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Risk aggregation with FGM copulas

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We offer a new perspective on risk aggregation with FGM copulas. Along the way, we discover new results and revisit existing ones, providing simpler formulas than one can find in the existing literature. This paper builds on two novel representations of FGM copulas based on symmetric multivariate Bernoulli distributions and order statistics. First, we detail families of multivariate distributions with closed-form solutions for the cumulative distribution function or moments of the aggregate random variables. We provide methods to compute the cumulative distribution function of aggregate rvs when the marginals are discrete, then order aggregate random variables under the convex order. Finally, we discuss risk-sharing and capital allocation, providing numerical examples for each.

1. Introduction

Insurance companies deal with a large number of heterogeneous and possibly dependent losses. For enterprise risk management purposes, it is important to understand the risks in one's portfolio at the individual level, but also at the company-wide level. For this reason, one is interested in the aggregate risk of the portfolio.

In this paper, we aim to provide a comprehensive treatment of risk aggregation of positive random variables (rvs) when the dependence structure is a Farlie-Gumbel-Morgenstern (FGM) copula. The family of FGM copulas has a long history in copula theory (see, for instance, Johnson and Kott (1975), Cambanis (1977), Kotz and Drouet (2001, Chapter 5), Kotz et al. (2004, Section 44.10), Nelsen (2007), Durante and Sempi (2015)). The family of FGM copulas is a popular copula since its simple shape enables analytic results, see, for instance, Genest and Favre (2007). One finds applications of the FGM family of copulas in actuarial science (for instance, Cossette et al. (2008); Bargès et al. (2009, 2011); Cossette et al. (2012, 2013); Woo and Cheung (2013); Chadjiconstantinidis and Vrontos (2014)). A FGM copula admits weak dependence, both positive and negative. For instance, the range of bivariate Spearman's rho for FGM copulas is $[-1/3, 1/3]$.

Within a large portfolio of diversified insurance risks, one does not expect to observe high dependence across every risk. Indeed, essential conditions for insurability include having a large number of similar exposure units and limited exposure to catastrophically large losses. An insurance company would actively avoid insuring two risks that exhibit significant positive dependence. For this reason, many insurance companies limit their exposures in regions where a single event could cause multiple claims. However, insurers do not refuse a risk simply because they have another risk that is positively correlated with a potential customer; weak positive dependence may be acceptable within the underwriting guidelines of an insurance company. A FGM copula, therefore, seems appropriate for a large portfolio of insurance risks because one expects underwriters to limit positive dependence, and a FGM copula lets one select flexible dependence structures between risks within the portfolio, under parameter constraints that the underlying FGM copula exists.

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Risk aggregation under FGM dependence has already been studied in the actuarial science literature (see, for instance, Bargès et al. (2009); Cossette et al. (2013, 2015); Navarro and Sarabia (2020)), but we consider the problem from a new perspective. In the past, FGM copulas did not have a genuine probabilistic interpretation (see, for instance, Durante et al. (2012)). This paper builds on two alternate representations of the FGM family of copulas that provide probabilistic interpretations. The first representation provides a method to construct FGM copulas, interpret the copula parameters, and enables the stochastic comparison of different FGM copulas. The second representation, for a given set of copula parameters, leads to new results on risk aggregation and rediscovers some cited in the literature above.

- The first representation is based on a one-to-one correspondence between the class of FGM copulas and symmetric multivariate Bernoulli random vectors, explored in Blier-Wong et al. (2022). By constructing a d-variate FGM copula from a d-variate symmetric multivariate Bernoulli random vector $\boldsymbol{I}$, we will see in Section 2 that the dependence structure of $\boldsymbol{I}$ governs the dependence structure of the FGM copula. One significant advantage of this representation is that the dependence structure of Bernoulli rvs is easier to interpret than a set of $2^d - d - 1$ central mixed moments between k-tuples, for $2 \leq k \leq d$, which is what one has with the natural formulation of the FGM copula. Another advantage of this representation is that it enables one to answer such questions as (i) what is the most positive and negative dependence structures attainable under FGM dependence; (ii) what is the effect of increasing a certain dependence parameter on the resulting aggregate distribution; (iii) how are two aggregate distributions with different FGM copulas ordered under the convex order. It turns out that trying to answer these questions using the natural representation of the FGM copula is tedious but becomes simple when using the stochastic representation.
- The second representation is based on order statistics. In Baker (2008), the author constructs multivariate distributions based on order statistics and finds that the simplest case consists of mixing the order statistics from two independent and identically distributed (iid) rvs corresponding to a FGM distribution. See also Section 8.3 of Bladt and Nielsen (2017) for the construction of multivariate models based on order statistics. If the order statistics of the marginal distributions have convenient forms, the aggregate distribution of the risks under FGM dependence may also have convenient forms. Instead of approaching the problem or risk aggregation from a purely mathematical point of view, we approach it using a probabilistic argument that simplifies the formulas and provides a more straightforward interpretation of the resulting expressions.

While FGM copulas only admit a moderate strength of dependence, we show that the dependence structure still has a significant impact on the distribution of the aggregate rv. Another advantage is that FGM copulas admit a wide variety of shapes; that is, a d-dimensional copula has $2^d - d - 1$ copula parameters, and each parameter controls the moments between k-tuples of the random vector, for $k \in \{2, \dots, d\}$. Hence, we may study the effect of mild negative and positive dependence on the behavior of the aggregate rv within the family of FGM copulas. Also, FGM copulas are the most simple case of Bernstein copulas, introduced in Sancetta and Satchell (2004). Bernstein copulas of dimension d are interesting from a practical point of view since they are dense on the hypercube $[0, 1]^d$. One may use Bernstein copulas to approximate other types of copulas. The results from this paper, covering FGM copulas, consist of important groundwork to study risk aggregation under a dependence structure induced by Bernstein copulas. See, e.g., Marri and Moutanabbir (2021) for related research for risk aggregation with mixed Bernstein copulas.

The remainder of this paper is structured as follows. In Section 2, we provide the preliminary notions of copulas and order statistics required for the main results of the paper. Section 3 outlines the general method to identify the Laplace-Stieltjes transform or the mth moments for the aggregate rv. In Section 4, we develop closed-form expressions for the cdf and mth moments for some continuous rvs, detail a method to compute the probability mass function (pmf) of the aggregate rv when each marginal is a discrete rv. We follow by proposing a method to approximate the cumulative distribution function (cdf) of continuous rvs using discretization methods and construct bounds for the risk measures of the aggregate rv by their discrete counterparts. In Section 5, we identify the lower and upper bounds of the aggregate rv under the convex order for the special case of exchangeable FGM copulas. Section 6 discusses TVaR-based risk allocation when the marginals are mixed Erlang rvs. In Section 7, we discuss the results and present some openings for further research.

