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Research paper

Managing ridesharing with incentives in a bottleneck model

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ABSTRACT

Ridesharing is recognized as an environmentally friendly alternative to daily transportation that can potentially reduce vehicular carbon emissions. Transportation Network Companies (TNC) have been providing ridesharing services. However, generalized costs need to be balanced between rideshare drivers and passengers via financial instruments to reach a sustainable equilibrium. In this paper, we demonstrate the effectiveness of Incentive-Based Travel Demand Management (IBTDM) strategies in managing morning commutes with ridesharing. Separate time-varying incentive schemes for both rideshare drivers and passengers are proposed to maintain the user equilibrium. Since IBTDM strategies are usually non-mandatory, it is found that the optimal incentives vary with respect to market penetration. Subsequently, we took the difficulty of scheme implementation, system cost reduction, and net utility as the criteria, and obtained the optimal incentive schemes under different criteria. Finally, we verified the effectiveness of the above incentive scheme in promoting ridesharing and reducing system costs by means of experimental economics.

1. Introduction

Economic development has put tremendous pressure on urban transportation systems. Expanding road capacity to meet ever-growing travel demand has grown increasingly difficult, especially for dense metropolitan areas. Instead of changing available supply, decision makers are now turning to Travel Demand Management (TDM) solutions instead. TDM refers to the guiding role of traffic policies to enable commuters to shift travel behavior to redistribute demand across time and space, or to shift demand from personal cars to sustainable travel modes such as transit and ridesharing. Ridesharing has been recognized as a cost-effective alternative that can reduce the demand for personal cars and the overall use of resources, such as gasoline and parking spaces, while meeting the travel needs of commuters. Many TDM solutions are based on encouraging ridesharing.

Aided by the proliferation of smart phones equipped with GPS devices, the ridesharing market (e.g., DiDi Hitch) has exploded in recent years. There were 16 billion ridesharing trips completed in 2017 globally, and the projected number of yearly ridesharing trips by 2030 is 35 billion (DMR, 2022). However, the imbalance between the number of drivers and passengers is usually seen in ridesharing services. Dida Chuxing is the earliest ridesharing platform in China. Now, its business has covered 359 cities, with more than 130 million users and 15 million car owners in 2019 (Hu & Creutzig, 2022). In addition to the extra fuel costs and vehicle depreciation losses that the drivers need to bear, when compared with driving on their own, rideshare drivers perceive psychological inconveniences and a lack of privacy. This utility loss is generally quantified as an inconvenience cost (Liu & Li, 2017). If no financial instrument is adopted, these cost discrepancies remain imbalanced, meaning that drivers have no motivation to share their vehicles.

Current practices involve making passengers pay drivers to eliminate this cost imbalance (Liu & Li, 2017; Ma & Zhang, 2017). However, many passengers exhibit strong resistance to charging, this discourages passengers from ridesharing to some extent as well. As a result, true ridesharing demand cannot be fully revealed and fulfilled. Without enough participating drivers and passengers, it is difficult to guarantee the efficiency and success of a ridesharing program. In order to eliminate the cost imbalance between drivers and passengers, and encourage users to rideshare, DiDi spends a significant portion of its revenue on subsidies for drivers and passengers each year. From the 2021 annual financial report of DiDi, the subsidy received by drivers made up 4.7% of the total fare income, and the subsidy received by passengers made up 10.9% of the total fare income. Therefore, to wisely allocate fine incentives and

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subsidies to achieve the most social welfare becomes an important research topic.

1.1. Incentives in Travel Demand Management

Positive incentives and subsidies can be used to encourage passengers to take part in ridesharing (Brownstone & Golob, 1992). They found that ridesharing subsidies reduced drive-alone commuting. Furthermore, Agatz et al. (2012) pointed out that subsidized ridesharing systems could provide a relatively inexpensive way to increase the capacity and efficiency of a transportation system. Therefore, managing travel demand to encourage ridesharing through incentives is an important component in the domain of Active Travel Demand Management (ATDM), which leads to the concept of IBTDM.

IBTDM strategies have gained increasing attention in recent years, as they are more attractive to commuters. IBTDM adopts incentives, such as gift cards or other rewards, to demotivate driving trips and to redistribute demand across space and time. Unlike punitive strategies such as tolling, IBTDM approaches have higher levels of social acceptance and have far lower equipment and facility costs.

In Principles of Economics (Mankiw, 2014), it is mentioned that rewards and tolls are both strategies that align with economic principles. Taxes are one of the sources of the reward budget. Using only tax revenue as funds for IBTDM may raise fairness issues, as it would come from all taxpayers, including those who don't even drive. However, in addition to taxes, IBTDM has multiple sources of income to balance the budget, such as in-app advertisements, sponsorship of in-kind rewards, and so on.

In addition to the evident benefits of alleviating traffic congestion and lowering travel costs, incentive schemes offer utility that extends beyond these straightforward evaluations. The reduction in travel time can generate various intangible external advantages, such as improved environmental conditions and enhanced traffic safety. These benefits are enjoyed by everyone, regardless of whether they travel by car or not. Therefore, the government has sufficient will to implement this strategy.

Bauer et al. (2018) summarized most existing pilots and mainstreamed IBTDM programs that aim to illustrate how to encourage travelers to consider alternative travel modes, routes or departure times through positive incentives. In the Hollandse Brug project (a pilot in the famous Spitsmijden program), 40% of the 2975 participants were willing to change their travel behavior, of which 17.5% were changing travel modes (Bliemer et al., 2009, ; Ettema et al., 2010). The city of Bengaluru implemented a project to incentivize commuters to travel during off-peak times by employing a pyramidal structure of rewards (Merugu et al., 2009). Stanford University launched the Congestion and Parking Relief Incentives (CAPRI) program with a $3 million grant from the Federal Highway Administration (Zhu et al., 2015). These initiatives resulted in a significant reduction in congestion and commuting time.

In recent years, IBTDM has also received more and more attention in theoretical research. Rouwendal et al. (2012) and Sun et al. (2020) proposed optimal time-varying incentive schemes to alleviate congestion during morning peak hours. Cenedese et al. (2022) proposed a dynamically discounted energy price as an incentive to decrease bottleneck congestion during rush hours. Ghafelebashi et al. (2023) provided personalized incentives to travelers to shift their routes, so as to achieve the purpose of reducing traffic congestion. Additionally, Wu et al. (2023) analyzed the relationship between private organizations and public organizations at various stages of IBTDM.

1.2. Knowledge gaps and research goals

It should be noted that the incentive is not simply a symmetrical version of tolling, and the theoretical research on the tolling cannot be copied to the incentive scheme. The different constraints and concerns between incentives and fees are two-fold:

First, IBTDM programs are typically voluntary. In reality, it is impossible for such programs to have market penetration of 100%. The penetration is more like the awareness rate of the IBTDM program within society. Program administrators can manipulate this rate by carrying out marketing campaigns and spreading the word. As the Hollandse Brug project shows, there are 10,900 travelers invited to participate in the project. However, there are only 2975 participants, so the market penetration is only 27% (Bliemer et al., 2009). Limited by the constraint of penetration, the charging scheme cannot be directly applied in IBTDM. Currently, there is no sufficient theoretical framework for managing ridesharing programs considering penetration.

Second, unlike tolling system, which actually collect tolls to maintain the system, IBTDM programs need to invest budget to alleviate congestion. Therefore, it is necessary to verify the effectiveness of the incentive scheme through its net utility, that is, whether the reduction of the system cost is greater than the budget investment. Although an unlimited budget input holds advantages in reducing system cost, it may lead to the situation that system cost reduction being lower than budget input due to the decreasing marginal utility of budget. In such case, the incentive scheme is ineffective. Therefore, to avoid the waste of resources due to blind budget investment, it is crucial to consider the difference between the system cost reduction and the budget investment when selecting the incentive scheme.

With rapidly evolving smartphone technology, incentives can now be fully time-dependent, personalized, and customized via smartphone apps. When compared with coarse incentive schemes, the time-varying incentive scheme allows more flexibility and higher efficiency for the administrator to achieve desirable management objectives. Additionally, administrators are gradually realizing that travelers have a higher demand for refined incentive schemes (time-varying incentive schemes).

In this paper, we aim to investigate the effectiveness of time-varying incentives for managing a ridesharing program in a bottleneck model. Since there is a single road connecting only one OD pair in the bottleneck model, the ridesharing in this paper can be regarded as a peer-to-peer ridesharing (hereinafter referred to as ridesharing).

We plan to determine the optimal time-varying incentive schemes for drivers and passengers in a way that minimizes the total system cost (sum of total schedule delay costs and total travel time costs). The analytical solutions are often obtained by making a set of assumptions that are often not realistic (e.g., assuming all users receive the information and will behave rationally). However, in reality, there may be irrational behaviors among travelers, leading to the problem that analytical solutions cannot be applied to real-world applications. Therefore, in this paper, we first obtain the optimal solution through theoretical derivation, and then collect the user's real travel behavior under the optimal TDM solution through experimental economics to verify that the analytical solution is still applicable to real life.

Specifically, we take the following steps to address the previously mentioned issues when utilizing incentives to promote ridesharing. First, since the total system cost is related to all commuters' departure patterns, we find optimal departure patterns with minimal system costs under market penetrations. Second, since the incentive provider can control the commuters' departure patterns through the incentive profile (with an assumption of user equilibrium), we derive optimal incentive profiles that lead to the optimal departure patterns. Third, we verify the effectiveness of the ridesharing-based incentive scheme for alleviating traffic congestion by means of experimental economics.

The contributions of this paper are mainly threefold. 1). This study optimizes a time-varying incentive scheme for ridesharing, providing a theoretical reference for administrators to set the optimal incentive scheme. 2). This study considers market penetration in ridesharing research. Due to the voluntary nature of IBTDM, it is impossible for all travelers to participate in the incentive project. If market penetration is not considered (the situation of 100% market penetration), it will cause a waste of resources and inefficiency in the incentive project. 3). This study verifies the effectiveness of the ridesharing-based incentive scheme in alleviating traffic congestion through experimental economics.

The rest of this paper is organized as follows: Section 2 reviews relevant past literature related to the application of bottleneck models in ridesharing. Section 3 presents the traditional bottleneck model with and without a ridesharing program. Section 4 presents the departure patterns when the total system cost is minimized with certain penetration rates. Section 5 presents the optimal incentive functions to achieve an equilibrium state and gives a numerical example. Section 6 validates our analysis through experimental economics, and Section 7 concludes the paper.

2. Literature review

Vickrey (1969) first proposed the concept of the bottleneck model to depict the commuting problem during peak hours. Based on this, many researchers have studied and expanded the bottleneck model. Daganzo (1985) proved that the model's equilibrium state is unique even with more general distributions of target arrival times. Cohen (1987) proposed the concept of the fine toll, which increased linearly from zero at the beginning of the queue to a maximum at the target arrival time, and subsequently decreased back to zero at the end of the queue. More generally, Arnott et al. (1988) calculated a time-varying toll profile for the User Equilibrium (UE) state and System Optimum (SO) state with heterogeneous commuters. In addition, Arnott et al. (1990) investigated a coarse toll based on the bottleneck model. Comprehensive reviews of the bottleneck model can be found in Small (2015) and more recently Li et al. (2020). Since the bottleneck model can well describe the dynamic process of queue formation and dissipation, it is widely adopted to study the spatiotemporal allocation of HOV lanes, tolls and subsidy schemes, and other dynamic problems related to ridesharing.

For instance, Qian and Zhang (2011) proposed a bottleneck model-based multi-modal morning commute model, including transit, driving alone and ridesharing. They found that beneficial measures, such as the presence of an HOV lane, encouraged commuters to rideshare. Xiao et al. (2016) investigated the impact of internal factors (such as commuters' travel costs) on ridesharing behavior in the bottleneck congestion. They proposed a model containing a ridesharing lane and a general-purpose lane only for solo individual drivers. However, the rideshare commuters needed to bear an additional constant cost in addition to the cost of travel time and schedule delay, but shorter experienced rush hours.

Furthermore, Liu and Li (2017) conducted a study on the impact of travel cost differences between different roles on ridesharing behavior. Commuters were divided into three roles: solo drivers, rideshare drivers, and passengers. The value of travel time units is equal for the three types of commuters. However, drivers need to pay additional vehicle operating expenses such as fuel, vehicle depreciation, and insurance, and rideshare commuters need to bear additional inconvenience costs. Both out-of-pocket expenses and inconvenience costs are linear functions of travel time and route length.

Different from Liu and Li (2017), Ma and Zhang (2017) internalized the difference in travel cost between passengers and drivers in the unit value of travel time. They found that properly designed parking and rideshare fees can reduce Vehicle Miles Traveled (VMT), Vehicle Hours Traveled (VHT), and total costs. Ma and Zhang (2017) refined the ridesharing model in detail with many variables introduced such as the ridesharing ratio, ridesharing payments, and parking fees. With all these additional decision variables, the authors concluded that an equilibrium could not be found explicitly.

