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Robust retirement and life insurance with inflation risk and model ambiguity

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ABSTRACT

We study a robust consumption-investment problem with retirement and life insurance decisions for an agent who is concerned about inflation risk and model ambiguity. Assuming that an inflation-linked index bond and a stock are available in the market, this paper considers a comprehensive setup of ambiguity in the return, volatility, and correlation parameters in the joint dynamics of their market prices. With a finite planning horizon, the agent has a general utility function with different marginal utilities of consumption before and after retirement. Combining the classical dual approach and the G-stopping time theory, we derive the novel robust strategies using integral equation representations. We numerically and extensively investigate the effects of ambiguity from different sources on the robust decisions. While model ambiguity generally leads the ambiguity- and risk-averse agent to decrease the consumption rate, life insurance purchase, and investment demands, it also generates contrasting effects on robust retirement time and wealth level. Specifically, model ambiguity lowers the target wealth level to immediate retirement of a young agent but increases the retirement time of an older agent compared to the case of known parameters. A rich agent takes ambiguity more seriously than a poor agent in the sense of adjusting the strategies on a more significant scale. Our simulation and comparison study demonstrate the significance of addressing the ambiguity in volatility and correlation in addition to the ambiguity in return.

1. Introduction

Contemporary comprehensive financial planning involves managing financial, mortality, and inflation risk, as well as deciding when to retire. It is not uncommon that wage earners invest in the stock market to benefit from market growth, purchase life insurance to prevent losses from premature death, trade inflation-linked index bonds to hedge against inflation risk, and determine the best balance between consumption and working time. The COVID-19 pandemic has led to unprecedented uncertainties in financial market fluctuations, inflation measures, public health, and consumption patterns. This increases the importance of investigating lifetime financial planning with this inescapable issue—the ambiguity risk.

The Ellsberg paradox experimentally distinguishes model ambiguity (uncertainty) from risk and ambiguity aversion from risk aversion. In this paper, we consider a comprehensive setup for an agent or wage earner by studying the consumption–investment–retirement problem with life insurance and inflation risk under a general utility objective. We completely address model ambiguity in return, volatility, and correlation within the joint stochastic movement of the inflation rate and stock price, and study their effects on the agent's robust decisions.

In our setup, the agent encounters ambiguity from three sources: stock price, inflation rate, and the correlation between them. Model ambiguity in stock returns is frequent challenge in the literature. Practitioners encounter difficulty in estimating stock returns accurately,

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Table 1
A comparison of key publications about life insurance, inflation risk and retirement.

	Mortality risk management	Inflation risk management	Early retirement option
Kwak and Lim (2014), Han and Hung (2017), Menoncin and Regis (2017)	✓	✓	×
Lim and Kwak (2016)	✓	×	✓
This paper	✓	✓	✓

Table 2
A comparison of key publications about inflation ambiguity.

	Non-equivalent measures	Ambiguity in drift	Ambiguity in diffusion
Ulrich (2013), Guan and Liang (2019)	×	✓	×
This paper	✓	✓	✓

for both their mean and variance (Cont, 2001). This motivates the development of calibration techniques for stock volatility but it does not resolve the considerable volatility ambiguity in practice (Yi et al., 2013). Model ambiguity is also a problem when estimating the inflation rate. Although stochastic models for inflation trends and volatility are available (Ulrich, 2013; Singor et al., 2013), Ulrich (2013) shows empirically that the agent faces model ambiguity in estimating inflation rate trends, which partially accounts for the upward-sloping term premium in U.S. Treasury bonds. As stock price and inflation data are collected at different time points or not synchronized, estimating the correlation between the samples becomes less promising. Although some empirical studies use a time-varying correlation (Lee, 2010), recent advances propose the use of correlation ambiguity (Fouque et al., 2016; Han and Wong, 2020). For all of these reasons, together with the long-term decision-making setup, we relax the assumption that the agent is fully aware of the underlying probability law governing the market. We therefore address ambiguity using a set of alternative models considered by the agent for robust decision-making.

In this paper, alternative models are those with different parameter combinations in which each parameter falls into a bounded interval, possibly obtained from the confidence interval estimation. These parameters are possibly time-varying and stochastic but bounded. This model ambiguity setup induces the set of non-dominated probability measures (priors) in robust optimization problems. This contrasts sharply with models that assume all possible priors to be equivalent to a reference measure (Chen and Epstein, 2002; Maenhout, 2004; Han et al., 2021). Accordingly, our robust optimization is represented by a max–min expected utility over two sets: the set of non-dominated priors and the set of nontrivial admissible controls (investment demands and the consumption rate) and stopping times (retirement times) subject to the non-negative consumption and bequest rates. Before detailing our mathematical and economic contributions, we provide a literature review.

1.1. Literature review

Richard (1975) pioneers studies of the dynamic demand for life insurance using a continuous-time consumption–investment problem. There are various extensions of this problem in the literature, such as mean-reverting stock returns (Pirvu and Zhang, 2012), a multivariate jump-diffusion model with a contagion effect (Liu et al., 2021), a family consisting of a wage-earner and a dependent (Kwak et al., 2011), and a wage-earning couple (Wei et al., 2020). Our paper is motivated by Kwak and Lim (2014), who introduce inflation risk into the analysis. They find that inflation-linked bonds can play various roles in an agent's decisions and that the demand for life insurance is significantly affected by the expected rate and volatility of inflation. We are also aware of the extensions to stochastic mortality intensity by Menoncin and Regis (2017) and stochastic differential utility by Han and Hung (2017) in the literature. The aforementioned studies about life insurance under inflation risk focus on pre-retirement decisions when the retirement time is deterministic and exogenous, whereas this paper permits the retirement time to be a component of the agent's decisions.

Research on retirement decision can be divided into three groups: exogenous mandatory retirement as mentioned above, a voluntary retirement time (Choi and Shim, 2006; Dybvig and Liu, 2010), and an early retirement option with a mandatory retirement date (Yang and Koo, 2018; Jeon et al., 2022; Chen et al., 2022). Although realistic retirement decision naturally considers inflation, the literature has not yet investigated its effect. Moreover, there are few studies about a retirement problem under model ambiguity. To the best of our knowledge, the study by Park and Wong (2023) is the first of its type to use the G-expectation theory.

Inflation ambiguity has also drawn attention in the academic literature, such as in Ulrich (2013); Munk and Rubtsov (2014); Guan and Liang (2019); Wang et al. (2021). These studies apply the framework of equivalent priors proposed by Maenhout (2004). Specifically, Ulrich (2013) studies the trend of inflation ambiguity in a representative agent asset pricing model. Guan and Liang (2019) consider reinsurance-investment strategies with ambiguity in surplus, inflation rate, interest rate, and stock price. Wang et al. (2021) study an investment problem for a DC pension (defined contribution pension) plan member under ambiguous inflation and a mean-reverting risk premium. A more realistic model assumes stochastic volatility and correlation on top of drift ambiguity (Pun and Wong, 2015; Yan et al., 2020), but the problem associated with ambiguity in volatility and correlation remains because the assumption of an equivalent prior set rules out the general ambiguity considerations in volatility and correlation.

1.2. Our contributions

Our first contribution is the theoretical treatment of the robust controller-stopper problem associated with the practical actuarial concerns of a robust consumption–investment–retirement decision subject to inflation risk. Inspired by Park and Wong (2023), we utilize a combination of the classical dual approach (Karatzas and Shreve, 1998; Karatzas and Wang, 2000) and the G-stopping time theory (Hu and Peng, 2013; Li and Peng, 2020). Unlike Park and Wong (2023), who only investigate the return ambiguity in stock return with equivalent priors type, we consider general ambiguity involving non-dominated priors and a nontrivial admissible set. Specifically, our ambiguity priors set in the primal space comprises ambiguity in return, volatility and correlation parameters. Classical techniques usually focus on return ambiguity and encounter extreme difficulties in handling all together of the ambiguity in the three aspects (e.g. Maenhout

(2004)). Meanwhile, G-expectation is an advanced theory designed for purely volatility ambiguity problem. Although a direct use of G-expectation is not allowed in our ambiguity setup, by resorting to the dual formulation, the dual process has volatility ambiguity only and then the dual problem can be solved completely using G-stopping time theory. Due to the nontrivial admissible set, we have to extend the optimal G-stopping time theory in Park and Wong (2023) by proposing a new verification theorem to ensure no duality gap as well as the admissibility of the robust strategy. In particular, our ambiguity priors set requires a nontrivial verification on the integrability of the candidate strategy, which is not explored by Park and Wong (2023) who only consider equivalent probability measures. By articulating the G-expectation toolkit, we overcome the mathematical difficulty. We characterize the robust retirement time in the sense of a robust stopping rule under Dybvig and Liu (2010)'s utility, and provide the integral equation representations for the robust consumption, life insurance, and investment strategies.

Such a theoretical development is also a complementary actuarial contribution to the literature. Table 1 shows the comprehensiveness of our economic considerations compared to the literature. Table 2 compares the ambiguity setup of our paper against others. Therefore, we can analyze the effect of ambiguity from different sources on the decision of a wage-earner for further actuarial and economic analysis. This constitutes our second contribution.

Our third contribution is the discovery of new economic insights from the comprehensive ambiguity setting. We highlight the unique feature from our model that the robust retirement decision has a cross-point pattern in the retirement wealth boundary under the ambiguity from any one of the following sources: stock price, inflation rate, and their correlation. Such a pattern is not realized by Park and Wong (2023), who studies a robust retirement decision under return ambiguity. This pattern implies that ambiguity leads to a polarization effect. In other words, wealthy agents tend to retire earlier while poor agents work for longer, consistent with the findings of Baliga et al. (2013). In addition, ambiguity makes the robust wage-earner self-regulated in terms of consumption and investment. Self-regulated consumption has a greater effect on young agents, as it decreases their retirement wealth boundary for future consumption needs compared with young agents with parameter certainty. Reduced investment demand affects older agents more, as it increases the retirement wealth boundary for the loss from potential investment gains.

Like other ambiguity setups, our ambiguity model generally decreases the demand for consumption, life insurance purchase, and investment. However, the impact on the consumption and life insurance purchase is more pronounced for stock demand than for inflation bond's. We also recognize a subtle difference in adjusting strategies to ambiguity according to the agents' wealth level. Wealthy agents take ambiguity more seriously than poor agents by adjusting their strategies at a significant scale.

Furthermore, in a realistic parameter setup, we conduct a simulation-based comparison study to examine the performance of the robust strategy and the significance of considering comprehensive model ambiguities. Our simulation study shows that, in addition to return ambiguity, managing ambiguity in volatility and correlation parameters can further improve the performance of the strategy and affects the consumption and life insurance purchase significantly.

The rest of the paper is organized as follows. Section 2 presents the problem formulation. This involves the introduction of the investment–consumption–retirement problem with life insurance and inflation-linked index bonds and our ambiguity setup. A primal max–min problem is defined under a general utility preference. In Section 3, the primal problem is transformed into a dual problem, which is further linked to an auxiliary G-stopping problem. We completely solve the G-stopping problem and provide a verification theorem for the primal problem. For the CRRA (constant relative risk-aversion) utility, we derive the agent's robust decisions in an explicit representation as a solution of an integral equation. We conduct numerical analyses in Section 4 and investigate the separate effects of stock price ambiguity, inflation ambiguity, and the ambiguous correlation between two on the agent's decisions. Furthermore, we conduct simulation experiments and robustness comparisons with other strategies under the unknown market environment. Concluding remarks are given in Section 5.

2. The model

2.1. The economy

Consider a continuous-time economy with a finite planning horizon $[0, T^1]$ with $T^1 < \infty$. The economy comprises a financial market and an insurance market with inflation risk. A representative agent of wage earners has *ambiguous* information about the inflation rate, typically represented by the Consumer Price Index (CPI) $P \equiv (P_t)_{t=0}^{T^1}$, and the stock price $S \equiv (S_t)_{t=0}^{T^1}$. Following the inflation risk literature (Zhang and Ewald, 2010; Han and Hung, 2012; Kwak and Lim, 2014), $(P, S)^\top$ is assumed to follow a bivariate Itô process such as the one defined in (4).

Setup for ambiguity. The agent's beliefs about the two ambiguous factors consist of probability measures that can be generated using the parameters of the bivariate Itô process. Formally, we adopt a canonical space $\Omega \equiv \{\omega = (\omega_t)_{t=0}^{T^1} \in C([0, T^1], \mathbb{R}^2) \mid \omega_0 = (P_0, S_0)^\top\}$ with corresponding Borel σ-algebra $\mathcal{B}(\Omega)$ and filtration $\mathbb{F} \equiv (\mathcal{F}_t)_{t=0}^{T^1}$ generated by $(P, S)^\top$, which allows for a set of all 2×2 matrix processes, defined by

$$\mathcal{S} \equiv \left\{ \Sigma \middle| \exists (\sigma_P, \sigma_S, \rho)^\top \in \mathcal{K} \text{ s.t. } \Sigma_t(\omega) = \Xi((\sigma_{P,t}(\omega), \sigma_{S,t}(\omega), \rho_t(\omega))^\top), \ \forall \omega \in \Omega, \ t \in [0, T^1] \right\}, \tag{1}$$

where $\mathcal{K}$ is the set of $\mathbb{R}^3$-valued and $\mathbb{F}$-progressively measurable processes, defined by

$$\mathcal{K} \equiv \left\{ (\sigma_P, \sigma_S, \rho)^\top \middle| (\sigma_{P,t}(\omega), \sigma_{S,t}(\omega), \rho_t(\omega))^\top \in \mathfrak{D}_{vol}, \ \forall \omega \in \Omega, t \in [0, T^1] \right\},$$

with $\mathfrak{D}_{vol} \equiv [\underline{\sigma}_P, \overline{\sigma}_P] \times [\underline{\sigma}_S, \overline{\sigma}_S] \times [\underline{\rho}, \overline{\rho}] \subset \mathbb{R}^3$ such that $[\underline{\sigma}_i, \overline{\sigma}_i] \subset (0, +\infty)$ for $i = P, S$ and $[\underline{\rho}, \overline{\rho}] \subset (-1, 1)$. The matrix function $\Xi : \mathfrak{D}_{vol} \to \mathbb{R}^{2\times 2}$ is given by

$$\Xi(x) \equiv \begin{pmatrix} x_1 & 0 \\ x_2 x_3 & x_2\sqrt{1 - x_3^2} \end{pmatrix} \in \mathbb{R}^{2\times 2} \quad \text{for} \quad x = (x_1, x_2, x_3)^\top \in \mathfrak{D}_{vol}. \tag{2}$$

The specification of the set $\mathcal{S}$ ensures non-degeneracy on the matrices in $\mathcal{S}$. In other words, there is a positive constant κ such that $\xi^\top \Sigma \xi \geq \kappa \|\xi\|^2$ for any $\Sigma \in \mathcal{S}$ and $\xi \in \mathbb{R}^2$.

To construct a plausible set of probability measures, we define a set of all density characteristics comprising $\Sigma \in \mathcal{S}$ and an $\mathbb{F}$-progressively measurable process $\mu = (\mu_1, \mu_2)^\top$:

$$\Theta \equiv \left\{ (\Sigma, \mu) \middle| \Sigma \in \mathcal{S} \text{ and } \mu_t(\omega) = (\mu_{P,t}(\omega), \mu_{S,t}(\omega))^\top \in \mathfrak{D}_{return}, \ \forall \omega \in \Omega, \ t \in [0, T^1] \right\}, \tag{3}$$

with $\mathfrak{D}_{return} = [\underline{\mu}_P, \overline{\mu}_P] \times [\underline{\mu}_S, \overline{\mu}_S] \subset \mathbb{R}^2$. The process μ stands for the drift term and Σ for the volatility of the bivariate process $(P, S)^\top$. We often call $\Sigma\Sigma^\top$ the variance-covariance matrix.

Definition 1. Let $\mathcal{P}$ denote the collection of all probability measures $\mathbb{P}$ on $(\Omega, \mathcal{B}(\Omega))$ such that $(P, S)^\top$ is the unique strong solution of the bivariate Itô process

$$(dP_t, dS_t)^\top = \text{diag}((P_t, S_t)^\top)(\mu_t dt + \Sigma_t dW_t^{\mathbb{P}}), \tag{4}$$

where $W_t^{\mathbb{P}} = (W_{1,t}^{\mathbb{P}}, W_{2,t}^{\mathbb{P}})^\top$ denotes a standard two dimensional $\mathbb{P}$-Brownian motion with $(\Sigma, \mu) \in \Theta$, and $\text{diag}(A)$ denotes the diagonal matrix with components of a vector A.

The drift and volatility of the CPI are typically time-dependent for a periodic term period, although constant parameters are assumed in the literature for portfolio selection with inflation risk (Kwak and Lim, 2014; Zhang and Ewald, 2010; Chen et al., 2017). The density characteristics in Θ relax such a restriction to allow for general ambiguity in the CPI and stock simultaneously. The consideration of $\mathcal{D}_{vol}$, i.e., the alternative volatilities and correlations, makes the robust optimization more comprehensive. This marks a significant difference from Chen and Epstein (2002); Pun and Wong (2015); Guan and Liang (2019) of focusing on the drift ambiguity. Furthermore, the set $\mathcal{P}$, encompassing the ambiguous volatility model (Avellaneda et al., 1995; Lyons, 1995; Peng, 2007), represents a larger set of measures covering comprehensive ambiguities, while it is subsumed into the semimartingales measures considered by Denis and Kervarec (2013); Bartl et al. (2021). The measures in $\mathcal{P}$ are not necessarily dominated, as many of them are mutually singular due to the ambiguous volatility and correlation.

Financial market. We assume that the financial market comprises three types of freely tradable asset: a nominal inflation-linked index bond $I \equiv (I_t)_{t=0}^{T^1}$, the stock $S = (S_t)_{t=0}^{T^1}$, and a nominal money market account $M = (M_t)_{t=0}^{T^1}$. For a given $\mathbb{P} \in \mathcal{P}$ with $(\Sigma, \mu) \in \Theta$, such that $\Sigma = \Xi((\sigma_P, \sigma_S, \rho)^\top)$ and $\mu = (\mu_P, \mu_S)^\top$, and $\mathbb{P}$-Brownian motion $W^{\mathbb{P}} = (W_1^{\mathbb{P}}, W_2^{\mathbb{P}})^\top$, the nominal inflation-linked index bond price I depending on the CPI, and the stock price S jointly follow,

$$\begin{aligned} dI_t/I_t &= r_I dt + dP_t/P_t = (r_I + \mu_{P,t})dt + \sigma_{P,t} dW_{1,t}^{\mathbb{P}}, \\ dS_t/S_t &= \mu_{S,t} dt + \sigma_{S,t}\big(\rho_t dW_{1,t}^{\mathbb{P}} + \sqrt{1-\rho_t^2} dW_{2,t}^{\mathbb{P}}\big), \end{aligned} \tag{5}$$

with the constant real interest rate $r_I > 0$. Note that there is ample evidence for the correlation between the stock price and the inflation rate (Kessel, 1956; Firth, 1979; Fama, 1981; Boudhouch and Richarson, 1993; Anari and Kolari, 2001). According to the definition of Θ and $\mathcal{S}$, $|\rho| \neq 1$, which means that inflation cannot be perfectly hedged by the stock.

The nominal money market account M earns a constant nominal interest rate $r_M > 0$ so that

$$dM_t/M_t = r_M dt. \tag{6}$$

Insurance market. As in Dybvig and Liu (2010); Lim and Kwak (2016); Park et al. (2021), who consider the uncertain lifetime feature in retirement problem, we assume that for any given $\mathbb{P} \in \mathcal{P}$ with $\mathbb{P}$-Brownian motion $W^{\mathbb{P}} = (W_1^{\mathbb{P}}, W_2^{\mathbb{P}})^\top$, the agent's uncertain lifetime τ_D, which is independent of $W^{\mathbb{P}}$, follows an exponential distribution with a certain rate $1/\lambda > 0$, i.e.,

$$\mathbb{P}(\tau_D \leq t) = \int_0^t \lambda e^{-\int_0^s \lambda du} ds = 1 - e^{-\lambda t}, \tag{7}$$

where $\lambda > 0$ represents the constant hazard rate of the agent.[1] To hedge the lifetime risk, the agent purchases life insurance in the insurance market, paying the premium continuously at a nominal rate $p_N = (p_{N,t})_{t=0}^{\tau_D}$ during the lifetime. In a frictionless market, as a coverage for the insurance contract, the amount for the agent at the time of death $t > 0$ is $p_{N,t}/\lambda$. We note that p_N can be negative. As explained by Dybvig and Liu (2010), $p_N < 0$ resembles a pension annuity. Specifically, the agent trades the amount $-p_{N,t}/\lambda$ at the death time t for cash inflow at the nominal rate p_N while living. Accordingly, the nominal value of the agent's bequest $\mathfrak{B}_{N,t}$ at the death time t is given by

$$\mathfrak{B}_{N,t} = X_{N,t} + \frac{p_{N,t}}{\lambda}, \tag{8}$$

where $X_{N,t}$ is the nominal value of agent's wealth at time t.

Inflation-adjusted wealth. We adopt the framework of a retirement option with a finite time horizon as considered by Yang and Koo (2018); Chen et al. (2022); Jeon et al. (2022); Park and Wong (2023). The agent who is currently working receives labor income at a

[1] The hazard rate is set to be constant for the tractability of the model.

nominal rate $w_N = (w_{N,t})_{t=0}^{\tau_D}$ and is required to retire no later than a certain time T, which is called the mandatory retirement date and assumed to be $0 < T < T^1$. The agent can also choose early retirement $\tau_R \in [0, T]$.

Let $(c_N, p_N, \pi_N, \tau_R)$ be a quadruple of the agent's nominal values of consumption rate $c_N = (c_{N,t})_{t=0}^{\tau_D}$, the life insurance purchase $p_N = (p_{N,t})_{t=0}^{\tau_D}$, the dollar amount invested in money market account, index bond, and stock $\pi_N = ((\pi_{N,t}^M, \pi_{N,t}^I, \pi_{N,t}^S)^\top)_{t=0}^{\tau_D}$, and the retirement time τ_R. In the event that $\{t < \tau_D\}$, the agent's nominal wealth $X_{N,t}$ at the time t evolves

$$dX_{N,t} = \pi_{N,t}^M \frac{dM_t}{M_t} + \pi_{N,t}^I \frac{dI_t}{I_t} + \pi_{N,t}^S \frac{dS_t}{S_t} - \left(c_{N,t} + p_{N,t} - w_{N,t}\mathbf{1}_{\{t<\tau_R\}}\right)dt. \tag{9}$$

As the agent derives utility, which is defined in Section 2.2, from the real (not nominal) values of the consumption and wealth rates, the nominal wealth rate (9) is adjusted by the inflation rate (i.e., the CPI price). For this, we denote the real values of the consumption rate, life insurance premium, and dollar amount invested in three traded assets by $c = (c_t)_{t=0}^{\tau_D}$, $p = (p_t)_{t=0}^{\tau_D}$, $\pi = (\pi_t^M, \pi_t^I, \pi_t^S)_{t=0}^{\tau_D}$ respectively, i.e., the corresponding nominal variable discounted by the CPI. As in Kwak and Lim (2014), the real income process $(w_t)_{t=0}^{\tau_D}$ is assumed to be a constant with a rate $w > 0$, i.e., $w_{N,t}/P_t = w_t = w > 0$ for all $t \le \tau_D$. Under any probability measure $\mathbb{P} \in \mathcal{P}$ with $\mathbb{P}$-Brownian motion $W^{\mathbb{P}}$ and the event $\{t < \tau_D\}$, the real wealth rate $X_t^{x;c,p,\pi,\tau_R}$ follows:

$$\begin{aligned} dX_t^{x;c,p,\pi,\tau_R} &= \left(r_I X_t^{x;c,p,\pi,\tau_R} - c_t - p_t + w\mathbf{1}_{\{t<\tau_R\}}\right)dt + \pi_t^M\left[-(\mu_{P,t} + r_I - r_M - \sigma_{P,t}^2)dt - \sigma_{P,t}dW_{1,t}^{\mathbb{P}}\right] \\ &\quad + \pi_t^S\left[(\mu_{S,t} - r_I - \mu_{P,t} - \rho_t\sigma_{P,t}\sigma_{S,t} + \sigma_{P,t}^2)dt + (\rho_t\sigma_{S,t} - \sigma_{P,t})dW_{1,t}^{\mathbb{P}} + \sigma_{S,t}\sqrt{1-\rho_t^2}\,dW_{2,t}^{\mathbb{P}}\right]. \end{aligned} \tag{10}$$

2.2. Primal max–min problem

The agent derives utility from the adjusted real values of the consumption and bequest rates. For the priors set $\mathcal{P}$ in Definition 1, the ambiguity-averse agent's preference is defined as follows: for a given endowment x,

$$\begin{aligned} U(0,x) \equiv \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\Bigg[&\int_0^{\tau_D\wedge T\wedge\tau_R} e^{-\delta t}u(c_t)dt + \mathbf{1}_{\{\tau_D\ge T\wedge\tau_R\}}\int_{T\wedge\tau_R}^{T^1\wedge\tau_D} e^{-\delta t}u(k_R c_t)dt \\ &+ \mathbf{1}_{\{\tau_D\le T^1\}}e^{-\delta\tau_D}u(k_B\mathfrak{B}_{\tau_D}) + \mathbf{1}_{\{\tau_D>T^1\}}e^{-\delta T^1}u(k_X X_{T^1}^{x;c,p,\pi,\tau_R})\Bigg], \end{aligned} \tag{11}$$

where $\delta > 0$ is a subjective discount rate, $u(\cdot)$ is the utility function of the real values of consumption rate c_t, bequest rates $\mathfrak{B}_{\tau_D}$ and $X_{T^1}^{x;c,p,\pi,\tau_R}$ on the uncertain lifetime τ_D and planning horizon time T^1, respectively, and τ_R is the retirement time. Following Dybvig and Liu (2010); Jeon et al. (2022), we denote the agent's preference for not working by $k_R > 1$.[2] The parameters $k_B, k_X > 0$ denote the agent's bequest motive on the uncertain lifetime and the planning horizon time, respectively. The difference in the marginal utility of consumption before and after retirement induces the retirement-consumption puzzle, which cannot be captured in the robust retirement problem of Park and Wong (2023), where the agent's preference is assumed to be a disutility setup of Yang and Koo (2018).

We impose the following assumption, which covers commonly used utility functions in consumption-investment problems, such as the CRRA, log, and mixture of CRRA utilities.

Assumption 1. The utility function $u(\cdot)$ satisfies the following conditions:

(i) $u(\cdot)$ is in $C^\infty((0,\infty))$ and takes value in $\mathbb{R}$. It is strictly concave and increasing, and it satisfies the Inada conditions, i.e., $\lim_{x\to0^+} u'(x) = \infty$ and $\lim_{x\to+\infty} u'(x) = 0$;
(ii) Define $I(y)$, the inverse of $u'(\cdot)$, i.e., $I(y) \equiv (u')^{-1}(y)$ for $y > 0$. There exist positive constants $\kappa_0, \kappa_1 \ge 1$ and $C_0, C_1 > 0$ such that $0 < I(y) \le C_0(1 + y^{-\kappa_0})$ and $-C_1(1 + y^{-\kappa_1}) \le I'(y) \le 0$ for all $y > 0$;
(iii) Denote by $-\frac{u'(x)}{u''(x)}$ the risk tolerance of the utility functions. The risk tolerance is non-decreasing, i.e., $\left(-\frac{u'(x)}{u''(x)}\right)' = -\frac{(u''(x))^2 - u'(x)u'''(x)}{(u''(x))^2} \ge 0$ for all $x > 0$.