2. Preliminaries

We begin by introducing general notation. Let $\boldsymbol{x}$ denote a vector $(x_1, \dots, x_d) \in \mathbb{R}^d$. All expressions such as $\boldsymbol{x} + \boldsymbol{y}$, $\boldsymbol{x} \times \boldsymbol{y}$ and $\boldsymbol{x} \leq \boldsymbol{y}$ represent component-wise operations. Let $\boldsymbol{X}$ represent a random vector on $\mathbb{R}_+^d$ with joint cdf $F_{\boldsymbol{X}}$ with $F_{\boldsymbol{X}}(\boldsymbol{x}) = \Pr(X_1 \leq x_1, \dots, X_d \leq x_d)$. Define also the Laplace-Stieltjes transform (LST) as $\mathcal{L}_{\boldsymbol{X}}(\boldsymbol{t}) = E[\exp\{-(t_1 X_1 + \cdots + t_d X_d)\}]$, for $\boldsymbol{t} \in \mathbb{R}_+^d$. Also, for a univariate cdf F_X, we define the generalized inverse $F_X^{-1}(u) = \inf\{x \in \mathbb{R}, F_X(x) \geq u\}$. Let the cdf of a symmetric Bernoulli distribution be denoted by $F_I(x) = 0.5 \times 1_{[0,\infty)}(x) + 0.5 \times 1_{[1,\infty)}(x), x \geq 0$, where $1_A(x) = 1$, if $x \in A$ and 0, otherwise. We denote $\mathcal{B}_d$ as the Fréchet class with d univariate marginals $F_1, \dots, F_d$ with $F_1 = \cdots = F_d = F_I$. Finally, we let $\mathbb{N}_0$ be the set of non-negative integers and $\mathbb{N}_1 = \mathbb{N}_0 \setminus \{0\}$ be the set of strictly positive integers.

2.1. Order statistics

This paper leverages the order statistic representation of the FGM copula, presented in Baker (2008) and revisited in Section 8.3 of Bladt and Nielsen (2017) and more recently in Blier-Wong et al. (2022). The current section provides preliminary results for order statistics. The interested reader can refer to the standard references on order statistics, for example, Casella and Berger (2002, Section 5.4), David and Nagaraja (2003) and Arnold et al. (2008), for more details.

Let (X_1, X_2) be a vector of two continuous iid rvs with marginal cdf F_X and probability density function (pdf) f_X. Define the vector $(X_{[1]}, X_{[2]})$ as the vector of order statistics of (X_1, X_2), that is, $X_{[1]} = \min(X_1, X_2)$ and $X_{[2]} = \max(X_1, X_2)$. The cdfs and pdfs of the order statistics are

$$F_{X_{[1]}}(x) = 1 - \overline{F}_X(x)^2; \quad f_{X_{[1]}}(x) = 2\overline{F}_X(x) f_X(x); \tag{1}$$

$$F_{X_{[2]}}(x) = F_X(x)^2; \quad f_{X_{[2]}}(x) = 2F_X(x) f_X(x), \tag{2}$$

where x takes values in the same support as those of F_X or f_X.

The following example presents the well-known order statistics of exponential rvs, first derived in Rényi (1953).

Example 1. Let (X_1, X_2) be two independent and exponentially distributed rvs with mean $1/\beta$. Let $Z_i, i \in \{1, 2\}$ be independent exponentially distributed rvs with mean 1. The associated order statistics $X_{[1]}$ and $X_{[2]}$ admit the representation

$$X_{[1]} = \min(X_1, X_2) \overset{\mathcal{D}}{=} \frac{Z_1}{2\beta};$$
$$X_{[2]} = \max(X_1, X_2) \overset{\mathcal{D}}{=} \frac{Z_1}{2\beta} + \frac{Z_2}{\beta},$$

where $\overset{\mathcal{D}}{=}$ means equality in distribution. It follows that $X_{[1]} \sim Exp(2\beta)$ and that $X_{[2]}$ follows a generalized Erlang distribution with parameters β and 2β.

Another useful representation of order statistics, due to Scheffe and Tukey (1945), is

$$(X_{[1]}, X_{[2]}) \overset{\mathcal{D}}{=} \left(F_X^{-1}(U_{[1]}), F_X^{-1}(U_{[2]})\right). \tag{3}$$

One can also write the pdf of two order statistics as

$$f_{X_{[j]}}(x) = 2F_X(x)^{j-1}\overline{F}_X(x)^{2-j} f_X(x) = (-1)^j 2 f_X(x) F_X(x) + (2-j) f_X(x), \tag{4}$$

for $j \in \{1, 2\}$. From the second equality in (4), we have

$$f_{X_{[j]}}(x) = (-1)^j f_{X_{[2]}}(x) + 2(2-j) f_X(x), \quad j \in \{1, 2\}. \tag{5}$$

Define $\mu^{(m)}_{X_{[j]}}$ as the mth moment of the jth order statistic of X. From (5), we have

$$\mu^{(m)}_{X_{[j]}} = (-1)^j \mu^{(m)}_{X_{[2]}} + 2(2-j)E[X^m], \quad m \in \mathbb{N}_1, \quad j \in \{1, 2\}. \tag{6}$$

Replacing $j = 1$ in (6), the relationship between moments of order statistics is

$$E[X^m] = \frac{1}{2}\left(\mu^{(m)}_{X_{[1]}} + \mu^{(m)}_{X_{[2]}}\right), \quad m \in \mathbb{N}_1. \tag{7}$$

In this paper, we will construct dependent random vectors by defining their joint cdfs with copulas. Standard references on copula theory include, for example, Kotz and Drouet (2001), Trivedi and Zimmer (2006), Nelsen (2007), Mai and Scherer (2014), Joe (2014), or Durante and Sempi (2015). Copulas are multivariate cdfs whose marginals are uniformly distributed on the interval $[0, 1]$, and the copula studied in this paper is constructed by pairs of order statistics. It is, therefore, useful to recall that for the special case where (U_1, U_2) is a pair of iid uniform rvs, then the order statistics $U_{[1]}$ and $U_{[2]}$ satisfy $U_{[j]} \sim Beta(j, 3-j)$, for $j \in \{1, 2\}$.