Wang et al. (2019) introduced a platform that balanced the generalized costs between rideshare drivers and passengers. The passengers needed to pay for the platform, and some of the money was used to compensate the drivers.

Zhong et al. (2020) proposed a revised bottleneck model that analyzed dynamic ridesharing for a morning commute with a HOV or a High Occupancy Toll (HOT) lane. Similar to Liu and Li (2017), Zhong et al. (2020) assumed that rideshare drivers needed to bear an additional Vehicle Operation Expense (VOE) and inconvenience cost. To maintain a state of equilibrium, passengers needed to pay drivers a ridesharing fee. Unlike Liu and Li (2017), Zhong et al. (2020) treated vehicle operating expenses and inconvenience costs as constants, while the ridesharing ratio was modeled as a time-varying decision variable. They found that a HOV lane was able to promote ridesharing and boost welfare, while a HOT lane brought additional welfare gains with only a modest toll.

Based on the ridesharing model of Xiao et al. (2016), Wei et al. (2022) investigated the temporal road capacity allocation scheme and step tolling-rewarding scheme for ridesharing. An innovative highlight is that Wei et al. (2022) is the first to study the effect of heterogeneous inconvenience costs on ridesharing and found that travelers with low inconvenience costs are more receptive to ridesharing.

Li et al. (2022) applied the auction system to the ridesharing management scheme, providing a new way to encourage ridesharing. Specifically, they devised an auction-based permit allocation and sharing system (A-PASS), which eliminates congestion by ensuring that the number of commuters arriving at the bottleneck does not exceed the capacity. Commuters can choose to rideshare and share permits in order to promote ridesharing. Li et al. (2022) found that when ride-sharing increases, all stake holders, including ridesharing platform, commuters, and society, benefit from the system.

Existing research on ridesharing in bottleneck model has mostly focused on time-varying pricing, de Palma et al. (2022) innovatively studied the effect of dynamic congestion with scheduling preferences on ridesharing matching. Different from the traditional bottleneck model with a single OD, they considered discrete origins of commuters and heterogeneous desired arrival times. It has been found that the optimal arrival time for rideshare vehicles is the lowest desired arrival time of the matched couple, so matching induces earliness and reduces lateness.

In this section, we reviewed some existing research on ridesharing, which mainly focuses on the spatiotemporal allocation of HOV lanes and the ridesharing charging scheme. What these studies have in common is that all the models are based on the traditional bottleneck model, and the researchers improve their own models by refining the travel roles and travel costs of travelers.

However, all the above studies do not consider the constraints of market penetration and net utility, which are two of the most critical factors for the effectiveness of IBTDM projects. If the penetration (the penetration defaults to 100%) and net utility are not considered, it is possible to overestimate the number of participants and the implementation effect of the project, resulting in a waste of resources due to blind investment, and even worse, the company will go bankrupt.

In addition, research on incentives, especially fully time-varying incentives is limited. For instance, Wang et al. (2019) proposed a partial time-varying charging-compensation scheme, with charges and compensation pre-defined as piecewise linear functions dependent on departure time. Wei et al. (2022) proposed a step tolling-rewarding scheme. The optimal time-varying incentive scheme can make the system more efficient, thereby generating greater utility with the same investment, which is beneficial to saving resources.

Furthermore, the above studies mainly focus on theoretical derivations and numerical examples, which means that the above conclusions are based on some certain assumptions. However, whether analytically optimal solutions can produce expected effects in real world has not been verified.

Therefore, this paper is aimed at solving the optimal time-varying incentive scheme considering market penetration and net utility, and proving the effectiveness of the incentive scheme proposed in this paper in promoting ridesharing and reducing system costs through experimental economics. This paper also improves the ridesharing model based on the traditional bottleneck model. Specifically, the similarities and differences between this paper and existing related works are listed in the following table (Table 1).

Based on bottleneck model, Sun et al. (2020) derived a fine incentive

Table 1
The similarities and differences between this paper and existing related studies.

Paper	Ridesharing ratio	Market Penetration	Budget	Traveler types	Toll or incentive	Travel cost			
						Schedule delay cost	Operational cost	Inconvenience cost	Travel time cost
Xiao et al. (2016)	1	No	No	Two	Toll	Yes	No	Constant	Homogeneous
Liu and Li (2017)	1	No	No	Three	Toll	Yes	Linear	Linear	Homogeneous
Ma and Zhang (2017)	Variable	No	No	Two	Toll	Yes	No	No	Heterogeneous
Wang et al. (2019)	1	No	No	Three	Toll/ incentive	Yes	Linear	Linear	Homogeneous
Yu et al. (2019)	1	No	No	Two	Toll	Yes	Constant	Constant	Heterogeneous
Zhong et al. (2020)	Variable	No	No	Three	Toll	Yes	Linear	Linear	Homogeneous
Wei et al. (2022)	Fixed	No	No	Two	Toll/ Incentive	Yes	No	Variable	Homogeneous
Li et al. (2022)	1	No	No	Three	Auction	Bidding price			
de Palma et al. (2022)	1	Elastic	No	Three	Matching	Yes	No	No	Heterogeneous
This paper	Fixed	Yes	Yes	Three	Incentive	Yes	No	No	Heterogeneous

schedule considering three dedicated constraints that are highly likely to be encountered when implementing IBTDM: 1) an incentive budget constraint, 2) a market penetration constraint, and 3) user heterogeneity. They found that, in the case where everyone drives alone, the optimal incentive profile is "U-shaped" when the penetration rate is less than 100% or the budget is constrained. This means the incentive should be set at ends of the peak hours first. Under the incentive scheme, participants would choose to depart earlier or later without queueing, while non-participants would tend to depart in the middle of the peak hours. When considering heterogeneous commuters, it is observed that the wealthier individuals, who have a higher unit time cost due to their aversion to queueing, will prefer to depart at both ends of the peak hour. On the other hand, individuals with lower unit time costs, will still choose to depart in the middle of the peak hour despite facing queues.

Different from Sun et al. (2020) who only considered solo drivers, this paper considers three types of travel roles, including solo drivers, rideshare drivers and passengers, and aims to determine the optimal time-varying incentive schemes for rideshare drivers and passengers respectively. In essence, Sun et al. (2020) is an extreme case in this paper when the ridesharing ratio (the ratio of the number of passengers to drivers in the same vehicle) is set to 0.

To conclude, we propose an improved ridesharing model. One of the major additions in our model is the consideration of penetration and net utility, which are not considered in other papers. More importantly, we also conduct economic experiments to verify the effectiveness of the incentive scheme.

3. Model setting

A notational glossary for reference in this paper is provided in Table 2.

In the bottleneck model, there are N commuters traveling from home to work during the morning rush hours through a road with a single bottleneck along the way. It is assumed that the bottleneck has a fixed capacity c, and that a queue will form if the arrival rate of vehicles at the bottleneck exceeds this capacity c. The travel cost $\varphi(t)$ for a commuter who departs at t is:

$$\varphi(t) = \begin{cases} \alpha\left(T(t)+T_f\right)+\beta\left[t^*-\left(t+T(t)+T_f\right)\right], \text{if } t^* \geq \left(t+T(t)+T_f\right) \\ \alpha\left(T(t)+T_f\right)+\gamma\left[\left(t+T(t)+T_f\right)-t^*\right], \text{if } t^* < \left(t+T(t)+T_f\right) \end{cases} \tag{1}$$

where α is the unit value of travel time, the total travel time is composed of queueing time $T(t)$ and the fixed free-flow travel time T_f. The queueing time $T(t)$ is the ratio of the queueing length to the bottleneck capacity. t^* is the target arrival time, commuters experience losses regardless of if they are earlier or later than t^*, β and γ are the unit value of early and late arrivals, respectively (Arnott et al., 1993; Cohen, 1987).

Table 2
Notational glossary.

Notation	Interpretation
α_i	Unit value of travel time for role i
β	Unit value of schedule early delay
γ	Unit value of schedule late delay
δ	$\delta \overset{\text{def}}{=} \beta\gamma/(\beta+\gamma)$
$\varphi(t)$	Travel cost excluding the incentive of a commuter departing at t
c	Capacity of the bottleneck
i	$i=1$ solo driver, $i=2$ rideshare driver, $i=3$ passenger
p	Market penetration
t'	Departure time of the vehicle arriving at t^*
t_b	Departure time of the first vehicle
t_e	Departure time of the last vehicle
t^*	Target arrival time
C_s	Total system cost
C_e	Total system cost of ridesharing participants departing at the fringes of $[t_b, t_e]$
C_m	Total system cost of ridesharing participants departing in the middle of $[t_b, t_e]$
C_1	Total system cost of solo drivers
$C_d(t)$	Schedule delay cost of the commuter departing at t
$C_t(t)$	Travel time cost of the commuter departing at t
$D_v(t)$	Departure rate of vehicles departing at t
$G_i(t)$	Generalized travel cost of a role i commuter departing at t
$I_i(t)$	Incentive for a role i commuter departing at t
M^*	Minimum budget required to achieve UE
N	Number of commuters
N_i	Number of commuters for role i
N_c	Number of ridesharing participants
N_v	Number of vehicles
N_e	Number of vehicles departing at the fringes of $[t_b, t_e]$
N_m	Number of vehicles departing in the middle of $[t_b, t_e]$
R	Ridesharing ratio
$S(t)$	Ridesharing payment to drivers departing at t
$T(t)$	Queueing time of a commuter departing at t
T_f	Free flow travel time
$W_p(t)$	Maximum willingness-to-pay of passengers departing at t
$W_d(t)$	Minimum willingness-to-accept of rideshare drivers departing at t

Before introducing the ridesharing model framework, the main assumptions in this study are listed herein.

Assumption 1. The choice of travelers between ridesharing and solo driving is exogenous. In addition, similar to de Palma et al. (2022), we assume that a central operator determines the matching and this can be enforced on the ridesharing participants. Therefore, in practice, ridesharing participants can depart at the theoretically optimal departure times.

Assumption 2. All commuters are fully rational, and seek to minimize individual disutility.

Assumption 3. There are three roles in the ridesharing program: solo drivers, rideshare drivers, and passengers (Liu & Li, 2017). Similar to Ma and Zhang (2017), we internalized the exogenous parameters such as inconvenience and operational costs into the unit value of travel time of the different roles. We denote α_1, α_2 and α_3 as the unit value of travel time of solo drivers, rideshare drivers, and passengers, respectively, where $\alpha_2 > \alpha_1 > \alpha_3$.

Assumption 4. The ridesharing ratio R is defined as the ratio of the number of passengers to drivers in the same vehicle. To develop an intuition for a more general problem from mathematical and economic perspectives, we assume the ridesharing ratio is an aggregated and continuous variable, which differs from assigning an individual traveler to a specific vehicle as in microscopic simulation (Ma & Zhang, 2017). It is important to note that the continuous ridesharing ratio encompasses the case of an integer ridesharing ratio. Therefore, if we obtain the optimal incentive function for the continuous ridesharing ratio, we can simply substitute the integer ridesharing ratio into the function to obtain the optimal incentive scheme for that specific case. However, if we don't resubstitute the integer ridesharing ratio into the function, the results obtained when R is a continuous variable will hold for integer values only if the results do not depend on the value of R.

Assumption 5. Rideshare is known to effectively reduce vehicle demand and reduce fuel consumption (Caulfield, 2009; Kelly, 2007). Therefore, we assume that rideshare can effectively reduce the average unit value of travel time of ridesharing participants, i.e., $(1+R)\alpha_1 > \alpha_2 + R\alpha_3$, otherwise no one would like to share ride. Specifically, if the inequality does not hold, it means that driving alone is less expensive than ridesharing at the same departure time. Therefore, although travelers can try to reduce their schedule delay costs by shifting their departure time, their optimal choice at any given moment is to drive alone. To simplify the model, we introduce the notation $\theta \equiv \alpha_2 + R\alpha_3$, which represents the unit travel time cost of rideshare vehicles.

Since the purpose of incentive schemes is to facilitate ridesharing, incentives are only provided to ridesharing participants, and not to solo drivers. Therefore, the generalized travel cost $G_i(t)$ of the commuter who departs at time t is:

$$G_i(t) = \alpha_i\left(T(t) + T_f\right) + C_d(t) - I_i(t), i = 1, 2, 3 \tag{2}$$

where α_i denotes the unit value of time of different roles, $C_d(t)$ denotes the schedule delay cost of the commuter departing at t, $I_i(t)$ denotes the incentive provided to the commuter who departs at t, and $I_1(t) = 0$.

No individual can reduce their own travel costs by unilaterally changing their travel decisions at UE. Thus, the generalized travel costs of commuters with the same role stay the same even when departing at different times. Essentially, $\dot{G}_i(t) = 0, i = 1, 2, 3$, for $t \in [t_b, t_e]$, where t_b is the departure time of the first vehicle, and t_e is the departure time of the last vehicle.