For a given $\mathbb{P} \in \mathcal{P}$ with $(\Sigma, \mu) \in \Theta$, such that $\Sigma = \Xi(\sigma_P, \sigma_S, \rho)$ and $\mu = (\mu_P, \mu_S)^\top$, we denote the (inflation-adjusted) real value of the future labor income by

$$m(t;w) \equiv \mathbb{E}_t^{\mathbb{P}}\left[\int_t^T w\frac{\mathcal{H}_s}{\mathcal{H}_t}ds\right] = \frac{1 - e^{-(r_I+\lambda)(T-t)}}{r_I + \lambda}w\mathbf{1}_{\{0\le t\le T\}}, \quad \text{for} \quad t \in [0, T^1], \tag{12}$$

where $(\mathcal{H}_t)_{t=0}^{T^1}$ is a state-price-density (i.e., pricing kernel) process, given by

$$\mathcal{H}_t \equiv e^{-(r_I+\lambda)t}\left.\frac{d\mathbb{Q}^{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathcal{F}_t} \quad \text{with} \quad \left.\frac{d\mathbb{Q}^{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathcal{F}_t} \equiv \exp\left(-\int_0^t (\Sigma_u^{-1}\zeta_u^{\Sigma,\mu})^\top dW_u^{\mathbb{P}} - \frac{1}{2}\int_0^t \|\Sigma_u^{-1}\zeta_u^{\Sigma,\mu}\|^2 du\right).$$

[2] $k_R > 1$ implies that the marginal utility of consumption is greater after retirement than before retirement.

It follows from the boundedness of all of the components of $(\Sigma, \mu) \in \Theta$ and the non-degeneracy of $\Sigma \in \mathcal{S}$ (satisfying Novikov's condition) that $\frac{d\mathbb{Q}^{\mathbb{P}}}{d\mathbb{P}}|_{\mathcal{F}_t}$ is an exponential $\mathbb{P}$-martingale, and the $\mathbb{Q}^{\mathbb{P}}$-Brownian motion $W_t^{\mathbb{Q}^{\mathbb{P}}} = W_t^{\mathbb{P}} + \int_0^t \Sigma_u^{-1}\zeta_u^{\Sigma,\mu}du = (W_{1,t}^{\mathbb{Q}^{\mathbb{P}}}, W_{2,t}^{\mathbb{Q}^{\mathbb{P}}})^\top$ is given by

$$\begin{pmatrix} W_{1,t}^{\mathbb{Q}^{\mathbb{P}}} \\ W_{2,t}^{\mathbb{Q}^{\mathbb{P}}} \end{pmatrix} = \begin{pmatrix} W_{1,t}^{\mathbb{P}} + \int_0^t \frac{1}{\sigma_{P,u}}\zeta_{1,u}^{\Sigma,\mu}du \\ W_{2,t}^{\mathbb{P}} + \int_0^t \Big(-\frac{\rho_u}{\sigma_{P,u}\sqrt{1-\rho_u^2}}\zeta_{1,u}^{\Sigma,\mu} + \frac{1}{\sigma_{S,u}\sqrt{1-\rho_u^2}}\zeta_{2,u}^{\Sigma,\mu}\Big)du \end{pmatrix}, \tag{13}$$

and $\zeta_u^{\Sigma,\mu} = (\zeta_{1,u}^{\Sigma,\mu}, \zeta_{2,u}^{\Sigma,\mu})^\top$ with $\zeta_{1,u}^{\Sigma,\mu} \equiv \mu_{P,u} - \sigma_{P,u}^2 + r_I - r_M$ and $\zeta_{2,u}^{\Sigma,\mu} \equiv \mu_{S,u} - \rho_u\sigma_{P,u}\sigma_{S,u} - r_M$ for $u \in [0, T^1]$. Note that the expected real value of the future income is invariant with the choice of probability measures in $\mathcal{P}$. It is worth noting that for each $\mathbb{P} \in \mathcal{P}$, the Itô diffusion market in (4) induces the unique martingale measure $\mathbb{Q}^{\mathbb{P}}$ due to the integral representation theorem and the Girsanov theorem (Kramkov and Schachermayer, 1999, example 5.2 in section 5).

Then, for the measure $\mathbb{P} \in \mathcal{P}$, we call a quadruple (c, p, π, τ_R) with the initial endowment x and the real value of the labor income rate $w > 0$ $\mathbb{P}$-*admissible* if the following conditions hold:

- τ_R belongs to $\mathcal{T}_{0,T}$ which is the collection of stopping rules $\tau_R \in \mathcal{T}$ with values in $[0, T]$, where $\mathcal{T}$ comprises measurable mappings $\tau_R : \Omega \to [0, T^1]$ such that $\{\tau_R \le t\} \in \mathcal{F}_t$ for all $t \in [0, T^1]$;
- c_t, p_t, and $\pi_t = (\pi_t^M, \pi_t^I, \pi_t^S)^\top$ are $\mathcal{F}_t$-progressively measurable processes and satisfy $\int_0^{T^1}(c_t + |p_t| + \|(\pi_t^M, \pi_t^S)^\top\|^2)dt < \infty$, $\mathbb{P}$-a.s., subject to $c_t \ge 0$ for all $t \in [0, T^1]$;
- $X_t^{x;c,p,\pi,\tau_R} > -m(t; w)\mathbf{1}_{\{0\le t\le\tau_R\}}$, $\mathbb{P}$-a.s., for all $t \in [0, T^1]$, which is called the natural borrowing limit of the wealth.

We denote by $\mathcal{A}_{0,T^1}^{\mathbb{P}}(x, w)$ the collection of all $\mathbb{P}$-admissible controls for a given endowment $x > -m(0; w)$ on $[0, T^1]$, which allows us to define a *robust* admissible set by

$$\mathcal{A}_{0,T^1}(x, w) \equiv \cap_{\mathbb{P}\in\mathcal{P}}\mathcal{A}_{0,T^1}^{\mathbb{P}}(x, w).$$

Note that the set $\mathcal{A}_{0,T^1}(x, w)$ aligns with $\mathcal{P}$-quasi-surely admissible controls in non-dominated priors (see Soner et al. (2011); Denis and Kervarec (2013); Bartl et al. (2021)).

Problem 1 *(Primal problem).* For a given $x > -m(0; w)$, the ambiguity-averse agent's problem is to determine a robust strategy $(c^*, p^*, \pi^*, \tau_R^*) \in \mathcal{A}_{0,T^1}(x)$ such that

$$U(0, x; c^*, p^*, \pi^*, \tau_R^*) = V(0, x) \equiv \sup_{(c,p,\pi,\tau_R)\in\mathcal{A}_{0,T^1}(x,w)} U(0, x; c, p, \pi, \tau_R),$$

where the agent's preference U in (11) can be rewritten as follows[3]:

$$U(0, x; c, p, \pi, \tau_R) = \inf_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\left[\int_0^{T^1} e^{-(\delta+\lambda)t}\big(u(c_t)\mathbf{1}_{\{t<\tau_R\}} + u(k_R c_t)\mathbf{1}_{\{t\ge\tau_R\}} + \lambda u(k_B\mathfrak{B}_t)\big)dt + e^{-(\delta+\lambda)T^1}u\big(k_X X_{T^1}^{x;c,p,\pi,\tau_R}\big)\right]. \tag{14}$$

Remark 1. Although the current max–min expected utility problem involves the non-dominated measures and nontrivial admissible controls, in Section 3.1 we show through the dual formulation that it can be transformed into a problem with ambiguity only on the volatility of the dual process, which is consistent with Park and Wong (2022, 2023). As we consider the ambiguous volatility and correlation in addition to the ambiguous return, verifications for no duality gap and admissibility cannot be directly replaced by the results of Park and Wong (2022, 2023), who consider only the return ambiguity.

3. Robust strategy: solution analysis

In Section 3.1, we transform the primal max–min problem (Problem 1) into a dual problem (Problem 2) using a standard martingale method. In the dual problem, all of the parameters of the CPI and stock price processes are recast as a volatility parameter of a dual process. This leads to another max–min problem that includes volatility ambiguity and a stopping time. In Section 3.2, we tackle this problem by introducing an auxiliary problem (Problem 3) that hinges on the G-expectation framework. In Section 3.3, we provide the dynamic principle results of the auxiliary problem to construct the explicit solution to the dual max–min problem. To characterize the robust strategy, in Section 3.4, we complete the verification proof for the duality relationship between the two max–min problems.

3.1. Dual max–min problem

For any given $\mathbb{P} \in \mathcal{P}$, we transform the dynamic budget constraint (10) into a static budget constraint using the martingale method developed by Karatzas and Shreve (1998); Cox and Huang (1989). Then, we incorporate all of the static budget constraints for all of the probability measures in $\mathcal{P}$ to derive the corresponding dual problem.

[3] If we take the condition (7) of the uncertain time of death τ_D on the agent's preference (11), the equivalent representation (14) can be obtained.

Recalling the exponential martingale (12) under a given $\mathbb{P}\in\mathcal{P}$ with $(\Sigma,\mu)\in\Theta$, such that $\Sigma=\Xi((\sigma_P,\sigma_S,\rho)^\top)$ and $\mu=(\mu_P,\mu_S)^\top$, we represent the real wealth dynamics (10) under $\mathbb{Q}^{\mathbb{P}}$ by

$$
\begin{aligned}
dX_t^{x;c,p,\pi,\tau_R} &= \Big((r_I+\lambda)X_t^{x;c,p,\pi,\tau_R} - c_t - \lambda\mathfrak{B}_t + w\mathbf{1}_{\{t<\tau_R\}}\Big)dt \\
&\quad - \pi_t^M\sigma_{P,t}dW_{1,t}^{\mathbb{Q}^{\mathbb{P}}} + \pi_t^S\left((\rho_t\sigma_{S,t}-\sigma_{P,t})dW_{1,t}^{\mathbb{Q}^{\mathbb{P}}} + \sigma_{S,t}\sqrt{1-\rho_t^2}\,dW_{2,t}^{\mathbb{Q}^{\mathbb{P}}}\right),
\end{aligned}
$$

where we use the relation $\lambda\mathfrak{B}_t=\lambda X_t+p_t$ for all $t\le T^1$. Thus, based on Fatou's lemma and Bayes' rule, we have the following static budget constraint:

$$
\mathbb{E}^{\mathbb{P}}\left[\int_0^{T^1} e^{-(r_I+\lambda)t}\left.\frac{d\mathbb{Q}^{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathcal{F}_t}\big(c_t+\lambda\mathfrak{B}_t-w\mathbf{1}_{\{t<\tau_R\}}\big)dt + e^{-(r_I+\lambda)T^1}\left.\frac{d\mathbb{Q}^{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathcal{F}_{T^1}} X_{T^1}^{x;c,p,\pi,\tau_R}\right]\le x,
$$

which is equivalent to the dynamic constraint (10).

Considering the static constraint for all of the probability measures in $\mathcal{P}$, we have the following transformation: for any $y>0$,

$$
\begin{aligned}
&U(0,x;c,p,\pi,\tau_R) \\
&\le \inf_{\mathbb{P}\in\mathcal{P}}\Bigg\{\mathbb{E}^{\mathbb{P}}\left[\int_0^{T^1} e^{-(\delta+\lambda)t}\big(u(c_t)\mathbf{1}_{\{t<\tau_R\}}+u(k_Rc_t)\mathbf{1}_{\{t\ge\tau_R\}}+\lambda u(k_B\mathfrak{B}_t)\big)dt + e^{-(\delta+\lambda)T^1}u\big(k_XX_{T^1}^{x;c,p,\pi,\tau_R}\big)\right] \\
&\quad - y\mathbb{E}^{\mathbb{P}}\left[\int_0^{T^1} e^{-(r_I+\lambda)t}\left.\frac{d\mathbb{Q}^{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathcal{F}_t}\big(c_t+\lambda\mathfrak{B}_t-w\mathbf{1}_{\{t<\tau_R\}}\big)dt + e^{-(r_I+\lambda)T^1}\left.\frac{d\mathbb{Q}^{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathcal{F}_{T^1}}X_{T^1}^{x;c,p,\pi,\tau_R}\right]\Bigg\} + yx \\
&= \inf_{\mathbb{P}\in\mathcal{P}}\mathbb{E}^{\mathbb{P}}\Bigg[\int_0^{\tau_R} e^{-(\delta+\lambda)t}\Big(u(c_t)-Y_tc_t+\lambda u(k_B\mathfrak{B}_t)-\lambda Y_t\mathfrak{B}_t+wY_t\Big)dt \\
&\quad + \int_{\tau_R}^{T^1} e^{-(\delta+\lambda)t}\Big(u(k_Rc_t)-Y_tc_t+\lambda u(k_B\mathfrak{B}_t)-\lambda Y_t\mathfrak{B}_t\Big)dt + e^{-(\delta+\lambda)T^1}\Big(u\big(k_XX_{T^1}^{x;c,p,\pi,\tau_R}\big)-Y_{T^1}X_{T^1}^{x;c,p,\pi,\tau_R}\Big)\Bigg] + yx,
\end{aligned}
\tag{15}
$$

where $Y=(Y_t)_{t=0}^{T^1}$ with $Y_0=y>0$, which is the *dual process*, is defined by

$$
Y_t \equiv ye^{(\delta-r_I)t}\left.\frac{d\mathbb{Q}^{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathcal{F}_t} = ye^{\int_0^t(\delta-r_I-\frac{1}{2}\|\Sigma_u^{-1}\zeta_u^{\Sigma,\mu}\|^2)du-\int_0^t(\Sigma_u^{-1}\zeta_u^{\Sigma,\mu})^\top dW_u^{\mathbb{P}}}. \tag{16}
$$

Then, let us introduce the convex conjugate function of u by

$$
\widetilde{u}(y)\equiv\sup_{x>0}\Big(u(x)-xy\Big)=u(I(y))-I(y)y, \quad \text{for} \quad y>0,
$$

with $I(y)=(u')^{-1}(y)$.

Given $x>-m(0;w)$ and $\tau_R\in\mathcal{T}_{0,T}$, we denote by $\mathcal{A}_{0,T^1}(x;\tau_R)$ the class of all admissible triplets (c,p,π) such that $(c,p,\pi,\tau_R)\in\mathcal{A}_{0,T^1}(x,w)$. We temporarily consider

$$
\mathcal{V}(0,x;\tau_R)\equiv\sup_{(c,p,\pi)\in\mathcal{A}_{0,T^1}(x;\tau_R)}U(0,x;c,p,\pi,\tau_R).
$$

Using $\widetilde{u}(\cdot)$ and $\mathcal{V}(0,x;\tau_R)$, we further transform (15) into

$$
\begin{aligned}
&\mathcal{V}(0,x;\tau_R) \\
&\le \inf_{\mathbb{P}\in\mathcal{P}}\mathbb{E}^{\mathbb{P}}\Bigg[\int_0^{\tau_R} e^{-(\delta+\lambda)t}\left(\sup_{c>0}\Big\{u(c)-Y_tc\Big\}+\lambda\sup_{\mathfrak{B}>0}\Big\{u(k_B\mathfrak{B})-Y_t\mathfrak{B}\Big\}+wY_t\right)dt \\
&\quad + \int_{\tau_R}^{T^1} e^{-(\delta+\lambda)t}\left(\sup_{c>0}\Big\{u(k_Rc)-Y_tc\Big\}+\lambda\sup_{\mathfrak{B}>0}\Big\{u(k_B\mathfrak{B})-Y_t\mathfrak{B}\Big\}\right)dt + e^{-(\delta+\lambda)T^1}\sup_{X>0}\Big\{u(k_XX)-Y_{T^1}X\Big\}\Bigg] + yx \\
&= \inf_{\mathbb{P}\in\mathcal{P}}\mathbb{E}^{\mathbb{P}}\left[\int_0^{\tau_R} e^{-(\delta+\lambda)t}\Big(\widetilde{u}(Y_t)+\lambda\widetilde{u}(\frac{Y_t}{k_B})+wY_t\Big)dt + \int_{\tau_R}^{T^1} e^{-(\delta+\lambda)t}\Big(\widetilde{u}(\frac{Y_t}{k_R})+\lambda\widetilde{u}(\frac{Y_t}{k_B})\Big)dt + e^{-(\delta+\lambda)T^1}\widetilde{u}(\frac{Y_{T^1}}{k_X})\right] + yx \\
&\equiv \inf_{\mathbb{P}\in\mathcal{P}}\mathcal{J}(0,y;\mathbb{P},\tau_R)+yx,
\end{aligned}
$$

where the first-order conditions imply the *candidate* real consumption rate $\hat{c}$ and the bequest rates $\hat{\mathfrak{B}}$ and $\hat{X}_{T^1}$, given by

$$\hat{c}_t \equiv I(Y_t)\mathbf{1}_{\{t<\tau_R\}} + \frac{1}{k_R} I(\frac{Y_t}{k_R})\mathbf{1}_{\{t\geq\tau_R\}}, \quad \hat{\mathfrak{B}}_t \equiv \frac{1}{k_B} I(\frac{Y_t}{k_B}) \quad \text{for} \quad t\in[0,T^1], \quad \text{and} \quad \hat{X}_{T^1} = \frac{1}{k_X} I(\frac{Y_{T^1}}{k_X}). \tag{17}$$

From the successive transformations, we have the following weak duality:

$$\begin{aligned} V(0,x) = \sup_{\tau_R\in\mathcal{T}_{0,T}} \mathcal{V}(0,x;\tau_R) &\leq \sup_{\tau_R\in\mathcal{T}_{0,T}} \inf_{y>0} \left\{ \inf_{\mathbb{P}\in\mathcal{P}} \mathcal{J}(0,y;\mathbb{P},\tau_R) + yx \right\} \\ &\leq \inf_{y>0} \left\{ \sup_{\tau_R\in\mathcal{T}_{0,T}} \inf_{\mathbb{P}\in\mathcal{P}} \mathcal{J}(0,y;\mathbb{P},\tau_R) + yx \right\}. \end{aligned} \tag{18}$$

Now we are ready to define the dual problem as follows.

Problem 2 *(Dual problem).* Consider the following dual optimization problem: for $y>0$

$$J(0,y) = \sup_{\tau_R\in\mathcal{T}_{0,T}} \inf_{\mathbb{P}\in\mathcal{P}} \mathcal{J}(0,y;\mathbb{P},\tau_R),$$

where the dynamics of the dual process $(Y_t)_{t=0}^{T^1}$ in (16) are given by $dY_t = (\delta - r_I)Y_t dt - (\Sigma_t^{-1}\zeta_t^{\Sigma,\mu})^\top Y_t dW_t^{\mathbb{P}}$ under $\mathbb{P}\in\mathcal{P}$ with $(\Sigma,\mu)\in\Theta$.

Problem 2 has two features. First, the dual process Y (16) has ambiguous volatility, which is connected to the G-expectation framework of Peng (2007, 2008, 2010). Second, the problem is not solely a minimization of probability measures but a max–min problem with an optimal stopping time feature, which is closely connected to a zero-sum stochastic differential game of controller and stopper Karatzas and Zamfirescu (2008); Bayraktar and Huang (2013); Bayraktar and Yao (2014). As $\mathcal{P}$ comprises priors for comprehensive ambiguities, it cannot directly substitute a representing priors set for the G-expectation, which is designed for volatility ambiguity (not for return ambiguity) and comprises non-dominated priors purely.

Following the optimal G-stopping framework in Li and Peng (2020); Park and Wong (2023), we construct an appropriate set of non-dominated measures to define a G-expectation and introduce the notion of G-stopping time (Hu and Peng (2013); Li and Peng (2020)) for an auxiliary problem in the following subsection.

3.2. Auxiliary problem: optimal G-stopping approach

Following Denis et al. (2011), we construct a representing set of priors for a G-expectation space corresponding to the ambiguous volatility of Y (16). Throughout this subsection, we fix any (primal) probability measure $\mathbb{P}\in\mathcal{P}$ with Brownian motion $W^{\mathbb{P}}$. We denote by Λ^Θ the set of all feasible and $\mathbb{F}$-progressively measurable processes $M^{(\widetilde{\Sigma},\widetilde{\mu})} \equiv \left(M_t^{(\widetilde{\Sigma},\widetilde{\mu})}\right)_{t=0}^{T^1} \in \mathbb{R}^{2\times 2}$ such that there exist $(\widetilde{\Sigma},\widetilde{\mu})\in\Theta$ with $\widetilde{\Sigma} = \Xi((\widetilde{\sigma}_P,\widetilde{\sigma}_S,\widetilde{\rho})^\top)$ and $\widetilde{\mu} = (\widetilde{\mu}_P,\widetilde{\mu}_S)^\top$ satisfying

$$M_t^{(\widetilde{\Sigma},\widetilde{\mu})} \equiv \mathrm{diag}\left(f((\widetilde{\sigma}_{P,t},\widetilde{\sigma}_{S,t},\widetilde{\rho}_t,\widetilde{\mu}_{P,t},\widetilde{\mu}_{S,t})^\top)\right)\left(\widetilde{\Sigma}^{-1}\right)^\top \in \mathbb{R}^{2\times 2}, \tag{19}$$

where $f:\mathbb{R}^5\to\mathbb{R}^2$ is given as follows: for $x=(x_1,x_2,x_3,x_4,x_5)^\top\in\mathbb{R}^5$,

$$f(x) = (f_1(x), f_2(x))^\top \in \mathbb{R}^2 \quad \text{with} \quad f_1(x) = x_4 - x_1^2 + r_I - r_M, \quad f_2(x) = x_5 - x_1x_2x_3 - r_M. \tag{20}$$

For the set Λ^Θ, we denote the set of non-dominated priors by

$$\mathcal{Q}^0 \equiv \left\{ \widetilde{\mathbb{P}} \equiv \mathbb{P}\circ B^{-1} \middle| \exists\, M^{(\widetilde{\Sigma},\widetilde{\mu})}\in\Lambda^\Theta \text{ such that } B_t \equiv \int_0^t M_s^{(\widetilde{\Sigma},\widetilde{\mu})} dW_s^{\mathbb{P}},\ t\geq 0,\ \mathbb{P}\text{-a.s.} \right\}. \tag{21}$$

We denote by $\mathcal{Q}$ the closure of $\mathcal{Q}^0$ under the topology of weak convergence, i.e., $\mathcal{Q}\equiv\overline{\mathcal{Q}^0}$, which is a representing probability set of the G-expectation. For a given $\widetilde{\mathbb{P}}\in\mathcal{Q}$ with $M^{(\widetilde{\Sigma},\widetilde{\mu})}\in\Lambda^\Theta$, the canonical process B is a martingale with a quadratic variation that is an $\mathbb{S}(d)$-valued process given by $\langle B\rangle_t = M_t^{(\widetilde{\Sigma},\widetilde{\mu})}(M_t^{(\widetilde{\Sigma},\widetilde{\mu})})^\top$, $t\in[0,T^1]$, where $\mathbb{S}(d)$ denotes the collection of $d\times d$ symmetric matrices.

We define a G-expectation $\hat{\mathbb{E}}$ using $\mathcal{Q}$: for any $\xi\in\mathcal{B}(\Omega)$ such that $\sup_{\widetilde{\mathbb{P}}\in\mathcal{Q}}\mathbb{E}^{\widetilde{\mathbb{P}}}[\xi]<\infty$,

$$\hat{\mathbb{E}}[\xi] \equiv \sup_{\widetilde{\mathbb{P}}\in\mathcal{Q}} \mathbb{E}^{\widetilde{\mathbb{P}}}[\xi],$$

and the process B in (21) is called the G-Brownian motion. The corresponding sublinear function $G:\mathbb{S}(d)\to\mathbb{R}$ is given by

$$G(A) = \frac{1}{2}\sup_{\gamma\in\Gamma}\mathrm{tr}\left[A\gamma\gamma^\top\right], \tag{22}$$

with $\Gamma \equiv \left\{ \mathrm{diag}(f(x))(\Xi^{-1}((x_1,x_2,x_3)^\top))^\top \mid x=(x_1,x_2,x_3,x_4,x_5)^\top \in \mathfrak{D}_{vol}\times\mathfrak{D}_{return} \right\}$.

Appendix B provides more detailed notions and fundamental results, including the G-expectation spaces for some processes Hu et al. (2014a,b) and G-stopping time Hu and Peng (2013); Li and Peng (2020). Using Proposition 3.1.5 of Peng (2010), we further construct the following 1-dimensional G-Brownian motion $\widetilde{B} = (\widetilde{B}_t)_{t=0}^{T^1}$ with the sublinear function $\widetilde{G}(\cdot)$ on the same space $(\Omega, L_G^1(\Omega), \hat{\mathbb{E}})$:

$$\widetilde{B}_t \equiv B_{1,t} + B_{2,t}, \quad \text{with} \quad \widetilde{G}(\alpha) \equiv \frac{1}{2}\hat{\mathbb{E}}[\alpha \widetilde{B}_1^2] = \frac{1}{2}(\bar{\Upsilon}^2\alpha^+ - \underline{\Upsilon}^2\alpha^-), \tag{23}$$

with $\alpha^\pm \equiv \max\{0, \pm\alpha\}$ for $\alpha \in \mathbb{R}$, where $\bar{\Upsilon}^2$ and $\underline{\Upsilon}^2$ are given by

$$\bar{\Upsilon}^2 \equiv 2G\Big(1_2 1_2^\top\Big) = \sup_{x\in\mathfrak{D}_{vol}\times\mathfrak{D}_{return}} \Big\|\Xi^{-1}((x_1,x_2,x_3)^\top)f(x)\Big\|^2,$$

$$\underline{\Upsilon}^2 \equiv -2G\Big(-1_2 1_2^\top\Big) = \inf_{x\in\mathfrak{D}_{vol}\times\mathfrak{D}_{return}} \Big\|\Xi^{-1}((x_1,x_2,x_3)^\top)f(x)\Big\|^2,$$

and $\|\Xi^{-1}((x_1,x_2,x_3)^\top)f(x)\|^2$ is given as follows: for $x = (x_1,x_2,x_3,x_4,x_5)^\top \in \mathfrak{D}_{vol}\times\mathfrak{D}_{return}$,

$$\Big\|\Xi^{-1}((x_1,x_2,x_3)^\top)f(x)\Big\|^2 = \frac{1}{1-x_3^2}\left(\Big(\frac{f_1(x)}{x_1}\Big)^2 - 2x_3\frac{f_1(x)f_2(x)}{x_1x_2} + \Big(\frac{f_2(x)}{x_2}\Big)^2\right),$$

with $f(\cdot)$ in (20).

Clearly, $\bar{\Upsilon}^2 \geq 0$ and $\underline{\Upsilon}^2 \geq 0$. Moreover, from the closedness and boundedness of $\mathfrak{D}_{vol}\times\mathfrak{D}_{return}$, there exist maximizer and minimizer to $\bar{\Upsilon}^2$ and $\underline{\Upsilon}^2$, respectively. We are particularly interested in the worst-case scenarios and define

$$(\sigma_P^*, \sigma_S^*, \rho^*, \mu_P^*, \mu_S^*) \equiv \underset{x\in\mathfrak{D}_{vol}\times\mathfrak{D}_{return}}{\operatorname{argmin}} \Big\|\Xi^{-1}((x_1,x_2,x_3)^\top)f(x)\Big\|^2, \tag{24}$$

$$\Sigma^* \equiv \Xi((\sigma_P^*, \sigma_S^*, \rho^*)^\top), \qquad \mu^* \equiv (\mu_P^*, \mu_S^*)^\top, \tag{25}$$

$$M^{(\Sigma^*,\mu^*)} \equiv \operatorname{diag}(\zeta^{\Sigma^*,\mu^*})((\Sigma^*)^{-1})^\top, \quad \zeta^{\Sigma^*,\mu^*} \equiv (\zeta_1^{\Sigma^*,\mu^*}, \zeta_2^{\Sigma^*,\mu^*})^\top = f((\sigma_P^*, \sigma_S^*, \rho^*, \mu_P^*, \mu_S^*)^\top). \tag{26}$$

Under the G-expectation space $(\Omega, L_G^1(\Omega), \hat{\mathbb{E}})$ in which the volatility ambiguity of $(Y_t)_{t=0}^{T^1}$ in Problem 2 is transformed by the following 1-dimensional G-Brownian motion $\widetilde{B} = (\widetilde{B}_t)_{t=0}^{T^1}$ (23), i.e., for $t \in [0, T^1]$,

$$dY_t = (\delta - r_I)Y_t dt + Y_t d\widetilde{B}_t \quad \text{with} \quad Y_0 = y > 0. \tag{27}$$

Problem 3 *(Auxiliary problem under the G-expectation space).* Consider the following optimal G-stopping problem on $(\Omega, L_G^1(\Omega), \hat{\mathbb{E}})$: for $(t, y) \in \overline{\mathcal{D}}_T \equiv [0, T] \times (0, \infty)$,

$$\widetilde{J}(t,y) \equiv \sup_{\widetilde{\tau}\in\widetilde{\mathcal{T}}_{t,T}} \left\{-\hat{\mathbb{E}}_t\left[-\int_t^{\widetilde{\tau}} e^{-(\delta+\lambda)(s-t)}\Big(\widetilde{u}(Y_s) + \lambda\widetilde{u}(\frac{Y_s}{k_B}) + wY_s\Big)ds - e^{-(\delta+\lambda)(\widetilde{\tau}-t)}\underline{\widetilde{J}}(\widetilde{\tau}, Y_{\widetilde{\tau}})\Big| Y_t = y\right]\right\}, \tag{28}$$

with the conditional G-expectation $\hat{\mathbb{E}}_t[\cdot]$ and the set $\widetilde{\mathcal{T}}_{t,T}$ of G-stopping times for $t \in [0, T]$,[4] where the function $\underline{\widetilde{J}}(t, y)$ is a G-expectation value on $(t, y) \in \overline{\mathcal{D}}_{T^1} \equiv [0, T^1] \times (0, \infty)$, given by

$$\underline{\widetilde{J}}(t,y) \equiv -\hat{\mathbb{E}}_t\left[-\left(\int_t^{T^1} e^{-(\delta+\lambda)(s-t)}\Big(\widetilde{u}(\frac{Y_s}{k_R}) + \lambda\widetilde{u}(\frac{Y_s}{k_B})\Big)ds + e^{-(\delta+\lambda)(T^1-t)}\widetilde{u}(\frac{Y_{T^1}}{k_X})\right)\Big| Y_t = y\right], \tag{29}$$

and the dual process $Y_s = ye^{(\delta-r_I)(s-t)-\int_t^s d\widetilde{B}_u - \frac{1}{2}\int_t^s d\langle\widetilde{B}\rangle_u}$, which is the solution of (27) with $Y_t = y$.

