test, in Figure 5, which is more or less the same as under the null, is meaningless and entirely explained by overall over-rejection under the null. Quite on the contrary, the empirical powers of the rank-based tests remain quite stable and van der Waerden with  $m = 6$  remains the best choice (note that Spearman with  $m = 14$  slightly over-rejects). Rank-based portmanteau tests, thus, unlike the traditional ones, are robust to the fact that 30 observations still belonging to the crisis period have been included in the analysis (the end-of-crisis date being unknown).

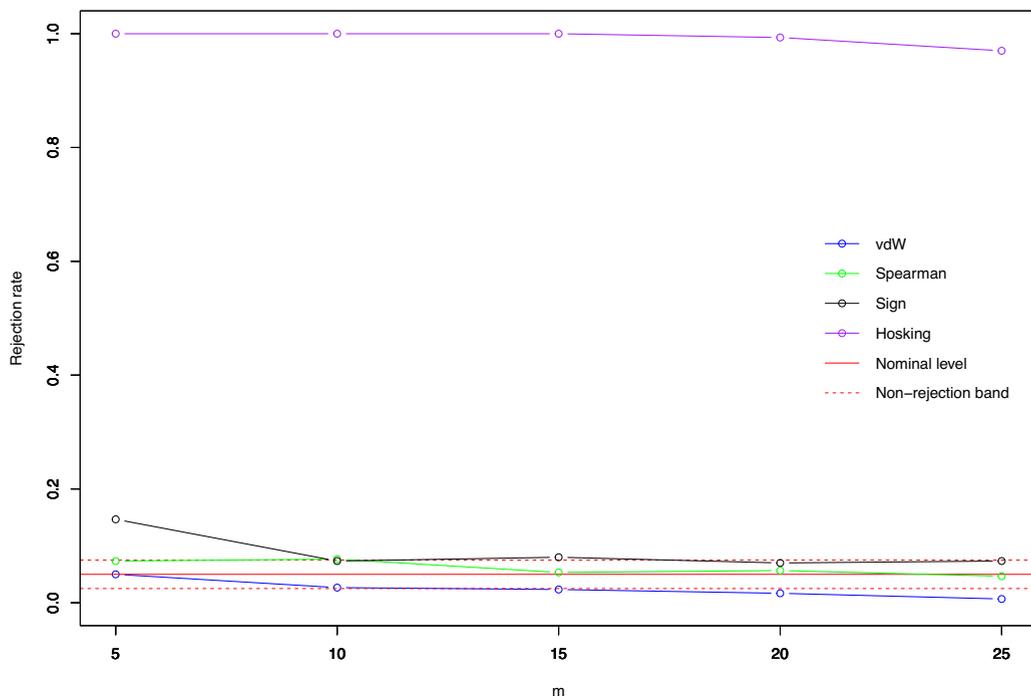
### 5.2.2. Resistance to patchy outliers

Another form of robustness is resistance to the presence of patchy innovation outliers. Here, at  $t = 500, 501$  and  $t = 502, 503$  respectively, shocks of sizes  $(20, 20)'$  and  $(-20, -20)'$  are added to the Gaussian innovations for the same VARMA(1,1) and VARMA(1,2) data-generating processes as in Section 5.1 (same observation period,  $n = 1000$ ).