Besides, the generalized travel costs for the rideshare drivers and passengers departing at the same time are equal at equilibrium, which can be expressed as $G_2(t) = G_3(t)$, for $t \in [t_b, t_e]$. Substituting $G_2(t) = G_3(t)$ into (2), we have:

$$I_2(t) - I_3(t) = (\alpha_2 - \alpha_3)\left[T(t) + T_f\right] \tag{3}$$

In addition, according to (1), $\dot{G}_i(t) = 0$ can be expressed as:

$$I_i(t) = \begin{cases} (\alpha_i - \beta)D_v(t)/c - \alpha_i, for\ t \in [t_b, t^{'}] \\ (\alpha_i + \gamma)D_v(t)/c - \alpha_i, for\ t \in (t^{'}, t_e] \end{cases}, i = 2, 3 \tag{4}$$

where $D_v(t)$ is the vehicle departure rate. It can be seen from (4) that the incentive profile $I_i(t)$ can be expressed as a function of $D_v(t)$, so that the departure rate $D_v(t)$ is an inverse function of the incentive scheme $I_i(t)$. Therefore, administrators can control the departure rate of vehicles through incentive schemes, in order to achieve the purpose of alleviating traffic congestion.

In addition to providing rideshare drivers and passengers with differentiated incentives as shown in (3), the travel cost difference between them can be eliminated by implementing a ride fee system, where passengers pay the rideshare drivers for the ride. Let $S(t)$ denote the ride fee that passengers need to pay to the rideshare drivers at time t. In this case, providing the same time-varying incentive scheme $I_3(t)$ to rideshare drivers and passengers is sufficient to compensate for the differences in travel costs over time.

To maintain the coherence of the paper, the calculation process and detailed discussion about $S(t)$ can be found in Appendix A. Based on the findings in Appendix A, it is evident that the optimal travel pattern of travelers under the two schemes is the same, and one scheme can be easily derived from the other. Since the topic of this paper is on the optimal incentive scheme in the ridesharing mode, we only consider the optimal differentiated incentive scheme for rideshared drivers and passengers in the following content.

4. Optimal departure patterns under the constraint of market penetration

Existing research on bottleneck model-based ridesharing mostly focuses on toll schemes and capacity allocation. All of these schemes are mandatory management strategies based on punishment. Different from these mandatory management strategies, IBTDM usually encourage travelers to change their inherent travel behavior using positive financial incentives. Potential travelers have the right to choose to participate in the program or not, which makes it necessary to consider market penetration.

In reality, the concept of "market penetration" means that there are Np commuters amongst all N commuters who know about the IBTDM program and have been given the permit to download the IBTDM app (assuming the program is carried out via mobile apps), where p denotes the market penetration. In addition, incentive programs typically have limited funds (e.g., Metropia, IncenTrip, RideAmigos, etc.) and thus cannot inform everyone about the IBTDM program. Therefore, administrators need to consider the constraints of market penetration when formulating incentive schemes. If penetration is too low, it may result in additional budgetary investment. On the contrary, it may affect the effectiveness of the incentive scheme.

Assume that there are a total of N_c commuters who choose to rideshare and a total of N_1 commuters who choose to drive alone. This is represented by $N_c + N_1 = N, 0 \le N_c, N_1 \le N$ and $N_c = Np$. Therefore, the total number of vehicles N_v is then given by:

$$N_v = N_c / (1 + R) + N_1 \tag{5}$$

It should be noted that the difference between t_b and t_e is only related to the total number of vehicles N_v when the total system cost is minimized, i.e., $N_v = c(t_e - t_b)$ (Arnott et al., 1988). Furthermore, the value of N_v depends on R and p. Therefore, the total system travel cost is influenced by the ridesharing ratio and penetration.

The total system cost is also related to the commuters' departure pattern. Thus, our goal is to investigate the departure patterns that correspond with the minimum total system cost for any given market penetration rate. Solving for this optimal departure pattern is equivalent to solving for the instantaneous departure rate of each commuter role at each moment within the interval $[t_b, t_e]$. If we want to solve for the instantaneous departure rate of the various roles, we need to know the departure time interval for each role.

In order to complete this logic, we first try to determine whether solo drivers and ridesharing participants will depart together when the total system cost is minimized. Next, in order to further determine the departure time interval of the solo drivers and ridesharing participants, we try to determine whether the departure time interval of the solo drivers is continuous. Finally, we determine the critical time of each interval, so that the departure rates of the solo drivers and the ridesharing participants at any time can be obtained. To prove whether the solo drivers and ridesharing participants depart together, we have the following proposition:

Proposition 1. When the total system cost is minimized rideshares and solo drivers will not depart at the same time.

The detailed proof can be found in Appendix B. Proposition 1 states that ridesharing participants and solo drivers will not depart together. This implies that there is a sequence of departures for ridesharing participants and solo drivers. In order to further determine the departure sequence, we have the following proposition:

Proposition 2. The departure time of solo drivers is continuous during $[t_b, t']$ and $[t', t_e]$, respectively, when the total system cost is minimized.

The detailed proof can be found in Appendix C. According to Proposition 1 and Proposition 2, there are three possibilities for the optimal departure order of ridesharing participants and solo drivers.

Case 1. Solo drivers depart in the middle of the peak period, while ridesharing participants depart at both ends of the peak period.

Case 2. Ridesharing participants depart from home in the middle of the peak period, while solo drivers depart at both ends of the peak period.

Case 3. Some ridesharing participants depart at both ends of the peak period, and the remaining ridesharing participants depart in the middle of the peak period.

After determining the departure sequence of solo drivers and rideshare vehicles, the next step is to determine the departure rate of each moment, so as to determine the entire departure patterns.

For Case 1: Sun et al. (2020) proved that when participants depart at both ends and non-participants depart in the middle, the participants do not need to experience queueing in an optimal state. Solo drivers cannot receive incentives, so their departure rate is equal to the rate under the equilibrium state of the traditional bottleneck model (Arnott et al., 1988). According to the departure order and departure rate of rideshare vehicles and non-rideshare vehicles, the departure pattern of Case 1 is shown in Fig. 1(a).

For Case 2: Ridesharing participants depart from home in the middle of the peak period, while solo drivers depart at both ends of the peak period. The departure pattern of Case 2 is shown in Fig. 1(b).

Proposition 3. In Case 2, the optimal cumulative departure curve of rideshare vehicles is the blue dashed line as shown in Fig. 1(b).

A detailed proof can be found in Appendix D. In summary, if the true cumulative departure curve appears below the blue dashed line, it contradicts the UE state. Conversely, if the true cumulative departure curve appears above the blue dashed line, it contradicts the optimal state with the minimum total system cost.

For Case 3: Part of the ridesharing participants depart at the ends of the peak period, and others depart in the middle of the peak period. For ridesharing participants who depart at the ends of the peak period, similar to case 1, they do not need to experience queueing in an optimal state. For ridesharing participants who depart in the middle of the peak period, similar to Case 2, the departure rate of the early arrivals is $\alpha_1 c/(\alpha_1 - \beta)$, and the departure rate of the late arrivals is $\alpha_1 c/(\alpha_1 + \gamma)$. The departure pattern of Case 3 is shown in Fig. 1(c).

In essence, these three situations are based on a tradeoff between the queueing time cost and the schedule delay cost for rideshare vehicles. For each rideshare vehicle, departing in the middle of the peak period would lead to queueing but a lower schedule delay cost. Conversely, there is no queueing if they depart at both ends of the peak period, but there is a higher schedule delay cost. Next, we will investigate which departure pattern has the smallest total system cost under different market penetration rates.

Without loss of generality, we assume that the number of rideshare vehicles departing at both ends of the peak period is N_e and the number of rideshare vehicles departing in the middle of the peak period is N_m. Case 1 corresponds to $N_m = 0$, and Case 2 corresponds to $N_e = 0$. In addition to $N_m, N_e \geq 0$, the constraint of penetration rate needs to be satisfied:

$$(1+R)(N_m + N_e) = N_c = Np \tag{6}$$

In addition, we use N_i to denote the number of commuters in category i, where $i = 1, 2, 3$. With the equality constraints $N_c + N_1 = N$ and Eq. (5), we obtain the following constraints:

$$N_m + N_e = N_2 \tag{7}$$

$$N_2 + N_3 = N_c \tag{8}$$

The total system cost is composed of three parts: the travel time costs and schedule delay costs of N_e, N_1 and N_m. According to Eq. (1) and equality constraints (6)–(8), the travel costs of N_e, N_1 and N_m can be calculated as:

$$C_e = (1+R)(2N_1 + N_e + 2N_m)N_e\delta / (2c) + \theta N_e T_f \tag{9}$$

$$C_1 = (N_1 + 2N_m)N_1\delta / (2c) + {N_1}^2\delta / (2c) + \alpha_1 N_1 T_f \tag{10}$$

$$C_m = \theta N_m T_f + \left[\alpha_1(1+R){N_m}^2\delta + \left({N_m}^2 + 2N_1N_m\right)\theta\delta\right] / (2\alpha_1 c) \tag{11}$$

The detailed derivation process can be found in Appendix E. Therefore, the total system cost C_s can be expressed as:

$$C_s = C_e + C_1 + C_m \tag{12}$$

It can be seen from Eqs. (9)–(12) that C_s is related to N_1, N_e and N_m. In order to show that there is a monotonic relationship between C_s and N_1, we substitute (5), and (7)–(11) into (12). This gives us the total system cost C_s expressed as a function of N_1:

$$C_s = A_1{N_1}^2 + B_1N_1 + C_1 \tag{13}$$

where A_1 and B_1 can be expressed as:

$$A_1 = \delta / [2(1+R)c] \tag{14}$$

$$B_1 = N_m\delta(\theta - R\alpha_1) / (\alpha_1 c) + RN\delta / [(1+R)c] + \alpha_1 T_f - \theta T_f / (1+R) \tag{15}$$

It is worth noting that (13) is a univariate quadratic function with respect to N_1, so the monotonicity of C_s with respect to N_1 is independent of C_1. Therefore, we do not show the exact value of C_1. It can be found that $A_1 > 0$. According to $N \geq N_m$ and (5), we have the following inequality:

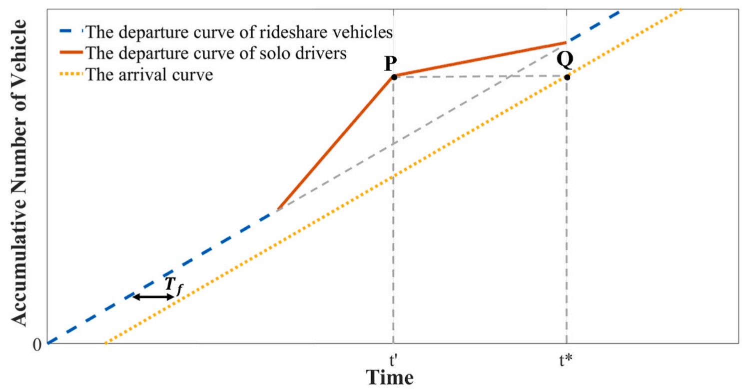
$$N_m\delta(\theta - R\alpha_1) / (\alpha_1 c) + RN\delta / [(1+R)c] \geq N_m\delta\theta / (\alpha_1 c) > 0 \tag{16}$$

In addition, according to Assumption 5, one can obtain that:

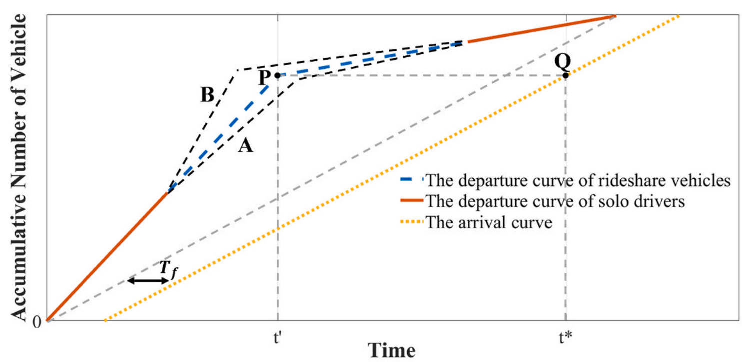
$$\alpha_1 T_f - \theta T_f / (1+R) > 0 \tag{17}$$

Combining (16) and (17), we find that $B_1 > 0$. Therefore, according to $A_1, B_1 > 0$, it can be calculated that C_s increases with respect to N_1 across the interval $[0, N]$.