Note that as $[T, T^1]$ is after the post mandatory retirement time horizon, we consider $\widetilde{J}$ only in $\bar{\mathcal{D}}_T$ rather than $\bar{\mathcal{D}}_{T^1}$. Extending $\widetilde{J}$ as $\widetilde{J}(t,y) = \underline{\widetilde{J}}(t,y)$ for $(t,y) \in (T, T^1] \times (0,\infty)$ allows for a complete solution to Problem 3. We show in Section 3.4 that this optimal value function under the G-expectation space coincides with that of the dual problem (Problem 2) and has a duality relationship with the optimal value function in the primal problem (Problem 1). The original combined control and stopping problem with model ambiguity is linked to a pure G-stopping problem.

3.3. *Characterization of optimal G-stopping*

As a max–min problem, Problem 3.1 can be represented by a reflected G-backward stochastic differential equation (reflected G-BSDE) with an upper obstacle. Li and Peng (2020) discuss the existence of a maximal solution (Definition 5 in Appendix B) to reflected G-BSDEs with an upper obstacle, and then Park and Wong (2023) demonstrate a dynamic representation of reflected G-BSDEs by using the arguments with an *approximate Skorohod condition* (Li and Song (2019); Li and Peng (2020)). To obtain the dynamic representation of Problem 3.1 (in Proposition 3), we utilize the arguments in Park and Wong (2022, 2023).

To this end, following the arguments provided in Lemma 1 of Park and Wong (2022), we first characterize a solution to $\underline{\widetilde{J}}$ that corresponds to the upper obstacle function (refer to Appendix A.1 for the proof).

[4] The definitions for the two notions are provided in Appendix B.

Lemma 3.1. *Under Assumption 1, the following statements are true:*

(a) *For $(t,y)\in\overline{\mathcal{D}}_{T^1}$, $\underline{\widetilde{J}}(t,y)$ is the unique viscosity solution of the following PDE (partial differential equation):*

$$\begin{cases}\partial_t\underline{\widetilde{J}}-\widetilde{G}\big(-y^2\partial_{yy}\underline{\widetilde{J}}\big)+(\delta-r_I)y\partial_y\underline{\widetilde{J}}-(\delta+\lambda)\underline{\widetilde{J}}+\widetilde{u}(\frac{y}{k_R})+\lambda\widetilde{u}(\frac{y}{k_B})=0, & (t,y)\in\mathcal{D}_{T^1},\\ \underline{\widetilde{J}}(T^1,y)=\widetilde{u}(\frac{y}{k_X}), & y\in(0,\infty),\end{cases} \tag{30}$$

with $\mathcal{D}_{T^1}\equiv[0,T^1)\times(0,+\infty)$ and $\widetilde{G}:\mathbb{R}\to\mathbb{R}$ in (23).

(b) *The solution $\underline{\widetilde{J}}$ is in $C^\infty(\overline{\mathcal{D}}_{T^1})$, strictly convex with respect to $y>0$, and thus explicitly represented by*

$$\begin{aligned}\underline{\widetilde{J}}(t,y)=&\mathbb{E}_t^{\widetilde{\mathbb{P}}^*}\left[\int_t^{T^1}e^{-(\delta+\lambda)(\xi-t)}\Big(\widetilde{u}(\frac{Y_\xi}{k_R})+\lambda\widetilde{u}(\frac{Y_\xi}{k_B})\Big)d\xi+e^{-(\delta+\lambda)(T^1-t)}\widetilde{u}(\frac{Y_{T^1}}{k_X})\;\Big|\;Y_t=y\right]\\ =&\int_t^{T^1}e^{-(\delta+\lambda)(\xi-t)}\int_0^\infty\Big(\widetilde{u}(\frac{\eta}{k_R})+\lambda\widetilde{u}(\frac{\eta}{k_B})\Big)\Phi_{\underline{\Upsilon}}(t,y;\xi,\eta)d\eta d\xi+e^{-(\delta+\lambda)(T^1-t)}\int_0^\infty\widetilde{u}(\frac{\eta}{k_X})\Phi_{\underline{\Upsilon}}(t,y;T^1,\eta)d\eta,\end{aligned} \tag{31}$$

for $(t,y)\in\overline{\mathcal{D}}_{T^1}$, with the prior $\widetilde{\mathbb{P}}^*\in\mathcal{Q}$ with the generator $M^{(\Sigma^*,\mu^*)}$ in (26), and the transition probability density function $\Phi_{\underline{\Upsilon}}(\cdot)$ of the dual process $(Y_\xi)_{\xi=t}^{T^1}$ under $\widetilde{\mathbb{P}}^*$, i.e.,

$$\Phi_{\underline{\Upsilon}}(t,y;\xi,\eta)=\frac{1}{\eta\sqrt{2\pi\underline{\Upsilon}^2(\xi-t)}}\exp\left\{-\frac{(\ln\eta-\ln y-(\delta-r_I-\frac{1}{2}\underline{\Upsilon}^2)(\xi-t))^2}{2\underline{\Upsilon}^2(\xi-t)}\right\}. \tag{32}$$

The following proposition is demonstrated by slightly extending the approach used in Proposition 2 of Park and Wong (2023). We give the proof in Appendix A.2 for completeness.

Proposition 3.1. *Under Assumption 1, the following statements are true:*

(a) *For $(t,y)\in\overline{\mathcal{D}}_T$, $\widetilde{J}(t,y)$ is the unique viscosity solution of the following obstacle problem:*

$$\begin{cases}\max\Big\{\partial_t\widetilde{J}-\widetilde{G}\big(-y^2\partial_{yy}\widetilde{J}\big)+(\delta-r_I)y\partial_y\widetilde{J}-(\delta+\lambda)\widetilde{J}+\widetilde{u}(y)+\lambda\widetilde{u}(\frac{y}{k_B})+wy,\;\underline{\widetilde{J}}-\widetilde{J}\Big\}=0, & (t,y)\in\mathcal{D}_T,\\ \widetilde{J}(T,y)=\underline{\widetilde{J}}(T,y), & y\in(0,\infty),\end{cases} \tag{33}$$

with $\mathcal{D}_T\equiv[0,T)\times(0,+\infty)$ and $\widetilde{G}:\mathbb{R}\to\mathbb{R}$ in (23).

(b) *$\widetilde{J}$ is the unique strong solution to Problem 3 such that $\widetilde{J}\in\mathcal{W}^{2,1}_{p,loc}(\mathcal{D}_T)\cap C(\overline{\mathcal{D}}_T)$ for any $p\geq1$, $\partial_t\widetilde{J},\partial_y\widetilde{J}\in C(\overline{\mathcal{D}}_T)$, and $\partial_{yy}\widetilde{J}>0$ a.e. in $\overline{\mathcal{D}}_T$.*[5]

(c) *The optimal G-stopping time $\widetilde{\tau}^*\in\widetilde{\mathcal{T}}_{t,T}$ to Problem 3 is given by*

$$\widetilde{\tau}^*\equiv\widetilde{\tau}^*(t,y)=\inf\Big\{s\geq t\mid Y_s\leq\hat{z}_{\underline{\Upsilon}}(s)\text{ with }Y_t=y\Big\}\wedge T, \tag{34}$$

where a free boundary $(\hat{z}_{\underline{\Upsilon}}(t))_{t=0}^T$, which is in $C^\infty([0,T))\cap C([0,T])$, is deterministic and time-dependent and satisfies the following non-linear integral equation:

$$0=\int_t^T\mathcal{I}_{\underline{\Upsilon}}(t,\hat{z}_{\underline{\Upsilon}}(t);s,\hat{z}_{\underline{\Upsilon}}(s))ds, \tag{35}$$

where $\mathcal{I}_{\underline{\Upsilon}}(t,y;s,z)$ is an expectation value given by

$$\begin{aligned}\mathcal{I}_{\underline{\Upsilon}}(t,y;s,z)\equiv&\mathbb{E}_t^{\widetilde{\mathbb{P}}^*}\left[e^{-(\delta+\lambda)(s-t)}\Big(\widetilde{u}(Y_s)-\widetilde{u}(\frac{Y_s}{k_R})+wY_s\Big)\mathbf{1}_{\{Y_s>z\}}\;\Big|\;Y_t=y\right]\\ =&e^{-(\delta+\lambda)(s-t)}\int_z^\infty\Big(\widetilde{u}(\eta)-\widetilde{u}(\frac{\eta}{k_R})+w\eta\Big)\Phi_{\underline{\Upsilon}}(t,y;s,\eta)d\eta,\end{aligned}$$

with $\Phi_{\underline{\Upsilon}}(\cdot)$ in (32).

(d) *The solution $\widetilde{J}$ is given by*

$$\widetilde{J}(t,y)=\int_t^T\mathcal{I}_{\underline{\Upsilon}}(t,y;s,\hat{z}_{\underline{\Upsilon}}(s))ds+\underline{\widetilde{J}}(t,y)\quad\text{for}\quad(t,y)\in\overline{\mathcal{D}}_T. \tag{36}$$

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[5] We provide the formal definition of the Sobolev space $\mathcal{W}^{2,1}_{p,loc}(\mathcal{D}_T)$ in Appendix A.2.

By extending $\widetilde{J}(t,y)=\underline{J}(t,y)$ for $(t,y)\in\overline{\mathcal{D}}_{T^1}\setminus\overline{\mathcal{D}}_T$, we have a complete solution $\widetilde{J}\in\mathcal{W}^{2,1}_{p,loc}(\overline{\mathcal{D}}_{T^1})\cap C(\overline{\mathcal{D}}_{T^1})$. Furthermore, it is C^∞-smooth in the region $\{(t,y)\in\mathcal{D}_T\,|\,y>\hat{z}(t)\}$.

The properties on the reflected G-BSDEs and some of the arguments in the obstacle problem are consistent with the paper by Park and Wong (2023). However, while the viscosity solution of the obstacle problem in Park and Wong (2023) turns out to be the solution to the variational inequality in Yang and Koo (2018) under worst-case probability measures, the viscosity solution to (33) is a strong solution (i.e., $W^{1,2}_{p,loc}(\mathcal{D}_{T^1})\cap C(\overline{\mathcal{D}}_{T^1})$) to the variational inequality in Jeon et al. (2022) under worst-case measures.

3.4. Verification theorem and robust strategy

To eventually obtain the solution to the dual max-min problem (Problem 2), in the following theorem, we demonstrate the equivalence between the optimal G-stopping time problem and the original dual problem (refer to Appendix A.3 for the proof).

Theorem 3.1. *The solution $J(0,y)$ to Problem 2 is the same as the solution $\widetilde{J}(0,y)$ to Problem 3, where the probability measure $\mathbb{P}^*\in\mathcal{P}$ such that $(P,S)^\top$ is the unique strong solution of the bivariate Itô process with the characteristics $\Sigma^*=\Xi((\sigma_P^*,\sigma_S^*,\rho^*)^\top)$ and $\mu^*=(\mu_P^*,\mu_S^*)^\top$ in (25), i.e., $(dP_t,dS_t)^\top=\mathrm{diag}((P_t,S_t)^\top)(\mu^*dt+\Sigma^*dW_t^{\mathbb{P}^*})$, and the stopping time $\hat{\tau}_R\in\mathcal{T}_{0,T}$, given by*

$$\hat{\tau}_R\equiv\hat{\tau}_R(0,y)=\inf\Big\{t\geq 0\mid Y_t\leq\hat{z}_{\underline{\Upsilon}}(t),\ Y_0=y\Big\}\wedge T, \tag{37}$$

realizes the worst-case optimal stopping scenario in Problem 2, i.e.,

$$\widetilde{J}(0,y)=J(0,y)=\mathcal{J}(0,y;\mathbb{P}^*,\hat{\tau}_R).$$

According to Epstein and Ji (2014), the set $\mathcal{P}$, which has dynamic consistency, implies the conditional representation of the dual problem, i.e., for each $(t,y)\in\overline{\mathcal{D}}_{T^1}$

$$\widetilde{J}(t,y)=J(t,y)=\mathcal{J}(t,y;\mathbb{P}^*,\hat{\tau}_R(t,y)),$$

where $J(t,y)$ is given by $J(t,y)=\sup_{\tau_R\in\mathcal{T}_{t,T}}\inf_{\mathbb{P}\in\mathcal{P}}\mathcal{J}(t,y;\mathbb{P},\tau_R)$ with

$$\begin{aligned}\mathcal{J}(t,y;\mathbb{P},\tau_R)=\mathbb{E}^{\mathbb{P}}\Bigg[&\int_t^{\tau_R}e^{-(\delta+\lambda)(s-t)}\Big(\widetilde{u}(Y_s)+\lambda\widetilde{u}(\frac{Y_s}{k_B})+wY_s\Big)ds\\&+\int_{\tau_R}^{T^1}e^{-(\delta+\lambda)(s-t)}\Big(\widetilde{u}(\frac{Y_s}{k_R})+\lambda\widetilde{u}(\frac{Y_s}{k_B})\Big)ds+e^{-(\delta+\lambda)(T^1-t)}\widetilde{u}(\frac{Y_{T^1}}{k_X})\,\Big|\,\mathcal{F}_t\Bigg],\end{aligned}$$

and $\mathcal{T}_{t,T}=\{\tau_R\in\mathcal{T}|\tau_R$ having values in $[t,T]\}$, and the conditional worst-case stopping time in $\mathcal{T}_{t,T}$ is given by $\hat{\tau}_R(t,y)=\inf\{s\geq t|Y_s\leq\hat{z}_{\underline{\Upsilon}}(s),\ Y_t=y\}\wedge T$.

From Proposition 3.1 and Theorem 3.1, we define the working region $\mathbf{WR}_y$ and retirement region $\mathbf{WR}_y$ in the dual domain $\overline{\mathcal{D}}_{T^1}$ as follows:

$$\mathbf{WR}_y\equiv\Big\{(t,y)\in\overline{\mathcal{D}}_T\mid y>\hat{z}_{\underline{\Upsilon}}(t)\Big\},\quad \mathbf{RR}_y\equiv\Big\{(t,y)\in\overline{\mathcal{D}}_T\mid 0<y\leq\hat{z}_{\underline{\Upsilon}}(t)\Big\}\cup\Big(\overline{\mathcal{D}}_{T^1}\setminus\overline{\mathcal{D}}_T\Big). \tag{38}$$

To establish the conjugate between Problem 1 and 2, two verifications are required. The first is the robust admissibility of the candidate strategies in (17) and the second is the worst-case realization of them. Under the equivalent probability measures of Park and Wong (2023), the admissibility can be shown straightforwardly, while in our non-dominated priors setup, the integrability of the candidate strategies is nontrivial. We further analyze this using the G-expectation toolkit and characterize the corresponding robust strategies in analytic forms. The following theorem summarizes our main results (refer to A.4 for the proof).

Theorem 3.2 *(Duality theorem and robust strategies). Given $x>-m(0;w)$, the value function V in Problem 1 and the dual value function J in Problem 2 satisfy the following duality relationship:*

$$V(0,x)=\min_{y>0}\Big(J(0,y)+yx\Big)=J(0,y^*)+y^*x,$$

where $J(0,y^*)=\widetilde{J}(0,y^*)$ is given by (36) with the obstacle function (31), $y^*\equiv I_J(-x)>0$ is the unique minimizer, and $I_J(\cdot)$ is the inverse function of $\partial_y J(0,\cdot)$. This implies that the worst-case scenario is under $\mathbb{P}^*\in\mathcal{P}$ with $(\Sigma^*,\mu^*)\in\Theta$ as given in (25).

Furthermore, the value function V is represented by

$$V(0,x)=\mathbb{E}^{\mathbb{P}^*}\left[\int_0^{T^1}e^{-(\delta+\lambda)t}\Big(u(c_t^*)\mathbf{1}_{\{t<\tau_R^*\}}+u(k_Rc_t^*)\mathbf{1}_{\{t\geq\tau_R^*\}}+\lambda u(k_B\mathfrak{B}_t^*)\Big)dt+e^{-(\delta+\lambda)T^1}u\big(k_XX_{T^1}^{x;*}\big)\right], \tag{39}$$

where τ_R^* is the robust retirement time, given by

$$\tau_R^* = \hat{\tau}_R(0, y^*) = \inf\left\{t \geq 0 \mid Y_t^* \leq \hat{z}_{\underline{\Upsilon}}(t),\ Y_0^* = y^*\right\} \wedge T, \tag{40}$$

with the deterministic boundary $\hat{z}_{\underline{\Upsilon}}$ solving the integral equation (35).

$(c_t^*)_{t=0}^{T^1}$, $(p_t^*)_{t=0}^{T^1}$, and $(X_t^{x;*})_{t=0}^{T^1}$ are the robust consumption, life insurance, and wealth of the agent, respectively, given by

$$\begin{aligned} c_t^* &\equiv \hat{c}_t(y^*) = I(Y_t^*)\mathbf{1}_{\{t<\tau_R^*\}} + \frac{1}{k_R} I(\frac{Y_t^*}{k_R})\mathbf{1}_{\{t\geq\tau_R^*\}}, \\ p_t^* &\equiv \lambda\hat{\mathfrak{B}}_t(y^*) - \lambda X_t^* = \frac{\lambda}{k_B} I(\frac{Y_t^*}{k_B}) + \lambda\partial_y J(t, Y_t^*), \\ X_t^{x;*} &\equiv X_t^{x;c^*,p^*,\pi^*,\tau_R^*}(y^*) \equiv -\partial_y J(t, Y_t^*), \end{aligned} \tag{41}$$

and the robust investment strategies $\pi_t^* = (\pi_t^{M,*}, \pi_t^{I,*}, \pi_t^{S,*})^\top$ are given by

$$\begin{aligned} \pi_t^{M,*} &= \frac{1}{1-(\rho^*)^2}\left[\frac{\rho^*\sigma_P^* - \sigma_S^*}{(\sigma_P^*)^2\sigma_S^*}\zeta_1^{\Sigma^*,\mu^*} + \frac{\rho^*\sigma_S^* - \sigma_P^*}{(\sigma_S^*)^2\sigma_P^*}\zeta_2^{\Sigma^*,\mu^*}\right] Y_t^*\partial_{yy} J(t, Y_t^*), \\ \pi_t^{I,*} &= X_t^{x;*} + \frac{1}{1-(\rho^*)^2}\left[\frac{1}{(\sigma_P^*)^2}\zeta_1^{\Sigma^*,\mu^*} - \frac{\rho^*}{\sigma_P^*\sigma_S^*}\zeta_2^{\Sigma^*,\mu^*}\right] Y_t^*\partial_{yy} J(t, Y_t^*), \\ \pi_t^{S,*} &= \frac{1}{1-(\rho^*)^2}\left[-\frac{\rho^*}{\sigma_P^*\sigma_S^*}\zeta_1^{\Sigma^*,\mu^*} + \frac{1}{(\sigma_S^*)^2}\zeta_2^{\Sigma^*,\mu^*}\right] Y_t^*\partial_{yy} J(t, Y_t^*), \end{aligned} \tag{42}$$

with $Y_t^* = y^* e^{(\delta - r_I - \frac{1}{2}\underline{\Upsilon}^2)t - ((\Sigma^*)^{-1}\zeta^{\Sigma^*,\mu^*})^\top W_t^{\mathbb{P}^*}}$ and $\zeta^{\Sigma^*,\mu^*} = (\zeta_1^{\Sigma^*,\mu^*}, \zeta_2^{\Sigma^*,\mu^*})^\top$ in (26).

From the duality theorem (Theorem 3.2), we define the robust retirement wealth boundary by

$$\overline{X}(t) \equiv -\partial_y J(t, \hat{z}_{\underline{\Upsilon}}(t)), \quad \text{for} \quad t \in [0, T], \tag{43}$$

which implies the following characterization of the robust retirement time τ_R^* with respect to the robust retirement wealth boundary:

$$\tau_R^* = \inf\left\{t \geq 0 \mid X_t^{x;*} \geq \overline{X}(t)\right\} \wedge T.$$

To fully characterize the robust strategies, it is first necessary to solve the deterministic retirement boundary from the integral equation (35), substitute it into (36) and (31) and calculate the dual value function J and its derivatives $\partial_y J, \partial_{yy} J$. Remarks 2, 3, and 4 partially provide some interpretations from the solution forms. A detailed numerical investigation is conducted in Section 4 for an agent with the CRRA utility.

Remark 2. The robust consumption strategy c^* admits a discontinuous point right before and after the retirement time τ_R^*, consistent with the result in Jeon et al. (2022), although here, the parameters are chosen for the worst case. The direction of the jump can be related to the agent's preference for not working (k_R) and risk-aversion level but is independent of model ambiguity. To be specific, we consider the CRRA utility, i.e., $u(x) = \frac{1}{1-\gamma}x^{1-\gamma}$ with the positive risk aversion parameter $\gamma \neq 1$. Then $\frac{1}{k_R} I(\frac{Y_t^*}{k_R}) = I(Y_t^*)k_R^{1/\gamma - 1}$. Therefore, $\gamma < 1$ implies $c_{\tau_R^*-}^* < c_{\tau_R^*+}^*$ while $\gamma > 1$ implies $c_{\tau_R^*-}^* > c_{\tau_R^*+}^*$. The latter case (i.e., $\gamma > 1$) explains the retirement-consumption puzzle (see Dybvig and Liu (2010) and the references therein).

Following Kwak and Lim (2014), we refer to $\zeta_1^{\Sigma^*,\mu^*}$ and $\zeta_2^{\Sigma^*,\mu^*}$ as the inflation-adjusted excess return on inflation bond and stock, respectively, under the worst-case scenario. They are in the same form as those in Kwak and Lim (2014) but expressed using the worst-case parameters derived from the optimization problem (24). They are critical for the construction of the robust investment strategies.

Remark 3. The investment strategy in the inflation bond $\pi^{I,*}$ consists of three parts: the demand for hedging inflation risk, the demand for investment and the demand for diversification. This is consistent with Kwak and Lim (2014), in which model ambiguity is not addressed. The whole financial capital $X^{x;*}$ is invested to hedge inflation risk, supporting the argument in Viceira (2007) that the long-term inflation bond is a riskless long-term investment vehicle for conservative agents. However, the dependency of model parameters on the remaining two parts is much more involved than in Kwak and Lim (2014). The ambiguity-averse agent needs to solve the optimization problem (24) to figure out the worst-case scenario. Because $\partial_{yy} J(0, y^*) > 0$ from Proposition 3.1 and Theorem 3.2, the signs of the worst-case inflation-adjusted excess returns together with the worst-case correlation ρ^* determine whether the inflation bond, from the perspective of the agent, is an attractive instrument for investment and diversification.

Remark 4. Similar to the analysis in Remark 3, the robust investment strategy in stock $\pi^{S,*}$ has two components that represent, respectively, the demand for diversification and the demand for investment. Again, the agent has to solve the worst-case parameters to determine the strength of these demands. The robust money market strategy satisfies $\pi_t^{M,*} = X_t^{x;*} - \pi_t^{I,*} - \pi_t^{S,*}$. Therefore, it finances the amount of money required by the other two investment strategies or deposits the excess money amount to realize the whole capital $X^{x;*}$.

3.5. The CRRA characterization

In this section, we apply our results to derive robust strategies in an integral equation representation for the CRRA utility. It is clear that Assumption 1 holds for the CRRA. From Proposition 3.1 and Theorem 3.1 and 3.2, we obtain the following CRRA characterization results.