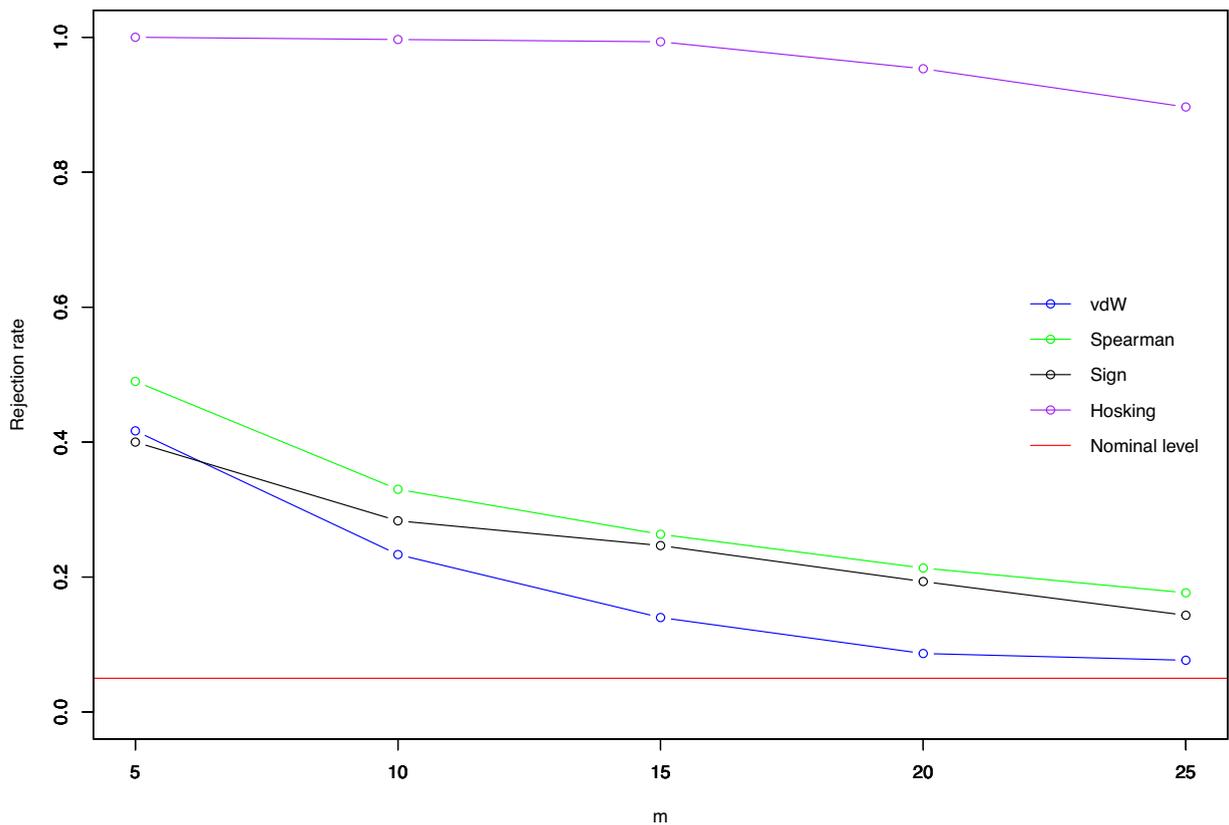
The rejection frequencies under the VARMA(1,1) null hypothesis and VARMA(1,2) alternative are shown in Figures 6 and 7, respectively. The plots clearly indicate that the rank-based tests are considerably more robust than their Gaussian competitor: the empirical size of the Hosking test (value 1 for  $m = 5, 10, 15$ , values 0.993 and 0.970 for  $m = 20, 25$ ) is ex-



**Fig. 5.** Rejection frequencies (nominal level 5%; asymptotic chi-square critical values), for  $m = 5, 10, \dots, 25$ , of the Gaussian, van der Waerden, Spearman, and sign portmanteau tests for unspecified VARMA(1,1) model, under the VARMA(1,2) alternative (5.3), with spherical normal innovations contaminated by 30 points of skew- $t_2$  additive outliers. Number of observations  $n = 1030$ ; number of replications  $N = 300$ . The solid and dashed horizontal lines indicate the nominal level  $\alpha = 5\%$ .



**Fig. 6.** Rejection frequencies (nominal level 5%; asymptotic chi-square critical values), for  $m = 5, 10, \dots, 25$ , of the Gaussian, van der Waerden, Spearman, and sign portmanteau tests for unspecified VARMA(1,1) model, under the VARMA(1,1) model (5.1) with spherical normal innovations contaminated by 4 points of additive outliers of size 20. Number of observations  $n = 1000$ ; number of replications  $N = 300$ . The solid and dashed horizontal lines indicate the nominal level  $\alpha = 5\%$  and the rejection limits of the 5% two-sided test of the hypothesis that the actual level indeed is 5%.



**Fig. 7.** Rejection frequencies (nominal level 5%; asymptotic chi-square critical values), for  $m = 5, 10, \dots, 25$ , of the Gaussian, van der Waerden, Spearman, and sign portmanteau tests for unspecified VARMA(1,1) model, under the VARMA(1,2) alternative (5.3), with spherical normal innovations contaminated by 4 points of additive outliers of size 20. Number of observations  $n = 1030$ ; number of replications  $N = 300$ . The solid and dashed horizontal lines indicate the nominal level  $\alpha = 5\%$ .

ploding and its empirical power, which is close to the empirical size, is therefore meaningless. In stark contrast, Figure 6 and the top panel of Figure 2 remain relatively similar: the empirical sizes of the rank-based tests are not affected by outliers (e.g., for  $m = 5$  or 6, the van der Waerden tests yield the nominal 5% size) and their empirical powers remain approximately the same as under the uncontaminated case.