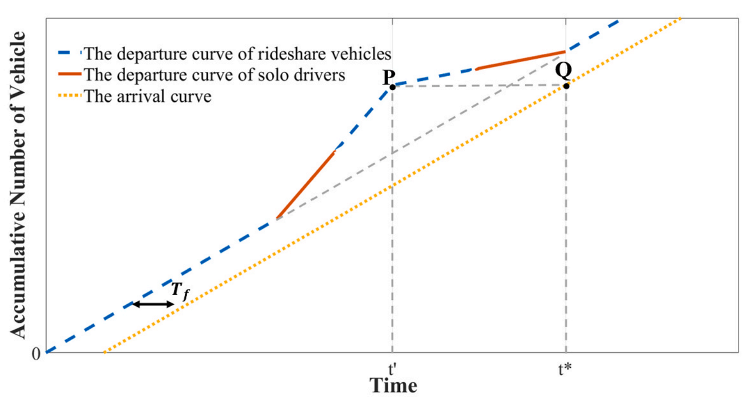
Next, in order to explore whether ridesharing participants depart at both ends of the peak period or in the middle, we derive the monotonic



(a): The departure pattern of Case 1.



(b): The departure pattern of Case 2, with the hypothetical departure curves of



(c): The departure pattern of Case 3.

Fig. 1. The departure pattern, where T_f is the free flow travel time and t^* is the target arrival time. Vehicles departing at time t' will arrive at t^* with the longest queueing time, as shown by the gray auxiliary line PQ.

relationship between C_s and N_m. Substituting $N_c = Np$ and $N_c + N_1 = N$ into (12), the total system cost C_s can be expressed as a function of N_m:

$$C_s = A_2 {N_m}^2 + B_2 N_m + C_2 \tag{18}$$

where A_2 and B_2 can be expressed as:

$$A_2 = (\alpha_2 + \alpha_3 R)\delta / (2\alpha_1 c) \tag{19}$$

$$B_2 = N(1-p)(\alpha_2 + R\alpha_3 - \alpha_1 R)\delta / (\alpha_1 c) \tag{20}$$

In order to maintain simplicity, we do not show the exact value of C_2.

Once again, it can be found that $A_2 > 0$. Therefore, the monotonicity of C_s with respect to N_m depends on the value of B_2. The value of B_2 varies depending on three cases:

Case 1. If $\alpha_2 + R\alpha_3 - \alpha_1 R \geq 0$, i.e., $R \leq \alpha_2 / (\alpha_1 - \alpha_3)$, we have $B_2 \geq 0$. Thus, C_s increases with respect to N_m across the interval $[0, Np/(1+R)]$. In other words, the total system cost is minimized when $N_m = 0$. This means that when $R \leq \alpha_2 / (\alpha_1 - \alpha_3)$, the ridesharing participants will depart at both ends of the peak interval, and the solo drivers will depart in the middle. This relationship is shown in Fig. 1(a). In addition, $R \leq \alpha_2 / (\alpha_1 - \alpha_3)$ is always true when $R = 1$.

If $\alpha_2 + R\alpha_3 - \alpha_1 R < 0$, i.e., $R > \alpha_2/(\alpha_1 - \alpha_3)$, $B_2 < 0$. Thus, C_s decreases with respect to N_m over the interval $[0, -B_2/(2A_2)]$ and increases for the interval $[-B_2/(2A_2), \infty]$. It is worth noting that the maximum value of N_m is constrained by the penetration, i.e., $N_m < Np/(1+R)$. Therefore, according to the value of $-B_2/(2A_2)$, the optimal value of N_m has two cases that minimize the total system cost.

Case 2. $-B_2/(2A_2) \geq Np/(1+R)$, according to (19) and (20), this is equivalent to $1-\theta/[(\alpha_1-\alpha_2)R + (\alpha_1-\alpha_3)R^2] \geq p$. For ease of representation, we let $p^* = 1-\theta/[(\alpha_1-\alpha_2)R + (\alpha_1-\alpha_3)R^2]$. C_s decreases with respect to N_m across the interval $[0, Np/(1+R)]$ when $p^* \geq p$, and the optimal value of N_m can be expressed as:

$$N_m = Np/(1+R) \tag{21}$$

According to (6), (21) is equivalent to $N_e = 0$. This means that when the ridesharing ratio R is relatively high and the penetration rate p is relatively low, all ridesharing participants will depart in the middle of the peak period, while solo drivers will depart at both ends of the peak period. This is seen in Fig. 1(b).

Case 3. $-B_2/(2A_2) < Np/(1+R)$. According to (19) and (20), this is equivalent to $p^* < p$. C_s is minimized when $N_m = -B_2/(2A_2)$. According to (19) and (20), we can calculate N_m as:

$$N_m = N(1-p)[(\alpha_1-\alpha_3)R - \alpha_2]/\theta \tag{22}$$

Therefore, when the ridesharing ratio R and the penetration p are both relatively high, some ridesharing participants will depart at both ends of the peak period and some will depart in the middle. This case is demonstrated in Fig. 1(c).

To conclude this section, we aimed to investigate the departure patterns that minimized the total system cost for any given market penetration. The optimal departure patterns under different market penetration and ridesharing ratios can be found in Table 3:

It can be found that the ridesharing ratio and penetration affected the departure order of ridesharing participants and solo drivers. When the ridesharing ratio was low, all ridesharing participants departed at both ends of the peak period. When the ridesharing ratio was high and the penetration was low, all ridesharing participants departed in the middle of the peak period. When the ridesharing ratio and penetration were both sufficiently high, some ridesharing participants departed at both ends of the peak and the remaining ridesharing participants departed in the middle.

Essentially, the departure pattern is a tradeoff between the queueing time and the schedule delay for rideshare vehicles. According to Assumption 5, it can be found that the greater the ridesharing ratio, the lower the average unit value of travel time for ridesharing participants. When the ridesharing ratio is low, the average unit value of travel time for ridesharing participants is high, so the travel time of ridesharing participants needs to be reduced as much as possible in order to minimize the total system cost. Therefore, ridesharing participants will tend to have a shorter queue time at SO, rather than a lessened schedule delay, meaning they will leave at the edge of peak periods. Conversely, ridesharing participants will choose a longer queueing time from the tradeoff between the schedule delay and the queueing delay when the ridesharing ratio is high, so that some of the ridesharing participants depart during the peak period at SO.

Table 3
The optimal departure patterns with different market penetration and ridesharing ratios.

	$R \leq \alpha_2/(\alpha_1-\alpha_3)$	$R > \alpha_2/(\alpha_1-\alpha_3)$
$p \leq p^*$	Case1 (shown as Fig. 1 (a))	Case 2 (shown as Fig. 1(b))
$p > p^*$		Case 3 (shown as Fig. 1(c))

5. Optimal incentive profiles to reach equilibrium

In the previous section, we obtained the optimal departure patterns with different ridesharing ratios and market penetrations. However, due to the difference in travel costs between drivers and passengers, the above travel behavior patterns (i.e., the departure pattern and the traveling mode) are not in equilibrium. In order to let travelers travel according to the optimal travel behavior pattern, it is necessary to eliminate the difference in travel cost between different travel roles through a time-varying incentive scheme. According to (4), the incentive schemes can be derived from the departure pattern. With that, the relationship between the optimal incentive schemes, budget, market penetration, and ridesharing ratio can be obtained. In this section, we derive the specific form of the incentive profile with different ridesharing ratios and penetrations.

To facilitate our calculations, it is assumed that $t^* = 0$ without a loss of generality. According to the proportionally symmetric nature of the bottleneck model, one can find that $-\beta(t_b + T_f) = \gamma(t_e + T_f)$. By combining $-\beta(t_b + T_f) = \gamma(t_e + T_f)$ and $N_v = c(t_e - t_b)$, we see that the departure time of the first and last commuters can be expressed as:

$$t_b = -\delta[Np/(1+R) + N(1-p)]/c\beta - T_f \tag{23}$$

$$t_e = \delta[Np/(1+R) + N(1-p)]/c\gamma - T_f \tag{24}$$

In order to describe the departure sequence conveniently, we divide $[t_b, t_e]$ into five intervals without a loss of generality. According to the proportional symmetry of the bottleneck model, there are $N_e\gamma/(\beta+\gamma)$ rideshare vehicles departing within the interval $[t_b, t_l^1]$, $N_1\gamma/(\beta+\gamma)$ non-rideshare vehicles departing within $[t_l^1, t_l^2]$, N_m rideshare vehicles departing within the interval $[t_l^2, t_r^2]$, $N_1\beta/(\beta+\gamma)$ non-rideshare vehicles departing within $[t_r^2, t_r^1]$, and $N_e\beta/(\beta+\gamma)$ rideshare vehicles departing within $[t_r^1, t_e]$. Specifically, $[t_l^2, t_r^2] = \varnothing$ when $N_m = 0$ and $[t_b, t_l^1] \cup [t_r^1, t_e] = \varnothing$ when $N_e = 0$.

According to the findings in the previous section, different market penetration and ridesharing ratios correspond to different optimal departure patterns. Therefore, administrators should formulate appropriate incentive schemes and ridesharing ratios based on the existing market penetration. There are three situations as follows:

5.1. Case 1: $R \leq \alpha_2/(\alpha_1 - \alpha_3)$

For this case, when the total system cost is minimized, we have shown that all ridesharing participants will depart at both ends of the peak interval and all solo drivers will depart during the peak period. Thus, according to Fig. 1(a), the departure rate $D_v(t)$ can be expressed as:

$$D_v(t) = \begin{cases} c, for\ t \in [t_b, t_l^1] \cup [t_r^1, t_e] \\ \alpha_1 c/(\alpha_1 - \beta), for\ t \in [t_l^1, t'] \\ \alpha_1 c/(\alpha_1 + \gamma), for\ t \in [t', t_r^1] \end{cases} \tag{25}$$

Substituting (25) into (4), the incentive profile of rideshare drivers and passengers can be written as:

$$I_2(t) = \begin{cases} -\beta(t - t_l^1) + (\alpha_2 - \alpha_3)T_f + k, for\ t \in [t_b, t_l^1] \\ \gamma(t - t_r^1) + (\alpha_2 - \alpha_3)T_f + k, for\ t \in [t_r^1, t_e] \end{cases} \tag{26}$$

$$I_3(t) = \begin{cases} -\beta(t - t_l^1) + k, for\ t \in [t_b, t_l^1] \\ \gamma(t - t_r^1) + k, for\ t \in [t_r^1, t_e] \end{cases} \tag{27}$$

where $t_l^1 = -\delta N(1-p)/c\beta - T_f$ and $t_r^1 = \delta N(1-p)/c\gamma - T_f$.

Considering that the IBTDM program adopts incentives to manage demand, we have the constraint $I_2(t), I_3(t) \geq 0$. Therefore, it is more intuitive to express the incentive function in the form of (26) and (27), where k is a non-negative constant. Thus, the total budget of the program can be minimized when $k = 0$. The minimum budget required to achieve this equilibrium state can be expressed as:

$$M^* = \int_{t_b}^{t_e} D_v(u)[I_2(u) + RI_3(u)]du \tag{28}$$

By substituting (25)–(27) into (28), the relationship between the minimum budget and the penetration rate when $R \le \alpha_2 / (\alpha_1 - \alpha_3)$ can be expressed as:

$$M^* = N^2 p^2 \delta / [2c(1+R)] + Np(\alpha_2 - \alpha_3)T_f / (1+R) \tag{29}$$

According to (29), it can be found that M^* increases with respect to p and decreases with respect to R. Obviously, higher market penetration means that more travelers are in the program, which in turn requires a higher incentive budget. In addition, the unit value of travel time for rideshare drivers is higher than for passengers, so drivers are rewarded more than passengers to make up for the difference in travel costs. Therefore, a higher ridesharing ratio means that a single rideshare driver can pick up more passengers, thereby reducing the required budget. If the incentive provider does not have a large enough budget to support the current market penetration, they can try to raise the ridesharing ratio to reduce the required budget. The incentive profiles are shown in Fig. 2:

It can be seen from Fig. 2 that regardless of whether the incentives are provided to the driver or passenger, the profile remains U-shaped. Commuters departing far earlier or later than the target arrival time receive higher incentives. Under these incentive profiles, no commuter will change their departure time separately. Therefore, this is an equilibrium state.

5.2. Case 2: $R > \alpha_2 / (\alpha_1 - \alpha_3)$ and $p \le p^*$

For this case, we have shown that all solo drivers will depart at both ends of the peak period and all ridesharing participants will depart in the middle of the peak period. All vehicles maintain a fixed departure rate for the intervals $[t_b, t']$ and $[t', t_e]$. Similarly, the incentive scheme and required budget corresponding to the equilibrium state can be expressed as Table 4:

Although the incentive scheme seems to be complicated in terms of expression, they are all piecewise linear functions dependent on t. The specific incentive profiles for Case 2 are shown in Fig. 3:

It can be seen from Fig. 3 that the incentives for passengers decrease as the queueing time increases, and the incentives for rideshare drivers increase as the queueing time increases. The longer the queueing time, the greater the difference in travel costs between drivers and passengers. The incentive scheme in this case has an unparalleled advantage that the incentive profile is continuous, which is easy to implement and avoids the cluster effects caused by the ending and restarting of incentive profiles.