2.2. FGM copulas

In this paper, we focus on FGM copulas, whose expression is given by

$$C(\boldsymbol{u}) = \prod_{k=1}^{d} u_k \left(1 + \sum_{n=2}^{d} \sum_{1 \le j_1 < \cdots < j_n \le d} \theta_{j_1 \ldots j_n} \overline{u}_{j_1}\overline{u}_{j_2} \ldots \overline{u}_{j_n}\right), \quad \boldsymbol{u} \in [0, 1]^d, \tag{8}$$

where $\overline{u}_j = 1 - u_j$, for $j \in \{1, \ldots, d\}$. The constraints on the parameters for the copula in (8), as derived by Cambanis (1977), are

$$\mathcal{T}_d = \left\{(\theta_{12}, \ldots, \theta_{1\ldots d}) \in \mathbb{R}^{2^d-d-1} : 1 + \sum_{n=2}^{d} \sum_{1 \le j_1 < \cdots < j_n \le d} \theta_{j_1 \ldots j_n} \varepsilon_{j_1}\varepsilon_{j_2} \ldots \varepsilon_{j_n} \ge 0\right\}, \tag{9}$$

for $\{\varepsilon_{j_1}, \varepsilon_{j_2}, \ldots, \varepsilon_{j_n}\} \in \{-1, 1\}^n$ and $n \in \{2, \ldots, d\}$. When $d = 2$, (8) becomes the well-known expression of the bivariate FGM copula with one parameter and is given by

$$C(u_1, u_2) = u_1 u_2 + \theta_{12} u_1 u_2 \overline{u}_1 \overline{u}_2, \quad (u_1, u_2) \in [0, 1]^2, \tag{10}$$

with $\theta_{12} \in \mathcal{T}_2 = [-1, 1]$. The association measures such as Kendall's tau and Spearman's rho for the bivariate FGM copula are respectively given by $\tau = 2\theta_{12}/9$ and $\rho = \theta_{12}/3$. We use the notation $C \in \mathcal{C}_d^{FGM}$ to denote that C is a d-variate FGM copula.

The following lemma combines the stochastic representation based on multivariate symmetric Bernoulli random vectors of FGM copulas proposed in Blier-Wong et al. (2022) along with the stochastic representation based on order statistics that is discussed in Baker (2008) and Section 8.3.2 of Bladt and Nielsen (2017).

Lemma 1 *(Remark 3.5 of Blier-Wong et al. (2022)). The copula in (8) has the equivalent representation*

$$C(\boldsymbol{u}) = E_{\boldsymbol{I}}\left[\prod_{k=1}^{d} F_{U_{[I_k+1]}}(u_k)\right], \tag{11}$$

for $\boldsymbol{u} \in [0,1]^d$, where $f_{\boldsymbol{I}}$ is the pmf of $\boldsymbol{I}$, a symmetric multivariate Bernoulli random vector, that is, $F_{\boldsymbol{I}} \in \mathcal{B}_d$. The dependence parameters are proportional to central mixed moments as follows:

$$\theta_{j_1\ldots j_n} = (-2)^n E_{\boldsymbol{I}}\left\{\prod_{\ell=1}^{n}\left(I_{j_\ell} - \frac{1}{2}\right)\right\}, \tag{12}$$

for $n \in \{2,\ldots,d\}$ and $1 \le j_1 < \cdots < j_n \le d$.

Lemma 2. *Let $\boldsymbol{U}_{[j]} = (U_{1,[j]},\ldots,U_{d,[j]})$ be a d-variate vector of iid rvs satisfying $U_{k,[j]} \sim Beta(j, 3-j)$ for $k \in \{1,\ldots,d\}$ and $j \in \{1,2\}$. Define the random vector*

$$\boldsymbol{U} \overset{\mathcal{D}}{=} (\boldsymbol{1} - \boldsymbol{I})\boldsymbol{U}_{[1]} + \boldsymbol{I}\boldsymbol{U}_{[2]}, \tag{13}$$

where $\boldsymbol{1}$ is a d-variate vector of ones. Then, we have that $F_{\boldsymbol{U}}(\boldsymbol{u}) \in \mathcal{C}_d^{FGM}$. More generally, fix some marginal cdfs $F_{X_1},\ldots,F_{X_d}$ and let $\boldsymbol{X}_{[j]} = (X_{1,[j]},\ldots,X_{d,[j]})$ be vectors of independent rvs with respective marginal cdf $F_{X_{k,[j]}}$, as defined in (1) and (2), for $k \in \{1,\ldots,d\}$ and $j \in \{1,2\}$. Define the random vector

$$\boldsymbol{X} \overset{\mathcal{D}}{=} (\boldsymbol{1} - \boldsymbol{I})\boldsymbol{X}_{[1]} + \boldsymbol{I}\boldsymbol{X}_{[2]}. \tag{14}$$

Then, we have $F_{\boldsymbol{X}}(\boldsymbol{x}) = C(F_{X_1}(x_1),\ldots,F_{X_d}(x_d))$, where $C \in \mathcal{C}_d^{FGM}$.

Proof. We prove the statement about the random vector $\boldsymbol{X}$. We have

$$\Pr(\boldsymbol{X} \le \boldsymbol{x}) = E_{\boldsymbol{I}}\left[\Pr\left((1-I_1)X_{1,[1]} + I_1 X_{1,[2]} \le x_1,\ldots,(1-I_d)X_{d,[1]} + I_d X_{d,[2]} \le x_d\right)\right],$$

which becomes

$$\begin{aligned}\Pr(\boldsymbol{X} \le \boldsymbol{x}) &= E_{\boldsymbol{I}}\left[\prod_{k=1}^{d}\Pr\left((1-I_k)X_{k,[1]} + I_k X_{k,[2]} \le x_k\right)\right]\\ &= E_{\boldsymbol{I}}\left[\prod_{k=1}^{d}\Pr\left(X_{k,[I_k+1]} \le x_k\right)\right] = E_{\boldsymbol{I}}\left[\prod_{k=1}^{d}F_{X_{k,[I_k+1]}}(x_k)\right]\\ &= E_{\boldsymbol{I}}\left[\prod_{k=1}^{d}F_{U_{[I_k+1]}}(F_{X_k}(x_k))\right].\end{aligned}$$

The proof for the random vector $\boldsymbol{U}$ holds by replacing $F_{X_k}(x) = x$, for $k = 1,\ldots,d$. $\square$

The representation of Lemmas 1 and 2 are convenient to develop the results of the current paper and will help us understand the dependence structure behind $C \in \mathcal{C}_d^{FGM}$. Lemma 1 states that, conditional on $\boldsymbol{I}$, the copula C is the product of independent cdfs of $U_{[1]}$ or $U_{[2]}$. Lemma 2 constructs random vectors $\boldsymbol{U}$ and $\boldsymbol{X}$ which have the same joint cdfs as the ones we are investigating in this paper. The authors of Blier-Wong et al. (2022) call (8) the natural representation of the FGM copula since the parameters in (12) are central mixed moments. They also refer to (11) as the stochastic representation of the FGM copula since it relies on the stochastic nature based on $\boldsymbol{I}$ and order statistics.