## 6. Conclusions

This paper achieves two objectives: a rigorous statement of the asymptotic behavior of the portmanteau test statistic under the null hypothesis, and the construction of rank-based versions of the same. Simulations indicate that the latter, thanks to a faster convergence, both in terms of the number  $n$  of observations and the number  $m$  of lags involved, bring substantial potential gains of power when compared to their classical counterparts—particularly so under skewed and heavy-tailed innovation densities.

The same simulations also provides empirical evidence of a better resistance of rank-based tests to the presence of contaminated innovations. A more thorough analysis, on the model of Hampel et al. (1986), Ronchetti and Yen (1986), or Heritier and Ronchetti (1994) (all restricted to models with independent observations), of the robustness properties of rank tests in the time-series context is highly desirable, and should be left for further research.

## Acknowledgement

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Appendix

**Proof of Proposition 3.1.** The result follows from deriving the asymptotic joint distribution, under  $P_{\theta, f}^{(n)}$ , of

$$(n-i)^{1/2} \text{vec}(\Gamma_{i:\mathcal{N}}^{(n)}(\boldsymbol{\theta})), \quad (n-j)^{1/2} \text{vec}(\Gamma_{j:\mathcal{N}}^{(n)}(\boldsymbol{\theta})), \quad \text{and} \quad \Delta_f^{(n)}(\boldsymbol{\theta})$$

along the same lines as in the proof of Lemma B.1 in Hallin et al. (2022b). An application of Le Cam's third Lemma yields the asymptotic shifts in (3.2) and concludes. Details are left to the reader.  $\square$

**Proof of Lemma 3.2.** Since  $\Delta_{\mathcal{N}}^{(n)}(\hat{\boldsymbol{\theta}}_{\mathcal{N}}^{(n)}) = \mathbf{0}$ , we have, in view of (3.4),  $\Gamma_{i:\mathcal{N}}^{(n)}(\hat{\boldsymbol{\theta}}_{\mathcal{N}}^{(n)}) = \Gamma_{i:\mathcal{N}}^{(n)*}(\hat{\boldsymbol{\theta}}_{\mathcal{N}}^{(n)})$ . Moreover, (3.3) entails

$$\begin{aligned} & (n-i)^{1/2} \text{vec}(\Gamma_{i:\mathcal{N}}^{(n)*}(\hat{\boldsymbol{\theta}}_{\mathcal{N}}^{(n)}) - \Gamma_{i:\mathcal{N}}^{(n)*}(\boldsymbol{\theta})) \\ &= (n-i)^{1/2} \text{vec}(\Gamma_{i:\mathcal{N}}^{(n)}(\hat{\boldsymbol{\theta}}_{\mathcal{N}}^{(n)}) - \Gamma_{i:\mathcal{N}}^{(n)}(\boldsymbol{\theta})) - (\boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma}^{-1}) \mathbf{c}'_{i,\boldsymbol{\theta}} \left( \sum_{i=1}^{n-1} \mathbf{c}_{i,\boldsymbol{\theta}} (\boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma}^{-1}) \mathbf{c}'_{i,\boldsymbol{\theta}} \right)^{-1} \\ & \quad \times \left( \sum_{i=1}^{n-1} \mathbf{c}_{i,\boldsymbol{\theta}} (n-i)^{1/2} \text{vec}(\Gamma_{i:\mathcal{N}}^{(n)}(\hat{\boldsymbol{\theta}}_{\mathcal{N}}^{(n)}) - \Gamma_{i:\mathcal{N}}^{(n)}(\boldsymbol{\theta})) \right) + o_P(1) \\ &= \left( (\boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma}^{-1}) \mathbf{c}'_{i,\boldsymbol{\theta}} - (\boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma}^{-1}) \mathbf{c}'_{i,\boldsymbol{\theta}} \left( \sum_{i=1}^{n-1} \mathbf{c}_{i,\boldsymbol{\theta}} (\boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma}^{-1}) \mathbf{c}'_{i,\boldsymbol{\theta}} \right)^{-1} \right. \\ & \quad \left. \times \left( \sum_{i=1}^{n-1} \mathbf{c}_{i,\boldsymbol{\theta}} (\boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma}^{-1}) \mathbf{c}'_{i,\boldsymbol{\theta}} \right) \right) n^{1/2} (\hat{\boldsymbol{\theta}}_{\mathcal{N}}^{(n)} - \boldsymbol{\theta}) + o_P(1) \\ &= o_P(1). \end{aligned}$$