Table 4
The optimal incentive scheme and required budget in case 2.

$$I_2(t) = \begin{cases} \dfrac{[(\alpha_3 - \alpha_1)(t - t') + (\alpha_2 - \alpha_3)(t - t_b)]\beta}{\alpha_1 - \beta} + (\alpha_2 - \alpha_3)T_f, \mathit{for}\ t \in [t_l^2, t'] \\ \dfrac{[(\alpha_3 - \alpha_1)(t' - t) + (\alpha_2 - \alpha_3)(t_e - t)]\gamma}{\alpha_1 + \gamma} + (\alpha_2 - \alpha_3)T_f, \mathit{for}\ t \in [t', t_r^2] \end{cases}$$

$$I_3(t) = \begin{cases} (\alpha_3 - \alpha_1)\beta(t - t')/(\alpha_1 - \beta), \mathit{for}\ t \in [t_l^2, t'] \\ (\alpha_3 - \alpha_1)\gamma(t' - t)/(\alpha_1 + \gamma), \mathit{for}\ t \in [t', t_r^2] \end{cases}$$

$$t_l^2 = t_b + N(1-p)\delta(\alpha_1 - \beta) / (\alpha_1 \beta c)$$

$$t_r^2 = t_e - N(1-p)\delta(\alpha_1 + \gamma) / (\alpha_1 \gamma c)$$

$$t' = t_l^2 + Np\delta(\alpha_1 - \beta) / [\alpha_1 \beta c(1+R)]$$

$$M^* = A_3 p^2 + B_3 p$$

$$A_3 = N^2 \delta[(1+R)\alpha_1 - (2R+1)\alpha_2 + R\alpha_3] / [2\alpha_1 c(1+R)^2]$$

$$B_3 = N^2 \delta(\alpha_2 - \alpha_3) / [\alpha_1 c(1+R)] + N(\alpha_2 - \alpha_3)T_f / (1+R)$$

5.3. Case 3: $R > \alpha_2 / (\alpha_1 - \alpha_3)$ and $p > p^*$

This is the most complicated case because some ridesharing participants depart at ends of the peak period and some depart in the middle. According to the departure patterns shown in Fig. 1(c), the incentive scheme corresponding to the equilibrium state is shown in Table 5:

The specific incentive profiles for Case 3 are shown in Fig. 4:

It can be seen from Fig. 4 that the incentive profiles are divided into three segments across the entire peak period, and the slope of the incentive is different for each part.

In this section, we solved for the incentive profiles required to reach equilibrium. In other words, this section provides a reference for incentive profiles that providers can utilize. In addition, different penetration rates and ridesharing ratios correspond to different optimal incentive profiles, which are presented in Table 6:

The optimal incentive scheme corresponding to case 2 (large ridesharing ratio and low penetration) is the simplest, while the optimal incentive scheme corresponding to case 3 (large ridesharing ratio and high penetration) is the most complex. In case 3, the optimal incentive scheme is divided into three separate pieces during the entire peak period, and the monotonicity of each segment is different. Firstly, the famous behavior pattern in prospect theory called complexity avoidance may trigger participants to behave more irrationally or even quit the program in order to avoid such complexity. In such a case, the expected equilibrium and the expected total travel cost reduction are unlikely to be achieved.

Secondly, separate incentive periods might lead to a clustering effect where quite a few commuters wait for the incentive periods to depart,

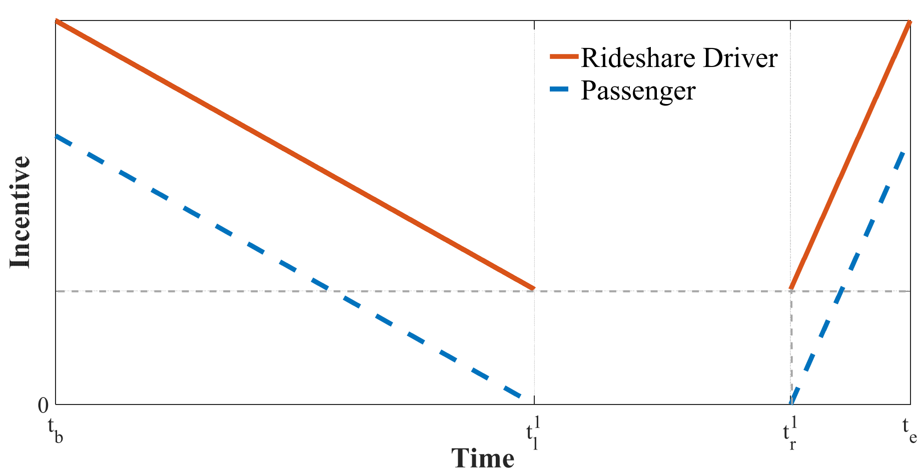


Fig. 2. The optimal incentive A, when the ridesharing ratio is low. Ridesharing participants depart in the interval $[t_b, t_l^1] \cup [t_r^1, t_e]$ and solo drivers depart within $[t_l^1, t_r^1]$.

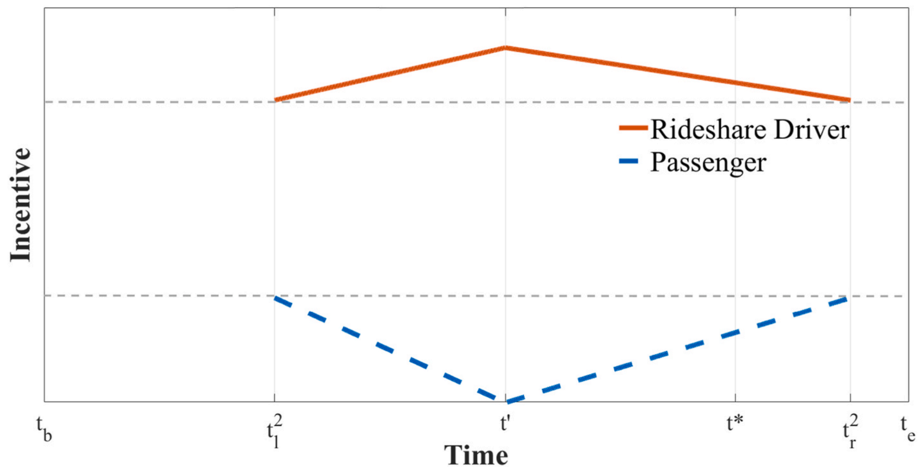


Fig. 3. The optimal incentive B, when the ridesharing ratio is high and the market penetration is low. Ridesharing participants depart in the interval $[t_l^2, t_r^2]$. $t^{'}$ is the departure time of a commuter who arrives at t^*, t^* is the target arrival time.

Table 5
The optimal incentive scheme and specific values of time nodes in case 3.

$$I_2(t) = \begin{cases} \dfrac{(\alpha_1-\alpha_3)\delta[N(1-p)+N_m]}{\alpha_1 c} - \beta(t-t_l^1) + (\alpha_2-\alpha_3)T_f, for\ t \in [t_b, t_l^1] \\ \dfrac{[(\alpha_3-\alpha_1)(t-t^{'}) + (\alpha_2-\alpha_3)(t-t_l^1)]\beta}{\alpha_1-\beta} + (\alpha_2-\alpha_3)T_f, for\ t \in [t_l^2, t^{'}] \\ \dfrac{[(\alpha_3-\alpha_1)(t^{'}-t) + (\alpha_2-\alpha_3)(t_r^1-t)]\gamma}{\alpha_1+\gamma} + (\alpha_2-\alpha_3)T_f, for\ t \in [t^{'}, t_r^2] \\ \dfrac{(\alpha_1-\alpha_3)\delta[N(1-p)+N_m]}{\alpha_1 c} + \gamma(t-t_r^1) + (\alpha_2-\alpha_3)T_f, for\ t \in [t_r^1, t_e] \end{cases}$$

$$I_3(t) = \begin{cases} -\beta(t-t_l^1) + (\alpha_1-\alpha_3)\delta[N(1-p)+N_m]/(\alpha_1 c), for\ t \in [t_b, t_l^1] \\ (\alpha_3-\alpha_1)\beta(t-t^{'})/(\alpha_1-\beta), for\ t \in [t_l^2, t^{'}] \\ (\alpha_3-\alpha_1)\gamma(t^{'}-t)/(\alpha_1+\gamma), for\ t \in [t^{'}, t_r^2] \\ \gamma(t-t_r^1) + (\alpha_1-\alpha_3)\delta[N(1-p)+N_m]/(\alpha_1 c), for\ t \in [t_r^1, t_e] \end{cases}$$

$t_l^1 = t_b + N_e\delta/(c\beta)$
$t_l^2 = t_l^1 + N(1-p)(\alpha_1-\beta)\delta/(\alpha_1\beta c)$
$t^{'} = t_l^2 + N_m(\alpha_1-\beta)\delta/(\alpha_1\beta c)$
$t_r^2 = t_r^1 - N(1-p)(\alpha_1+\gamma)\delta/(\alpha_1\gamma c)$
$t_r^1 = t_e - N_e\delta/(c\gamma)$

thus causing unexpected congestion at the boundary of incentive and non-incentive periods.

Therefore, from the perspective of implementation difficulty, when the penetration is low ($p \le p^*$), it is better to set a higher ridesharing ratio ($R > \alpha_2/(\alpha_1-\alpha_3)$), and the corresponding optimal incentive at this time is incentive B. When the penetration is at a high level ($p > p^*$), it is better to set a lower ridesharing ratio ($R \le \alpha_2/(\alpha_1-\alpha_3)$), and the corresponding optimal incentive at this time is incentive A.

In terms of the difficulty of scheme implementation, such a reference on how to choose an appropriate incentive scheme under different penetrations is shown as Fig. 5:

It is worth noting that although the penetration can be manipulated by carrying out marketing campaigns, it is still difficult to accurately control the penetration rate in practice, unless the incentive provider can accurately estimate all the critical parameters and shut down the channel for new participants to join the program immediately once the optimal penetration is reached. The incentive profiles are continuous when the ridesharing ratio is high and the penetration rate is low. If the participation channel cannot be shut down immediately, more

Table 6
The optimal incentive profiles with different market penetration and ridesharing ratios.

	$R \le \alpha_2/(\alpha_1-\alpha_3)$	$R > \alpha_2/(\alpha_1-\alpha_3)$
$p \le p^*$	Incentive A (shown as Fig. 2)	Incentive B (shown as Fig. 3)
$p > p^*$		Incentive C (shown as Fig. 4)

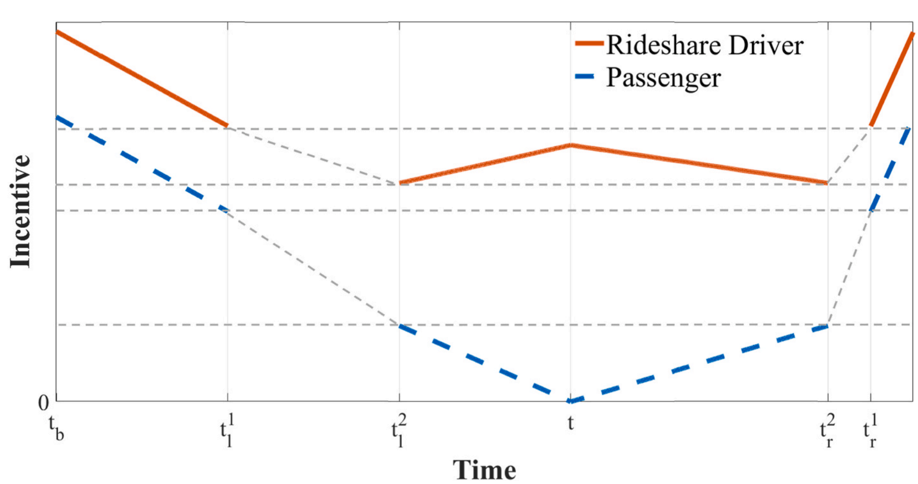


Fig. 4. The optimal incentive C, when the ridesharing ratio and market penetration are both high. Ridesharing participants depart in the interval $[t_b, t_l^1] \cup [t_l^2, t_r^2] \cup [t_r^1, t_e]$ and solo drivers depart in the interval $[t_l^1, t_l^2] \cup [t_r^2, t_r^1]$.