3. Risk aggregation with FGM copulas: the general method

For this section, we consider a vector of non-negative rvs $\boldsymbol{X} = (X_1,\ldots,X_d)$ with cdf

$$F_{\boldsymbol{X}}(\boldsymbol{x}) = C(F_{X_1}(x_1),\ldots,F_{X_d}(x_d)), \quad \boldsymbol{x} \in \mathbb{R}_+^d, \tag{15}$$

where $C \in \mathcal{C}_d^{FGM}$. Using the representation of the FGM copula in (3) and (11), the joint cdf of $\boldsymbol{X}$ becomes

$$F_{\boldsymbol{X}}(\boldsymbol{x}) = E_{\boldsymbol{I}}\left[\prod_{k=1}^{d}F_{U_{[I_k+1]}}(F_{X_k}(x_k))\right] = E_{\boldsymbol{I}}\left[\prod_{k=1}^{d}F_{X_{k,[I_k+1]}}(x_k)\right], \quad \boldsymbol{x} \in \mathbb{R}^d. \tag{16}$$

It follows that the joint LST of $\boldsymbol{X}$ is

$$\mathcal{L}_{\boldsymbol{X}}(\boldsymbol{t}) = \int_{\mathbb{R}_+^d}\prod_{k=1}^{d}e^{-t_k x_k}\,\mathrm{d}F_{\boldsymbol{X}}(\boldsymbol{x}) = E_{\boldsymbol{I}}\left[\prod_{k=1}^{d}\int_{\mathbb{R}_+}e^{-t_k x_k}\,\mathrm{d}F_{X_{k,[I_k+1]}}(x_k)\right] = E_{\boldsymbol{I}}\left[\prod_{k=1}^{d}\mathcal{L}_{X_{k,[I_k+1]}}(t_k)\right], \tag{17}$$

for $\boldsymbol{t} \in \mathbb{R}_+^d$. Let S be the rv representing the sum of the components of the random vector $\boldsymbol{X}$, that is, $S = X_1 + \cdots + X_d$. From Lemma 2, we also have

$$S \stackrel{\mathcal{D}}{=} \sum_{k=1}^{d} \left\{(1 - I_k)X_{k,[1]} + I_k X_{k,[2]}\right\}. \tag{18}$$

We are now in a position to state the following theorem.

Theorem 1. *The LST of S is*

$$\mathcal{L}_S(t) = E_{\boldsymbol{I}}\left[\prod_{k=1}^{d} \mathcal{L}_{X_{k,[I_k+1]}}(t)\right], \quad t \geq 0. \tag{19}$$

Proof. The result follows directly from the definition of $\mathcal{L}_S(t)$ and (17). □

Theorem 1 is the main tool to identify the distribution of the aggregate risk rv S. In some cases, we obtain exact results to compute the cdf of S. In others, we are only able to obtain the moments of S.

Theorem 2. *For $m \in \mathbb{N}_1$, and assuming that $E[X_k^m]$ exists for $k = 1, \ldots, d$, we have*

$$E\left[S^m\right] = \sum_{j_1+\cdots+j_d=m} \frac{m!}{j_1! \cdots j_d!} \left\{\prod_{k=1}^{d} E[X_k^{j_k}]\right\} E_{\boldsymbol{I}}\left[\prod_{k=1}^{d}\left\{1 + (-1)^{I_k}\left(1 - \frac{\mu^{(j_k)}_{X_{k,[2]}}}{E[X_k^{j_k}]}\right)\right\}\right]. \tag{20}$$

Proof. Applying the multinomial theorem, we have

$$E\left[S^m\right] = E\left[\left(\sum_{k=1}^{d} X_k\right)^m\right] = E\left[\sum_{j_1+\cdots+j_d=m} \binom{m!}{j_1! \ldots j_d!} X_1^{j_1} \ldots X_d^{j_d}\right].$$

We condition on $\boldsymbol{I}$ to obtain

$$\begin{aligned} E[S^m] &= E_{\boldsymbol{I}}\left[E\left[\sum_{j_1+\cdots+j_d=m} \frac{m!}{j_1! \cdots j_d!} X_{1,[I_1+1]}^{j_1} \cdots X_{d,[I_d+1]}^{j_d} \middle| \boldsymbol{I}\right]\right] \\ &= \sum_{j_1+\cdots+j_d=m} \frac{m!}{j_1! \cdots j_d!} E_{\boldsymbol{I}}\left[\mu^{(j_1)}_{X_{1,[I_1+1]}} \cdots \mu^{(j_d)}_{X_{d,[I_d+1]}}\right]. \end{aligned}$$

Inserting the last equality into (6), the mth moment of S becomes

$$E\left[S^m\right] = \sum_{j_1+\cdots+j_d=m} \frac{m!}{j_1! \cdots j_d!} E_{\boldsymbol{I}}\left[\prod_{k=1}^{d}\left\{(-1)^{1+I_k}\mu^{(j_k)}_{X_{k,[2]}} + 2(1 - I_k)E[X_k^{j_k}]\right\}\right].$$

Factoring out the expected values of the original marginals yields the desired result. □

Using the relation in (7), one finds

$$E\left[S^m\right] = \sum_{j_1+\cdots+j_d=m} \frac{m!}{j_1! \cdots j_d!} \left\{\prod_{k=1}^{d} E[X_k^{j_k}]\right\} E_{\boldsymbol{I}}\left[\prod_{k=1}^{d}\left\{1 + (-1)^{I_k}\left(\frac{\mu^{(j_k)}_{X_{k,[1]}}}{E[X_k^{j_k}]} - 1\right)\right\}\right]. \tag{21}$$

One obtains exact results for the mth moment of S if one has exact results for the jth moment of each marginal and either the minimum or maximum of two marginals, with $j \in \{1, \ldots, m\}$. Alternatively, we can write the moments in terms of the natural representation of the FGM copula.

Corollary 1. *For $m \in \mathbb{N}_1$, we have*

$$E\left[S^m\right] = \sum_{j_1+\cdots+j_d=m} \frac{m!}{j_1! \cdots j_d!} \left\{\prod_{k=1}^{d} E[X_k^{j_k}]\right\} A_l(j_1, \ldots, j_d), \tag{22}$$

for either $l \in \{1, 2\}$, with

$$A_1(j_1,\ldots,j_d)=1+\sum_{n=2}^{d}\sum_{1\le\ell_1<\cdots<\ell_n\le d}\theta_{\ell_1\ldots\ell_n}\left(\frac{\mu^{(j_{\ell_1})}_{X_{\ell_1,[1]}}}{E\left[X_{\ell_1}^{j_{\ell_1}}\right]}-1\right)\cdots\left(\frac{\mu^{(j_{\ell_n})}_{X_{\ell_n,[1]}}}{E\left[X_{\ell_n}^{j_{\ell_n}}\right]}-1\right);$$

$$A_2(j_1,\ldots,j_d)=1+\sum_{n=2}^{d}\sum_{1\le\ell_1<\cdots<\ell_n\le d}\theta_{\ell_1\ldots\ell_n}\left(1-\frac{\mu^{(j_{\ell_1})}_{X_{\ell_1,[2]}}}{E\left[X_{\ell_1}^{j_{\ell_1}}\right]}\right)\cdots\left(1-\frac{\mu^{(j_{\ell_n})}_{X_{\ell_n,[2]}}}{E\left[X_{\ell_n}^{j_{\ell_n}}\right]}\right).$$

The special case where C is the independence copula yields $A_1(j_1,\ldots,j_d)=A_2(j_1,\ldots,j_d)=1$ for all $(j_1,\ldots,j_d)\in\{0,\ldots,m\}^d$. Note that, although Corollary 1 does not rely on the stochastic formulation of FGM copulas, it is a new result whose proof is straightforward due to Lemma 1. Further, both (20) and (22) have a similar complexity from a computational standpoint. However, some subfamilies of symmetric multivariate Bernoulli distributions will lead to few non-zero values for pmfs, such that the computation of the last expectation in (20) will typically be faster.