The result follows.  $\square$

**Proof of Proposition 3.3.** Due to Lemma 3.2 and the exponential decrease, as  $i \rightarrow \infty$ , of  $\|\mathbf{c}_{i,\boldsymbol{\theta}}\|$ ,

$$\left\| (n-i)^{1/2} \text{vec} \left( \Gamma_{i:\mathcal{N}}^{(m,n)**}(\boldsymbol{\theta}) - \Gamma_{i:\mathcal{N}}^{(n)}(\hat{\boldsymbol{\theta}}_{\mathcal{N}}^{(n)}) \right) \right\|$$

with probability one converges to zero exponentially fast as  $m$  increases and  $n \rightarrow \infty$ . Part (i) then follows.

Turning to Part (ii), let

$$\Gamma_{\mathcal{N}}^{(m,n)**}(\boldsymbol{\theta}) := n^{-1/2} \left( (n-1)^{1/2} (\text{vec}(\Gamma_{1:\mathcal{N}}^{(m,n)**}(\boldsymbol{\theta})))', \dots, (n-m)^{1/2} (\text{vec}(\Gamma_{m:\mathcal{N}}^{(m,n)**}(\boldsymbol{\theta})))' \right).$$

Note that  $\Gamma_{\mathcal{N}}^{(m,n)**}(\boldsymbol{\theta})$  can be written as  $\Gamma_{\mathcal{N}}^{(m,n)**}(\boldsymbol{\theta}) = \mathbf{E}_{\mathcal{N}}^{(m)}(\boldsymbol{\theta}) \Gamma_{\mathcal{N}}^{(m,n)}(\boldsymbol{\theta})$ , where

$$\begin{aligned} \mathbf{E}_{\mathcal{N}}^{(m)}(\boldsymbol{\theta}) &:= \mathbf{I}_{md^2} - (\mathbf{I}_m \otimes \boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma}^{-1}) \mathbf{C}_{\boldsymbol{\theta}}^{(m+1)'} \\ & \quad \times \left( \mathbf{C}_{\boldsymbol{\theta}}^{(m+1)} (\mathbf{I}_m \otimes \boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma}^{-1}) \mathbf{C}_{\boldsymbol{\theta}}^{(m+1)'} \right)^{-1} \mathbf{C}_{\boldsymbol{\theta}}^{(m+1)} \end{aligned}$$

is an idempotent matrix. Then it follows from Proposition 3.1 that  $\Gamma_{\mathcal{N}}^{(m,n)**}(\boldsymbol{\theta})$  is asymptotically normal with mean zero and covariance  $\mathbf{E}_{\mathcal{N}}^{(m)}(\boldsymbol{\theta}) (\mathbf{I}_m \otimes \boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma})$  under  $P_{\theta, f}^{(n)}$ . It remains to prove that  $\text{tr}(\mathbf{E}_{\boldsymbol{\theta}}^{(m)}) = (m-p-q)d^2$ , where  $\text{tr}(\mathbf{E}_{\boldsymbol{\theta}}^{(m)})$  denotes the trace of  $\mathbf{E}_{\boldsymbol{\theta}}^{(m)}$ . Using  $\text{tr}(\mathbf{ABC}) = \text{tr}(\mathbf{CAB})$ , we obtain

$$\begin{aligned} \text{tr}(\mathbf{E}_{\boldsymbol{\theta}}^{(m)}) &= \text{tr}(\mathbf{I}_{md^2}) - \text{tr} \left( \mathbf{C}_{\boldsymbol{\theta}}^{(m+1)} (\mathbf{I}_m \otimes \boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma}^{-1}) \mathbf{C}_{\boldsymbol{\theta}}^{(m+1)'} \right. \\ & \quad \left. \times \left( \mathbf{C}_{\boldsymbol{\theta}}^{(m+1)} (\mathbf{I}_m \otimes \boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma}^{-1}) \mathbf{C}_{\boldsymbol{\theta}}^{(m+1)'} \right)^{-1} \right) \\ &= md^2 - \text{tr}(\mathbf{I}_{(p+q)d^2}) = (m-p-q)d^2. \end{aligned}$$