Theorem 3.3 *(Dual value function and robust strategies with the CRRA utility).*

(a) *$J(t,y)$ has the following integral equation representation: for $(t,y) \in \overline{\mathcal{D}}_{T^1}$*

$$\begin{aligned} J(t,y) &= \frac{\gamma}{1-\gamma}\left(k_R^{-1+\frac{1}{\gamma}} + \lambda k_B^{-1+\frac{1}{\gamma}}\right)\left(\frac{1-e^{-(K_{\underline{\Upsilon}}+\lambda)(T^1-t)}}{K_{\underline{\Upsilon}}+\lambda}\right) y^{1-\frac{1}{\gamma}} + \frac{\gamma}{1-\gamma} k_X^{-1+\frac{1}{\gamma}} e^{-(K_{\underline{\Upsilon}}+\lambda)(T^1-t)} y^{1-\frac{1}{\gamma}} \\ &+ \mathbf{1}_{\{(t,y)\in \mathbf{WR}_y\}} \left\{ \frac{\gamma}{1-\gamma}\left(1-k_R^{-1+\frac{1}{\gamma}}\right) y^{1-\frac{1}{\gamma}} \int_t^T e^{-(K_{\underline{\Upsilon}}+\lambda)(\xi-t)} \mathcal{N}\left(d_{\underline{\Upsilon}}^{-\gamma}\left(\xi-t, \frac{y}{\hat{z}_{\underline{\Upsilon}}(\xi)}\right)\right) d\xi \right. \\ &\left. + wy \int_t^T e^{-(r_I+\lambda)(\xi-t)} \mathcal{N}\left(d_{\underline{\Upsilon}}^{+}\left(\xi-t, \frac{y}{\hat{z}_{\underline{\Upsilon}}(\xi)}\right)\right) d\xi \right\}, \end{aligned} \tag{44}$$

with $\mathbf{WR}_y$ in (38), where $K_{\underline{\Upsilon}} \equiv r + \frac{\delta - r}{\gamma} + \frac{\gamma-1}{2\gamma^2}\underline{\Upsilon}^2$ and $\mathcal{N}(\cdot)$ is the standard normal cumulative distribution function,

$$d_{\underline{\Upsilon}}^{\pm}(t,y) \equiv \frac{\log(y) + (\delta - r_I \pm \frac{1}{2}\underline{\Upsilon}^2)t}{|\underline{\Upsilon}|\sqrt{t}}, \quad d_{\underline{\Upsilon}}^{\pm\gamma}(t,y) \equiv \frac{\log(y) + (\delta - r_I \pm \frac{1}{2}\underline{\Upsilon}^2 \pm \frac{1-\gamma}{\gamma}\underline{\Upsilon}^2)t}{|\underline{\Upsilon}|\sqrt{t}},$$

and the free boundary $\hat{z}_{\underline{\Upsilon}}(\cdot)$ solves the following integral equation: for $t \in [0,T]$,

$$\begin{aligned} 0 &= \frac{\gamma}{1-\gamma}\left(1-k_R^{-1+\frac{1}{\gamma}}\right)\left(\hat{z}_{\underline{\Upsilon}}(t)\right)^{-\frac{1}{\gamma}} \int_t^T e^{-(K_{\underline{\Upsilon}}+\lambda)(\xi-t)} \mathcal{N}\left(d_{\underline{\Upsilon}}^{-\gamma}\left(\xi-t, \frac{\hat{z}_{\underline{\Upsilon}}(t)}{\hat{z}_{\underline{\Upsilon}}(\xi)}\right)\right) d\xi \\ &+ w \int_t^T e^{-(r_I+\lambda)(\xi-t)} \mathcal{N}\left(d_{\underline{\Upsilon}}^{+}\left(\xi-t, \frac{\hat{z}_{\underline{\Upsilon}}(t)}{\hat{z}_{\underline{\Upsilon}}(\xi)}\right)\right) d\xi. \end{aligned} \tag{45}$$

Using the duality theorem (in Theorem 3.2), for a given initial $X_0 = x > -m(0;w)$, we can have the unique minimizer y^* such that $x = -\partial_y J(0,y^*)$, where $\partial_y J(t,y)$ with $(t,y) \in \overline{\mathcal{D}}_{T^1}$ is explicitly given by

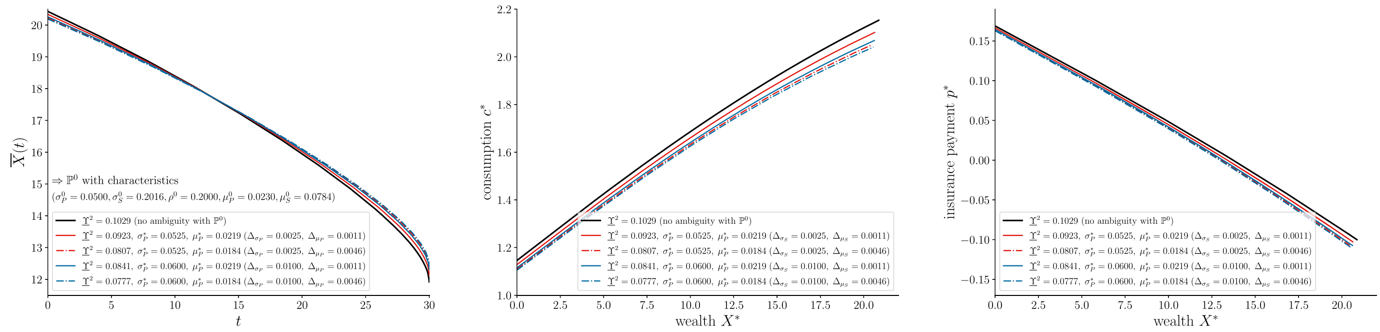
$$\begin{aligned} \partial_y J(t,y) &= -\left(k_R^{-1+\frac{1}{\gamma}} + \lambda k_B^{-1+\frac{1}{\gamma}}\right)\left(\frac{1-e^{-(K_{\underline{\Upsilon}}+\lambda)(T^1-t)}}{K_{\underline{\Upsilon}}+\lambda}\right) y^{-\frac{1}{\gamma}} - k_X^{-1+\frac{1}{\gamma}} e^{-(K_{\underline{\Upsilon}}+\lambda)(T^1-t)} y^{-\frac{1}{\gamma}} \\ &+ \mathbf{1}_{\{(t,y)\in \mathbf{WR}_y\}} \left\{ -\left(1-k_R^{-1+\frac{1}{\gamma}}\right) y^{-\frac{1}{\gamma}} \int_t^T e^{-(K_{\underline{\Upsilon}}+\lambda)(\xi-t)} \mathcal{N}\left(d_{\underline{\Upsilon}}^{-\gamma}\left(\xi-t, \frac{y}{\hat{z}_{\underline{\Upsilon}}(\xi)}\right)\right) d\xi \right. \\ &+ \frac{\gamma}{1-\gamma}\left(1-k_R^{-1+\frac{1}{\gamma}}\right) y^{-\frac{1}{\gamma}} \int_t^T e^{-(K_{\underline{\Upsilon}}+\lambda)(\xi-t)} \mathbf{n}\left(d_{\underline{\Upsilon}}^{-\gamma}\left(\xi-t, \frac{y}{\hat{z}_{\underline{\Upsilon}}(\xi)}\right)\right) \frac{1}{|\underline{\Upsilon}|\sqrt{\xi-t}} d\xi \\ &\left. + w \int_t^T e^{-(r_I+\lambda)(\xi-t)} \mathcal{N}\left(d_{\underline{\Upsilon}}^{+}\left(\xi-t, \frac{y}{\hat{z}_{\underline{\Upsilon}}(\xi)}\right)\right) d\xi + w \int_t^T e^{-(r_I+\lambda)(\xi-t)} \mathbf{n}\left(d_{\underline{\Upsilon}}^{+}\left(\xi-t, \frac{y}{\hat{z}_{\underline{\Upsilon}}(\xi)}\right)\right) \frac{1}{|\underline{\Upsilon}|\sqrt{\xi-t}} d\xi \right\}. \end{aligned} \tag{46}$$

(b) *The robust retirement time τ_R^* in (40) is characterized by the boundary $\hat{z}_{\underline{\Upsilon}}(\cdot)$ solving (45). The robust consumption $(c_t^*)_{t=0}^{T^1}$ and insurance purchase $(p^*)_{t=0}^{T^1}$ are given by*

$$c_t^* = (Y_t^*)^{-\frac{1}{\gamma}} \mathbf{1}_{\{t<\tau_R^*\}} + (k_R)^{\frac{1}{\gamma}-1} (Y_t^*)^{-\frac{1}{\gamma}} \mathbf{1}_{\{t \ge \tau_R^*\}}, \quad p_t^* = \lambda k_B^{\frac{1}{\gamma}-1} (Y_t^*)^{-\frac{1}{\gamma}} - \lambda X_t^{x;*},$$

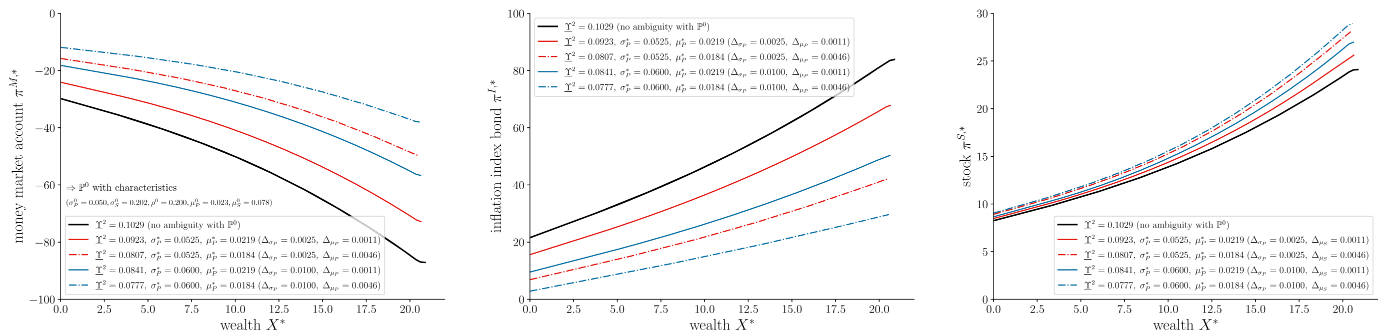
where the robust wealth $X_t^{x;*}$ in (41) is characterized by the representation in (46), and the robust investment strategies $\pi_t^* = (\pi_t^{M,*}, \pi_t^{I,*}, \pi_t^{S,*})^\top$ are characterized by the following integral equation representation of $y\partial_{yy}J$:

$$\begin{aligned} y\partial_{yy} J(t,y) &= \left(k_R^{-1+\frac{1}{\gamma}} + \lambda k_B^{-1+\frac{1}{\gamma}}\right)\left(\frac{1-e^{-(K_{\underline{\Upsilon}}+\lambda)(T^1-t)}}{K_{\underline{\Upsilon}}+\lambda}\right) \frac{1}{\gamma} y^{-\frac{1}{\gamma}} + k_X^{-1+\frac{1}{\gamma}} e^{-(K_{\underline{\Upsilon}}+\lambda)(T^1-t)} \frac{1}{\gamma} y^{-\frac{1}{\gamma}} \\ &+ \mathbf{1}_{\{(t,y)\in \mathbf{WR}_y\}} \left\{ (1-k_R^{-1+\frac{1}{\gamma}}) \frac{1}{\gamma} y^{-\frac{1}{\gamma}} \int_t^T e^{-(K_{\underline{\Upsilon}}+\lambda)(\xi-t)} \mathcal{N}\left(d_{\underline{\Upsilon}}^{-\gamma}\left(\xi-t, \frac{y}{\hat{z}_{\underline{\Upsilon}}(\xi)}\right)\right) d\xi \right. \\ &- \frac{\gamma}{1-\gamma}\left(1-k_R^{-1+\frac{1}{\gamma}}\right) y^{-\frac{1}{\gamma}} \int_t^T e^{-(K_{\underline{\Upsilon}}+\lambda)(\xi-t)} \mathbf{n}\left(d_{\underline{\Upsilon}}^{-\gamma}\left(\xi-t, \frac{y}{\hat{z}_{\underline{\Upsilon}}(\xi)}\right)\right) d_{\underline{\Upsilon}}^{+\gamma}\left(\xi-t, \frac{y}{\hat{z}_{\underline{\Upsilon}}(\xi)}\right) \frac{d\xi}{\underline{\Upsilon}^2(\xi-t)} \\ &\left. - w \int_t^T e^{-(r_I+\lambda)(\xi-t)} \mathbf{n}\left(d_{\underline{\Upsilon}}^{+}\left(\xi-t, \frac{y}{\hat{z}_{\underline{\Upsilon}}(\xi)}\right)\right) d_{\underline{\Upsilon}}^{-}\left(\xi-t, \frac{y}{\hat{z}_{\underline{\Upsilon}}(\xi)}\right) \frac{d\xi}{\underline{\Upsilon}^2(\xi-t)} \right\}. \end{aligned} \tag{47}$$



(a) Wealth boundary for retirement (b) Consumption rate. (c) Life insurance purchase .

Fig. 1. Robust retirement boundary, consumption, and insurance purchase under inflation ambiguity. (For interpretation of the colors in the figures, the reader is referred to the web version of this article.)



(a) Money market acount. (b) Inflation index bond. (c) Stock.

Fig. 2. Robust investments under inflation ambiguity.

By the C^1-condition of J at $\hat{z}_{\underline{\Upsilon}}(\cdot)$, i.e., $\partial_y J(t, \hat{z}_{\underline{\Upsilon}}(t)) = \partial_y \widetilde{J}(t, \hat{z}_{\underline{\Upsilon}}(t)) = \partial_y \widetilde{\underline{J}}(t, \hat{z}_{\underline{\Upsilon}}(t))$ for $t \in [0, T]$, the robust retirement wealth boundary is explicitly derived as follows: for $t \in [0, T]$,

$$\overline{X}(t) = \left(k_R^{-1+\frac{1}{\gamma}} + \lambda k_B^{-1+\frac{1}{\gamma}}\right)\left(\frac{1 - e^{-(K_{\underline{\Upsilon}}+\lambda)(T^1 - t)}}{K_{\underline{\Upsilon}} + \lambda}\right)\left(\hat{z}_{\underline{\Upsilon}}(t)\right)^{-\frac{1}{\gamma}} + k_X^{-1+\frac{1}{\gamma}} e^{-(K_{\underline{\Upsilon}}+\lambda)(T^1 - t)}\left(\hat{z}_{\underline{\Upsilon}}(t)\right)^{-\frac{1}{\gamma}}. \tag{48}$$

4. Properties of the robust strategies

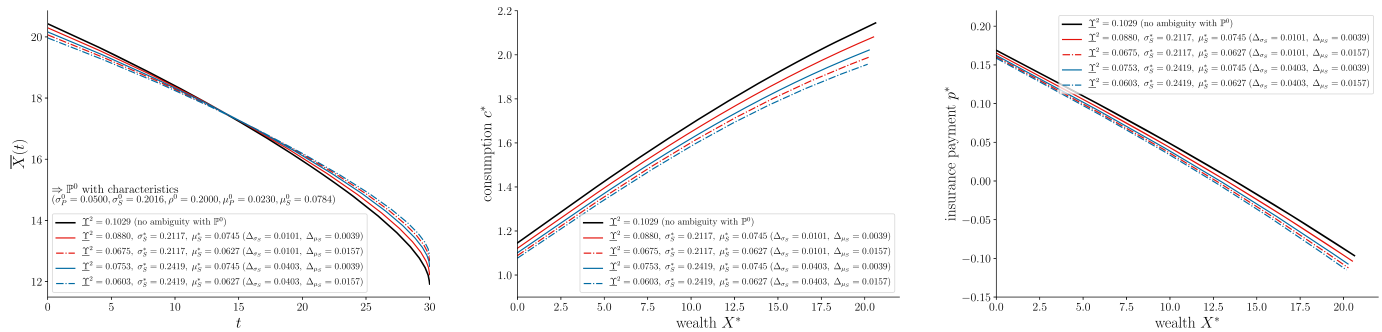
In this section, we provide the numerical solutions of the robust strategies for the CRRA utility studied in Section 3.5. This entails two computational procedures: (i) characterization of the worst-case measures $\mathbb{P}^*$ with $\Sigma^* = \Xi((\sigma_P^*, \sigma_S^*, \rho^*)^\top)$ and $\mu^* = (\mu_P^*, \mu_S^*)^\top$ in (25) and (ii) integrations in the equation of the free boundary (45) and derivatives of the dual value function (46) and (47). We adopt the Python open library SciPy to solve the worst-case parameters in the atemporal optimization problem (24) for several ambiguous regions $\mathfrak{D}_{vol} \times \mathfrak{D}_{return}$. To evaluate the integral equation representation for the free boundary and robust strategies, we use the recursive integration method developed by Huang et al. (1996).

In the following subsections, we separately study the ambiguity risk from three sources: the inflation rate (CPI), the stock price, and the correlation between them. In particular, whenever the ambiguity from one of the sources is considered, the agent is assumed to ignore the ambiguity issues in the others and use the baseline parameters to be specified later as true parameters. The real and nominal interest return rates are $r_I = 0.007$ and $r_M = 0.014$, respectively. The agent's planning horizon and mandatory retirement times are $T^1 = 50$ and $T = 30$. Following the CPI parameter values in Kwak and Lim (2014) and the financial market values in Bansal et al. (2012), the baseline parameter values for the three sources are assumed to be $\sigma_P^0 = 0.05$, $\sigma_S^0 = 0.2016$, $\rho^0 = 0.2$, $\mu_P^0 = 0.023$, and $\mu_S^0 = 0.0784$. We choose the values of the other parameters as follows: $\delta = 0.07$, $\gamma = 3$, $k_R = 3$, $k_B = 0.05$, $k_X = 0.05$, $w = 1$, and $\lambda = 0.02$.

4.1. The effect of inflation ambiguity

We suppose that the agent suspects that the inflation-linked index bond price model is misspecified and ignores the ambiguity in the other parameters. Formally, the agent's beliefs about return μ_P and volatility σ_P are assumed to lie in the intervals $[\mu_P^0 - \Delta_{\mu_P}, \mu_P^0 + \Delta_{\mu_P}]$, $[\sigma_P^0 - \Delta_{\sigma_P}, \sigma_P^0 + \Delta_{\sigma_P}]$ for $\Delta_{\mu_P}, \Delta_{\sigma_P} > 0$. A larger Δ_{μ_P} (resp. Δ_{σ_P}) indicates that the agent is less confident in the information about μ_P^0 (resp. σ_P^0) and faces higher ambiguity risk. We let $\Delta_{\mu_P} \in \{0.05\mu_P^0, 0.2\mu_P^0\} = \{0.00115, 0.0046\}$, $\Delta_{\sigma_P} \in \{0.05\sigma_P^0, 0.2\sigma_P^0\} = \{0.0025, 0.01\}$, and their combinations lead to four cases with different ambiguous levels for the return and volatility of the index bond. The case without ambiguity risk ($\Delta_{\mu_P} = \Delta_{\sigma_P} = 0$) is considered as the baseline case.

Figs. 1 and 2 show the robust strategies in the aforementioned four cases (red solid and dotted lines and blue solid and dotted lines in the two figures). Fixing one of $\Delta_{\mu_P}, \Delta_{\sigma_P}$ and varying the other, we conclude that return ambiguity and volatility ambiguity affect



(a) Wealth boundary for retirement. (b) Consumption rate. (c) Life insurance purchase.

Fig. 3. Robust retirement boundary, consumption, and insurance purchase under stock price ambiguity.

the decisions in the same direction. Figs. 1(b) and 1(c) indicate that inflation ambiguity slightly reduces consumption and life insurance purchase. In Fig. 1(a), a robust retirement wealth boundary demonstrates a downward slope, indicating a shrinkage of the working region (i.e., $\{(t,x)|0 \le t \le T,\ -m(t;w) < x < \overline{X}(t)\}$) as the mandatory retirement time $T = 30$ is approached, which is consistent with the optimal retirement wealth boundary of no ambiguity case in Jeon et al. (2022). A unique feature is the mild cross-point pattern among different ambiguity levels (possibly multiple non-overlapping cross points). We provide interpretations on this pattern as follows.

Model ambiguity makes the index bond less effective in hedging inflation risk and less attractive as an investment opportunity and instrument for diversifying portfolio risk. As a result, the demand for the index bond decreases, which is also reported in Fig. 2(b). In the early time period ($0 \le t \le 5$ in Fig. 1(a)), with a high wealth level slightly below the baseline retirement boundary, the agent facing inflation ambiguity turns his/her attention from purchasing inflation bond to consumption planning. As a robust decision, motivated by the preference for not working ($k_R > 1$), the agent prefers to retire earlier by lowering the retirement boundary. However, as the mandatory retirement time ($t \ge 20$ in Fig. 1(a)) is approached, the agent who has not accumulated enough wealth for early retirement starts to worry about his/her wealth level. An ambiguous index bond makes the agent more desperate for non-ambiguous (i.e., certain) labor income. As a result, inflation ambiguity induces the opposite effect compared to the early time period by improving the retirement wealth boundary and postponing retirement. This pattern implies that ambiguity leads to a polarization effect in the sense that wealthy people retire earlier and poor people work for longer. Indeed, Baliga et al. (2013) demonstrate that polarization can occur as an optimal response to ambiguity aversion.

Huang and Yu (2021) consider a robust stopping time in the α-maxmin nonlinear expectation and demonstrate theoretically that an ambiguous environment induces a tendency to choose earlier stopping time as a robust decision. Their result seems to contrast with our robust retirement time feature. This is because the ambiguity risk in our problem exists both before and after the stopping time, whereas their optimization involves ambiguity risk only before the stopping time. This structural difference in optimization leads to different characteristics of the two stopping time decisions. Even so, they are explained by the same motivation of avoiding ambiguity.

The robust investment strategies are significantly affected by inflation ambiguity, particularly for the first two. Fig. 2 demonstrates the result. Generally, as the degree of the inflation ambiguity increases, the agent's behavior becomes more conservative because the money market account finances less. The index bond demand decreases because of model ambiguity and, counter-intuitively, stock demand increases, with both becoming more significant as the wealth level increases. This implies that to satisfy the consumption needs, the agent desires a certain level of risk exposure and investment return and therefore adjusts the portfolio weights by improving the stock holding (note that the correlation between these two assets $\rho^0 > 0$), which is free from inflation ambiguity.

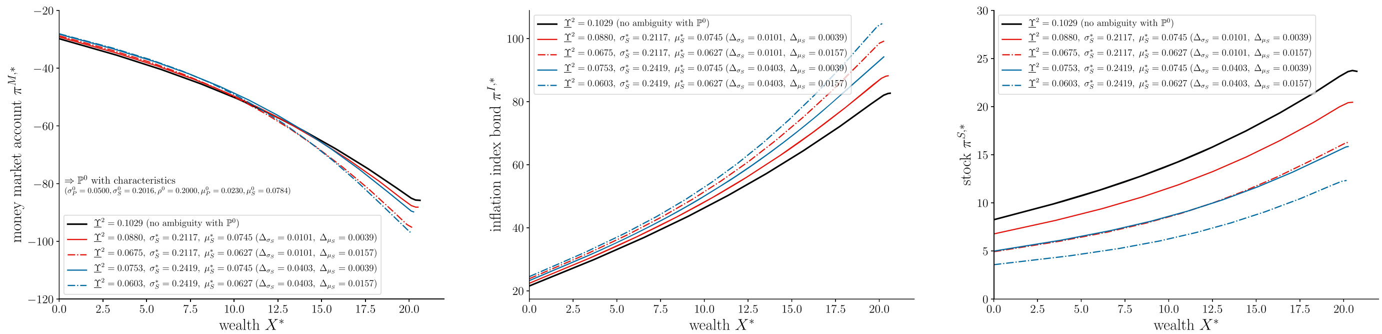
Furthermore, Figs. 1(b), 1(c), and 2 show that the effect of inflation ambiguity becomes more significant as the wealth level increases. This implies that wealthy agents take inflation ambiguity more seriously than poor agents and adjust the consumption rate, the demand for life insurance, and investment portfolio in a larger amount.

4.2. The effect of stock price ambiguity

We then investigate the sensitivity of the agent's decisions regarding stock price ambiguity. The agent's beliefs about σ_S and μ_S are assumed to lie in the intervals $[\mu_S^0 - \Delta_{\mu_S}, \mu_S^0 + \Delta_{\mu_S}]$ and $[\sigma_S^0 - \Delta_{\sigma_S}, \sigma_S^0 + \Delta_{\sigma_S}]$, where we set $\Delta_{\mu_S} \in \{0.05\mu_S^0, 0.2\mu_S^0\} = \{0.00394, 0.01573\}$ and $\Delta_{\sigma_S} \in \{0.05\sigma_S^0, 0.2\sigma_S^0\} = \{0.01008, 0.04032\}$. This setup leads to four different ambiguity situations. We still plot the benchmark case for as an illustration.

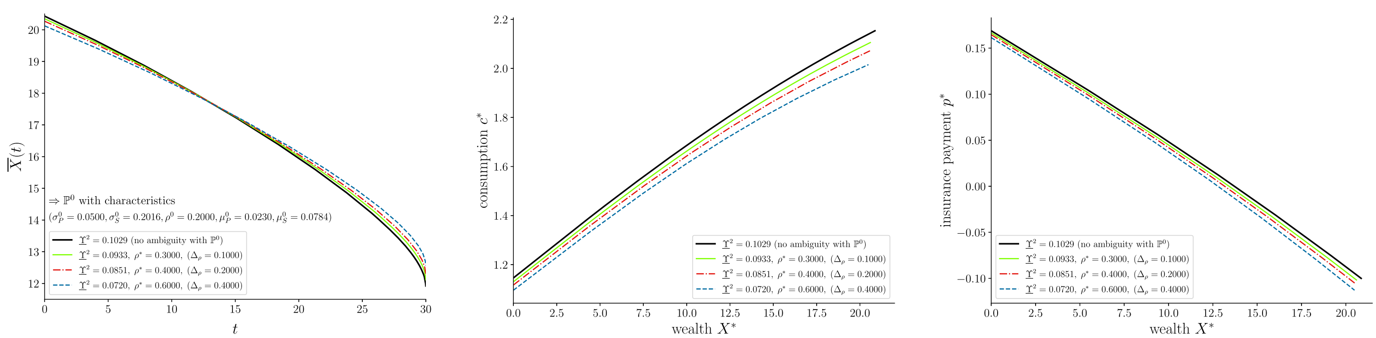
Fig. 3 presents the robust retirement boundary, consumption, and insurance purchase under stock price ambiguity. The consumption rate and life insurance purchase decrease as ambiguity risk improves. The cross-point pattern appears again in Fig. 3(a). We can interpret it in a similar manner as in Section 4.1. Ambiguity in the stock price reduces stock demand. With less interest in trading the stock, an agent in the early working period prefers to retire earlier to enjoy consumption with higher marginal utility. As t moves closer to the mandatory retirement time T, for an agent who has not yet exercised their retirement options, non-ambiguous labor income becomes a more important staple, and the agent increases the retirement boundary and extends the working time. Moreover, Figs. 3(b) and 3(c) have similar patterns as Figs. 1(b) and 1(c) in Section 4.1 and indicate more conservative behavior as stock price ambiguity increases. We note that stock price ambiguity has relatively more significant effects on the robust retirement, consumption, and insurance purchase strategies than inflation ambiguity.

Fig. 4 illustrates the robust investment strategies. Figs. 4(b) and 4(c) can be interpreted in a similar manner to the arguments in Section 4.1. The agent trades fewer ambiguous stocks and more index bonds to maintain a certain financial risk exposure and gains for the satisfaction of the consumption. Interestingly, the money market strategy exhibits a cross-point pattern in Fig. 4(a), which is a particular



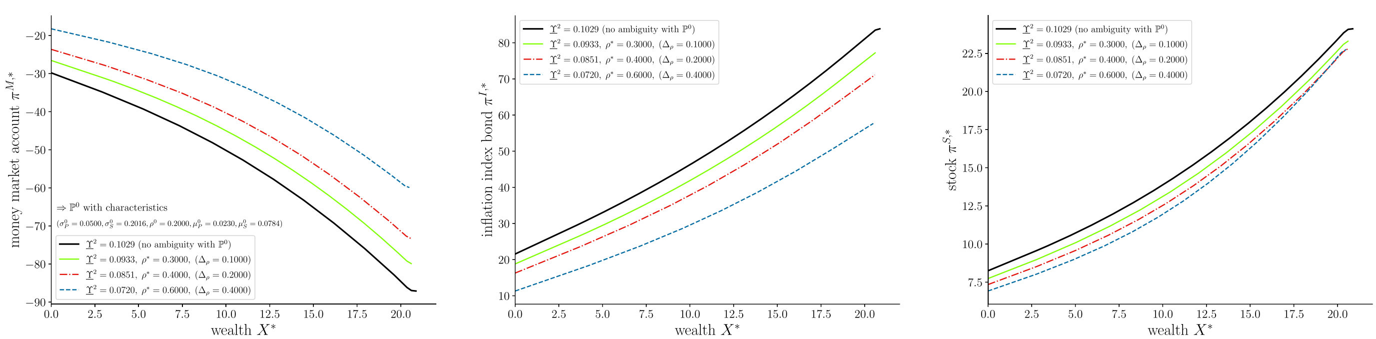
(a) Money market acount. (b) Inflation index bond. (c) Stock.

Fig. 4. Robust investments under stock price ambiguity.



(a) Wealth boundary for retirement. (b) Consumption rate. (c) Life insurance purchase .

Fig. 5. Robust retirement boundary, consumption and insurance purchase under correlation ambiguity.



(a) Money market acount. (b) Inflation index bond. (c) Stock.

Fig. 6. Robust investments under correlation ambiguity.

feature caused by stock price ambiguity. This indicates that, for an agent with a low wealth level, the decrease in the demand for ambiguous stocks is greater than the increase in the demand for the index bonds, leading to conservative investment behavior. However, for wealthy agents, the opposite is true because they can afford more financial risk and have a greater need to hedge inflation risk.

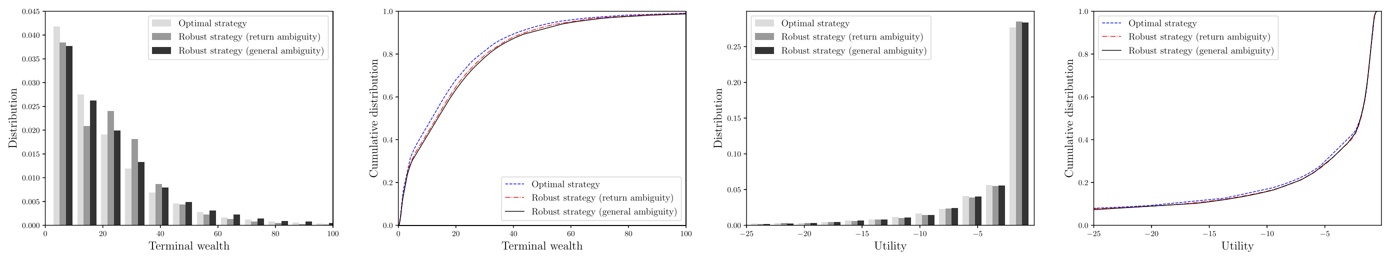
4.3. The effect of correlation ambiguity

Lastly, we conduct a sensitivity analysis to study the effect of the ambiguous correlation between the stock and index bond on the robust strategies. We let $\Delta_\rho \in \{0.5\rho^0, \rho^0, 2\rho^0\} = \{0.1, 0.2, 0.4\}$. Together with the benchmark ($\Delta_\rho = 0$) case, they represent four situations with different correlation ambiguity levels.