4. Implications for specific families of distributions

4.1. Aggregation of some continuous rvs

This section investigates special cases of distributions for positive continuous rvs which are closed under convolution when the dependence structure is a FGM copula or which admit closed-form solutions for the mth moments. The guiding principle is that when rvs have closed-form representations for (i) the cdfs of their order statistics, or (ii) the mth moments of their order statistics, then one may derive equivalent results for the aggregate rvs.

4.1.1. Mixed Erlang distributions

Let X_k, for $k=1,\ldots,d$, follow mixed Erlang distributions, parametrized by vectors of probabilities $\{q_{k,j}, j\in\mathbb{N}_1\}$, a common rate parameter β, and cdfs

$$F_{X_k}(x)=\sum_{j=1}^{\infty}q_{k,j}H(x;j,\beta),\quad x\ge 0,$$

where $H(x,\alpha,\beta)$ is the cdf of an Erlang distribution with shape α and rate β. Also, let L_k be the discrete rv with pmf $\Pr(L_k=j)=q_{k,j}$, for $j\in\mathbb{N}_1$ and $k\in\{1,\ldots,d\}$. Letting $\mathcal{P}_Y(t)$ denote the probability generating function (pgf) of a rv Y, the LST of X_k is given by $\mathcal{L}_{X_k}(t)=\mathcal{P}_{L_k}(\beta/(\beta+t))$, for $k\in\{1,\ldots,d\}$ and $t\ge 0$.

In Landriault et al. (2015), the authors show that when a rv X is mixed Erlang distributed, then $X_{[1]}$ and $X_{[2]}$ are also mixed Erlang distributed. We briefly recall their result and provide expressions for the new parameters of the distributions for the rvs $X_{[1]}$ and $X_{[2]}$. Let $Q_{k,j}=\sum_{m=1}^{j}q_{k,m}$, for $j\in\mathbb{N}_1$ and $Q_{k,0}=0$. One has

$$F_{X_{k,[i+1]}}(x)=\sum_{j=1}^{\infty}q_{k,j,\{i+1\}}H(x;j,2\beta),\quad k=1,\ldots,d,\quad i\in\{0,1\},\quad x>0,$$

with

$$q_{k,j,\{i+1\}}=\begin{cases}\frac{1}{2^{j-1}}\sum_{m=0}^{j-1}\binom{j-1}{m}q_{k,m+1}\left(1-Q_{k,j-1-m}\right), & \text{for } i=0\\ \frac{1}{2^{j-1}}\sum_{m=0}^{j-1}\binom{j-1}{m}q_{j,m+1}Q_{k,j-1-m}, & \text{for } i=1\end{cases},\tag{23}$$

for $j\in\mathbb{N}_1$, where (23) is a special case of equation (2.7) from Landriault et al. (2015).

Note that $q_{k,j,\{i+1\}}$ does necessarily correspond to $\Pr(L_{k,[i+1]}=j)$, for $k\in\{1,\ldots,d\}$ and $j\in\mathbb{N}_1$, it is for this reason that we use the braces notation instead of the brackets notation. We denote $L_{k,\{i+1\}}$ the rv with probability masses $\left\{q_{k,j,\{i+1\}}, j\in\mathbb{N}_1\right\}$, for $k\in\{1,\ldots,d\}$ and $i\in\{0,1\}$.

From Theorem 1, the Laplace-Stieltjes transform of S is

$$\mathcal{L}_S(t)=E_{\boldsymbol{I}}\left[\prod_{k=1}^{d}\sum_{j=1}^{\infty}q_{k,j,\{I_k+1\}}\left(\frac{2\beta}{2\beta+t}\right)^j\right]=E_{\boldsymbol{I}}\left[\prod_{k=1}^{d}\mathcal{P}_{L_{k,\{I_k+1\}}}\left(\frac{2\beta}{2\beta+t}\right)\right],\quad t\ge 0.$$

Noticing that $S|\boldsymbol{I}$ is the sum of d independent compound distributed rvs, we rearrange the LST as

$$\mathcal{L}_S(t)=\sum_{\boldsymbol{i}\in\{0,1\}^d}f_{\boldsymbol{I}}(\boldsymbol{i})\mathcal{P}_{M_{\boldsymbol{i}}}\left(\frac{2\beta}{2\beta+t}\right)=\sum_{\boldsymbol{i}\in\{0,1\}^d}f_{\boldsymbol{I}}(\boldsymbol{i})\sum_{j=1}^{\infty}\Pr(M_{\boldsymbol{i}}=j)\left(\frac{2\beta}{2\beta+t}\right)^j,\tag{24}$$

where $M_{\boldsymbol{i}}=L_{1,\{i_1+1\}}+\cdots+L_{d,\{i_d+1\}}$ for $\boldsymbol{i}\in\{0,1\}^d$ and $t\ge 0$. We suggest using the fast Fourier transform algorithm of Cooley and Tukey (1965) to compute the probability masses of $M_{\boldsymbol{i}}$. From the expression in (24), we conclude that S also follows a mixed Erlang distribution. Indeed, we deduce from (24) that

$$F_S(x) = \sum_{\boldsymbol{i}\in\{0,1\}^d} f_{\boldsymbol{I}}(\boldsymbol{i}) \sum_{j=1}^{\infty} \Pr(M_{\boldsymbol{i}} = j) H(x; j, 2\beta) = \sum_{j=1}^{\infty} q_{S,j} H(x; j, 2\beta), \quad x \geq 0, \tag{25}$$

with

$$q_{S,j} = \sum_{\boldsymbol{i}\in\{0,1\}^d} f_{\boldsymbol{I}}(\boldsymbol{i}) \Pr(M_{\boldsymbol{i}} = j), \quad j \in \mathbb{N}_1. \tag{26}$$

From (26), one may compute risk measures of the aggregate rv S. For instance, from (25), we have that the TVaR of S at level $\kappa \in (0,1)$ is

$$\mathrm{TVaR}_\kappa(S) = \sum_{j=1}^{\infty} q_{S,j} \frac{j}{2\beta} \overline{H}(\mathrm{VaR}_\kappa(S); j+1, 2\beta), \tag{27}$$

where $\overline{H}(x; \alpha, \beta) = 1 - H(x; \alpha, \beta)$, and where $\mathrm{VaR}_\kappa(S)$ is obtained by numerical inversion of (25).

Exponential distributions are special cases of mixed Erlang distributions, hence the above results also hold. However, a simpler proof is possible in that case, which we present in Appendix A.1.