The result follows.  $\square$

**Proof of Lemma 4.2.** It follows from (4.10) and Lemma 4.4 in Kreiss (1987) that, under  $P_{\theta, f}^{(n)}$ ,

$$\Delta_{\sim J_1, J_2}^{(n)}(\boldsymbol{\theta}_{\sim J_1, J_2}^{(n)}) = \Delta_{\sim J_1, J_2}^{(n)}(\boldsymbol{\theta}) - n^{1/2} \Upsilon_{J_1, J_2, f}^{(n)}(\boldsymbol{\theta}) \left( \boldsymbol{\theta}_{\sim J_1, J_2}^{(n)} - \boldsymbol{\theta} \right) + o_P(1).$$

Hence, letting  $\Upsilon_{\tilde{J}_1, \tilde{J}_2, f}(\boldsymbol{\theta}) := \lim_{n \rightarrow \infty} \Upsilon_{\tilde{J}_1, \tilde{J}_2, f}^{(n)}(\boldsymbol{\theta})$ , it follows from the definition of  $\boldsymbol{\theta}_{\tilde{J}_1, \tilde{J}_2}^{(n)}$  in (4.11) that

$$\begin{aligned} \Delta_{\tilde{J}_1, \tilde{J}_2}^{(n)}(\boldsymbol{\theta}_{\tilde{J}_1, \tilde{J}_2}^{(n)}) &= \Delta_{\tilde{J}_1, \tilde{J}_2}^{(n)}(\boldsymbol{\theta}) - n^{1/2} \left( \Upsilon_{\tilde{J}_1, \tilde{J}_2, f}(\boldsymbol{\theta}) \bar{\boldsymbol{\theta}}^{(n)} + n^{-1/2} \Delta_{\tilde{J}_1, \tilde{J}_2}^{(n)}(\bar{\boldsymbol{\theta}}^{(n)}) - \Upsilon_{\tilde{J}_1, \tilde{J}_2, f}(\boldsymbol{\theta}) \boldsymbol{\theta} \right) + o_P(1) \\ &= \Delta_{\tilde{J}_1, \tilde{J}_2}^{(n)}(\boldsymbol{\theta}) - n^{1/2} \Upsilon_{\tilde{J}_1, \tilde{J}_2, f}(\boldsymbol{\theta}) (\bar{\boldsymbol{\theta}}^{(n)} - \boldsymbol{\theta}) - \Delta_{\tilde{J}_1, \tilde{J}_2}^{(n)}(\bar{\boldsymbol{\theta}}^{(n)}) + o_P(1) \\ &= n^{1/2} \Upsilon_{\tilde{J}_1, \tilde{J}_2, f}(\boldsymbol{\theta}) (\bar{\boldsymbol{\theta}}^{(n)} - \boldsymbol{\theta}) - n^{1/2} \Upsilon_{\tilde{J}_1, \tilde{J}_2, f}(\boldsymbol{\theta}) (\bar{\boldsymbol{\theta}}^{(n)} - \boldsymbol{\theta}) + o_P(1) = o_P(1). \end{aligned}$$

This establishes part (i) of the lemma. Part (ii) follows as a corollary of Part (i), since  $\Gamma_{\tilde{J}_1, \tilde{J}_2, f}^{(n)*}(\boldsymbol{\theta}_{\tilde{J}_1, \tilde{J}_2}^{(n)})$  is the residual of the regression of  $\Gamma_{\tilde{J}_1, \tilde{J}_2, f}^{(n)}(\boldsymbol{\theta})$  with respect to  $\Delta_{\tilde{J}_1, \tilde{J}_2}^{(n)}(\boldsymbol{\theta})$  computed at  $\boldsymbol{\theta} = \boldsymbol{\theta}_{\tilde{J}_1, \tilde{J}_2}^{(n)}$ . Asymptotic linearity, the asymptotic discreteness of  $\boldsymbol{\theta}_{\tilde{J}_1, \tilde{J}_2}^{(n)}$  and Lemma 4.4 of Kreiss (1987) entail (iii).  $\square$