As shown in Figs. 5 and 6, all of the strategies except for the retirement boundary become more conservative as ambiguity in correlation increases. The ambiguous correlation has mild effects on the robust retirement boundary, consumption, and insurance purchase but a significant effect on the agent's investment strategy. We note that the cross-point pattern also appears in Fig. 5(a). The ambiguous correlation makes the financial market less attractive as the agent has difficulty diversifying the portfolio risk and therefore decreases the investment demand substantially, as shown in Fig. 6. Following similar arguments in Section 4.1 and 4.2, we interpret the cross-section point as a time point when the agent starts to be concerned about the wealth level at the terminal time, i.e., the death time or the mandatory retirement time, than about post-retirement consumption.

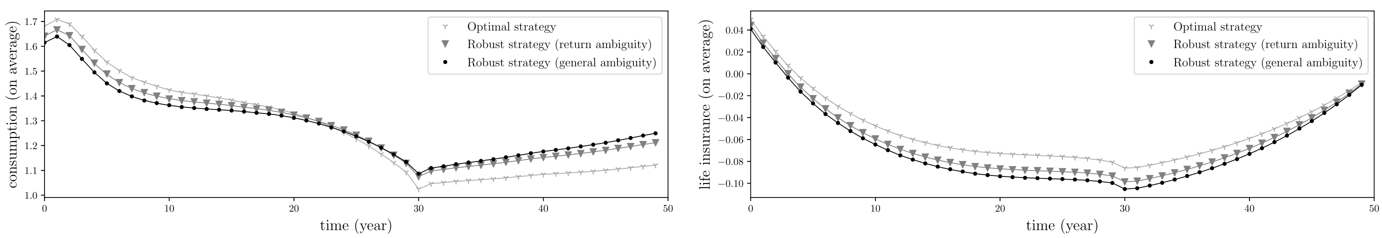
Table 3
Average values of bequest rate, lifetime aggregated utility (agg. utility), and retirement time pattern under the unknown market parameters.

	Terminal bequest (X_{T^1})	Aggregated utility ($U(0,x)$)	Retirement time (τ_R)
Optimal strategy (ignoring ambiguity)	17.4624	-7.1454	16.82
Robust strategy (return ambiguity)	18.9618	-6.9916	16.11
Robust strategy (general ambiguity)	19.6285	-6.7935	15.87



(a) Dist. of bequest rate (b) C.dist. of bequest rate (c) Dist. of agg. utility (d) C.dist. of agg. utility

Fig. 7. Distributional/Cumulative-distributional (Dist./C.dist.) analysis of bequest rate and lifetime aggregated utility (agg. utility) under the unknown market parameters.



(a) Consumption rate in time horizon (b) Life insurance purchase in time horizon

Fig. 8. Consumption rate and life-insurance purchase pattern under the unknown market parameters.

4.4. Robustness test

We examine the robustness of our proposed strategies in Section 3.5 through simulation experiments. We assume that the agent's beliefs about all the (unknown) market parameters $(\mu_P, \mu_S, \sigma_P, \sigma_S, \rho)$ lie in certain intervals (e.g., $\mu_P \in [\mu_P^0 - \Delta_{\mu_P}, \mu_P^0 + \Delta_{\mu_P}]$ with $\Delta_{\mu_P} > 0$). The midpoints of these intervals $\mu_P^0, \mu_S^0, \sigma_P^0, \sigma_S^0, \rho^0$ together with the other model parameters are given at the beginning of Section 4. For simplicity, all the intervals are taken to be $\pm 5\%$ error from the baseline value (i.e., $\Delta_{\mu_P} = 0.05\mu_P^0$, similarly defined for the other market parameters).

We simulate 40000 episodes where each episode derives a primal prior under market parameters randomly chosen (in uniformly distributed sense) from the intervals of the agent's beliefs. We contrast our robust strategy with the following two counterparts. The first one adopts the optimal strategy under the baseline market parameters without addressing any ambiguity (i.e., $\Delta_{\mu_P} = \Delta_{\mu_S} = \Delta_{\sigma_P} = \Delta_{\sigma_S} = \Delta_\rho = 0$). The second adopts the robust strategy with only consideration of ambiguity on the return parameters of the inflation-linked index bond price and the stock price (i.e., $\Delta_{\mu_P} = 0.05\mu_P^0$, $\Delta_{\mu_S} = 0.05\mu_S^0$, and $\Delta_{\sigma_P} = \Delta_{\sigma_S} = \Delta_\rho = 0$). We call these two strategies the optimal strategy ignoring ambiguity and the robust strategy under return ambiguity, respectively. Within each episode, we calculate three strategies and the corresponding wealth paths. Here, the initial wealth level is set to $x = 10$.

In Table 3, we summarize the performance metrics of the three strategy using the average values of terminal bequest rate (X_{T^1}), the lifetime aggregated utility ($U(0,x)$), and the retirement time (τ_R). For each episode, the lifetime aggregated utility is calculated by discretizing the integral in (39) and assigning the prior, the strategy and the wealth process. On average, the robust strategy with consideration of general ambiguity achieves better performance in the bequest rate and the aggregated utility than other strategies. In particular, the performance gap between the robust strategies addressing only return ambiguity and general ambiguity implies the significance of considering ambiguity in volatility and correlation parameters in the robust optimization. Nevertheless, the robust strategy with only return ambiguity still improves performance against the optimal strategy ignoring ambiguity. Furthermore, we observe that the robust strategy encourages the agents to retire more early in the unknown environment. This is consistent with the results of the robust retirement wealth boundary in Sections 4.1–4.3 for the relatively young agents (recall the mandatory time is $T = 30$ years). In addition, the consideration of general ambiguity including the volatility ambiguity strengthens the motive for the early retirement.

Fig. 7 shows the distributional analysis of the bequest rate and the aggregated utility generated by three strategies. We observe from Fig. 7(a) and 7(b) that the bequest rate can be significantly improved by considering the general ambiguity in the optimization. Moreover, addressing the return ambiguity improves the agents' aggregated utility. Although the differences between the aggregated utility distributions of two robust strategies are minor, we note that considering volatility ambiguity really improves the expected value of the aggregated utility, which is the agent's objective to optimize.

Fig. 8 illustrates the consumption and life insurance purchase curves for the three cases. Consistent with the findings in Dybvig and Liu (2010), the consumption curve in Fig. 8(a) shows a kink point nearby the mandatory retirement time $T = 30$, which is referred to as the consumption jumps at retirement. We note that the consideration of the model ambiguity relieves the downward and upward slopes around the kink point. This finding confirms the analysis in Sections 4.1–4.3. Specifically, the robust strategy proposes the young agents to decrease the consumption rate during their early lifetime and enjoy the enhanced and stable consumption after retirement. Moreover, as the negative life insurance purchase rate represents a pension annuity (Dybvig and Liu, 2010), we can deduce from Fig. 8(b) that the agents would utilize the life insurance product as an inflow annuity to manage the ambiguity risk from the financial market. In general, by comparing the robust strategies under general and return-only ambiguity, we conclude that considering volatility ambiguity in the robust optimizations can significantly affect both the consumption and life insurance strategies.

5. Conclusion

This paper has two types of contributions. First, we provide several theoretical analyses for a general ambiguity model to investigate how inflation and stock price ambiguity affect an agent's robust decision: (1) robust duality under non-dominated probability measures; (2) characterization of robust retirement time with the utility setup of Dybvig and Liu (2010), using the optimal G-stopping time approach; (3) integral equation representations for robust consumption, life insurance, and investment strategies. Second, using these theoretical analyses, we uncover some innovative findings in the robust strategies: (1) the demand reduction effects of ambiguity on consumption, life insurance purchase, and investment with the same ambiguity source; (2) the contrasting effects of ambiguity on the robust retirement depending on the remaining working period; (3) the subtle difference in the adjustment of strategies to ambiguity according to the agent's wealth level; and (4) the significant effect of model ambiguity in volatility and correlation on robust consumption, life insurance and retirement strategies.

For the model's tractability or simplicity, this paper assumes that there is no ambiguity regarding agent's mortality risk and income rate and that there is no variational-term for ambiguity-averse preference. The consideration of ambiguity in mortality and the unhedgeable income shock would be an interesting direction for future research (e.g., Han and Hung (2021)). A generalization of our duality theory by considering a variational ambiguity-averse term with Wasserstein metric would be a challenging problem (e.g., Backhoff-Veraguas et al. (2020)).

Declaration of competing interest

There is no competing interest.

Data availability

No data was used for the research described in the article.

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Appendix A. Proofs

A.1. *Proof of Lemma 3.1*

Proof of (a): Consider the following G-(forward) BSDE: given $(t,y)\in\overline{\mathcal{D}}_{T^1}$, for $u\in[t,T^1]$

$$\begin{cases} Y_u^{t,y} = y + \int_t^u (\delta - r_I) Y_s^{t,y} ds - \int_t^u d\widetilde{B}_s \quad \text{with} \quad Y_t^{t,y} = y, \\ \underline{\mathcal{G}}_u^{t,y} = -\widetilde{u}(\frac{Y_{T^1}^{t,y}}{k_X}) - \int_u^{T^1} \big(\widetilde{u}(\frac{Y_s^{t,y}}{k_R}) + \lambda\widetilde{u}(\frac{Y_s^{t,y}}{k_B}) + (\delta+\lambda)\underline{\mathcal{G}}_s^{t,y}\big)ds - \int_u^{T^1} \underline{\mathcal{M}}_s^{t,y} d\widetilde{B}_s - (\underline{\mathcal{K}}_{T^1}^{t,y} - \underline{\mathcal{K}}_u^{t,y}). \end{cases} \tag{49}$$

Under Assumption 1.(ii), it is clear that $\widetilde{u}(\cdot)\in C^\infty((0,\infty))$ satisfies the following properties: there exist $\kappa\geq 1$ and $C>0$ such that

$$|\widetilde{u}(y)| \leq C(y^\kappa + y^{-\kappa}) \quad \text{for all } \ y>0. \tag{50}$$

It follows that the driver and terminal terms of $(\underline{\mathcal{G}}_u^{t,y})_{u=t}^{T^1}$ in (49) fit into the growth conditions set out in Section 3.2 of Park and Wong (2022).[6] Thus, it follows from Lemma 1 in Park and Wong (2022) that we have the existence and uniqueness of the solution $(\underline{\mathcal{G}}_u^{t,y}, \underline{\mathcal{M}}_u^{t,y}, \underline{\mathcal{K}}_u^{t,y})_{u=t}^{T^1} \in \mathfrak{S}_G^\alpha(0,T^1)$ with some $\alpha>1$,[7] and that $\underline{g}(t,y)\equiv\underline{\mathcal{G}}_t^{t,y}\in C(\overline{\mathcal{D}}_{T^1})$ is the *unique viscosity solution* to the following PDE:

[6] Park and Wong (2022) slightly extend the Feynman-Kac formula of the G-framework, proposed in Hu et al. (2014a,b), by relaxing the growth condition of the driver and terminal terms.

[7] Refer to Definition 4 for the definition of the solution to the G-BSDE.

$$\begin{cases} \partial_t \underline{g} + \widetilde{G}\big(y^2 \partial_{yy}\underline{g}\big) + (\delta - r_I) y \partial_y \underline{g} - (\delta+\lambda)\underline{g} - \widetilde{u}(\frac{y}{k_R}) - \lambda \widetilde{u}(\frac{y}{k_B}) = 0, & (t,y) \in \mathcal{D}_{T^1}, \\ \widetilde{\underline{J}}(T^1, y) = \widetilde{u}(\frac{y}{k_X}), & y \in (0,\infty). \end{cases} \tag{51}$$

By the relation $\widetilde{\underline{J}}(t,y) = -\underline{g}(t,y) = -\underline{\mathcal{G}}_t^{t,y}$ for all $(t,y) \in \overline{\mathcal{D}}_{T^1}$, we have the dynamic programming principle (30) for $\widetilde{\underline{J}}$.

Proof of (b): For $(\Sigma^*, \mu^*) \in \Theta$ in (25), we denote by $\widetilde{\mathbb{P}}^* \in \mathcal{Q}$ a prior induced by $M^{(\Sigma^*,\mu^*)} \in \Lambda^\Theta$. Then, we denote by $\underline{\Psi}(t,y)$ a value function under the classical expectation $\widetilde{\mathbb{P}}^*$, i.e.,

$$\underline{\Psi}(t,y) = \mathbb{E}_t^{\widetilde{\mathbb{P}}^*}\left[\int_t^{T^1} e^{-(\delta+\lambda)(\xi - t)} \Big(\widetilde{u}(\frac{Y_\xi}{k_R}) + \lambda \widetilde{u}(\frac{Y_\xi}{k_B}) \Big) d\xi + e^{-(\delta+\lambda)(T^1 - t)} \widetilde{u}(\frac{Y_{T^1}}{k_X}) \,\Big|\, Y_t = y \right].$$

It follows from Karatzas and Shreve (1998) and $dY_t = (\delta - r_I) Y_t dt - Y_t M^{(\Sigma^*,\mu^*)} dW_t^{\widetilde{\mathbb{P}}^*}$ for $t \in [0, T^1]$ $\widetilde{\mathbb{P}}^*$-a.s. that the linear expectation value $\underline{\Psi}$ is a strong solution to the following reduced PDE:

$$\begin{cases} \partial_t \underline{\Psi} + \mathcal{L}_{\underline{\Upsilon}} \underline{\Psi} + \widetilde{u}(\frac{y}{k_R}) + \lambda \widetilde{u}(\frac{y}{k_B}) = 0, & (t,y) \in \mathcal{D}_{T^1}, \\ \underline{\Psi}(T^1, y) = \widetilde{u}(\frac{y}{k_X}), & y \in (0,\infty), \end{cases} \tag{52}$$

where the operator $\mathcal{L}_{\underline{\Upsilon}}$ is given by $\mathcal{L}_{\underline{\Upsilon}} \equiv \frac{1}{2}\underline{\Upsilon}^2 y^2 \partial_{yy} + (\delta - r_I) y \partial_y - (\delta + \lambda)$.

Furthermore, from Theorem 2 in Yang and Koo (2018) and $\widetilde{u}(\cdot) \in C^\infty((0,\infty))$, we have $\underline{\Psi} \in C^\infty(\overline{\mathcal{D}}_{T^1})$. The strict convexity of $\widetilde{u}(\cdot)$ leads to $\partial_{yy}\underline{\Psi} > 0$. It follows from the fact that $\widetilde{G}(\alpha) = \frac{1}{2}\big(\bar{\Upsilon}^2 \alpha^+ - \underline{\Upsilon}^2 \alpha^-\big)$ $\forall \alpha \in \mathbb{R}$, that $-\widetilde{G}\big(-y^2 \partial_{yy}\underline{\Psi}\big) = \frac{1}{2}\underline{\Upsilon}^2 y^2 \partial_{yy}\underline{\Psi}$. Thus, we can conclude that the strong solution $\underline{\Psi} \in C^\infty(\overline{\mathcal{D}}_{T^1})$ is the solution to (30), i.e., $\widetilde{\underline{J}}(t,y) = \underline{\Psi}(t,y)$ on $\overline{\mathcal{D}}_{T^1}$.

A.2. Proof of Proposition 3.1

Proof of (a): As $\widetilde{\underline{J}}$ is an optimal stopping problem under adverse nonlinear expectation (i.e., $\inf_{\widetilde{\tau}} \sup_{\widetilde{\mathbb{P}}}$), a reflected G-BSDE with an *upper obstacle* is considered. Consider the following reflected G-(forward) BSDE: for $u \in [t, T]$,

$$\begin{cases} Y_u^{t,y} = y + \int_t^u (\delta - r_I) Y_s^{t,y} ds - \int_t^u Y_s^{t,y} d\widetilde{B}_s \quad \text{with} \quad Y_t^{t,y} = y > 0, \\ \mathcal{G}_u^{t,y} = \underline{\mathcal{G}}_T^{t,y} - \int_u^T \big(\widetilde{u}(Y_s^{t,y}) + \lambda \widetilde{u}(\frac{Y_s^{t,y}}{k_B}) + w Y_s^{t,y} + (\delta+\lambda)\mathcal{G}_s^{t,y} \big) ds - \int_u^T \mathcal{M}_s^{t,y} d\widetilde{B}_s + (\mathcal{K}_T^{t,y} - \mathcal{K}_u^{t,y}). \end{cases} \tag{53}$$

The upper obstacle, which is the unique solution to (49), is represented by

$$\underline{\mathcal{G}}_u^{t,y} = \underline{\mathcal{G}}_t^{t,y} + \int_t^u b_s^{t,y} ds + \int_t^u l_s^{t,y} d\langle \widetilde{B}\rangle_s + \int_t^u \sigma_s^{t,y} d\widetilde{B}_s, \tag{54}$$

with $\underline{\mathcal{G}}_t^{t,y} = -\widetilde{\underline{J}}(t,y)$, where by the strong regularity of the solution to G-BSDE (49) in Lemma 3.1 (i.e., $-\widetilde{\underline{J}} \in C^\infty(\overline{\mathcal{D}}_{T^1})$), we can apply the G-Itô's formula to obtain $(b_s^{t,y})_{s=t}^T$, $(l_s^{t,y})_{s=t}^T$, and $(\sigma_s^{t,y})_{s=t}^T$, given by

$$b_s^{t,y} \equiv -\partial_t \widetilde{\underline{J}}(s, Y_s^{t,y}) - (\delta - r_I) Y_s^{t,y} \partial_y \widetilde{\underline{J}}(s, Y_s^{t,y}), \quad l_s^{t,y} \equiv -\frac{1}{2}(Y_s^{t,y})^2 \partial_{yy} \widetilde{\underline{J}}(s, Y_s^{t,y}), \text{ and} \\ \sigma_s^{t,y} \equiv Y_s^{t,y} \partial_y \widetilde{\underline{J}}(s, Y_s^{t,y}). \tag{55}$$

Using the arguments in Park and Wong (2023), we show the existence of the solution to (53).

Lemma A.1. *The reflected G-BSDE (53) has a maximal solution in $\mathbb{S}_G^\alpha(0, T^1)$ with $\alpha \geq 2$.*[8]

Proof of Lemma A.1. The proof is similar to that of Lemma 1 in Park and Wong (2023). The existence of a maximal solution is established by showing that the reflected G-BSDE (53) satisfies the assumptions (A1-A4) of Theorem 4.2 in Li and Peng (2020).

As $(Y_s^{t,y})_{s=t}^T$ in (53) is a G-Itô process with constant drift and diffusion coefficients, $(Y_s^{t,y})_{s=t}^T \in M_G^\beta(0,T^1) \cap S_G^\beta(0,T^1)$ for any $\beta \geq 1$ (see Peng (2007)). Similarly, a process $(1/Y_u^{t,y})_{u=t}^T$ is also a G-Itô process, i.e., $1/Y_u^{t,y} = 1/y + \int_t^u (r_I - \delta) 1./Y_s^{t,y} ds + \int_t^u 1/Y_s^{t,y} d\widetilde{B}_s + \int_t^u 1/Y_s^{t,y} d\langle \widetilde{B}\rangle_s$, which implies that $(1/Y_u^{t,y})_{u=t}^T \in M_G^\beta(0,T^1) \cap S_G^\beta(0,T^1)$ for any $\beta \geq 1$.

Combining the above facts with the growth property of $\widetilde{u}$ in (50) yields

$$\Big(\widetilde{u}(Y_s^{t,y}) + \lambda \widetilde{u}(\frac{Y_s^{t,y}}{k_B}) + w Y_s^{t,y} + (\delta+\lambda) g \Big)_{s=t}^T \in M_G^\beta(0,T^1) \quad \forall \beta \geq 1, \ \forall g \in \mathbb{R},$$

and

$$\hat{\mathbb{E}}_t\Big[\sup_{s\in[t,T]} \Big| \widetilde{u}(Y_s^{t,y}) + \lambda \widetilde{u}(\frac{Y_s^{t,y}}{k_B}) + w Y_s^{t,y} + (\beta+\lambda) g \Big|^\beta \Big] \\ \leq C\Big(\hat{\mathbb{E}}_t\Big[\sup_{s\in[t,T]} (Y_s^{t,y})^{\kappa\beta} \Big] + \hat{\mathbb{E}}_t\Big[\sup_{s\in[t,T]} (Y_s^{t,y})^{\beta} \Big] + \hat{\mathbb{E}}_t\Big[\sup_{s\in[t,T]} (1/Y_s^{t,y})^{\kappa\beta} \Big] + |g|^\beta \Big) < \infty, \quad \forall \beta \geq 2, \ \forall g \in \mathbb{R},$$

[8] Refer to Definition 5 for the definition of the solution to the G-BSDE.

with some $\kappa \geq 1$ and $C > 0$ depending on β, where we use $|a+b|^\beta \leq 2^{\beta-1}(|a|^\beta + |b|^\beta)$ for any $\beta \geq 2$. Overall, we deduce that (A1) holds. (A2) is easily satisfied by the linearity of the driver term $\widetilde{u}(y) + \lambda\widetilde{u}(\frac{y}{k_B}) + wy + (\delta+\lambda)g$ with respect to $g \in \mathbb{R}$.

Note that under Assumption 1.(ii), $|\widetilde{u}(y)| + |\widetilde{u}'(y)| + |u(I(y))| \leq C(y^\kappa + y^{-\kappa})$ for $y > 0$ with some constants $\kappa \geq 1$ and $C > 0$. Combining the estimate with the argument of Theorem 1.(ii) of Yang and Koo (2018) in which suitable approximations belong to the C^∞ space on some bounded domains, we have the following estimate of $\underline{\widetilde{J}} = \underline{\mathcal{G}}$: for $(t, y) \in \overline{\mathcal{D}}_{T^1}$,

$$|\underline{\widetilde{J}}(t, y)| + |\partial_y \underline{\widetilde{J}}(t, y)| + |\partial_{yy}\underline{\widetilde{J}}(t, y)| \leq C(y^\kappa + y^{-\kappa}), \tag{56}$$

with some constants $\kappa \geq 1$ and $C > 0$.

The growth estimate (56) and the fact that $(Y_u^{t,y})_{u=t}^T, (1/Y_u^{t,y})_{u=t}^T \in M_G^\beta(0, T^1) \cap S_G^\beta(0, T^1)$ for any $\beta \geq 1$, imply $(\underline{\mathcal{G}}_u^{t,y})_{u=t}^{T^1} \in S_G^\beta(0, T^1)$, $(\sigma_u^{t,y})_{u=t}^{T^1} = (Y_u^{t,y}\partial_y \underline{\widetilde{J}}(u, Y_u^{t,y}))_{u=t}^{T^1} = (\underline{\mathcal{M}}_u^{t,y})_{u=t}^{T^1} \in H_G^\beta(0, T^1)$, and $\hat{\mathbb{E}}_t[\sup_{s\in[t,T]} |\sigma_s^{t,y}|^\beta] < \infty$ for any $\beta \geq 1$. Similarly, we have $(l_u^{t,y})_{u=t}^{T^1} = \big(-\frac{1}{2}(Y_u^{t,y})^2\partial_{yy}\underline{\widetilde{J}}(u, Y_u^{t,y})\big)_{u=t}^{T^1} \in M_G^\beta(0, T^1)$ with $\hat{\mathbb{E}}_t[\sup_{s\in[t,T]} |l_s^{t,y}|^\beta] < \infty$ for any $\beta \geq 1$.

As $\underline{\widetilde{J}}$ is the solution to linear PDE (52), $(b_u^{t,y})_{u=t}^{T^1}$ can be rewritten as

$$b_u^{t,y} = \frac{1}{2}\underline{\Upsilon}^2(Y_u^{t,y})^2\partial_{yy}\underline{\widetilde{J}}(u, Y_u^{t,y}) - (\delta+\lambda)\underline{\widetilde{J}}(u, Y_u^{t,y}) + \widetilde{u}(\frac{Y_u^{t,y}}{k_R}) + \lambda\widetilde{u}(\frac{Y_u^{t,y}}{k_B}).$$

Combining this with the growth estimate (56) yields $(b_u^{t,y})_{u=t}^{T^1} \in M_G^\beta(0, T^1)$ and $\hat{\mathbb{E}}_t[\sup_{s\in[t,T]} |b_s^{t,y}|^\beta] < \infty$ $\forall\beta \geq 1$. Overall, we deduce that (A3) holds.

Moreover, it follows from Corollary 2.15 in Hu et al. (2014b) that $\underline{\mathcal{G}}_T^{t,y} = -\underline{\widetilde{J}}(T, Y_T^{t,y}) \in L_G^\beta(\Omega_T)$ for any $\beta \geq 1$, where $\Omega_T \equiv \{\omega_{\cdot\wedge T} \mid \omega \in \Omega\}$. This implies that (A4) holds.

Because the reflected G-BSDE (53) satisfies (A1-A4) with any $\beta \geq 1$, Theorem 4.2 in Li (2020) allows a maximal solution $(\mathcal{G}_u^{t,y}, \mathcal{M}_u^{t,y}, \mathcal{K}_u^{t,y})_{u=t}^{T^1} \in \mathbb{S}_G^\alpha(0, T^1)$ with $2 \leq \alpha < \beta$, and Proposition 5.5 in Li (2020) implies that $\mathcal{G}_t^{t,y} = -\widetilde{J}(t, y)$ for $(t, y) \in \overline{\mathcal{D}}_T$. □

We now turn to **Proof of (a)**. So far, we have shown that the reflected G-BSDE (53) has a maximal solution such that $\mathcal{G}_t^{t,y} = -\widetilde{J}(t, y)$ for $(t, y) \in \overline{\mathcal{D}}_T$. To obtain a dynamic programming principle for $\widetilde{J}(t, y)$, the continuity of $\widetilde{J}(t, y)$ should be verified. As the driver and terminal terms of $(\mathcal{G}_u^{t,y})_{u=t}^{T^1}$ and the obstacle function $(\underline{\mathcal{G}}_u^{t,y})_{u=t}^T$ satisfy the same growth conditions of a reflected G-BSDE studied in Park and Wong (2023), we can utilize the priori estimate result in Lemma 4 of Park and Wong (2023) for the continuity of $\widetilde{J}(t, y)$, i.e., $\widetilde{J} \in C(\overline{\mathcal{D}}_T)$. Using the similar argument in Proposition 2 of Park and Wong (2023) in which the standard viscosity technique of Li (2020); Li et al. (2018) is proposed to have dynamic principle results, we deduce that the maximal solution $\mathcal{G}_t^{t,y} = -\widetilde{J}(t, y)$ satisfies the following upper obstacle PDE:

$$\begin{cases} \min\Big\{-\partial_t \widetilde{J} + \widetilde{G}\big(-y^2\partial_{yy}\widetilde{J}\big) - (\delta - r_I)y\partial_y\widetilde{J} + (\delta+\lambda)\widetilde{J} - \widetilde{u}(y) - \lambda\widetilde{u}(\frac{y}{k_B}) - wy,\ -\underline{\widetilde{J}} + \widetilde{J}\Big\} = 0, \quad (t, y) \in \mathcal{D}_T, \\ -\widetilde{J}(T, y) = -\underline{\widetilde{J}}(T, y), \quad y \in (0, \infty), \end{cases}$$

which is equivalent to (33). This is the end of the proof.