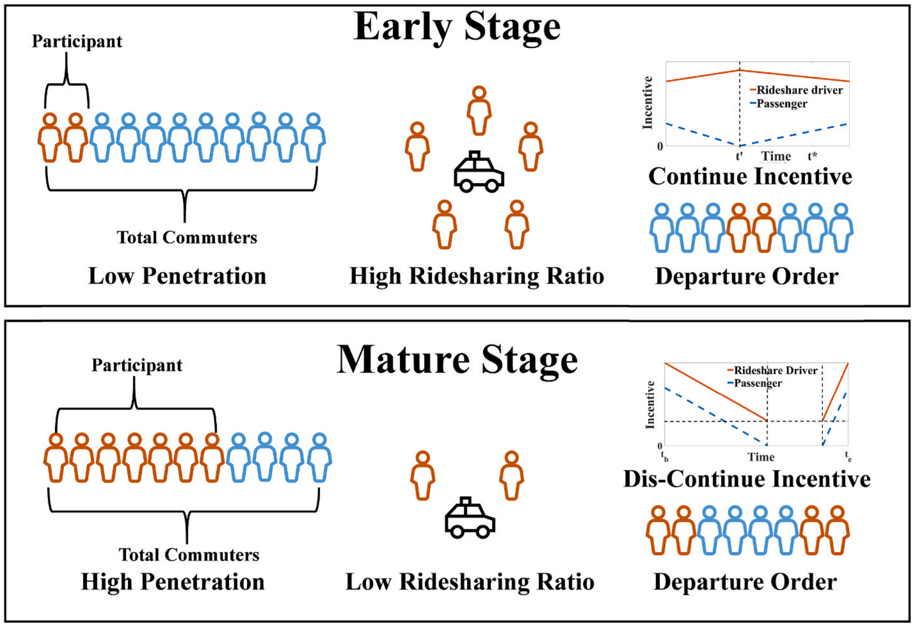


Fig. 5. The transition of the optimal incentive scheme with the promotion of the project when considering the scheme implementing difficulty. In the early stage (when the penetration is lower than p^*), administrators should set a high ridesharing ratio and a user-friendly incentive scheme (shown as Fig. 3) to attract participants. In the mature stage (when the penetration reaches above p^*), administrators should set incentive A (shown as Fig. 2) to avoid the most complicated scheme.

commuters might join the program than expected, making the penetration rate greater than the critical value, which leads to the situation in incentive C (with intermittent incentive profiles).

5.4. Numerical example of the most efficient incentive scheme under the budget

Administrators aim to alleviate traffic congestion and reduce the total system cost by setting up the ridesharing-based incentive scheme. Cohen-Blankshtain et al. (2022) summarized eight key design issues for incentive schemes, including efficiency, participation, administration, etc. With the widespread adoption of smartphones and the rise of TNCs, private organizations can also administer TDM programs. For private administrators, they are more concerned about whether the project can make the net utility positive. If the budget is regarded as the input of the project, and the system travel cost reduction is regarded as the output, then the measurement of effectiveness of the project is whether the reduction of the system cost is greater than the budget investment.

In reality, in order to boost market penetration, administrators can carry out marketing campaigns to acquire users in various ways. For instance, they can advertise, provide in-app referral bonuses, sponsor social media campaigns, and direct sales outreach efforts (Xiao et al., 2021). For the sake of simplicity, we did not consider the marketing expenditure needed to increase penetration in this work.

A numerical example is presented herein to more intuitively show the impact of budget investment on the total system cost reduction under different incentive schemes. The specific setup of the numerical example is shown in Table 7:

Table 7
Numerical experiment setup.

Notation	Interpretation
$N_{max} = 1,000\ veh$	Number of potential commuters
$t^* = 8:00$AM	Expected arrival time
$\alpha_1 = \$5/h$	Unit value of travel time of solo driver
$\alpha_2 = \$5.5/h$	Unit value of travel time of rideshare driver
$\alpha_3 = \$3/h$	Unit value of travel time of passenger
$\beta = \$2.5/h$	Unit value of schedule early delay
$\gamma = \$10/h$	Unit value of schedule late delay
$T_f = 5\ min$	Free flow travel time
$c = 20\ veh/min$	Capacity of the bottleneck

The values of α, β, and γ agree with past research in Small (1982) and Arnott et al. (1993). Specifically, Small (1982) estimated the ratios $\beta/\alpha = 0.5$ and $\gamma/\alpha = 2$, Arnott et al. (1993) took the travel time cost as \$5/h. In addition, based on the relationship of $\alpha_1 > \alpha_3 > \beta$ in Ma and Zhang (2017), we assume that the unit value of travel time of passengers is \$3/h. According to $(1+R)\alpha_1 > \alpha_2 + R\alpha_3$ in Assumption 5, we then set the unit value of travel time of rideshare drivers to \$5.5/h (see Fig. 5).

According to (12), the relationship between the system cost reduction and penetration under different ridesharing ratios are depicted in Fig. 6.

It can be seen from Fig. 6 that the system cost reduction is increase w.r.t. the penetration and the ridesharing ratio. According to Assumption 4, we treat the ridesharing ratio as a continuous variable, but in reality, the ridesharing ratio can only take integer values. Therefore, we projected the curve in Fig. 6, along which the ridesharing ratio is an integer, onto the plot of market penetration-system cost reduction. This allows us to obtain the relationship between market penetration and system cost reduction for integer ridesharing ratios. The relationship is shown in Fig. 7:

It can be seen from Fig. 7 that for a given market penetration, the system cost reduction monotonically increases with the ridesharing ratio. It is intuitive that a larger ridesharing ratio leads to a larger number of passengers and less vehicle demand, resulting in a smaller system cost.

If we solely consider the system cost reduction, the optimal ridesharing ratio would be 4. The point B represents the cut-off point of the optimal incentive scheme for this ratio. For market penetrations below 42%, incentive scheme B is the optimal choice. While incentive scheme C is more suitable when penetration rate reaches above 42%.

According to Eq. (28), the relationship between the net utility and penetration under different ridesharing ratios can be shown in Fig. 8.

It can be seen from Fig. 8 that no matter what the penetration is, the net utility of the project is positive, which means that administrators will have enough motivation to implement the project. Similarly, the

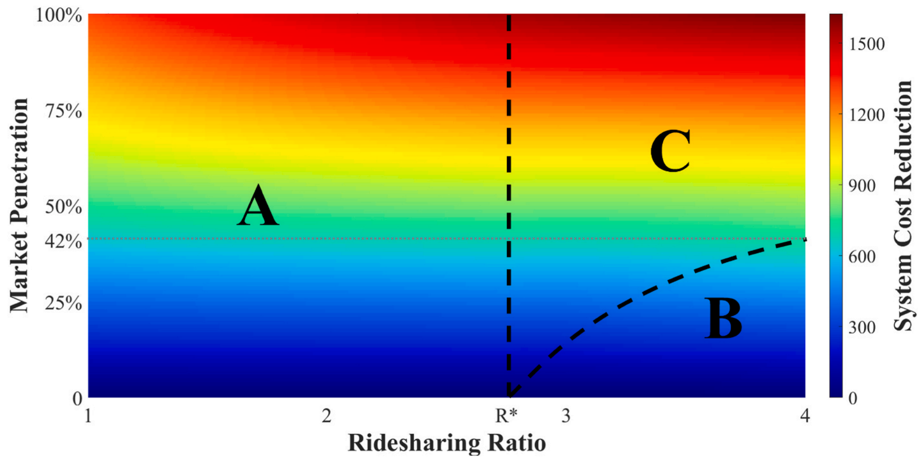


Fig. 6. The relationship between the system cost reduction and penetration under different ridesharing ratios, the black dashed lines represent the critical states of three different departure patterns. Area A corresponds to incentive scheme A (case of Fig. 2), area B corresponds to incentive scheme B (case of Fig. 3), and area C corresponds to incentive scheme C (case of Fig. 4). $R^* = \alpha_2/(\alpha_1 - \alpha_3)$.

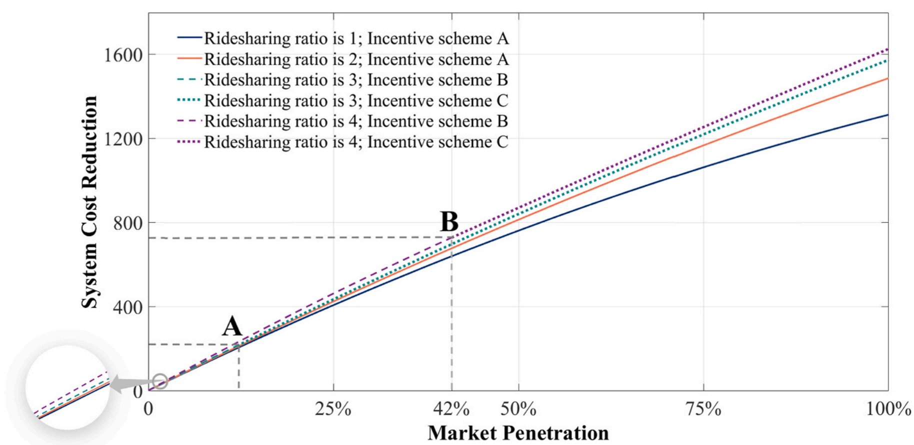


Fig. 7. The relationship between market penetration and system cost reduction for integer ridesharing ratios, where point A and B are the cut-off points of the optimal incentive scheme for ridesharing ratios 3 and 4, respectively.

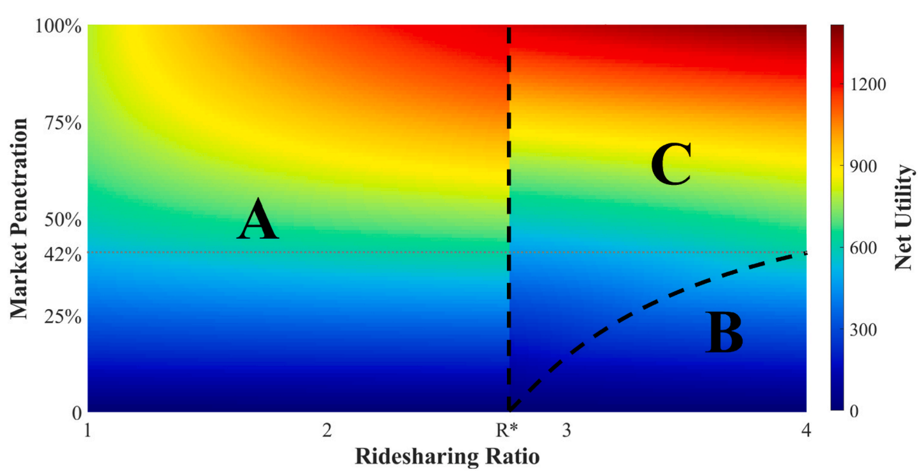


Fig. 8. The relationship between the net utility and penetration under different ridesharing ratios.

relationship between market penetration and net utility under the integer ridesharing ratio can be seen in Fig. 9:

It can be seen from Fig. 9 that point C represents the critical point for different optimal ridesharing ratios, if only the system cost reduction is considered. The optimal ridesharing ratio is 2 when market penetration is below 58%, corresponding to incentive scheme A, and it is 4 when market penetration is above 58%, corresponding to incentive scheme C.

The following table shows how administrators should choose the

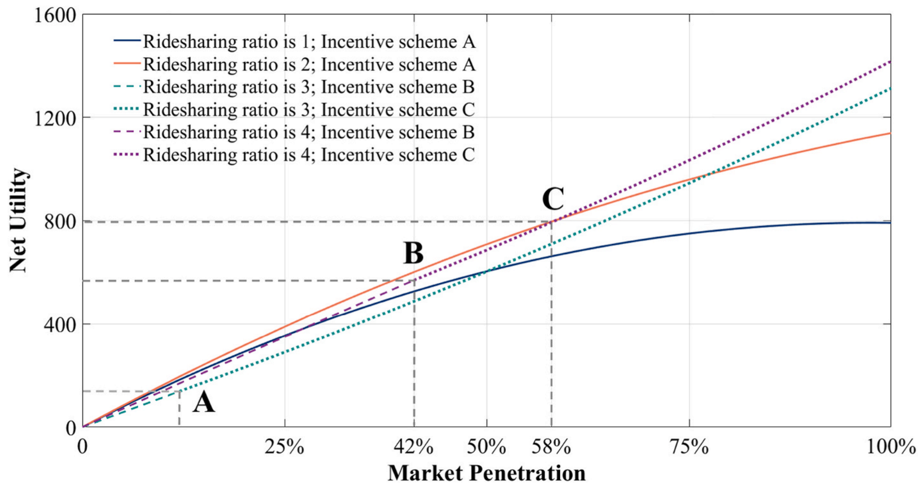


Fig. 9. The relationship between market penetration and net utility for integer ridesharing ratios, where point A and B are the cut-off points of the optimal incentive scheme for ridesharing ratios 3 and 4, respectively, point C is the intersection of the case when ridesharing ratio is 2 and 4.

appropriate ridesharing ratio and incentive scheme according to different objective functions:

It can be seen from Table 8 that none of the incentive schemes is strictly superior to the others under the three evaluation criteria. At different stages, administrators can formulate different schemes according to different criteria. For example, at the initial stage of the project, when the penetration is low, administrators can implement incentive B to attract more participants and increase penetration. Furthermore, incentive B can maximize the system cost reduction. As the project progresses and the market penetration increases, incentive B may no longer be suitable when the penetration exceeds 42%. In this case, administrators may consider implementing incentive A or incentive C.