Remark 1. The results from this subsection were previously shown, though stated differently, in Proposition 4.2 of Cossette et al. (2013) from a purely algebraic argument, under the natural representation of the FGM copula. The significant contribution from this subsection is that formulas are much simpler and more intuitive. In addition, the stochastic representation of FGM copulas breaks down the problem of computing (26) as a convolution or a mixture of the discrete probability masses in (23). From a programming standpoint, this is an important advantage since one can validate the proper computation of pmfs at all intermediate steps. Finally, one can obtain similar results for mixed Erlang distributions that do not share the same rate parameter using the strategy from Section 2 of Willmot and Woo (2007).

So far, we showed that when the random vector $\boldsymbol{X}$ has mixed Erlang marginals, and when the copula defining $F_{\boldsymbol{X}}$ is FGM, then the aggregate rv S is also mixed Erlang distributed. The conditions for this result are that each marginal distribution is closed under order statistics, finite mixture and convolution (for each marginal and across the random vector). As shown in Bladt and Nielsen (2017), phase-type and matrix-exponential distributions also satisfy these three closure properties. It follows that the aggregate rv of phase-type and matrix-exponential distributions under FGM dependence will also respectively follow phase-type and matrix-exponential distributions, though we defer investigating the implications of these statements to future research. See also Cheung et al. (2022) for related results on risk theory with multivariate matrix-exponential distributions.

4.1.2. Moments of other continuous rvs

Let X be Pareto or Weibull distributed, then one may compute the moments of X, of $X_{[1]}$ and of $X_{[2]}$, the latter requiring the identity in (7). We provide the expressions for the moments of the aggregate rv S in Appendices A.2 and A.3. Note that since Pareto and Weibull distributions do not have convenient closure properties for mixtures and convolution operations, we may not obtain exact expressions for the cdf of S as we have for mixed Erlang distributions.

The relationship in (21), which uses the moments of the minimum order statistic, is usually more useful: for survival functions defined as compositions of a first function with a power function, the survival function of the minimum will also be defined as compositions of a first function with another power function. That is, squaring a power function will yield another power function. This is why we obtain closed-form expressions for Pareto and Weibull marginals. Another example which satisfies this condition is when X follows a Gompertz-Makeham distribution, then $X_{[1]}$ also follows a Gompertz-Makeham distribution. However, contrarily to Pareto and Weibull distributions, computing the moments from Gompertz-Makeham distributions requires numerical integration.

On the other hand, when the cdf is defined as the composition of a first function with a power function, then squaring the cdf will yield a cdf in the same family as the original cdf, so the moment associated with $X_{[2]}$ has a preferable shape for computations. Simple examples include the standard power function distribution or the Gumbel distribution.

In Nadarajah (2008), the author presents expressions for the moments of order statistics for normal and log-normal distributions. These expressions are a function of a finite sum of Lauricella functions of type A. The moments of order statistics for log-normal distributions still require numerical integration. Since the expressions for these moments are tedious, we omit them in the current paper.

Finally, we note that (20), (21), and (22) do not require to assume the same marginal distributions; one can compute the exact value of a given mth moment for the sum of a combination of, for example, mixed Erlang, phase-type, matrix-exponential, Pareto and Weibull distributions with different parameters, provided the jth moments, $j = 1, \ldots, m$ of the maximum of each marginal distribution is finite.

4.2. Sum of discrete rvs

We now turn our attention to discrete non-negative rvs. We first provide preliminary results for the order statistics of discrete rvs, then present an efficient algorithm to compute the pmf of the aggregate rv S. Let $\boldsymbol{X} = (X_1, \ldots, X_d)$ be a vector of discrete non-negative rvs whose joint cdf is defined with a FGM copula. We note $p_{k,j} = \Pr(X_k = j)$, for $k \in \{1, \ldots, d\}$ and $j \in \mathbb{N}_0$. The pmf of the minimum and maximum of two iid discrete rvs are

$$p_{k,[1],j} := \Pr(X_{k,[1]} = j) = 2p_j \sum_{m=j+1}^{\infty} p_m + p_j^2 = 2p_j \left(1 - \sum_{m=0}^{j} p_m\right) + p_j^2; \tag{28}$$

$$p_{k,[2],j} := \Pr(X_{k,[2]} = j) = 2p_j \sum_{m=0}^{j-1} p_m + p_j^2 = 2p_j \sum_{m=0}^{j} p_m - p_j^2. \tag{29}$$

The identity $p_{j,k} = \left(p_{k,[1],j} + p_{k,[2],j}\right)/2$ holds for all $k \in \{1,\dots,d\}$ and $j \in \mathbb{N}_0$. Few discrete distributions admit neat representations for their order statistics; we illustrate this point with two examples.

Example 2. Let X be geometrically distributed with $\Pr(X = j) = q(1-q)^j$ and $\Pr(X > j) = (1-q)^{j+1}$, for $j \in \mathbb{N}_0$. Then, $\Pr(X_{[1]} > j) = (1-q)^{2j+2}$ and we conclude that $X_{[1]} \sim Geom(1-(1-q)^2)$. One has $\Pr(X_{[2]} \le j) = 1 - 2(1-q)^{j+1} - (1-q)^{2j+2}$. Letting $q^* = 1-(1-q)^2$, it follows that

$$\mathcal{P}_{X_{[i]}}(t) = (-1)^{i-1}\left(\frac{q^*}{1-(1-q^*)t} - \frac{q}{1-(1-q)t}\right) + \frac{q}{1-(1-q)t}, \quad i \in \{1,2\},$$

for $|t| \le 1$. Consider two identically distributed rvs X_1 and X_2 which follow geometric distributions, where $F_{X_1,X_2}(x_1,x_2) = C(F_{X_1}(x_1), F_{X_2}(x_2))$, for $(x_1,x_2) \in \mathbb{N}_0^2$ and $C \in \mathcal{C}_2^{FGM}$. Then, the pgf of $S = X_1 + X_2$ is

$$\begin{aligned}
\mathcal{P}_S(t) &= E_I\left[\prod_{k=1}^{2}\left\{(-1)^{I_k}\left(\frac{q^*}{1-(1-q^*)t} - \frac{q}{1-(1-q)t}\right) + \frac{q}{1-(1-q)t}\right\}\right] \\
&= \left(\frac{q}{1-(1-q)t}\right)^2 + \theta_{12}\left\{\frac{q^*}{1-(1-q^*)t} - \frac{q}{1-(1-q)t}\right\}^2 \\
&= (1+\theta_{12})\left(\frac{q}{1-(1-q)t}\right)^2 + \theta_{12}\left(\frac{q^*}{1-(1-q^*)t}\right)^2 - 2\theta_{12}\frac{q^*}{1-(1-q^*)t}\frac{q}{1-(1-q)t},
\end{aligned}$$

which is the pgf of a mixture of three distributions: the first two are negative binomial and the third one is the distribution associated with the sum of two independent geometric distributed rvs with different success probabilities. Since there are no simple formulas for the pmf of the third rv, we do not have a simple formula for the pmf of S, although one may show that S follows a mixture of Pascal distributions (studied in, for instance, Furman (2007); Mi et al. (2008); Zhao and Balakrishnan (2010); Badescu et al. (2015)) and, more generally, a matrix-geometric distribution (see Bladt and Nielsen (2017)).