Proof of (b). Recall the prior $\widetilde{\mathbb{P}}^* \in \mathcal{Q}$ with $M^{(\Sigma^*,\mu^*)} \in \Lambda^\Theta$. We denote by $\Psi(t, y)$ a G-stopping problem under the (classical) linear expectation on $\widetilde{\mathbb{P}}^*$, such that

$$\Psi(t, y) \equiv \sup_{\widetilde{\tau}\in\widetilde{\mathcal{T}}_{t,T}} \mathbb{E}_t^{\widetilde{\mathbb{P}}^*}\left[\int_t^{\widetilde{\tau}} e^{-(\delta+\lambda)(s-t)}\Big(\widetilde{u}(Y_s) + \lambda\widetilde{u}(\frac{Y_s}{k_B}) + wY_s\Big)ds + e^{-(\delta+\lambda)(\widetilde{\tau}-t)}\underline{\widetilde{J}}(\widetilde{\tau}, Y_{\widetilde{\tau}}) \,\Big|\, Y_t = y\right], \tag{57}$$

where it follows from Lemma 3.1(b) that for a given $\widetilde{\tau} \in \widetilde{\mathcal{T}}_{t,T}$, $\underline{\widetilde{J}}(\widetilde{\tau}, Y_{\widetilde{\tau}})$ has the following classical expectation representation under $\widetilde{\mathbb{P}}^*$:

$$\underline{\widetilde{J}}(\widetilde{\tau}, Y_{\widetilde{\tau}}) = \mathbb{E}_{\widetilde{\tau}}^{\widetilde{\mathbb{P}}^*}\left[\int_{\widetilde{\tau}}^{T^1} e^{-(\delta+\lambda)(s-\widetilde{\tau})}\Big(\widetilde{u}(\frac{Y_s}{k_R}) + \lambda\widetilde{u}(\frac{Y_s}{k_B})\Big)ds + e^{-(\delta+\lambda)(T^1-\widetilde{\tau})}\widetilde{u}(\frac{Y_{T^1}}{k_X})\right].$$

It follows from Fleming and Soner (2006); Pham (2009) and $dY_t = (\delta - r_I)Y_t dt - Y_t M^{(\Sigma^*,\mu^*)} dW_t^{\widetilde{\mathbb{P}}^*}$ for $t \in [0, T^1]$ $\widetilde{\mathbb{P}}^*$-a.s. that $\Psi(t, y)$ is the unique viscosity solution to the following (reduced) obstacle problem with the lower obstacle $\underline{\widetilde{J}}$, i.e.,

$$\begin{cases} \max\Big\{\partial_t\Psi + \mathcal{L}_{\underline{\Upsilon}}\Psi + \widetilde{u}(y) + \lambda\widetilde{u}(\frac{y}{k_B}) + wy,\ \underline{\widetilde{J}} - \Psi\Big\} = 0, \quad (t, y) \in \mathcal{D}_T, \\ \Psi(T, y) = \underline{\widetilde{J}}(T, y), \quad y \in (0, \infty), \end{cases} \tag{58}$$

recalling that $\mathcal{L}_{\underline{\Upsilon}} = \frac{1}{2}\underline{\Upsilon}^2 y^2\partial_{yy} + (\delta - r_I)y\partial_y - (\delta+\lambda)$.

Moreover, if we consider the following substitution:

$$\widetilde{\Psi}(t, y) \equiv \frac{1}{y}\Big(\Psi(t, y) - \underline{\widetilde{J}}(t, y)\Big) \quad \text{for} \quad (t, y) \in \overline{\mathcal{D}}_T, \tag{59}$$

it is clear that $\widetilde{\Psi}(t, y)$ satisfies the following obstacle problem with 0 obstacle:

$$\begin{cases}\max\left\{\partial_t\widetilde{\Psi}+\mathcal{L}^1_{\underline{\Upsilon}}\widetilde{\Psi}+\frac{1}{y}\left(\widetilde{u}(y)-\widetilde{u}(\frac{y}{k_R})\right)+w,\ -\widetilde{\Psi}\right\}=0, & (t,y)\in\mathcal{D}_T,\\ \widetilde{\Psi}(T,y)=0,\quad y\in(0,\infty),\end{cases} \tag{60}$$

where we use the fact that $\partial_t\underline{\widetilde{J}}+\mathcal{L}_{\underline{\Upsilon}}\underline{\widetilde{J}}+\widetilde{u}(\frac{y}{k_R})+\lambda\widetilde{u}(\frac{y}{k_B})=0$ on $\mathcal{D}_{T^1}$, and the linear operator $\mathcal{L}^1_{\underline{\Upsilon}}$ is given by $\mathcal{L}^1_{\underline{\Upsilon}}=\frac{1}{2}\underline{\Upsilon}^2y^2\partial_{yy}+\left(\delta-r_I+\underline{\Upsilon}^2\right)y\partial_y-(r_I+\lambda)$.

The obstacle problem (60) with the linear operator $\mathcal{L}^1_{\underline{\Upsilon}}$ and inhomogeneous term $\frac{1}{y}(\widetilde{u}(y)-\widetilde{u}(\frac{y}{k_R}))+w$ perfectly coincides with the one in Jeon et al. (2022) (refer to Appendix A in Jeon et al. (2022)). In terms of the problem (60), thus, we have the following properties of $\widetilde{\Psi}(t,y)$, which is based on Theorem A.1 and Lemma A.3 in Jeon et al. (2022):

Lemma A.2. *The unique viscosity solution $\widetilde{\Psi}(t,y)$ is actually a strong solution such that*

1. *$\widetilde{\Psi}\in\mathcal{W}^{1,2}_{p,loc}(\mathcal{D}_T)\cap C(\overline{\mathcal{D}}_T)$ for any $p>1$, $\partial_y\widetilde{\Psi}\in C(\mathcal{D}_T)$, and $\partial_t\widetilde{\Psi}\in C(\overline{\mathcal{D}}_T)$,*[9]
2. *$\partial_y\widetilde{\Psi}\geq 0$ on $\mathcal{D}_T$ and $\partial_t\widetilde{\Psi}\leq 0$ on $\overline{\mathcal{D}}_T$.*

From Lemma A.2, the relation (59) (i.e., $\Psi=y\widetilde{\Psi}+\underline{\widetilde{J}}$), and Lemma 3.1.(b) (i.e., $\underline{\widetilde{J}}\in C^\infty(\overline{\mathcal{D}}_{T^1})$), we have $\Psi\in\mathcal{W}^{2,1}_{p,loc}(\mathcal{D}_T)\cap C(\overline{\mathcal{D}}_T)$ for any $p\geq 1$, $\partial_t\Psi,\partial_y\Psi\in C(\overline{\mathcal{D}}_T)$.

Furthermore, it follows from the strict convexity of the integrand $\widetilde{u}(y)+\lambda\widetilde{u}(\frac{y}{k_B})+wy$ in (57) and the obstacle function $\underline{\widetilde{J}}$ (Lemma 3.1.(b)) that $\partial_{yy}\Psi(t,y)>0$ a.e. in $\overline{\mathcal{D}}_T$.

Thus, for $\widetilde{G}(\cdot)$ in (23), $-\widetilde{G}\left(-y^2\partial_{yy}\Psi\right)=\frac{1}{2}\underline{\Upsilon}^2y^2\partial_{yy}\underline{\Psi}$. Thus, we conclude that $\Psi\in\mathcal{W}^{2,1}_{p,loc}(\mathcal{D}_T)\cap C(\overline{\mathcal{D}}_T)$ with $p\geq 1$ is the unique solution to (33), i.e., $\widetilde{J}(t,y)=\Psi(t,y)$ on $\overline{\mathcal{D}}_T$, with $\partial_t\widetilde{J},\partial_y\widetilde{J}\in C(\overline{\mathcal{D}}_T)$, and $\partial_{yy}\widetilde{J}>0$ a.e. in $\overline{\mathcal{D}}_T$. This is the end of the proof.

Proof of (c) and (d). It follows from Lemma A.2, i.e., $\partial_y\widetilde{\Psi}\geq 0$ that the following free boundary $\hat{z}_{\underline{\Upsilon}}(t)$ is well defined:

$$\hat{z}_{\underline{\Upsilon}}(t)\equiv\inf\left\{y\geq 0\mid\widetilde{\Psi}(t,y)>0\right\}=\inf\left\{y\geq 0\mid\widetilde{J}(t,y)>\underline{\widetilde{J}}(t,y)\right\}\quad\text{for } t\in[0,T), \tag{61}$$

where the second equality follows from the relation (59) and $\Psi=\widetilde{J}$, i.e., $\widetilde{\Psi}(t,y)=\frac{1}{y}(\widetilde{J}(t,y)-\underline{\widetilde{J}}(t,y))$.

Using the similar argument in Lemma A.4 of Jeon et al. (2022), we have the following properties of $\hat{z}_{\underline{\Upsilon}}(t)$:

Lemma A.3. *The following statements hold:*

1. *Define the terminal point of the free boundary by $\hat{z}_{\underline{\Upsilon}}(T)\equiv\lim_{t\to T-}\hat{z}_{\underline{\Upsilon}}(t)$. Then, the free boundary is strictly increasing and satisfies*
$$z_L<\hat{z}_{\underline{\Upsilon}}(t)<z_U\quad\forall t\in[0,T);\quad\text{and}\quad\hat{z}_{\underline{\Upsilon}}(T)=z_U,$$
where z_U is the unique solution of the equation $h(z)=0$, given by $h(z)\equiv\frac{1}{z}(\widetilde{u}(z)-\widetilde{u}(\frac{z}{k_R}))+w$ on $z>0$, and z_L is the unique solution of the integral equation $H(z)$, given by
$$H(z)\equiv\int_z^\infty\xi^{-m_{\underline{\Upsilon},1}-1}h(\xi)d\xi\quad\text{on } z>0,$$
and $m_{\underline{\Upsilon},1}>0$ and $m_{\underline{\Upsilon},2}<-1$ are positive and negative roots of the following quadratic equation $q(m)=0$, respectively, given by $q(m)=\frac{1}{2}\underline{\Upsilon}^2m^2+(\delta-r_I+\frac{1}{2}\underline{\Upsilon}^2)m-(r_I+\lambda)$.
2. *The free boundary satisfies $\hat{z}_{\underline{\Upsilon}}(\cdot)\in C([0,T])\cap C^\infty([0,T))$.*

From (61) and the properties of $\hat{z}_{\underline{\Upsilon}}(\cdot)$ in Lemma A.3, the candidate G-stopping time $\widetilde{\tau}^*$ is given by

$$\widetilde{\tau}^*\equiv\widetilde{\tau}^*(t,y)=\inf\left\{s\geq t\mid\widetilde{J}(s,Y_s)=\underline{\widetilde{J}}(s,Y_s),\ Y_t=y\right\}=\inf\left\{s\geq t\mid Y_s\leq\hat{z}_{\underline{\Upsilon}}(s),\ Y_t=y\right\}\wedge T. \tag{62}$$

As $\hat{z}_{\underline{\Upsilon}}(\cdot)$ is smooth, i.e., $\hat{z}_{\underline{\Upsilon}}(\cdot)\in C([0,T])\cap C^\infty([0,T))$, $\widetilde{\tau}^*$ is the first exit time of a closed region with the regular boundary $\hat{z}_{\underline{\Upsilon}}(\cdot)$, which leads to $\widetilde{\tau}^*\in\widetilde{\mathcal{T}}_{t,T}$. Thus, $\widetilde{\tau}^*$ is the optimal G-stopping time to Problem 3.

Under the substitution $\widetilde{\Psi}$ in (59), the obstacle and terminal value terms in (60) are zero, and the source term $\frac{1}{y}(\widetilde{u}(y)-\widetilde{u}(\frac{y}{k_R}))$ is strictly increasing in $y>0$. Thus, we can use the arguments in Jeon et al. (2022) and the fact $\hat{z}_{\underline{\Upsilon}}(t)\equiv\inf\{y\geq 0|\widetilde{\Psi}(t,y)>0\}$ to show that $\widetilde{\Psi}(t,y)$ is the strong solution to the following free boundary problem:

$$\begin{cases}\partial_t\widetilde{\Psi}+\mathcal{L}^1_{\underline{\Upsilon}}\widetilde{\Psi}+\frac{1}{y}\left(\widetilde{u}(y)-\widetilde{u}(\frac{y}{k_R})\right)+w=0, & \text{if}\quad(t,y)\in\{(t,y)\in\mathcal{D}_T|y>\hat{z}_{\underline{\Upsilon}}(t)\},\\ \widetilde{\Psi}(t,y)=0, & \text{if}\quad(t,y)\in\{(t,y)\in\mathcal{D}_T|0<y\leq\hat{z}_{\underline{\Upsilon}}(t)\},\end{cases} \tag{63}$$

[9] $\mathcal{W}^{1,2}_{p,loc}(\mathcal{D}_T)$ with $p\geq 1$ is the set of all functions whose restrictions to the domain $\mathcal{K}$ belong to $\mathcal{W}^{1,2}_p(\mathcal{K})$ for any compact subset $\mathcal{K}\subseteq\overline{\mathcal{D}}_T$, where $\mathcal{W}^{1,2}_p(\mathcal{K})$ with $p\geq 1$ is the completion of $C^\infty(\mathcal{K})$ under the norm for V,

$$\|V\|_{\mathcal{W}^{1,2}_p(\mathcal{K})}=\left(\int_{\mathcal{K}}\left(|V|^p+|\partial_tV|^p+|\partial_yV|^p+|\partial_{yy}V|^p\right)dydt\right)^{1/p}.$$

with $\widetilde{\Psi}(T, y) = 0$ for $y > 0$. It follows from (Jeon et al., 2022, proposition 3.2(c)) that the solution to the free boundary problem is given by the following integral representation:

$$\begin{aligned}\widetilde{\Psi}(t, y) &= \frac{1}{y}\mathbb{E}_t^{\widetilde{\mathbb{P}}^*}\left[\int_t^T e^{-(\delta+\lambda)(s-t)}\Big(\widetilde{u}(Y_s) - \widetilde{u}(\frac{Y_s}{k_R}) + wY_s\Big)\mathbf{1}_{\{Y_s > \hat{z}_{\underline{\Upsilon}}(s)\}}ds \,\Big|\, Y_t = y\right] \\ &= \frac{1}{y}\int_t^T e^{-(\delta+\lambda)(s-t)}\left(\int_{\hat{z}_{\underline{\Upsilon}}(s)}^{\infty}\Big(\widetilde{u}(\eta) - \widetilde{u}(\frac{\eta}{k_R}) + w\eta\Big)\Phi_{\underline{\Upsilon}}(t, y; s, \eta)d\eta\right)ds \equiv \frac{1}{y}\int_t^T \mathcal{I}_{\underline{\Upsilon}}(t, y; s, \hat{z}_{\underline{\Upsilon}}(s))ds,\end{aligned} \tag{64}$$

with $\Phi_{\underline{\Upsilon}}$ in (32). By the smooth pasting condition (C^1-condition) on $y = \hat{z}_{\underline{\Upsilon}}(t) > 0$, i.e. $\widetilde{\Psi}(t, \hat{z}_{\underline{\Upsilon}}(t)) = 0$, we have the integral equation for the free boundary $\hat{z}_{\underline{\Upsilon}}(\cdot)$ in (35).

It follows from $\widetilde{J} = \Psi = y\widetilde{\Psi} + \underline{\widetilde{J}}$ with the representations in (57) and (64) and the optimal G-stopping time $\widetilde{\tau}^*$ in (62) that

$$\begin{aligned}\widetilde{J}(t, y) =&\mathbb{E}_t^{\widetilde{\mathbb{P}}^*}\left[\int_t^{\widetilde{\tau}^*} e^{-(\delta+\lambda)(s-t)}\Big(\widetilde{u}(Y_s) + \lambda\widetilde{u}(\frac{Y_s}{k_B}) + wY_s\Big)ds + e^{-(\delta+\lambda)(\widetilde{\tau}_R^*-t)}\underline{\widetilde{J}}(\widetilde{\tau}^*, Y_{\widetilde{\tau}^*})\right] \\ =&\mathbb{E}_t^{\widetilde{\mathbb{P}}^*}\left[\int_t^{\widetilde{\tau}^*} e^{-(\delta+\lambda)(s-t)}\Big(\widetilde{u}(Y_s) - \widetilde{u}(\frac{Y_s}{k_R}) + wY_s\Big)ds\right] \\ &+ \mathbb{E}_t^{\widetilde{\mathbb{P}}^*}\left[\int_t^{\widetilde{\tau}^*} e^{-(\delta+\lambda)(s-t)}\Big(\widetilde{u}(\frac{Y_s}{k_R}) + \lambda\widetilde{u}(\frac{Y_s}{k_B})\Big)ds + e^{-(\delta+\lambda)(\widetilde{\tau}^*-t)}\underline{\widetilde{J}}(\widetilde{\tau}^*, Y_{\widetilde{\tau}^*})\right] \\ =&\mathbb{E}_t^{\widetilde{\mathbb{P}}^*}\left[\int_t^T e^{-(\delta+\lambda)(s-t)}\Big(\widetilde{u}(Y_s) - \widetilde{u}(\frac{Y_s}{k_R}) + wY_s\Big)\mathbf{1}_{\{s \le \widetilde{\tau}^*\}}ds\right] + \underline{\widetilde{J}}(t, y) = \int_t^T \mathcal{I}_{\underline{\Upsilon}}(t, y; s, \hat{z}_{\underline{\Upsilon}}(s))ds + \underline{\widetilde{J}}(t, y).\end{aligned}$$

Since $\widetilde{u}(\cdot)$ is smooth, the local regularity theory for parabolic pde (e.g., Lieberman (1996)) implies that $\widetilde{J}$ is also smooth in the region $\{(t, y) \in \mathcal{D}_T \,|\, y > \hat{z}_{\underline{\Upsilon}}(t)\}$.

A.3. Proof of Theorem 3.1

The arguments of this proof are similar to those in Theorem 1 of Park and Wong (2023). Given any probability measure $\mathbb{P} \in \mathcal{P}$, if we consider the dual process $(Y_t)_{t=0}^T$ on the canonical space $(\Omega, \mathcal{B}(\Omega), \mathbb{P})$, then the stopping time, which is the first exit time with the smooth boundary $\hat{z}_{\underline{\Upsilon}}(\cdot)$, i.e.,

$$\hat{\tau}_R = \inf\Big\{t \ge 0 \mid Y_t \le \hat{z}_{\underline{\Upsilon}}(t),\ Y_0 = y\Big\} \wedge T, \tag{65}$$

is adapted with respect to $\mathbb{F}$. From Proposition 3.1.(c), it follows that $\hat{\tau}_R (= \widetilde{\tau}^*(0, y)) \in \mathcal{T}_{0,T} \cap \widetilde{\mathcal{T}}_{0,T}$ is the optimal stopping time under both probability measures $\mathbb{P}^* \in \mathcal{P}$ with $(\Sigma^*, \mu^*) \in \Theta$ and $\widetilde{\mathbb{P}}^* \in \mathcal{Q}$ with $M^{(\Sigma^*,\mu^*)} \in \Lambda^{\Theta}$.

Furthermore, under $\mathbb{P}^*$, as the dual process is given by

$$Y_t = ye^{(\delta - r_I)t}\left.\frac{d\mathbb{Q}^{\mathbb{P}^*}}{d\mathbb{P}^*}\right|_{\mathcal{F}_t} = y\exp\left((\delta - r_I - \frac{1}{2}\|(\Sigma^*)^{-1}\zeta^{\Sigma^*,\mu^*}\|^2)t - ((\Sigma^*)^{-1}\zeta^{\Sigma^*,\mu^*})^\top W_t^{\mathbb{P}^*}\right),$$

with $\zeta^{\Sigma^*,\mu^*} \equiv f((\sigma_P^*, \sigma_S^*, \rho^*, \mu_P^*, \mu_S^*)^\top)$ and $\mathbb{P}^*$-Brownian motion $W^{\mathbb{P}^*}$, the dual process under $\mathbb{P}^* \in \mathcal{P}$ has the same law as under $\widetilde{\mathbb{P}}^* \in \mathcal{Q}$. Thus, we have

$$\begin{aligned}J(0, y) &\le \sup_{\tau \in \mathcal{T}_{0,T}} \mathcal{J}(0, y, ; \mathbb{P}^*, \tau) \\ &= \mathbb{E}^{\mathbb{P}^*}\left[\int_0^{\hat{\tau}_R} e^{-(\delta+\lambda)t}\Big(\widetilde{u}(Y_t) + \lambda\widetilde{u}(\frac{Y_t}{k_B}) + wY_t\Big)dt + \int_{\hat{\tau}_R}^{T^1} e^{-(\delta+\lambda)t}\Big(\widetilde{u}(\frac{Y_t}{k_R}) + \lambda\widetilde{u}(\frac{Y_t}{k_B})\Big)dt + e^{-(\delta+\lambda)T^1}\widetilde{u}(\frac{Y_{T^1}}{k_X})\right] \\ &= \mathbb{E}^{\widetilde{\mathbb{P}}^*}\left[\int_0^{\hat{\tau}_R} e^{-(\delta+\lambda)t}\Big(\widetilde{u}(Y_t) + \lambda\widetilde{u}(\frac{Y_t}{k_B}) + wY_t\Big)dt + \int_{\hat{\tau}_R}^{T^1} e^{-(\delta+\lambda)t}\Big(\widetilde{u}(\frac{Y_t}{k_R}) + \lambda\widetilde{u}(\frac{Y_t}{k_B})\Big)dt + e^{-(\delta+\lambda)T^1}\widetilde{u}(\frac{Y_{T^1}}{k_X})\right] \\ &= \sup_{\widetilde{\tau} \in \widetilde{\mathcal{T}}_{0,T}} \mathbb{E}^{\widetilde{\mathbb{P}}^*}\left[\int_0^{\widetilde{\tau}} e^{-(\delta+\lambda)t}\Big(\widetilde{u}(Y_t) + \lambda\widetilde{u}(\frac{Y_t}{k_B}) + wY_t\Big)dt + e^{-(\delta+\lambda)(\widetilde{\tau}-t)}\underline{\widetilde{J}}(\widetilde{\tau}, Y_{\widetilde{\tau}})\right] = \widetilde{J}(0, y),\end{aligned}$$

where the last equality is from the optimality of the G-stopping time $\hat{\tau}_R (= \widetilde{\tau}^*(0, y)) \in \widetilde{\mathcal{T}}_{0,T}$ in (34).

For any $\mathbb{P}^o \in \mathcal{P}$ with $(\Sigma^o, \mu^o) \in \Theta$, there exists $\widetilde{\mathbb{P}}^o \in \mathcal{Q}^0 \subset \mathcal{Q}$ with $M^{(\Sigma^o,\mu^o)} \in \Lambda^\Theta$ such that the probability law of $(Y_t)_{t=0}^{T^1}$ under $\mathbb{P}^o$ is the same as under $\widetilde{\mathbb{P}}^o$, i.e. $\mathbb{P}^o(\{Y \in C\}) = \widetilde{\mathbb{P}}^o(\{Y \in C\})$ for all $C \in \mathbb{F}$. This implies

$$
\begin{aligned}
J(0,y) &\ge \inf_{\mathbb{P}^o\in\mathcal{P}} \mathcal{J}(0,y,;\mathbb{P}^o,\hat{\tau}_R)\\
&= \inf_{\mathbb{P}^o\in\mathcal{P}} \mathbb{E}^{\mathbb{P}^o}\left[\int_0^{\hat{\tau}_R} e^{-(\delta+\lambda)t}\Big(\widetilde{u}(Y_t)+\lambda\widetilde{u}(\frac{Y_t}{k_B})+wY_t\Big)dt + \int_{\hat{\tau}_R}^{T^1} e^{-(\delta+\lambda)t}\Big(\widetilde{u}(\frac{Y_t}{k_R})+\lambda\widetilde{u}(\frac{Y_t}{k_B})\Big)dt + e^{-(\delta+\lambda)T^1}\widetilde{u}(\frac{Y_{T^1}}{k_X})\right]\\
&\ge \inf_{\widetilde{\mathbb{P}}^o\in\mathcal{Q}^0} \mathbb{E}^{\widetilde{\mathbb{P}}^o}\left[\int_0^{\hat{\tau}_R} e^{-(\delta+\lambda)t}\Big(\widetilde{u}(Y_t)+\lambda\widetilde{u}(\frac{Y_t}{k_B})+wY_t\Big)dt + \int_{\hat{\tau}_R}^{T^1} e^{-(\delta+\lambda)t}\Big(\widetilde{u}(\frac{Y_t}{k_R})+\lambda\widetilde{u}(\frac{Y_t}{k_B})\Big)dt + e^{-(\delta+\lambda)T^1}\widetilde{u}(\frac{Y_{T^1}}{k_X})\right]\\
&\ge -\hat{\mathbb{E}}\left[-\left(\int_0^{\hat{\tau}_R} e^{-(\delta+\lambda)t}\Big(\widetilde{u}(Y_t)+\lambda\widetilde{u}(\frac{Y_t}{k_B})+wY_t\Big)dt - e^{-(\delta+\lambda)(\hat{\tau}_R-t)}\underline{\widetilde{J}}(\hat{\tau}_R, Y_{\hat{\tau}_R})\right)\right]\\
&= \sup_{\widetilde{\tau}\in\widetilde{\mathcal{T}}_{0,T}} \left\{-\hat{\mathbb{E}}\left[-\int_t^{\widetilde{\tau}} e^{-(\delta+\lambda)t}\Big(\widetilde{u}(Y_t)+\lambda\widetilde{u}(\frac{Y_t}{k_B})+wY_t\Big)dt - e^{-(\delta+\lambda)\widetilde{\tau}}\underline{\widetilde{J}}(\widetilde{\tau}, Y_{\widetilde{\tau}})\right]\right\} = \widetilde{J}(0,y),
\end{aligned}
$$

where we use the relation $\mathcal{Q}^0 \subset \mathcal{Q}$ for the third inequality and the optimality of $\hat{\tau}_R = \widetilde{\tau}^*(0,y)$ to Problem 3 (provided in Proposition 3.1) for the last two equations.

A.4. Proof of Theorem 3.2

We provide the following *saddle point conditions* under which the dual conjugate relationship between $V(0,x)$ and $J(0,y)$ holds.

Lemma A.4. *The inequality (18) holds as an equality if and only if an admissible strategy $(c^*, p^*, \pi^*, \tau_R^*) \in \mathcal{A}_{0,T}(x,w)$ and the corresponding wealth process $X^{x;*} \equiv X^{x;c^*,p^*,\pi^*,\tau_R^*}$ satisfy the following conditions: τ_R^* is given by (37), $(c_t^*)_{t=0}^{T^1}$, $(\mathfrak{B}_t^*)_{t=0}^{T^1}$ and $X_{T^1}^{x;*}$ are given by (17), and*

$$
\mathbb{E}^{\mathbb{P}^*}\left[\int_0^{T^1} e^{-(r_I+\lambda)t}\left.\frac{d\mathbb{Q}^{\mathbb{P}^*}}{d\mathbb{P}^*}\right|_{\mathcal{F}_t}(c_t^*+\lambda\mathfrak{B}_t^*-w\mathbf{1}_{\{t<\tau_R^*\}})dt + e^{-(r_I+\lambda)T^1}\left.\frac{d\mathbb{Q}^{\mathbb{P}^*}}{d\mathbb{P}^*}\right|_{\mathcal{F}_{T^1}} X_{T^1}^{x;*}\right] = x, \tag{66}
$$

and

$$
\begin{aligned}
&U(0,x;c^*,p^*,\pi^*,\tau_R^*)\\
&= \mathbb{E}^{\mathbb{P}^*}\left[\int_0^{T^1} e^{-(\delta+\lambda)t}\big(u(c_t^*)\mathbf{1}_{\{t<\tau_R^*\}} + u(k_Rc_t^*)\mathbf{1}_{\{t\ge\tau_R^*\}} + \lambda u(k_B\mathfrak{B}_t^*)\big)dt + e^{-(\delta+\lambda)T^1}u\big(k_XX_{T^1}^{x;*}\big)\right],
\end{aligned} \tag{67}
$$

where $\mathbb{P}^* \in \mathcal{P}$ with generator (Σ^*, μ^*) in (25).

Proof of Lemma A.4. For any $(c,p,\pi,\tau_R) \in \mathcal{A}_{0,T^1}(x,w)$ and $y > 0$, the following inequalities hold:

$$
\begin{aligned}
&J(0,y) + yx \ge \mathcal{J}(0,y;\mathbb{P}^*,\tau_R) + yx\\
&\ge \mathbb{E}^{\mathbb{P}^*}\Bigg[\int_0^{\tau_R} e^{-(\delta+\lambda)t}\Big(u(c_t) - Y_tc_t + \lambda u(k_B\mathfrak{B}_t) - \lambda Y_t\mathfrak{B}_t + wY_t\Big)dt\\
&+ \int_{\tau_R}^{T^1} e^{-(\delta+\lambda)t}\Big(u(k_Rc_t) - Y_tc_t + \lambda u(k_B\mathfrak{B}_t) - \lambda Y_t\mathfrak{B}_t\Big)dt + e^{-(\delta+\lambda)T^1}\Big(u\big(k_XX_{T^1}^{x;c,p,\pi,\tau_R}\big) - Y_{T^1}X_{T^1}^{x;c,p,\pi,\tau_R}\Big)\Bigg] + yx\\
&\ge \mathbb{E}^{\mathbb{P}^*}\left[\int_0^{T^1} e^{-(\delta+\lambda)t}\big(u(c_t)\mathbf{1}_{\{t<\tau_R\}} + u(k_Rc_t)\mathbf{1}_{\{t\ge\tau_R\}} + \lambda u(k_B\mathfrak{B}_t)\big)dt + e^{-(\delta+\lambda)T^1}u\big(k_XX_{T^1}^{x;c,p,\pi,\tau_R}\big)\right]\\
&\ge U(0,x;c,p,\pi,\tau_R),
\end{aligned}
$$

where the first inequality becomes equality if τ_R is given by (37), the second one does if $(c_t)_{t=0}^{T^1}$, $(\mathfrak{B}_t)_{t=0}^{T^1}$, and $X_{T^1}^{x;c,p,\pi,\tau_R}$ are given by (17), the third if (66) holds, and the last if (67) holds. □

We now turn to the proof of Theorem 3.2.