Additionally, it is evident that when the market penetration is at a low level (less than 58%), a low ridesharing ratio is advantageous for maximizing net utility, while a high ridesharing ratio is advantageous for maximizing system cost reduction. When the market penetration is at a high level (greater than 58%), a high ridesharing ratio holds advantages in both system cost reduction and net utility. However, it is important to note that the incentive scheme C, corresponding to the high ridesharing ratio, is the most complex. The ridesharing demand is now split into pieces, which could result in difficulties during management. Therefore, administrators need to make trade-offs between different goals when deciding on ridesharing ratios and incentive schemes.

6. Economic experiment to verify the effectiveness of the incentive scheme

In the previous section, we have demonstrated the optimal incentive schemes corresponding to different penetrations. However, in the process of theoretical derivation, we made some assumptions such as user homogeneity and behavioral rationality. To check if the optimal incentive solutions work as expected in the real world, we adopt the Experimental Economics (EE) approach to collect the real decision-making behavior of travelers under the theoretical optimal solutions, thus to verify the effectiveness of the optimal incentive scheme in alleviating congestion and promoting ridesharing. As an alternative approach for data collection, this EE-based approach is a more flexible way to understand individual and system performance and has been widely adopted in travel behavior (route choice, departure time choice, etc.) analysis (Aziz et al., 2015; Tian et al., 2022).

The experiment was conducted on the WeChat applet (the most popular social media app in China). It started on January 22, 2022 and lasted for a total of 31 days. A total of 206 volunteers were recruited nationwide in China for this experiment, and by the end of the experiment, 194 volunteers had completed all the experiments.

In this experiment, we assumed the penetration rate is 100%, meaning that, for each day every participant needs to decide whether to be a solo driver, rideshare driver, or passenger, and when to depart in a Vickrey's bottleneck network. It is worth noting that although we assume a penetration of 100%, due to the irrational behavior of users, not all experimenters will choose to rideshare. Users who choose to be rideshare drivers and passengers within the same time period will be automatically matched (the ridesharing ratio is 1), and others will be assigned as a solo driver.

There are three phases in the experiment. The first phase is the control group, and the second and third phases are the experimental group. The first phase lasts for 10 days, and no incentives are implemented during this phase. The second phase lasts for 14 days, and a time-varying incentive scheme is implemented. Since the ridesharing ratio is set to 1, and the penetration is 100%, the setting of the time-varying incentive scheme is similar to incentive A (as shown in Case 1). That is, the incentive scheme is distributed throughout the peak hours, and a higher reward will be obtained when departing at a time far away from the target arrival time. Drivers and passengers who successfully rideshare can get corresponding incentives. The third stage lasts for 7 days. In addition to the time-varying incentive scheme, a fixed amount of compensation is provided to travelers who are not successfully matched for ridesharing. It is expected that the number of ridesharing participants will be further boosted by reducing the risk of ridesharing failure. The result of the experiment is shown as follows:

One can tell from Fig. 10 that in phase 1 without incentive, although some people choose to rideshare, the success rate of ridesharing is not very high. This is because the travel cost of rideshare drivers is higher than that of passengers, so most of the experimenters would choose to be passengers, resulting in an imbalance in the number of rideshare drivers and passengers. Therefore, setting a high ridesharing ratio during the

Table 8
The appropriate incentive scheme with different criterion.

Penetration	Criterion		
	Easy to implement	System cost reduction	Net utility
[0, 42%]	Ridesharing ratio 4 Incentive scheme B	Ridesharing ratio 4 Incentive scheme B	Ridesharing ratio 2 Incentive scheme A
(42%, 58%]	Ridesharing ratio 2 Incentive scheme A	Ridesharing ratio 4 Incentive scheme C	Ridesharing ratio 2 Incentive scheme A
(58%, 100%]	Ridesharing ratio 2 Incentive scheme A	Ridesharing ratio 4 Incentive scheme C	Ridesharing ratio 4 Incentive scheme C

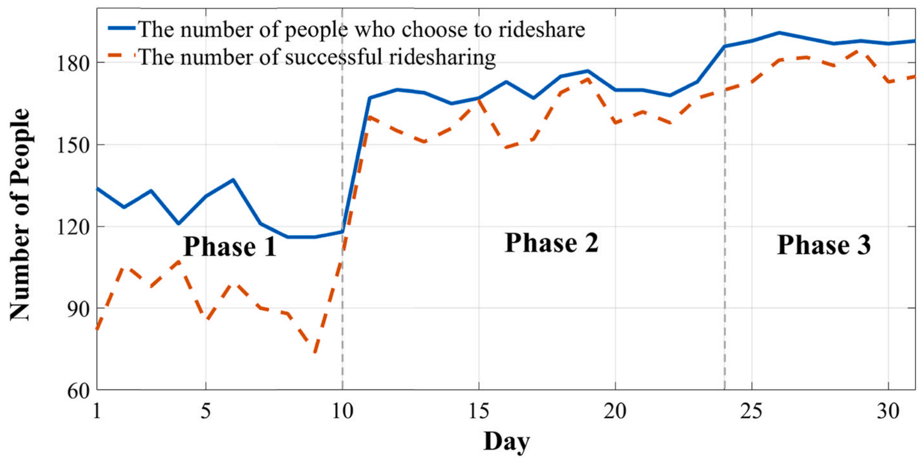


Fig. 10. The number of people who are willing to rideshare and the number of people who successfully rideshare. Where phase 1 is the control group (without incentive), and phases 2 and 3 are experimental groups.

implementation phase of the project can help to eliminate the imbalance between different travel roles. In addition, it can be found that compared with the first phase, the number and the success rate of ridesharing have been significantly improved in the second and third phases. This also confirms that the incentive scheme (shown as Fig. 3) can effectively increase market penetration.

From Fig. 11, it can be found that compared with the first stage, the traffic system has been significantly improved in the second and third stages, and the travel cost and queueing time have been greatly reduced. The average queueing time in the second stage decreased by 60.7% compared to the first stage, and the average queueing time in the third stage decreased by 71.9% compared with the first stage. In addition, compared with the first stage, the average travel cost in the second stage decreased by 71.3%, and the third stage decreased by 78.4% compared with the first stage. These results demonstrate the effectiveness of the incentive scheme in alleviating congestion.

In section 5.1, we proved that when the ridesharing ratio is low, ridesharing participants will be distributed at both ends of the peak hours, and solo drivers will depart in the middle. In order to verify whether travelers will depart in this order, we selected the last day of the second phase for analysis. The travel distribution of ridesharing participants and solo drivers is shown in Fig. 12.

It can be seen from Fig. 12 that the departure order of travelers obtained in the experiment is as expected. The majority of ridesharing participants choose to depart at the fringes, while solo drivers preferred to depart around the target arrival time. This indicates that incentives can attract ridesharing participants to advance or postpone their departure time, and travelers are more likely to accept early arrivals than late arrivals. Based on these observation, one can conclude that it is necessary to control ridesharing in the early peak hours, such as by implementing HOV lanes between 7:30–8:20, which would be sufficient to meet most of the ridesharing demands.

To sum up, this section reveals the effectiveness of incentives in promoting ridesharing and alleviating traffic congestion through experimental economics. However, it should be noted that due to the irrational behavior of users, the effect of the incentive scheme in real life will be different from the theoretical solution. Therefore, the numerical results of the experiment can be used to guide administrators in making adjustments to the incentive scheme obtained from the theoretical solution, such as spending most of the budget on attracting ridesharing participants to depart earlier rather than later.

7. Conclusions

In this paper, we explored the effectiveness of IBTDM strategies in managing morning commuters with ridesharing in a bottleneck model. It has been proven that incentive-based ridesharing programs can reduce total system costs with positive net utility at any given market penetration. We selected the implementation difficulty of the incentive scheme, the reduction of system cost, and the net utility as the potential goals of administrators. It is found that the optimal incentive schemes corresponding to different market penetrations are different.

In addition, administrators need to make trade-offs between different goals when choosing the optimal incentive scheme. Although the system

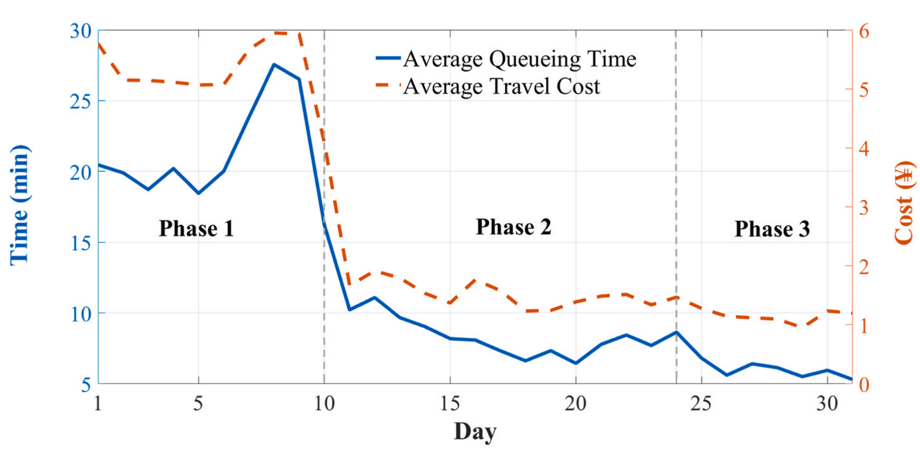


Fig. 11. The average queueing time and the average travel cost.

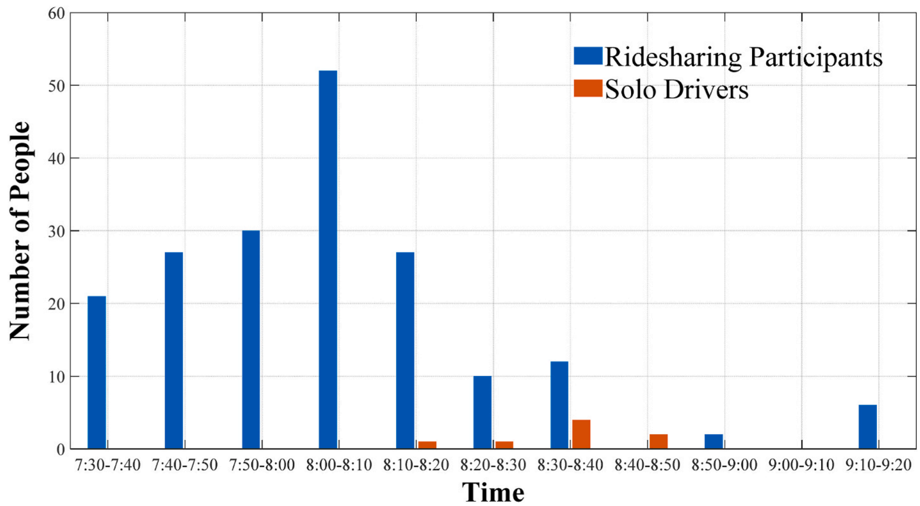


Fig. 12. The distribution of departure times of ridesharing participants and solo-drivers, where 8:40 is the target arrival time.

cost reduction and net utility decrease monotonically w.r.t penetration, it does not necessarily mean that higher penetration is always better in practice. In the case of high penetration, if administrators still set a high ridesharing ratio in order to maximize system cost reduction and net utility, then the optimal incentive scheme at this time is extremely complicated and inefficient. The ridesharing demand is now separated into pieces, so that the project implementation may suffer due to the complexity avoidance and cluster effect.

Another thing to note is that in the actual application of the incentive project, the theoretical optimal solution may have some deviations from the actual situation. This means that the theoretical optimal solution is only a reference for administrators to formulate incentive schemes, but administrators cannot completely follow the theoretical solution. Instead, some adjustments need to be made according to the actual situation.

In this study, we assumed that the ridesharing ratio is constant throughout the peak period. In future research, time-varying ridesharing ratios can be examined to replicate riders' demand. In addition, marketing expenditure, which was not discussed herein, is typically required to increase the penetration rate. Thus, another direction worth exploring is to investigate how to best allocate budgets while considering marketing costs. Finally, it is also necessary to extend the model beyond just a simple bottleneck to a multi-OD road network.

CRediT authorship contribution statement

Jiyan Wu: Conceptualization, and design, Methodology, Interpretation of results, Writing – original draft. **Ye Tian:** Conceptualization, and design, Interpretation of results, Writing – review & editing, Supervision. **Jian Sun:** Conceptualization, and design, Writing – review & editing.