Example 3. When X is Poisson distributed with intensity λ, (28) and (29) respectively become

$$\begin{aligned}
p_{j,[1]} &= \frac{2\lambda^j e^{-\lambda}}{j!}\left\{1 - \frac{\Gamma(j+1,\lambda)}{j!}\right\} + \left(\frac{\lambda^j e^{-\lambda}}{j!}\right)^2; \\
p_{j,[2]} &= \frac{2\lambda^j e^{-\lambda}}{j!}\frac{\Gamma(j,\lambda)}{(j-1)!} \times 1_{\{j\ge 1\}} + \left(\frac{\lambda^j e^{-\lambda}}{j!}\right)^2,
\end{aligned}$$

for $k \in \mathbb{N}_0$, where $\Gamma(x,\lambda)$ is the upper incomplete Gamma function, that is, $\Gamma(x,\lambda) = \int_\lambda^\infty t^{x-1}e^{-t}\,\mathrm{d}t$. There does not seem to have elegant representations for the sum of rvs $X_{[1]}$ and $X_{[2]}$ when X follows a Poisson distribution.

Examples 2 and 3 show that convenient forms for the pmfs of order statistics for discrete rvs aren't trivial. However, one can still compute the exact values of the pmf of S. Using the same arguments as in the proof of Theorem 1, the pgf of S for discrete marginal is

$$\mathcal{P}_S(t) = E_{\boldsymbol{I}}\left[\prod_{k=1}^{d}\mathcal{P}_{X_{k,[I_k+1]}}(t)\right], \quad |t| \le 1. \tag{30}$$

It follows that the representation in (30) enables an algorithmic approach to find the pmf of S. Suppose there is a number $\omega \in \mathbb{N}_0$ such that $p_{k,\ell} = 0$, for all $k \in \{1,\dots,d\}$ and $\ell \ge \omega$. The discrete Fourier transform of S forms a vector $\boldsymbol{\phi}_S$ with elements $\phi_{S,j} = \mathcal{P}_S(\exp\{-2\pi \mathrm{i} j/(d\times\omega)\})$, for $j = 0,\dots,d\times\omega-1$. Therefore, the values of the pmf of S are given by

$$p_{S,j} = \frac{1}{d\times\omega}\sum_{n=0}^{d\times\omega-1} E_{\boldsymbol{I}}\left[\prod_{k=1}^{d}\mathcal{P}_{X_{k,[I_k+1]}}\left(\exp\{-2\pi \mathrm{i} n/(d\times\omega)\}\right)\right]\exp\{2\pi \mathrm{i} nj/(d\times\omega)\}, \tag{31}$$

for $j = 0,\dots,d\times\omega-1$. Based on (31), we propose Algorithm 1 to compute the pmf for S when margins are discrete.

4.3. Numerical bounds

As stated in Section 4.1, the cdf or the moments of the aggregate rv S have convenient forms when the margins of the random vector are closed under order statistics. However, this is not the case for most continuous distributions. In these cases, one may discretize continuous rvs into discrete rvs, and use the numerical tools provided in Section 4.2 to study the approximate behavior of S. Some approximation methods are provided in Embrechts and Frei (2009) or appendix E of Klugman et al. (2018). When using an approximation method, it is important to obtain upper and lower bounds for the true cdf and risk measures of the aggregate rv S in order to quantify the accuracy of the approximation. To construct these bounds, we will require the following stochastic order.

Algorithm 1: Computing the pmf of S.

Input: Values of $p_{k,j}, j=0,\dots,\omega-1, k=1,\dots,d$, table $f_{\boldsymbol{I}}$
Output: pmf of S
for $k=1,\dots,d$ **do**
  Set $\boldsymbol{p}_k=(p_{k,0},\dots,p_{k,\omega-1},0,\dots,0)\in[0,1]^{d\times\omega}$;
  Compute $\boldsymbol{P}_k$ as the cumulative sum of $\boldsymbol{p}_k$;
  Compute $\boldsymbol{P}_{k,[2]}=\boldsymbol{P}_k^2$ (element-wise);
  Compute $\boldsymbol{p}_{k,[2]}$ as the difference vector of $\boldsymbol{P}_{k,[2]}$;
  Compute $\boldsymbol{p}_{k,[1]}=2\times\boldsymbol{p}_k-\boldsymbol{p}_{k,[2]}$ (element-wise) ;
  Use fft to compute the discrete Fourier transform $\boldsymbol{\phi}_{k,[1]}$ of $\boldsymbol{p}_{k,[1]}$;
  Use fft to compute the discrete Fourier transform $\boldsymbol{\phi}_{k,[2]}$ of $\boldsymbol{p}_{k,[2]}$;
Compute $\boldsymbol{\phi}_S=\sum_{\boldsymbol{i}\in\{0,1\}^d}f_{\boldsymbol{I}}(\boldsymbol{i})\prod_{k=1}^{d}\boldsymbol{\phi}_{k,[i_k+1]}$ (element-wise);
Use fft to compute the inverse discrete Fourier transform $\boldsymbol{p}_S$ of $\boldsymbol{\phi}_S$;
Return $\boldsymbol{p}_S$.

Definition 1 *(Usual stochastic order).* Let $\boldsymbol{Y}$ and $\boldsymbol{Y}'$ be two d-variate random vectors satisfying $E[f(\boldsymbol{Y})]\leq E[f(\boldsymbol{Y}')]$ for all bounded increasing function $f:\mathbb{R}^n\to\mathbb{R}$. Then, we say that $\boldsymbol{Y}$ is smaller than $\boldsymbol{Y}'$ under the usual stochastic order and denote this relation by $\boldsymbol{Y}\preceq_{st}\boldsymbol{Y}'$.

In the univariate case, the implications of the usual stochastic order between two rvs Y and Y' are that $E[Y]\leq E[Y']$, $\mathrm{VaR}_\kappa(Y)\leq \mathrm{VaR}_\kappa(Y')$ for all $\kappa\in(0,1)$ and $E[\phi(Y)]\leq E[\phi(Y')]$ for all increasing function ϕ, assuming that the expectations exist (including the TVaR). Within the context of the current paper, if we may construct cdfs for two rvs A and B such that $A\preceq_{st}S\preceq_{st}B$, then we may construct bounds on VaRs and certain expected values. One approach is to use the upper and lower methods, defined next.