Step 1. The dual value function $J(0,y)$ is strictly convex in $y > 0$, and for any given $x > -m(0)$, there exists a unique $y^* > 0$ such that $x = -\partial_y J(0,y^*)$.

Proof of Step 1. According to Theorem 3.1,

$$J(0,y)=\mathcal{J}(0,y;\mathbb{P}^*,\hat{\tau})=\sup_{\tau\in\mathcal{T}_{0,T}}\mathbb{E}^{\mathbb{P}^*}\left[\int_0^\tau e^{-(\delta+\lambda)t}\Big(\widetilde{u}(Y_t)+\lambda\widetilde{u}(\frac{Y_t}{k_B})+wY_t\Big)dt+e^{-(\delta+\lambda)\tau}\underline{J}(\tau,Y_\tau)\right],$$

where the stopping time $\hat{\tau}$ is given in (37) and the value function $\underline{J}(t,y)$ which is the same as $\widetilde{\underline{J}}$ in (31), is given as follows: for $(t,y)\in\overline{\mathcal{D}}_{T^1}$

$$\underline{J}(t,y)\equiv\mathbb{E}_t^{\mathbb{P}^*}\left[\int_t^{T^1} e^{-(\delta+\lambda)(\xi-t)}\Big(\widetilde{u}(\frac{Y_\xi}{k_R})+\lambda\widetilde{u}(\frac{Y_\xi}{k_B})\Big)d\xi+e^{-(\delta+\lambda)(T^1-t)}\widetilde{u}(\frac{Y_{T^1}}{k_X})\;\Big|\;Y_t=y\right].$$

For any $y_1,y_2>0$ with $y_1\neq y_2$, denote $y_3\equiv hy_1+(1-h)y_2$ with some $h\in(0,1)$. For $j=1,2$, and 3,

$$J(0,y_j)=\sup_{\tau\in\mathcal{T}_{0,T}}\mathbb{E}^{\mathbb{P}^*}\left[\int_0^\tau e^{-(\delta+\lambda)t}\Big(\widetilde{u}(Y_t^j)+\lambda\widetilde{u}(\frac{Y_t^j}{k_B})+wY_t^j\Big)dt+e^{-(\delta+\lambda)\tau}\underline{J}(\tau,Y_\tau^j)\right],$$

where $Y_t^j=y_je^{(\delta-r_I)t}\frac{d\mathbb{Q}^{\mathbb{P}^*}}{d\mathbb{P}^*}|_{\mathcal{F}_t}$. From the strict convexity of $\widetilde{u}(\cdot)$, it is clear that $\widetilde{u}(Y_t^3)<h\widetilde{u}(Y_t^1)+(1-h)\widetilde{u}(Y_t^2)$ and $\widetilde{u}(\frac{Y_t^3}{k_B})<h\widetilde{u}(\frac{Y_t^1}{k_B})+(1-h)\widetilde{u}(\frac{Y_t^2}{k_B})$, which leads to $\underline{J}(\tau,Y_\tau^3)<h\underline{J}(\tau,Y_\tau^1)+(1-h)\underline{J}(\tau,Y_\tau^2)$.

Using this, we have

$$\begin{aligned}J(0,y_3)<\sup_{\tau\in\mathcal{T}_{0,T}}&\left\{h\mathbb{E}^{\mathbb{P}^*}\left[\int_0^\tau e^{-(\delta+\lambda)t}\Big(\widetilde{u}(Y_t^1)+\lambda\widetilde{u}(\frac{Y_t^1}{k_B})+wY_t^1\Big)dt+e^{-(\delta+\lambda)\tau}\underline{J}(\tau,Y_\tau^1)\right]\right.\\&\left.+(1-h)\mathbb{E}^{\mathbb{P}^*}\left[\int_0^\tau e^{-(\delta+\lambda)t}\Big(\widetilde{u}(Y_t^2)+\lambda\widetilde{u}(\frac{Y_t^2}{k_B})+wY_t^2\Big)dt+e^{-(\delta+\lambda)\tau}\underline{J}(\tau,Y_\tau^2)\right]\right\}\\&\le h\sup_{\tau\in\mathcal{T}_{0,T}}\mathbb{E}^{\mathbb{P}^*}\left[\int_0^\tau e^{-(\delta+\lambda)t}\Big(\widetilde{u}(Y_t^1)+\lambda\widetilde{u}(\frac{Y_t^1}{k_B})+wY_t^1\Big)dt+e^{-(\delta+\lambda)\tau}\underline{J}(\tau,Y_\tau^1)\right]\\&+(1-h)\sup_{\tau\in\mathcal{T}_{0,T}}\mathbb{E}^{\mathbb{P}^*}\left[\int_0^\tau e^{-(\delta+\lambda)t}\Big(\widetilde{u}(Y_t^2)+\lambda\widetilde{u}(\frac{Y_t^2}{k_B})+wY_t^2\Big)dt+e^{-(\delta+\lambda)\tau}\underline{J}(\tau,Y_\tau^2)\right]\\&=hJ(0,y_1)+(1-h)J(0,y_2).\end{aligned}$$

Thus, $J(0,y)$ is strictly convex in $y>0$.

From the equivalence $J=\widetilde{J}$ (in Theorem 3.1) and the variational properties of $\widetilde{J}$ in Proposition 3.1, it follows that

$$\begin{cases}-\partial_tJ-\mathcal{L}_\Upsilon J=\widetilde{u}(y)+\lambda\widetilde{u}(\frac{y}{k_B})+wy\quad\text{for }\big\{(t,y)\in\mathcal{D}_T\,|\,y>\hat{z}_\Upsilon(t)\big\}\\ J(T,y)=\underline{J}(T,y),\quad\forall\,y\ge\hat{z}_\Upsilon(T),\quad J(t,\hat{z}_\Upsilon(t))=\underline{J}(t,\hat{z}_\Upsilon(t)),\quad\forall\,t\in[0,T].\end{cases}\tag{68}$$

Recalling $m(t;w)=\mathbb{E}_t^{\mathbb{P}^*}[\int_t^T w\frac{\mathcal{H}_s}{\mathcal{H}_t}ds]=\mathbb{E}_t^{\mathbb{P}^*}[\int_t^T e^{-(\delta+\lambda)(s-t)}w\frac{Y_s}{Y_t}ds]$, we temporarily denote $\mathcal{M}(t,y)\equiv\partial_yJ(t,y)-m(t;w)$ for $(t,y)\in\overline{\mathcal{D}}_T$, then $\mathcal{M}(t,y)$ satisfies

$$\begin{cases}-\partial_t\mathcal{M}-\mathcal{L}_\Upsilon^1\mathcal{M}=-I(y)-\frac{\lambda}{k_B}I(\frac{y}{k_B})\quad\text{for }\big\{(t,y)\in\mathcal{D}_T\,|\,y>\hat{z}_\Upsilon(t)\big\},\\ \mathcal{M}(T,y)=\partial_y\underline{J}(T,y),\ \forall\,y\ge\hat{z}_\Upsilon(T),\ \mathcal{M}(t,\hat{z}_\Upsilon(t))=\partial_y\underline{J}(t,\hat{z}_\Upsilon(t))-m(t;w),\ \forall\,t\in[0,T],\end{cases}\tag{69}$$

with $\mathcal{L}_\Upsilon^1=\frac{\Upsilon^2}{2}y^2\partial_{yy}+(\delta-r_I+\Upsilon^2)y\partial_y-(r_I+\lambda)$, and $\partial_y\underline{J}(t,y)$ satisfies

$$\begin{cases}-\partial_t(\partial_y\underline{J})-\mathcal{L}_\Upsilon^1(\partial_y\underline{J})=-\frac{1}{k_R}I(\frac{y}{k_R})-\frac{\lambda}{k_B}I(\frac{y}{k_B})\quad\text{for}\quad(t,y)\in\mathcal{D}_{T^1},\\ \partial_y\underline{J}(T^1,y)=-\frac{1}{k_X}I(\frac{y}{k_X}),\quad\forall\,y>0.\end{cases}$$

It follows from $I(\frac{y}{k_R})$, $I(\frac{y}{k_B})$, and $I(\frac{y}{k_X})>0$ for all $y>0$ that $\partial_y\underline{J}(t,y)\le0$ for all $(t,y)\in\overline{\mathcal{D}}_{T^1}$. Applying the comparison principles for PDE to (69) yields $\mathcal{M}(t,y)\le0$, which is equivalent to

$$-\partial_yJ(t,y)\ge-m(t;w)\quad\text{for }\big\{(t,y)\in\mathcal{D}_T\,|\,y>\hat{z}_\Upsilon(t)\big\}.\tag{70}$$

We consider a function Ξ given by $\Xi(t,y)\equiv\mathbb{E}_t^{\mathbb{Q}^{\mathbb{P}^*}}[\int_t^T e^{-(r_I+\lambda)(s-t)}I(\frac{Y_s}{k_R})ds\mid Y_t=y]$, which satisfies $-\partial_t\Xi(t,y)-\mathcal{L}_\Upsilon^1\Xi(t,y)=I(\frac{y}{k_R})$ for all $(t,y)\in\mathcal{D}_T$. We temporarily denote $\mathcal{M}_1(t,y)=\mathcal{M}(t,y)-\partial_y\underline{J}(t,y)+\Xi(t,y)$, then we have

$$\begin{cases} -\partial_t \mathcal{M}_1 - \mathcal{L}^1_{\underline{\Upsilon}} \mathcal{M}_1 = I(\frac{y}{k_R}) - I(y) + \frac{1}{k_R} I(\frac{y}{k_R}) \text{ for } \left\{(t,y) \in \mathcal{D}_T \,|\, y > \hat{z}_{\underline{\Upsilon}}(t)\right\}, \\ \mathcal{M}_1(T,y) = 0, \ \forall\, y \geq \hat{z}_{\underline{\Upsilon}}(T), \ \ \mathcal{M}_1(t, \hat{z}_{\underline{\Upsilon}}(t)) = \Xi(t, \hat{z}_{\underline{\Upsilon}}(t)) - m(t; w), \ \forall\, t \in [0,T]. \end{cases}$$

It follows from $k_R > 1$ (see (11) in Section 2.2) that $I(\frac{y}{k_R}) - I(y) + \frac{1}{k_R} I(\frac{y}{k_R}) > 0$ for any $y > 0$.

Furthermore, from the optimality of $\hat{\tau}_R(t, \hat{z}_{\underline{\Upsilon}}(t)) = \widetilde{\tau}^*(t, \hat{z}_{\underline{\Upsilon}}(t)) = t$ (see Proposition 3.1.(c) and Theorem 3.1), the inequality $u(I(\frac{y}{k_R})) - u(I(y)) < y(I(\frac{y}{k_R}) - I(y))$ $\forall y > 0$ (due to the strict convexity of $u(\cdot)$), and the relation $\widetilde{u}(y) = u(I(y)) - yI(y)$ $\forall y > 0$, we have that for $t \in [0,T]$,

$$\begin{aligned} 0 = J(t, \hat{z}_{\underline{\Upsilon}}(t)) - \underline{J}(t, \hat{z}_{\underline{\Upsilon}}(t)) &= \mathbb{E}^{\mathbb{P}^*}\left[\int_t^{\hat{\tau}_R(t,\hat{z}_{\underline{\Upsilon}}(t))} e^{-(\delta+\lambda)(s-t)} \Big(\widetilde{u}(Y_s) - \widetilde{u}(\frac{Y_s}{k_R}) + wY_s \Big) ds \Big| Y_t = \hat{z}_{\underline{\Upsilon}}(t) \right] \\ &\geq \mathbb{E}^{\mathbb{P}^*}\left[\int_t^{T} e^{-(\delta+\lambda)(s-t)} \Big(\widetilde{u}(Y_s) - \widetilde{u}(\frac{Y_s}{k_R}) + wY_s \Big) ds \Big| Y_t = \hat{z}_{\underline{\Upsilon}}(t) \right] \\ &> \mathbb{E}^{\mathbb{P}^*}\left[\int_t^{T} e^{-(\delta+\lambda)(s-t)} Y_s \Big(w - I(\frac{Y_s}{k_R}) \Big) ds \Big| Y_t = \hat{z}_{\underline{\Upsilon}}(t) \right] \\ &= -\hat{z}_{\underline{\Upsilon}}(t) \left(\Xi(t, \hat{z}_{\underline{\Upsilon}}(t)) - m(t; w) \right), \end{aligned}$$

which leads to $\Xi(t, \hat{z}_{\underline{\Upsilon}}(t)) - m(t; w) > 0$ for all $t \in [0,T]$. Thus, we can employ the comparison principles for PDE to have $\mathcal{M}_1(t,y) \geq 0$ for $\left\{(t,y) \in \mathcal{D}_T \,|\, y > \hat{z}_{\underline{\Upsilon}}(t)\right\}$, i.e.,

$$\partial_y J(t,y) - m(t; w) \geq \partial_y \underline{J}(t,y) - \Xi(t,y) \ \text{ for } \left\{(t,y) \in \mathcal{D}_T \,|\, y > \hat{z}_{\underline{\Upsilon}}(t)\right\}. \tag{71}$$

From (70) and (71), we have $\partial_y \underline{J}(t,y) - \Xi(t,y) \leq \partial_y J(t,y) - m(t; w) \leq 0$ for $\left\{(t,y) \in \mathcal{D}_T \,|\, y > \hat{z}_{\underline{\Upsilon}}(t)\right\}$.

From $\lim_{y \to +\infty} I(y) = 0$, it follows that $\lim_{y \to +\infty} \partial_y \underline{J}(t,y) = 0$ and $\lim_{y \to +\infty} \Xi(t,y) = 0$ for all $t \in [0,T]$. Thus, we have $\lim_{y \to +\infty} -\partial_y J(t,y) = -m(t; w)$.

From $\lim_{y \to 0+} I(y) = +\infty$, it is easy to show that $\lim_{y \to 0+} \partial_y \underline{J}(t,y) = -\infty$. This implies that $\lim_{y \to 0+} -\partial_y J(t,y) = \lim_{y \to 0+} -\partial_y \underline{J}(t,y) = +\infty$. Thus, we have Step 1.

Given $y^* > 0$ such that $x = -\partial_y J(0, y^*)$, denote by

$$\mathcal{X}(t, Y_t^*) \equiv \mathbb{E}_t^{\mathbb{Q}^{\mathbb{P}^*}}\left[\int_t^{T^1} e^{-(r_I+\lambda)(s-t)} \Big(\hat{c}_s(y^*) + \lambda \hat{\mathfrak{B}}_s(y^*) - w\mathbf{1}_{\{s < \hat{\tau}_R(t, Y_t^*)\}} \Big) ds + e^{-(r_I+\lambda)(T_1-t)} \hat{X}_{T^1}(y^*) \right], \tag{72}$$

with $Y_t^* = y^* e^{(\delta - r_I + \frac{1}{2}\underline{\Upsilon}^2)t - ((\Sigma^*)^{-1}\zeta^{\Sigma^*,\mu^*})^\top W_t^{\mathbb{Q}^{\mathbb{P}^*}}}$ and $\zeta^{\Sigma^*,\mu^*} = (\zeta_1^{\Sigma^*,\mu^*}, \zeta_2^{\Sigma^*,\mu^*})^\top$ in (26), where we use the candidate strategies $(\hat{c}_t(y^*))_{t=0}^{T^1}$, $(\hat{\mathfrak{B}}_t(y^*))_{t=0}^{T^1}$, and $\hat{X}_{T^1}(y^*)$ in (17). From $I(y) > 0$ for all $y > 0$, it follows that $-m(t; w) < \mathcal{X}(t, Y_t^*) < \infty$ for any $t \in [0, T^1]$.

Under $\mathbb{Q}^{\mathbb{P}^*}$, $(\hat{c}_t(y^*))_{t=0}^{T^1}$ and $(\hat{\mathfrak{B}}_t(y^*))_{t=0}^{T^1}$ are L^1 integrable, and $\mathbb{E}^{\mathbb{Q}^{\mathbb{P}^*}}[|\hat{X}_{T^1}(y^*)|^p] < \infty$ for any $p \geq 1$. Thus, we can use the classical BSDE argument in Lemma 2 of Yang and Koo (2018) to claim that under $\mathbb{Q}^{\mathbb{P}^*}$, there exists $(\hat{\pi}_t^M, \hat{\pi}_t^S)^\top$ satisfying the following dynamics with $X_0^{\mathcal{X}(0,y^*)} = \mathcal{X}(0, y^*) > -m(0; w)$:

$$\begin{aligned} dX_t^{\mathcal{X}(0,y^*)} &= \Big((r_I + \lambda) X_t^{\mathcal{X}(0,y^*)} - \hat{c}_t(y^*) - \lambda \hat{\mathfrak{B}}_t(y^*) + w\mathbf{1}_{\{t < \hat{\tau}_R(0,y^*)\}} \Big) dt \\ &\quad - \hat{\pi}_t^M X_t^{\mathcal{X}(0,y^*)} \sigma_P^* dW_{1,t}^{\mathbb{Q}^{\mathbb{P}^*}} + \hat{\pi}_t^S X_t^{\mathcal{X}(0,y^*)} \left((\rho^* \sigma_S^* - \sigma_P^*) dW_{1,t}^{\mathbb{Q}^{\mathbb{P}^*}} + \sigma_S^* \sqrt{1 - (\rho^*)^2} dW_{2,t}^{\mathbb{Q}^{\mathbb{P}^*}} \right), \end{aligned} \tag{73}$$

and $(\hat{\pi}_t^M, \hat{\pi}_t^S)^\top$ is L^2-integrable under $\mathbb{Q}^{\mathbb{P}^*}$ (equivalently, under $\mathbb{P}^*$).

From the relation, $I(y)y = u(I(y)) - \widetilde{u}(I(y))$ and Theorem 3.1, i.e., $J(0, y^*) = \mathcal{J}(0, y^*; \mathbb{P}^*, \hat{\tau}_R)$, we deduce

$$\begin{aligned} y^* \mathcal{X}(0, y^*) &= \mathbb{E}^{\mathbb{P}^*}\left[\int_0^{T^1} e^{-(\delta+\lambda)t} Y_t^* \Big(\hat{c}_t(y^*) + \lambda \hat{\mathfrak{B}}_t(y^*) - w\mathbf{1}_{\{t < \hat{\tau}_R(0,y^*)\}} \Big) dt + e^{-(\delta+\lambda)T_1} Y_{T^1}^* \hat{X}_{T^1}(y^*) \right] \\ &= \mathbb{E}^{\mathbb{P}^*}\Bigg[\int_0^{T^1} e^{-(\delta+\lambda)t} \Big(u(\hat{c}_t(y^*)) \mathbf{1}_{\{t < \hat{\tau}_R(0,y^*)\}} + u(k_R \hat{c}_t(y^*)) \mathbf{1}_{\{t \geq \hat{\tau}_R(0,y^*)\}} + \lambda u(k_B \hat{\mathfrak{B}}_t(y^*)) \Big) dt \\ &\quad + e^{-(\delta+\lambda)T^1} u\big(k_X \hat{X}_{T^1}(y^*)\big) \Bigg] - J(0, y^*) \\ &\equiv \mathcal{W}(0, y^*) - J(0, y^*). \end{aligned} \tag{74}$$

As $\hat{\tau}_R(0,y^*)$ in (37), $(\hat{c}_t(y^*))_{t=0}^{T^1}$ and $(\hat{\mathfrak{B}}_t(y^*))_{t=0}^{T^1}$ (hence, $\hat{p}_t(y^*)=\lambda\hat{\mathfrak{B}}_t(y^*)-\lambda\mathcal{X}(t,Y_t^*)$) in (17) are the optimal strategies under $\mathbb{P}^*$ with the dual value function $J(0,y^*)=\mathcal{J}(0,y^*;\mathbb{P}^*,\hat{\tau}_R)$, it follows from the classic duality (i.e., no ambiguity) in (Jeon et al., 2022, Theorem 3.1) that $\mathcal{W}(0,y^*)$ in (74) is the optimal value function under $\mathbb{P}^*$ and a given wealth $\mathcal{X}(0,y^*)>-m(0;w)$, i.e.,

$$J(0,y^*)+y^*\mathcal{X}(0,y^*)=\mathcal{W}(0,y^*)=\inf_{y>0}\left\{J(0,y)+y\mathcal{X}(0,y^*)\right\}. \tag{75}$$

The strict convexity of J (in Step 1) guarantees that $x=-\partial_y J(0,y^*)=\mathcal{X}(0,y^*)$.

Step 2. For a given endowment $x=\mathcal{X}(0,y^*)>-m(0;w)$, the strategies $(\hat{c}_t(y^*))_{t=0}^{T^1}$ given in (17), $(\hat{p}_t(y^*))_{t=0}^{T^1}$ given by $\hat{p}_t(y^*)=\lambda\hat{\mathfrak{B}}_t(y^*)-\lambda\mathcal{X}(t,Y_t^*)$, $(\hat{\pi}_t(y^*))_{t=0}^{T^1}$ given by

$$\begin{aligned}
\hat{\pi}_t^M(y^*)&=-\frac{1}{1-(\rho^*)^2}\left[\frac{\rho^*\sigma_P^*-\sigma_S^*}{(\sigma_P^*)^2\sigma_S^*}\zeta_1^{\Sigma^*,\mu^*}+\frac{\rho^*\sigma_S^*-\sigma_P^*}{(\sigma_S^*)^2\sigma_P^*}\zeta_2^{\Sigma^*,\mu^*}\right]Y_t^*\partial_y\mathcal{X}(t,Y_t^*),\\
\hat{\pi}_t^I(y^*)&=\mathcal{X}(t,Y_t^*)-\frac{1}{1-(\rho^*)^2}\left[\frac{1}{(\sigma_P^*)^2}\zeta_1^{\Sigma^*,\mu^*}-\frac{\rho^*}{\sigma_P^*\sigma_S^*}\zeta_2^{\Sigma^*,\mu^*}\right]Y_t^*\partial_y\mathcal{X}(t,Y_t^*),\\
\hat{\pi}_t^S(y^*)&=-\frac{1}{1-(\rho^*)^2}\left[-\frac{\rho^*}{\sigma_P^*\sigma_S^*}\zeta_1^{\Sigma^*,\mu^*}+\frac{1}{(\sigma_S^*)^2}\zeta_2^{\Sigma^*,\mu^*}\right]Y_t^*\partial_y\mathcal{X}(t,Y_t^*),
\end{aligned}$$

and $\hat{\tau}_R(0,y^*)$ given by $\hat{\tau}_R(0,y^*)=\inf\{t\geq 0\mid Y_t^*\leq\hat{z}_{\underline{\Upsilon}}(t),Y_0^*=y^*\}\wedge T$ (given in (37)) are *robust* admissible, i.e., $(\hat{c}(y^*),\hat{p}(y^*),\hat{\pi}(y^*),\hat{\tau}(0,y^*))\in\mathcal{A}_{0,T^1}(x,w)$.

Proof of Step 2. It follows from Theorem 3.1 that $\hat{\tau}_R(0,y^*)$ is adapted with respect to $\mathbb{F}$, which implies the robust admissibility of the stopping time.

By Assumption 1.(ii), i.e., $\exists\ C>0$ and $\kappa\geq 1$ such that $0<I(y)\leq C(1+y^{-\kappa})$ for all $y>0$, we can use the argument in the proof of Lemma A.1 to have $(I(Y_t^*))_{t=0}^{T^1}\in M_G^p(0,T^1)$ for any $p\geq 1$. Furthermore, using the argument that for any $\mathbb{P}\in\mathcal{P}$ with $(\Sigma,\mu)\in\Theta$, $\exists$ $\widetilde{\mathbb{P}}\in\mathcal{Q}^0\subset\mathcal{Q}$ with $M^{(\Sigma,\mu)}\in\Lambda^{\Theta}$ such that $\mathbb{P}(\{Y^*\in C\})=\widetilde{\mathbb{P}}(\{Y^*\in C\})$ for all $C\in\mathbb{F}$, we have

$$\sup_{\mathbb{P}\in\mathcal{P}}\mathbb{E}^{\mathbb{P}}\Big[\int_0^{T^1}\big(\hat{c}_t(y^*)^p+\hat{\mathfrak{B}}_t(y^*)^p\big)dt\Big]\leq\hat{\mathbb{E}}\Big[\int_0^{T^1}\big(\hat{c}_t(y^*)^p+\hat{\mathfrak{B}}_t(y^*)^p\big)dt\Big]<\infty,$$

which implies that $\hat{c}_t(y^*)$ and $\hat{\mathfrak{B}}_t(y^*)$ are L^1-integrable for all $\mathbb{P}\in\mathcal{P}$.