**Proof of Proposition 4.3.** Part (i) follows from (4.19) and the exponential decrease of  $\|\mathbf{c}_{i, \boldsymbol{\theta}}\|$ . Turning to Part (ii), let

$$\Gamma_{\tilde{J}_1, \tilde{J}_2, f}^{(m, n)**}(\boldsymbol{\theta}) := n^{-1/2} \left( (n-1)^{1/2} (\text{vec}(\Gamma_{\tilde{J}_1, \tilde{J}_2, f}^{(m, n)**}(\boldsymbol{\theta})))', \dots, (n-m)^{1/2} (\text{vec}(\Gamma_{\tilde{J}_1, \tilde{J}_2, f}^{(m, n)**}(\boldsymbol{\theta})))' \right)'$$

where  $\Gamma_{\tilde{J}_1, \tilde{J}_2, f}^{(m, n)**}(\boldsymbol{\theta})$  is defined in (4.21). Note that  $\Gamma_{\tilde{J}_1, \tilde{J}_2, f}^{(m, n)**}(\boldsymbol{\theta})$  can be written as

$$\Gamma_{\tilde{J}_1, \tilde{J}_2, f}^{(m, n)**}(\boldsymbol{\theta}) = \mathbf{E}_{\tilde{J}_1, \tilde{J}_2, f}^{(m)}(\boldsymbol{\theta}) \Gamma_{\tilde{J}_1, \tilde{J}_2}^{(m, n)}(\boldsymbol{\theta}).$$

Then, from the definition of  $Q_{\tilde{J}_1, \tilde{J}_2, f}^{(n)**}(\boldsymbol{\theta})$  in (4.20),

$$\begin{aligned} Q_{\tilde{J}_1, \tilde{J}_2, f}^{(n)**}(\boldsymbol{\theta}) &= n \left( \Gamma_{\tilde{J}_1, \tilde{J}_2, f}^{(m, n)**}(\boldsymbol{\theta}) \right)' (\text{diag}(\Omega_{\tilde{J}_1, \tilde{J}_2, f}^{(m)}(\boldsymbol{\theta}))_{1 \leq i \leq m})^{-1} \Gamma_{\tilde{J}_1, \tilde{J}_2, f}^{(m, n)**}(\boldsymbol{\theta}) \\ &= n \left( \Gamma_{\tilde{J}_1, \tilde{J}_2}^{(m, n)}(\boldsymbol{\theta}) \right)' \mathbf{E}_{\tilde{J}_1, \tilde{J}_2, f}^{(m)'}(\boldsymbol{\theta}) (\text{diag}(\Omega_{\tilde{J}_1, \tilde{J}_2, f}^{(m)}(\boldsymbol{\theta}))_{1 \leq i \leq m})^{-1} \\ &\quad \times \mathbf{E}_{\tilde{J}_1, \tilde{J}_2, f}^{(m)}(\boldsymbol{\theta}) \Gamma_{\tilde{J}_1, \tilde{J}_2}^{(m, n)}(\boldsymbol{\theta}) \\ &= n \left( \Gamma_{\tilde{J}_1, \tilde{J}_2}^{(m, n)}(\boldsymbol{\theta}) \right)' \mathbf{E}_{\tilde{J}_1, \tilde{J}_2, f}^{(m)'}(\boldsymbol{\theta}) (\mathbf{E}_{\tilde{J}_1, \tilde{J}_2, f}^{(m)}(\boldsymbol{\theta}) (\mathbf{I}_m \otimes \mathbf{D}_{\tilde{J}_1, \tilde{J}_2}) \\ &\quad \times \mathbf{E}_{\tilde{J}_1, \tilde{J}_2, f}^{(m)'}(\boldsymbol{\theta}))^{-1} \mathbf{E}_{\tilde{J}_1, \tilde{J}_2, f}^{(m)}(\boldsymbol{\theta}) \Gamma_{\tilde{J}_1, \tilde{J}_2}^{(m, n)}(\boldsymbol{\theta}), \end{aligned}$$