Declaration of competing interest

The authors declare that there is no conflict of interest regarding the publictiong of this paper.

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Appendix A

Denote $S(t)$ as the ride fee that passengers need to pay to the rideshare drivers at time t, then the generalized incentive obtained by passengers is $I_3(t) - S(t)$, and the generalized incentive obtained by rideshare drivers is $I_3(t) + RS(t)$. According to (3), $S(t)$ can be calculated as:

$$S(t) = (I_2(t) - I_3(t)) / (1 + R) = (\alpha_2 - \alpha_3)\left(T(t) + T_f\right) / (1 + R) \tag{A1}$$

For the passenger, the maximum willingness-to-pay (WTP) $W_p(t)$ is the difference between the cost of solo driving and the cost of traveling with matching. For the driver, the minimum willingness-to-accept (WTA) $W_d(t)$ is the extra-cost of matching compared with solo driving. Therefore, W_d and W_p can be expressed as:

$$W_d(t) = (\alpha_2 - \alpha_1)\left(T(t) + T_f\right) \tag{A2}$$

$$W_p(t) = (\alpha_1 - \alpha_3)\left(T(t) + T_f\right) \tag{A3}$$

Passengers and rideshare drivers can be matched only if the ride fee is not higher than the passenger's maximum WTP, nor be lower than the rideshare driver's minimum WTA (de Palma et al., 2022). Based on Assumption 5, which states that $(1 + R)\alpha_1 > \alpha_2 + R\alpha_3$, it can be easily proven that $S(t) < W_p(t)$ and $RS(t) > W_d(t)$. This indicates that all rideshare drivers and passengers have sufficient willingness to be matched.

Whether providing differentiated incentive schemes for rideshare drivers and passengers, or setting ride fees, the objective is to compensate for the difference in travel costs between drivers and passengers departing at the same time. The optimal travel pattern of travelers under the two schemes is the same. According to (A.1), the ride fee can be easily derived from the optimal incentive scheme $I_i(t), i = 2, 3$.

Appendix B

Proposition 1. When the total system cost is minimized ridesharing participants and solo drivers will not depart at the same time.

Proof. Using proof by contradiction, we assume that ridesharing participants and solo drivers will depart together during the interval $[t_\alpha, t_\beta], \forall t_\alpha, t_\beta \in [t_b, t']$.

It is worth noting that solo drivers cannot receive incentives. In order to achieve UE, the total departure rate of rideshare and non-rideshare vehicles at any time in $[t_\alpha, t_\beta]$ must be constant $\alpha_1 c/(\alpha_1 - \beta)$ (Arnott et al., 1988). If a rideshare vehicle adjusts its departure time from t_c to t_s, in order to ensure equilibrium, a solo driver will adjust their departure time from t_s to t_c, where $\forall t_c, t_s \in [t_\alpha, t_\beta]$ and $t_c > t_s$. Assuming that $t_c - t_s = \Delta t$. Thus, the change in travel time cost of the rideshare vehicle can be expressed as:

$$\Delta C_t = (\alpha_2 + R\alpha_3)[\alpha_1 c / (\alpha_1 - \beta) - c]\Delta t / c \tag{B1}$$

The change in schedule delay cost of the rideshare vehicle can be expressed as:

$$\Delta C_d = -(1 + R)\beta[\alpha_1 c / (\alpha_1 - \beta)]\Delta t / c \tag{B2}$$

According to Assumption 5 ($(1 + R)\alpha_1 > \alpha_2 + R\alpha_3$), it can be calculated that $\Delta C_t + \Delta C_d < 0$. This means solo drivers departing earlier than ridesharing participants during $[t_\alpha, t_\beta]$ can effectively reduce the total system cost.

Therefore, if the ridesharing participants and solo drivers depart separately, the total system cost will be lower than if some of them depart together. Similarly, the same conclusion can be drawn when $t_\alpha, t_\beta \in [t', t_e]$. Therefore, ridesharing participants and solo drivers will not depart at the same time when the total system cost is minimized.□

Appendix C

Proposition 2. The departure time of solo drivers is continuous during $[t_b, t']$ and $[t', t_e]$ respectively when the total system cost is minimized.

Proof. Using proof by contradiction, we assume that when the total system cost is minimized, the departure time of the solo drivers during the interval $[t_b, t']$ is disjointed. This is shown in Fig. C1:

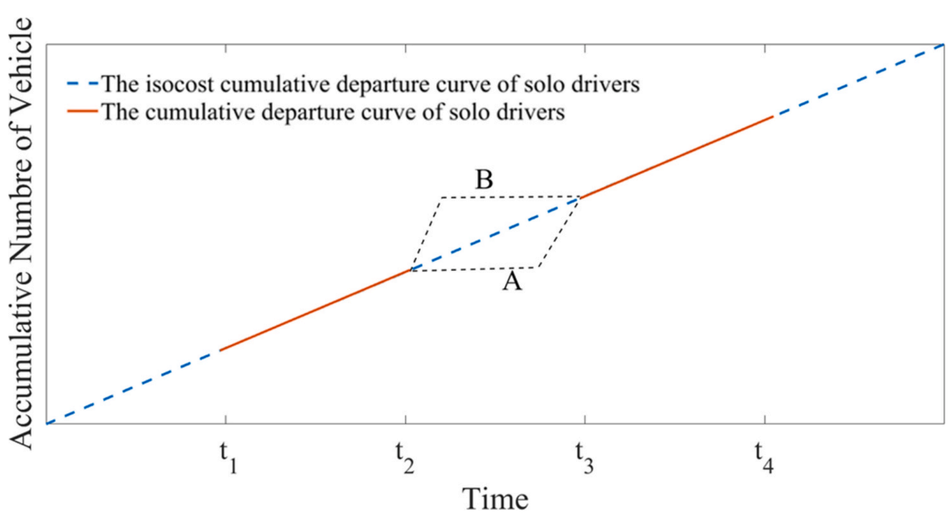

Fig. C1. The departure times of solo drivers when the total system cost is minimized. In the figure, $\exists[t_1, t_4] \subseteq [t_b, t']$.

The solid line in Fig. C1 represents the cumulative departure curve for solo drivers. We assume that some of the ridesharing participants will depart in the interval $[t_2, t_3]$, such that the departure times of the solo drivers in the interval $[t_b, t']$ are interrupted. As stated in Proposition 1, in order to ensure an equilibrium state, the departure rate of solo drivers needs to be maintained at the constant $\alpha_1 c/(\alpha_1 - \beta)$. The blue dashed line in Fig. C1 represents the isocost cumulative departure curve for individual departures. This means that if the cumulative departure curve of solo drivers is the blue dashed line, the travel cost for solo drivers is equal to the equilibrium cost (i.e., the travel cost of solo drivers departing within the interval $[t_1, t_2] \cup [t_3, t_4]$). The cumulative departure curve of rideshare vehicles in the interval $[t_2, t_3]$ can be divided into the following three cases:

Case 1. There exists a time interval within $[t_2, t_3]$ where the cumulative departure curve of the rideshare vehicles is below the isocost cumulative departure curve. This is shown by line A in Fig. C1. Solo drivers departing within the interval $[t_1, t_2] \cup [t_3, t_4]$ will change their departure time to $[t_2, t_3]$, since departing during the interval $[t_2, t_3]$ costs less. Therefore, Case 1 does not form an equilibrium state.

Case 2. There exists a time interval within $[t_2, t_3]$ where the cumulative departure curve of the rideshare vehicles is above the isocost cumulative departure curve. This is marked by line B in Fig. C1. Although this is an equilibrium state, line B clearly corresponds to a higher total system cost when compared to the isocost line. Thus, Case 2 does not meet the requirement of minimizing system cost.

Case 3. This case is where the cumulative departure curve of the rideshare vehicles coincides with the isocost line. According to the proof for Proposition 1, if the departure rates of rideshare vehicles and solo drivers are equal within a period of time, in order to minimize the total system cost, solo drivers will depart strictly earlier than ridesharing participants. Thus, Case 3 also does not meet the minimum system cost.

Similarly, the same conclusion is obtained when $[t_1, t_4] \subseteq [t^{'}, t_e]$. Therefore, in order to minimize the total system cost, the departure time of solo drivers is continuous across $[t_b, t^{'}]$ and $[t^{'}, t_e]$ respectively.□

Appendix D

Proposition 3. In case 2, the optimal cumulative departure curve of rideshare vehicles is the blue dashed line as shown in Fig. 1(b).

Proof. Using proof by contradiction. Assuming that the blue dashed line is not the optimal cumulative departure curve for ridesharing participants, it means that the optimal departure curve is not exactly coincident with the blue dashed line.

Suppose there exists a time interval during which the cumulative departure curve for ridesharing participants is below the blue dashed line. For the sake of clarity, we consider an extreme situation where the cumulative departure curve of rideshare vehicles is entirely below the blue dashed line. The situation is illustrated by the black dashed line A in Fig. 1(b).

For solo drivers, the departure rate of the early arrivals is $\alpha_1 c/(\alpha_1 - \beta)$ and $\alpha_1 c/(\alpha_1 + \gamma)$ of the late arrivals in the user equilibrium state. In other words, the blue dashed line in the figure is also the iso-cost line for solo drivers. Since the travel time corresponding to line A is shorter than the travel time corresponding to the blue dashed line, the travel cost of solo drivers in the middle of the peak period is less than that at the ends of the peak. Solo drivers would shift their departure times to the middle due to lower travel costs, thus breaking the equilibrium. Therefore, the cumulative departure curve of rideshare vehicles will definitely not have a part below the blue dashed line.

Suppose there exists a time interval during which the cumulative departure curve for ridesharing participants is above the blue dashed line. For the sake of clarity, we consider an extreme situation where the cumulative departure curve of rideshare vehicles is entirely above the blue dashed line. The situation is illustrated by the black dashed line B in Fig. 1(b).

Obviously, the travel time corresponding to line B is greater than the travel time corresponding to the blue dashed line. Although groups of solo drivers will not change their departure time to the middle of the peak period, it is contradicting the optimal state with the minimum total system cost. Administrators can make line B coincide with the blue dashed line by offering incentives.

Therefore, in Case 2, the cumulative departure curve of rideshare vehicles is shown by the blue dashed line.□

Appendix E

Assuming that a total of N_e rideshare vehicles depart at both ends of the peak interval, N_m rideshare vehicles depart in the middle of the peak period, and N_1 non-rideshare vehicles depart between the two groups of rideshare vehicles. For ease of understanding, a notational glossary for reference in this paper is provided in Table E. 1:

Table E1
Notational Glossary of Appendix E

Notation	Interpretation
TC_e	Travel time cost of ridesharing participants departing at both ends
ASC_e	Average schedule delay cost of ridesharing participants departing at both ends
ATC_1	Average travel time cost of solo drivers
ASC_1	Average schedule delay cost of solo drivers
ATC_m	Average travel time cost of ridesharing participants departing in the middle
ASC_m	Average schedule delay cost of ridesharing participants departing in the middle

First, in an equilibrium state, rideshare vehicles will depart at both ends of the peak period and there is no need to queue, then the total travel time cost of rideshare vehicles can be calculated as:

$$TC_e = \theta N_e T_f \quad \text{(E1)}$$

In addition, the schedule delay cost function is a linear profile, so the average schedule delay cost of the ridesharing participants departing at both ends of the peak period can be calculated as:

$$ASC_e = [(N_e + N_1 + N_m)(1 + R)\delta / C + (N_1 + N_m)(1 + R)\delta / C] / 2 \quad \text{(E2)}$$

Therefore, C_e can be calculated as $N_e \times ASC_e + TC_e$.

Second, the queue length is a linear function with respect to time, so the average travel time cost of N_1 commuters can be calculated as:

$$ATC_1 = \{2T_f + [1 / c - (\alpha_1 - \beta) / (\alpha_1 c)]N_1\gamma / (\beta + \gamma)\}\alpha_1 / 2 \quad \text{(E3)}$$

Similar to (E.2), the average schedule delay cost of N_1 commuters can be calculated as:

$$ASC_1 = [(N_1 + N_m)\delta / C + N_m\delta / C] / 2 \quad \text{(E4)}$$

Therefore, C_1 can be calculated as $N_1(ATC_1 + ASC_1)$.

Third, just like with (E.3), the average travel time cost of N_m rideshare vehicles departing in the middle of the peak period can be calculated as:

$$ATC_m = (\alpha_2 + R\alpha_3)\{2T_f + [\beta / (\alpha_1 c)][(2N_1 + N_m)\gamma / (\beta + \gamma)]\} / 2 \quad \text{(E5)}$$

Similarly, the average schedule delay cost of N_m rideshare vehicles departing in the middle of the peak period can be calculated as:

$$ASC_m = N_m\delta(1 + R) / 2c \quad \text{(E6)}$$

Therefore, C_m can be calculated as $N_m(ATC_m + ASC_m)$. □

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