The strong Markov property implies that under $\mathbb{Q}^{\mathbb{P}^*}$, $X_t^x=X_t^{\mathcal{X}(0,y^*)}=\mathcal{X}(t,Y_t^*)$, which satisfies $\partial_t\mathcal{X}+\mathcal{L}_{\underline{\Upsilon}}^1\mathcal{X}+I(y)+\frac{\lambda}{k_B}I(\frac{y}{k_B})-w=0$ in $\{(t,y)\in\mathcal{D}_T|y>\hat{z}_{\underline{\Upsilon}}(t)\}$. Applying the classical Itô's lemma to $\mathcal{X}(t,Y_t^*)$ for $t<\hat{\tau}_R(0,y^*)\wedge\tau_D$ under $\mathbb{Q}^{\mathbb{P}^*}$ leads to

$$\begin{aligned}
d\mathcal{X}(t,Y_t^*)&=\Big(\partial_t\mathcal{X}(t,Y_t^*)+\frac{\underline{\Upsilon}^2}{2}(Y_t^*)^2\partial_{yy}\mathcal{X}(t,Y_t^*)+(\delta-r_I+\underline{\Upsilon}^2)Y_t^*\partial_y\mathcal{X}(t,Y_t^*)\Big)dt\\
&\quad-\Big((\Sigma^*)^{-1}\zeta^{\Sigma^*,\mu^*}\Big)^{\top}Y_t^*\partial_y\mathcal{X}(t,Y_t^*)dW_t^{\mathbb{Q}^{\mathbb{P}^*}}.
\end{aligned}\tag{76}$$

Comparing the equation (76) with (73), we have the explicit representation of $(\hat{\pi}_t^M,\hat{\pi}_t^S)^{\top}$ by

$$\begin{aligned}
\hat{\pi}_t^M(y^*)&=-\frac{1}{1-(\rho^*)^2}\left[\frac{\rho^*\sigma_P^*-\sigma_S^*}{(\sigma_P^*)^2\sigma_S^*}\zeta_1^{\Sigma^*,\mu^*}+\frac{\rho^*\sigma_S^*-\sigma_P^*}{(\sigma_S^*)^2\sigma_P^*}\zeta_2^{\Sigma^*,\mu^*}\right]Y_t^*\partial_y\mathcal{X}(t,Y_t^*),\\
\hat{\pi}_t^S(y^*)&=-\frac{1}{1-(\rho^*)^2}\left[-\frac{\rho^*}{\sigma_P^*\sigma_S^*}\zeta_1^{\Sigma^*,\mu^*}+\frac{1}{(\sigma_S^*)^2}\zeta_2^{\Sigma^*,\mu^*}\right]Y_t^*\partial_y\mathcal{X}(t,Y_t^*),
\end{aligned}\tag{77}$$

where from the strong Markov property and the regularity of $J=\widetilde{J}$ in Proposition 3.1.(d), it follows that $y\partial_y\mathcal{X}=-y\partial_{yy}J$ satisfies

$$\begin{cases}\partial_t(y\partial_y\mathcal{X})+\mathcal{L}_{\underline{\Upsilon}}^1(y\partial_y\mathcal{X})+yI'(y)\mathbf{1}_{\{y\geq\hat{z}_{\underline{\Upsilon}}(t)\}}+\frac{y}{(k_R)^2}I'(\frac{y}{k_R})\mathbf{1}_{\{y<\hat{z}_{\underline{\Upsilon}}(t)\}}+\frac{\lambda y}{(k_B)^2}I'(\frac{y}{k_B})=0, & (t,y)\in\mathcal{D}_T,\\ y\partial_y\mathcal{X}(T,y)=y\partial_y\underline{\mathcal{X}}(T,y), & y\in(0,\infty),\end{cases}$$

where for any $Y_T=y>0$,

$$y\partial_y\underline{\mathcal{X}}(T,y)=\mathbb{E}_T^{\mathbb{Q}^{\mathbb{P}^*}}\left[\int_T^{T^1}e^{-(r+\lambda)(s-T)}\left(\frac{Y_s}{(k_R)^2}I'\left(\frac{Y_s}{k_R}\right)+\frac{\lambda Y_s}{(k_B)^2}I'(\frac{Y_s}{k_B})\right)ds+e^{-(r+\lambda)(T^1-T)}\frac{Y_{T^1}}{(k_X)^2}I'(\frac{Y_{T^1}}{k_X})\right].$$

It follows from Assumption 1.(ii), comparison principle for PDEs (see Tso (1985)), and the convexity of J that there exist $C>0$ and $\kappa\geq 1$ satisfying the following estimate:

$$0<-Y_t^*\partial_y\mathcal{X}(t,Y_t^*)\leq C\mathbb{E}_t^{\mathbb{Q}^{\mathbb{P}^*}}\left[\int_t^{T^1}e^{-(r+\lambda)(s-t)}Y_s^*(1+(Y_s^*)^{-\kappa})ds+e^{-(r+\lambda)(T^1-t)}Y_{T^1}^*(1+(Y_{T^1}^*)^{-\kappa})\right],\tag{78}$$

which guarantees that there exist $C_1>0$ and $\kappa_1\geq 1$ such that

$$0 < -Y_t^* \partial_y \mathcal{X}(t, Y_t^*) \le C_1\big((Y_t^*)^{\kappa_1} + (Y_t^*)^{-\kappa_1}\big).$$

Combining this estimate with the fact that $(Y_t^*)_{t=0}^{T^1}$, $(1/Y_t^*)_{t=0}^{T^1} \in M_G^\beta(0, T^1) \cap S_G^\beta(0, T^1)$ for any $\beta \ge 1$ (refer to Proof of Lemma A.1), we deduce that $(Y_t^* \partial_y \mathcal{X}(t, Y_t^*))_{t=0}^{T^1} \in M_G^p(0, T^1)$ for any $p \ge 1$, which leads to

$$\sup_{\mathbb{P}\in\mathcal{P}} \mathbb{E}^{\mathbb{P}}\left[\int_0^{T^1} \left|Y_t^*\partial_y\mathcal{X}(t, Y_t^*)\right|^p dt\right] \le \hat{\mathbb{E}}\left[\int_0^{T^1} \left|Y_t^*\partial_y\mathcal{X}(t, Y_t^*)\right|^p dt\right] < \infty.$$

From this, we deduce that the candidate investment $(\hat{\pi}_t^M, \hat{\pi}_t^S)^\top$ in (77) is L^2 integrable for all $\mathbb{P} \in \mathcal{P}$. Using the relation $X_t^x = \mathcal{X}(t, Y_t^*) = \hat{\pi}_t^I(y^*) + \hat{\pi}_t^M(y^*) + \hat{\pi}_t^S(y^*)$, $\hat{\pi}_t^I(y^*)$ is given by

$$\hat{\pi}_t^I(y^*) = \mathcal{X}(t, Y_t^*) - \frac{1}{1-(\rho^*)^2}\left[\frac{1}{(\sigma_P^*)^2}\zeta_1^{\Sigma^*,\mu^*} - \frac{\rho^*}{\sigma_P^*\sigma_S^*}\zeta_2^{\Sigma^*,\mu^*}\right] Y_t^*\partial_y\mathcal{X}(t, Y_t^*).$$

Using a similar argument, we show that $\mathcal{X}(t, Y_t^*)$ in (72) is L^1 integrable for all $\mathbb{P} \in \mathcal{P}$, which leads to the L^1 integrability of $\hat{p}_t(y^*) = \lambda\hat{\mathfrak{B}}_t(y^*) - \lambda\mathcal{X}(t, Y_t^*)$ for all $\mathbb{P} \in \mathcal{P}$. Thus, we have shown that all the candidate strategies are robust admissible.

Step 3. The candidate strategies $(\hat{c}_t(y^*))_{t=0}^{T^1}$, $(\hat{p}_t(y^*))_{t=0}^{T^1}$, $(\hat{\pi}_t(y^*))_{t=0}^{T^1}$, and $\hat{\tau}_R(0, y^*)$ satisfy the conditions (66) and (67) in Lemma A.4. Thus the duality relationship is established.

Proof of Step 3. To proceed, we claim that given endowment $x > -m(0; w)$, the probability measure $\mathbb{P}^*$ realizes the worst-case scenario of the candidate strategies, i.e.,

$$U\big(0, x; \hat{c}(y^*), \hat{p}(y^*), \hat{\pi}, \hat{\tau}_R(0, y^*)\big) = \mathcal{W}(0, y^*), \tag{79}$$

with $\mathcal{W}(0, y^*)$ in (74) and $y^* > 0$ such that $x = \mathcal{X}(0, y^*) = -\partial_y J(0, y^*)$ (in Step 1).

For this, we show that the expectation value $\mathcal{W}$ in (74) is equivalent to the following G-expectation value: for $y^* > 0$ such that $x = -\partial_y J(0, y^*)$,

$$\mathcal{W}(0, y^*) = -\hat{\mathbb{E}}\left[-\int_0^{\hat{\tau}_R(0,y^*)} e^{-(\delta+\lambda)t}\Big(u(I(Y_t^*)) + \lambda u(I(\frac{Y_t^*}{k_B}))\Big)dt - e^{-(\delta+\lambda)\hat{\tau}_R(0,y^*)}\underline{\mathcal{W}}(\hat{\tau}_R(0, y^*), Y^*_{\hat{\tau}_R(0,y^*)})\right], \tag{80}$$

where $\underline{\mathcal{W}}$ is given by

$$\underline{\mathcal{W}}(t, y_t^*) = -\hat{\mathbb{E}}_t\left[-\int_t^{T^1} e^{-(\delta+\lambda)(s-t)}\Big(u(I(\frac{Y_s^*}{k_R})) + \lambda u(I(\frac{Y_s^*}{k_B}))\Big)ds - e^{-(\delta+\lambda)(T^1-t)}u\big(I(\frac{Y_{T^1}^*}{k_X})\big)\right], \tag{81}$$

with $(t, Y_t^*) \in \overline{\mathcal{D}}_{T^1}$. From Assumption 1.(ii), $u(I(y))$ satisfies the following growth condition: $\exists\ C > 0$ and $\kappa \ge 1$ such that $|u(I(y))| \le C(y^\kappa + y^{-\kappa})$. This allows us to utilize Lemma 2 of Park and Wong (2023) to show that $\underline{\mathcal{W}}$ is the unique viscosity solution of the following PDE:

$$\partial_t\underline{\mathcal{W}} - \widetilde{G}(-y^2\partial_{yy}\underline{\mathcal{W}}) + (\delta - r_I)y\underline{\mathcal{W}} - (\delta+\lambda)\underline{\mathcal{W}} + u(I(\frac{y}{k_R})) + \lambda u(I(\frac{y}{k_B})) = 0, \quad (t, y) \in \mathcal{D}_{T^1},$$

with $\underline{\mathcal{W}}(T^1, y) = u(I(\frac{y}{k_X}))$ in $(0, \infty)$. It follows from Assumption 1.(iii) and the strict concavity of $u(\cdot)$ that $u(I(y))$ is convex for all $y > 0$, i.e., $(u(I(y)))'' \ge 0$ for all $y > 0$. Combining this with the argument in the proof of Lemma 3.1, we can claim that $\underline{\mathcal{W}}$ is convex and thus satisfies the reduced PDE, i.e., $\partial_t\underline{\mathcal{W}} + \mathcal{L}_{\underline{\Upsilon}}\underline{\mathcal{W}} + u(I(\frac{y}{k_R})) + \lambda u(I(\frac{y}{k_B})) = 0$, $(t, y) \in \mathcal{D}_T$ $\underline{\mathcal{W}}(T^1, y) = u(I(\frac{y}{k_X}))$, $y > 0$, recalling $\mathcal{L}_{\underline{\Upsilon}} \equiv \frac{1}{2}\underline{\Upsilon}^2 y^2\partial_{yy} + (\delta - r_I)y\partial_y - (\delta+\lambda)$. Hence, the G-expectation in (81) can be replaced by the expectation under $\mathbb{P}^*$.

To show (80), if we first consider the case $y^* \le \hat{z}_{\underline{\Upsilon}}(t)$, it follows from $\hat{\tau}_R(0, y^*) = \inf\{t \ge 0 \mid Y_t^* \le \hat{z}_{\underline{\Upsilon}}(t),\ Y_0^* = y^*\} \wedge T = 0$ that $\mathcal{W}(0, y^*) = \underline{\mathcal{W}}(0, y^*)$. This implies that $\mathcal{W}$ in (74) is equivalent to (81) at time $t = 0$. For the other case (i.e., $y^* > \hat{z}_{\underline{\Upsilon}}(t)$), we remark that from Proposition 3.1.(d) (i.e., J is C^∞-smooth in $\{(t, y) \in \mathcal{D}_T | y \ge \hat{z}_{\underline{\Upsilon}}(t)\}$), $\mathcal{X} = -\partial_y J = -\partial_y\widetilde{J}$ is C^∞-smooth in the same region. Thus, $\mathcal{W} = y\mathcal{X} + J$ in (74) is C^∞-smooth in $\{(t, y) \in \mathcal{D}_T | y \ge \hat{z}_{\underline{\Upsilon}}(t)\}$ and satisfies

$$\begin{cases}\partial_t\mathcal{W} + \mathcal{L}_{\underline{\Upsilon}}\mathcal{W} + u(I(y))\mathbf{1}_{\{y>\hat{z}_{\underline{\Upsilon}}(t)\}} + u(I(\frac{y}{k_R}))\mathbf{1}_{\{y\le\hat{z}_{\underline{\Upsilon}}(t)\}} + \lambda u(I(\frac{y}{k_B})) = 0, & (t, y) \in \mathcal{D}_T,\\ \mathcal{W}(T, y) = \underline{\mathcal{W}}(T, y), & y \in (0, \infty).\end{cases} \tag{82}$$

Clearly, the growth properties on $I(\cdot)$ and $I'(\cdot)$ in Assumption 1, the comparison principle for PDEs (see Tso (1985)), and the regularity of $\mathcal{W}$ guarantee that $\mathcal{W}$ has the following estimates: $\exists C > 0$, $\kappa \ge 1$ such that

$$|\partial_y\mathcal{W}(t, y)| + |\partial_{yy}\mathcal{W}(t, y)| \le C(1 + y^{-\kappa}) \quad \text{a.e. in } \overline{\mathcal{D}}_T. \tag{83}$$

To claim that $\mathcal{W}$ is convex, we consider any $y_1^*, y_2^* > 0$ with $y_1^* \neq y_2^*$ and denote $y_3^* \equiv h y_1^* + (1-h) y_2^*$ with some $h \in (0,1)$. Then,

$$\begin{aligned}
&\mathcal{W}(0, y_3^*) \\
&= \mathbb{E}^{\mathbb{P}^*}\left[\int_0^{T^1} e^{-(\delta+\lambda)t}\left(u(I(Y_t^{*,3}))\mathbf{1}_{\{t<\hat{\tau}_R(0,Y_t^{*,3})\}} + u(I(\frac{Y_t^{*,3}}{k_R}))\mathbf{1}_{\{t\geq\hat{\tau}_R(0,y_3^*)\}} + \lambda u(I(\frac{Y_t^{*,3}}{k_B}))\right)dt + e^{-(\delta+\lambda)T^1}u\big(I(\frac{Y_{T^1}^{*,3}}{k_X})\big)\right] \\
&\leq h\mathbb{E}^{\mathbb{P}^*}\left[\int_0^{T^1} e^{-(\delta+\lambda)t}\left(u(I(Y_t^{*,1}))\mathbf{1}_{\{t<\hat{\tau}_R(0,y_3^*)\}} + u(I(\frac{Y_t^{*,1}}{k_R}))\mathbf{1}_{\{t\geq\hat{\tau}_R(0,y_3^*)\}} + \lambda u(I(\frac{Y_t^{*,1}}{k_B}))\right)dt + e^{-(\delta+\lambda)T^1}u\big(I(\frac{Y_{T^1}^{*,1}}{k_X})\big)\right] \\
&\quad + (1-h)\mathbb{E}^{\mathbb{P}^*}\left[\int_0^{T^1} e^{-(\delta+\lambda)t}\left(u(I(Y_t^{*,2}))\mathbf{1}_{\{t<\hat{\tau}_R(0,y_3^*)\}} + u(I(\frac{Y_t^{*,2}}{k_R}))\mathbf{1}_{\{t\geq\hat{\tau}_R(0,y_3^*)\}} + \lambda u(I(\frac{Y_t^{*,2}}{k_B}))\right)dt + e^{-(\delta+\lambda)T^1}u\big(I(\frac{Y_{T^1}^{*,2}}{k_X})\big)\right] \\
&\leq h\mathcal{W}(0, y_1^*) + (1-h)\mathcal{W}(0, y_2^*),
\end{aligned} \tag{84}$$

with $Y_t^{*,i} = y_i^* e^{(\delta - r - \frac{1}{2}\underline{\Upsilon}^2)t - ((\Sigma^*)^{-1}\zeta^{\Sigma^*,\mu^*})^\top W_t^{\mathbb{P}^*}}$ for $i = 1, 2$, and 3, where the first inequality is from the convexity of $u(I(\cdot))$ and the second inequality is from the optimality of $\hat{\tau}_R(0, y_i)$ under $\mathbb{P}^*$ for each given $x_i = -\partial_y J(0, y_i^*)$ with $i = 1, 2$, and 3.

Hence, applying the G-Itô's formula (Peng, 2010, Theorem 8.3.4) to $\mathcal{W}$ implies that for $y^* > \hat{z}_{\underline{\Upsilon}}(t)$,

$$\begin{aligned}
-\mathcal{W}(0, y^*) &= -e^{-(\delta+\lambda)\hat{\tau}_R(0,y^*)}\underline{\mathcal{W}}\big(\hat{\tau}_R(0, y^*), Y^*_{\hat{\tau}_R(0,y^*)}\big) - \int_0^{\hat{\tau}_R(0,y^*)} e^{-(\delta+\lambda)t}\Big(u(I(Y_t^*)) + \lambda u(I(\frac{Y_t^*}{k_B}))\Big)dt \\
&\quad - \int_0^{\hat{\tau}_R(0,y^*)} e^{-(\delta+\lambda)t} Y_t^* \partial_y \mathcal{W}(t, Y_t^*) d\widetilde{B}_t - (\mathcal{K}^{\mathcal{W}}_{\hat{\tau}_R(0,y^*)} - \mathcal{K}^{\mathcal{W}}_0),
\end{aligned} \tag{85}$$

with $\mathcal{K}_t^{\mathcal{W}} \equiv \int_0^t e^{-(\delta+\lambda)u}(-\frac{1}{2}(Y_u^*)^2\partial_{yy}\mathcal{W}(u, Y_u^*)d\langle\widetilde{B}\rangle_u - \widetilde{G}(-(Y_u^*)^2\partial_{yy}\mathcal{W}(u, Y_u^*))du)$ for $0 \leq t \leq \hat{\tau}_R(0, y^*)$, where we have used that $\mathcal{W}(\hat{\tau}_R(0, y^*), Y^*_{\hat{\tau}_R(0,y^*)}) = \underline{\mathcal{W}}(\hat{\tau}_R(0, y^*), Y^*_{\hat{\tau}_R(0,y^*)})$, that $\partial_t \mathcal{W}(t, Y_t^*) + \mathcal{L}_{\underline{\Upsilon}}\mathcal{W}(t, Y_t^*) + u(I(Y_t^*)) + \lambda u(I(\frac{Y_t^*}{k_B})) = 0$ for $0 < t < \hat{\tau}_R(0, y^*)$, and that from the convexity in (84), it follows that $\frac{1}{2}\underline{\Upsilon}^2 y^2 \partial_{yy}\mathcal{W} = -\widetilde{G}(-y^2\partial_{yy}\mathcal{W})$.

The estimates in (83) and the fact that $(Y_u^{t,y})_{u=t}^T, (1/Y_u^{t,y})_{u=t}^T \in M_G^\beta(0, T^1) \cap S_G^\beta(0, T^1)$ for any $\beta \geq 1$ (see Appendix A.2) guarantee that $\int_0^{\hat{\tau}_R(0,y^*)} e^{-(\delta+\lambda)t} Y_t^* \partial_y \mathcal{W}(t, Y_t^*) d\widetilde{B}_t$ is a G-martingale with $(Y_t^*\partial_y\mathcal{W}(t, Y_t^*))_{t=0}^{\hat{\tau}_R(0,y^*)} \in H_G^\alpha(0, T^1)$ and that $(\mathcal{K}_t^{\mathcal{W}})_{t=0}^{\hat{\tau}_R(0,y^*)}$ is a non-increasing G-martingale with $\mathcal{K}_0^{\mathcal{W}} = 0$ and $\mathcal{K}^{\mathcal{W}}_{\hat{\tau}_R(0,y^*)} \in L_G^\alpha(\Omega_{\hat{\tau}_R(0,y^*)})$ for any $\alpha \geq 1$. Hence, taking G-expectation to (85) implies the result in (80).

From the argument that for any $\mathbb{P} \in \mathcal{P}$ with $(\Sigma, \mu) \in \Theta$, $\exists\ \widetilde{\mathbb{P}} \in \mathcal{Q}^0 \subset \mathcal{Q}$ with $M^{(\Sigma,\mu)}$ such that $\mathbb{P}(\{Y^* \in C\}) = \widetilde{\mathbb{P}}(\{Y^* \in C\})$ for all $C \in \mathbb{F}$ and the G-expectation value $\mathcal{W}$ in (80), it is clear that $U\big(0, x; \hat{c}(y^*), \hat{p}(y^*), \hat{\pi}, \hat{\tau}_R(0, y^*)\big) \geq \mathcal{W}(0, y^*)$. Furthermore, $\mathcal{W}$ in (80) is equivalent to the expectation value under $\mathbb{P}^* \in \mathcal{P}$, as in (74). This implies $\mathcal{W}(0, y^*) \geq U\big(0, x; \hat{c}(y^*), \hat{p}(y^*), \hat{\pi}, \hat{\tau}_R(0, y^*)\big)$. Hence, we establish the condition in (79).

Combining (79) with (74) and (75), we deduce that for $x > -m(0; w)$,

$$\begin{aligned}
&J(0, y^*) + y^*x = U\big(0, x; \hat{c}(y^*), \hat{p}(y^*), \hat{\pi}, \hat{\tau}_R(0, y^*)\big) \\
&\leq \sup_{(c,p,\pi,\tau_R)\in\mathcal{A}_{0,T^1}(x,w)} U(0, x; c, p, \pi, \tau_R) \leq \inf_{y>0}\{J(0, y) + yx\} \leq J(0, y^*) + y^*x,
\end{aligned}$$

with $x = \mathcal{X}(0, y^*) = -\partial_y J(0, y^*)$. Thus, the candidate strategies satisfy the conditions (66) and (67) in Lemma A.4, which establishes the robust duality.

Appendix B. Function and process spaces, and stopping time in the G-expectation

For each $p \geq 1$, we introduce the notion of G-expectation spaces for some processes and functions. We refer to Peng (2010); Hu et al. (2014a,b) for detailed notions and fundamental results, including the conditional G-expectation and stochastic integral of the spaces.

Definition 2 *(G-expectation spaces).* We denote by $\Omega_t \equiv \{\omega_{\cdot\wedge t} \mid \omega \in \Omega\}$, then

- $L_G^p(\Omega_t)$, the completion of $L_{ip}(\Omega_t)$ with the norm $(\hat{\mathbb{E}}[|\cdot|^p])^{\frac{1}{p}}$, where $L_{ip}(\Omega_t) \equiv \{\varphi(B_{t_1}, \cdots, B_{t_n}) \mid n \geq 1,\ t_1, \cdots, t_n \in [0, t],\ \varphi \in C_{b\cdot Lip}(\mathbb{R}^{2\times n})\}$ and $C_{b\cdot Lip}$ denotes the set of bounded Lipschitz functions on $\mathbb{R}^{d\times n}$;
- $\mathbb{L}^1(\Omega_t) \equiv \{X \in L^0(\Omega_t) \mid \hat{\mathbb{E}}[|X|] < \infty\}$ where $L^0(\Omega_t)$ is the collection of $\mathcal{B}(\Omega_t)$-measurable processes X such that $X: \Omega_t \to [-\infty, \infty]$;
- $M_G^p(0, T^1)$, the completion of $M_G^0(0, T^1)$ with $\|\cdot\|_{M_G^p} = (\hat{\mathbb{E}}[\int_0^{T^1}|\cdot|^p ds])^{\frac{1}{p}}$, where $M_G^0(0, T^1)$ is the collection of processes in the following form: for a given partition $\{t_0, \cdots, t_N\} = \pi_{T^1}$ of $[0, T^1]$, $\eta_t(\omega) = \sum_{j=0}^{N-1}\xi_j(\omega)I_{[t_j,t_{j+1})}(t)$ where $\xi_j \in L_{ip}(\Omega_{t_j})$ $(j = 0, 1, \cdots, N-1)$;
- $H_G^p(0, T^1)$, the completion of $M_G^0(0, T^1)$ with $\|\cdot\|_{H_G^p} = \{\hat{\mathbb{E}}[(\int_0^{T^1}|\cdot|^2 ds)^{\frac{p}{2}}]\}^{\frac{1}{p}}$;
- $S_G^p(0, T^1)$, the completion of $S_G^0(0, T^1)$ with $\|\cdot\|_{S_G^p} = \{\hat{\mathbb{E}}[\sup_{t\in[0,T^1]}|\cdot|^p]\}^{\frac{1}{p}}$, where $S_G^0(0, T^1) \equiv \{h(t, B_{t_1\wedge t}, ..., B_{t_n\wedge t}) | n \geq 1,\ t_1, \cdots, t_n \in [0, T^1],\ h \in C_{b,Lip}(\mathbb{R}^{1+n\times d})\}$.

Based on Hu and Peng (2013); Li and Peng (2020), we introduce the notion of G-stopping times appropriate for the space $(\Omega, L^1_G(\Omega), \hat{\mathbb{E}})$.

Definition 3 *(G-stopping time).* A random time $\widetilde{\tau} : \Omega_{T^1} \to [0, T^1]$ is called a $*$-stopping time if $\mathbf{1}_{\{\widetilde{\tau} \geq t\}} \in L^{1*}_G(\Omega_t)$ for each $t \in [0, T^1]$, where $L^{1*}_G(\Omega_t) \equiv \{X \in \mathbb{L}^1(\Omega_t) \mid \exists \{X_n\}_{n\in\mathbb{N}} \subset L^1_G(\Omega_t) \text{ such that } X_n \downarrow X \ q.s.\}$, with $\mathbb{L}^1(\Omega_t)$ in Definition 2. We denote by $\widetilde{\mathcal{T}}_{t,T}$ the collection of all random times $\widetilde{\tau}$ (G-stopping times) such that $\widetilde{\tau}$ takes values in $[t, T]$ and there exists a sequence of $*$-stopping times $\{\widetilde{\tau}_n\}_{n\in\mathbb{N}}$ such that $\widetilde{\tau}_n$ converges to $\widetilde{\tau}$, quasi-surely.

We introduce the definition of the solution to G-BSDE as follows:

Definition 4 *(Hu et al. (2014a)).* A triplet $(\underline{\mathcal{G}}^{t,y}_u, \underline{\mathcal{M}}^{t,y}_u, \underline{\mathcal{K}}^{t,y}_u)^{T^1}_{u=t}$ is called a solution to (49) if for some $1 < \alpha < \beta$ the following property holds:

(a) $(\underline{\mathcal{G}}^{t,y}_u, \underline{\mathcal{M}}^{t,y}_u, \underline{\mathcal{K}}^{t,y}_u)^{T^1}_{u=t} \in \mathfrak{S}^\alpha_G(0, T^1)$, i.e., $(\underline{\mathcal{G}}^{t,y}_u)^{T^1}_{u=t} \in S^\alpha_G(0, T^1)$, $(\underline{\mathcal{M}}^{t,y}_u)^{T^1}_{u=t} \in H^\alpha_G(0, T^1)$, and $(\underline{\mathcal{K}}^{t,y}_u)^{T^1}_{u=t}$ is a non-increasing G-martingale with $\underline{\mathcal{K}}^{t,y}_t = 0$ and $\underline{\mathcal{K}}^{t,y}_{T^1} \in L^\alpha_G(\Omega_{T^1})$;

(b) $\underline{\mathcal{G}}^{t,y}_u = -\widetilde{u}(\frac{Y^{t,y}_{T^1}}{k_X}) - \int_u^{T^1} \left(\widetilde{u}(\frac{Y^{t,y}_s}{k_R}) + \lambda\widetilde{u}(\frac{Y^{t,y}_s}{k_B}) + (\beta+\lambda)\underline{\mathcal{G}}^{t,y}_s\right)ds - \int_u^{T^1} \underline{\mathcal{M}}^{t,y}_s d\widetilde{B}_s - (\underline{\mathcal{K}}^{t,y}_{T^1} - \underline{\mathcal{K}}^{t,y}_u)$, $u \in [t, T^1]$.

We introduce the definition of the solution to the reflected G-BSDE with the upper obstacle.

Definition 5 *(Li and Peng (2020)).* A triple of processes $(\mathcal{G}^{t,y}_u, \mathcal{M}^{t,y}_u, \mathcal{K}^{t,y}_u)^T_{u=t}$ is called a solution to (53) if for some $2 \leq \alpha < \beta$ the following properties are satisfied:

(a) $(\mathcal{G}^{t,y}_u, \mathcal{M}^{t,y}_u, \mathcal{K}^{t,y}_u)^T_{u=t} \in \mathbb{S}^\alpha_G(0, T^1)$, i.e., $(\mathcal{G}^{t,y}_u)^T_{u=t} \in S^\alpha_G(0, T^1)$, $(\mathcal{M}^{t,y}_u)^T_{u=t} \in H^\alpha_G(0, T^1)$, and $(\mathcal{K}^{t,y}_u)^T_{u=t} \in S^\alpha_G(0, T^1)$ is a continuous process with finite variation satisfying $\mathcal{K}^{t,y}_t = 0$, and $(-\mathcal{K}^{t,y}_u)^T_{u=t}$ is a G-submartingale;

(b) $\mathcal{G}^{t,y}_u \leq \underline{\mathcal{G}}^{t,y}_u$ for all $u \in [t, T]$;

(c) $\mathcal{G}^{t,y}_u = \underline{\mathcal{G}}^{t,y}_T - \int_u^T \left(\widetilde{u}(Y^{t,y}_s) + \lambda\widetilde{u}(\frac{Y^{t,y}_s}{k_B}) + wY^{t,y}_s + (\delta+\lambda)\mathcal{G}^{t,y}_s\right)ds - \int_u^T \mathcal{M}^{t,y}_s d\widetilde{B}_s + (\mathcal{K}^{t,y}_T - \mathcal{K}^{t,y}_u)$ for $u \in [t, T]$;

(d) $(-\int_t^u (\underline{\mathcal{G}}^{t,y}_s - \mathcal{G}^{t,y}_s) d\mathcal{K}^{t,y}_s)^T_{u=t}$ is a non-increasing G-martingale.

The solution is called a maximal solution if $(\mathcal{G}'_u, \mathcal{M}'_u, \mathcal{K}'_u)^T_{u=t}$ is another solution, then $\mathcal{G}^{t,y}_u \geq \mathcal{G}'_u$ for all $u \in [t, T]$.

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