where the last equality follows from (4.16). Now, letting

$$\begin{aligned} \mathbf{M}_{\boldsymbol{\theta}} &:= \mathbf{M}_{\tilde{J}_1, \tilde{J}_2, f}^{(m)}(\boldsymbol{\theta}) \\ &:= (\mathbf{I}_m \otimes \mathbf{D}_{\tilde{J}_1, \tilde{J}_2}^{1/2}) \mathbf{E}_{\tilde{J}_1, \tilde{J}_2, f}^{(m)'}(\boldsymbol{\theta}) (\mathbf{E}_{\tilde{J}_1, \tilde{J}_2, f}^{(m)}(\boldsymbol{\theta}) (\mathbf{I}_m \otimes \mathbf{D}_{\tilde{J}_1, \tilde{J}_2}) \mathbf{E}_{\tilde{J}_1, \tilde{J}_2, f}^{(m)'}(\boldsymbol{\theta}))^{-1} \mathbf{E}_{\tilde{J}_1, \tilde{J}_2, f}^{(m)}(\boldsymbol{\theta}) (\mathbf{I}_m \otimes \mathbf{D}_{\tilde{J}_1, \tilde{J}_2}^{1/2}), \end{aligned}$$

we obtain

$$Q_{\tilde{J}_1, \tilde{J}_2, f}^{(n)**}(\boldsymbol{\theta}) = n \left( \Gamma_{\tilde{J}_1, \tilde{J}_2}^{(m, n)}(\boldsymbol{\theta}) \right)' (\mathbf{I}_m \otimes \mathbf{D}_{\tilde{J}_1, \tilde{J}_2}^{-1/2}) \mathbf{M}_{\boldsymbol{\theta}} (\mathbf{I}_m \otimes \mathbf{D}_{\tilde{J}_1, \tilde{J}_2}^{-1/2}) \Gamma_{\tilde{J}_1, \tilde{J}_2}^{(m, n)}(\boldsymbol{\theta}).$$

In view of Proposition 4.1,  $(\mathbf{I}_m \otimes \mathbf{D}_{\tilde{J}_1, \tilde{J}_2}^{-1/2}) \Gamma_{\tilde{J}_1, \tilde{J}_2}^{(m, n)}(\boldsymbol{\theta})$  is asymptotically normal with covariance  $\mathbf{I}_{md^2}$ . Moreover, since  $\mathbf{M}_{\boldsymbol{\theta}}$  is symmetric and idempotent (indeed,  $\mathbf{M}_{\boldsymbol{\theta}} \mathbf{M}_{\boldsymbol{\theta}} = \mathbf{M}_{\boldsymbol{\theta}}$ ), it only remains to show that  $\mathbf{M}_{\boldsymbol{\theta}}$  is of rank  $(m-p-q)d^2$ . Note that  $\mathbf{M}_{\boldsymbol{\theta}}$  has the same rank as  $\mathbf{E}_{\tilde{J}_1, \tilde{J}_2, f}^{(m)}(\boldsymbol{\theta})$  which is also idempotent, it suffices to prove that  $\text{tr}(\mathbf{E}_{\tilde{J}_1, \tilde{J}_2, f}^{(m)}(\boldsymbol{\theta})) = (m-p-q)d^2$ . Using again the fact that  $\text{tr}(\mathbf{ABC}) = \text{tr}(\mathbf{CAB})$ , we have

$$\begin{aligned} \text{tr}(\mathbf{E}_{\tilde{J}_1, \tilde{J}_2, f}^{(m)}(\boldsymbol{\theta})) &= \text{tr}(\mathbf{I}_{md^2}) - \text{tr} \left( (\mathbf{I}_m \otimes \mathbf{K}_{\tilde{J}_1, \tilde{J}_2, f}) \mathbf{C}_{\boldsymbol{\theta}}^{(m+1)'} \left( \sum_{i=1}^m \mathbf{c}_{i, \boldsymbol{\theta}} \mathbf{K}_{\tilde{J}_1, \tilde{J}_2, f} \mathbf{c}_{i, \boldsymbol{\theta}}' \right)^{-1} \mathbf{C}_{\boldsymbol{\theta}}^{(m+1)} \right) \\ &= md^2 - \text{tr} \left( \mathbf{C}_{\boldsymbol{\theta}}^{(m+1)} (\mathbf{I}_m \otimes \mathbf{K}_{\tilde{J}_1, \tilde{J}_2, f}) \mathbf{C}_{\boldsymbol{\theta}}^{(m+1)'} \left( \sum_{i=1}^m \mathbf{c}_{i, \boldsymbol{\theta}} \mathbf{K}_{\tilde{J}_1, \tilde{J}_2, f} \mathbf{c}_{i, \boldsymbol{\theta}}' \right)^{-1} \right) \\ &= md^2 - \text{tr}(\mathbf{I}_{(p+q)d^2}) = (m-p-q)d^2. \end{aligned}$$

The result follows.  $\square$

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