Definition 2 *(Lower and upper methods).* The pmf of a discretized rv $\tilde{X}^{(l,h)}$, under the *lower* method is $f_{\tilde{X}^{(l,h)}}(0)=0$ and $f_{\tilde{X}^{(l,h)}}(jh)=F_X(jh)-F_X((j-1)h)$, for $j\in\mathbb{N}_1$, where $h>0$. The pmf of a discretized rv $\tilde{X}^{(u,h)}$ under the *upper* method is $f_{\tilde{X}^{(u,h)}}(0)=F_X(h)$ and $f_{\tilde{X}^{(u,h)}}(jh)=F_X((j+1)h)-F_X(jh)$, for $j\in\mathbb{N}_1$.

Note that the authors of Embrechts and Frei (2009) call the lower and upper methods, respectively, the backward and forward differences. From the definitions of the lower and upper methods, we have (see Section 1.11 of Müller and Stoyan (2002)) that, for $0<h<h'<\infty$,

$$\tilde{X}^{(u,h')}\preceq_{st}\tilde{X}^{(u,h)}\preceq_{st}X\preceq_{st}\tilde{X}^{(l,h)}\preceq_{st}\tilde{X}^{(l,h')}. \tag{32}$$

It is useful to construct bounds as in (32) for the cdf of the aggregate rv S. To do so, we first discretize the cdfs of each marginal distribution, in particular, the cdfs of the order statistics of each marginal.

Remark 2. Let X_1 and X_2 be independent copies of a positive rv X with cdf F_X. Let $X_{[1]}=\min(X_1,X_2)$ and $X_{[2]}=\max(X_1,X_2)$. Then,

$$F_{\tilde{X}_{[1]}^{(m,h)}}(x)=1-(1-F_{\tilde{X}^{(m,h)}}(x))^2;\quad F_{\tilde{X}_{[2]}^{(m,h)}}(x)=F_{\tilde{X}^{(m,h)}}(x)^2,$$

for $m\in\{l,u\}$ and $x\geq 0$. That is, one can compute the cdf of $\tilde{X}_{[1]}$ and $\tilde{X}_{[2]}$ using the definitions of the lower and upper discretization methods or first discretize the rv X and then compute the cdf using the relationships in (1) and (2).

From (32) and Theorem 4.1 of Müller and Scarsini (2001) (see also Theorem 3.3.8 of Müller and Stoyan (2002)), we have that

$$\tilde{\boldsymbol{X}}^{(u,h')}\preceq_{st}\tilde{\boldsymbol{X}}^{(u,h)}\preceq_{st}\boldsymbol{X}\preceq_{st}\tilde{\boldsymbol{X}}^{(l,h)}\preceq_{st}\tilde{\boldsymbol{X}}^{(l,h')} \tag{33}$$

if all random vectors share the same copula.

For a fixed FGM copula, define the aggregate rv of the discretized marginals with the upper and lower methods, that is, $\tilde{S}^{(u,h)}$ and $\tilde{S}^{(l,h)}$ by $\tilde{S}^{(u,h)}=\tilde{X}_1^{(u,h)}+\dots+\tilde{X}_d^{(u,h)}$ and $\tilde{S}^{(l,h)}=\tilde{X}_1^{(l,h)}+\dots+\tilde{X}_d^{(l,h)}$. Since the usual stochastic order is preserved under monotone transformations (see Theorem 3.3.11 of Müller and Stoyan (2002)), it follows from (33) that

$$\tilde{S}^{(u,h')}\preceq_{st}\tilde{S}^{(u,h)}\preceq_{st}S\preceq_{st}\tilde{S}^{(l,h)}\preceq_{st}\tilde{S}^{(l,h')}, \tag{34}$$

for $0<h\leq h'$. The relationship in (34) is useful since one can construct bounds on risk measures.

In the following example, we consider a portfolio of log-normally distributed risks. Note that there are no closed-form expressions for the cdf of the minimum or maximum of two log-normal distributions, but one may still approximate the cdf with Algorithm 1, and the relation in (34) provides bounds on the (Tail-)Value-at-Risk of the aggregate risk rv S.

Example 4. Consider a portfolio of $n=3$ risks and $X_k\sim LNorm(\mu_k,\sigma_k)$ for $k=1,2,3$. We set (μ_k,σ_k), $k\in\{1,2,3\}$ such that $E[X_k]=10$ for $k=1,2,3$ and $Var(X_1)=20$, $Var(X_2)=50$ and $Var(X_3)=100$. The dependence structure is induced by a Markov-Bernoulli FGM copula, as introduced in Blier-Wong et al. (2022), whose expression is

$$C(\boldsymbol{u})=\prod_{k=1}^{d}u_k\left(1+\sum_{n=1}^{\left\lfloor\frac{d}{2}\right\rfloor}\sum_{1\leq j_1<\dots<j_{2n}\leq d}\alpha^{\gamma_{j_1\dots j_{2n}}}\overline{u}_{j_1}\dots\overline{u}_{j_{2n}}\right),\quad \boldsymbol{u}\in[0,1]^d,$$

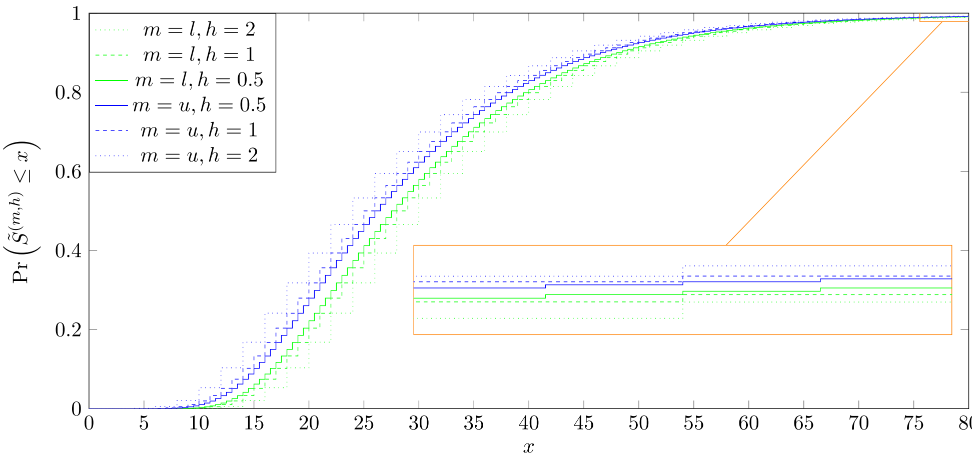

Fig. 1. Values of the cumulative distribution function of S for different discretization methods.

Table 1
Values of the TVaR of S from different discretization methods.

h	Upper				Lower			
	2	1	0.5	0.1	0.1	0.5	1	2
$\kappa = 0.9$	57.60	59.08	59.83	60.43	60.73	61.33	62.08	63.60
$\kappa = 0.99$	92.65	94.13	94.88	95.48	95.78	96.38	97.13	98.65
$\kappa = 0.999$	142.93	144.42	145.16	145.76	146.06	146.66	147.42	148.93

where $\gamma_{j_1 \ldots j_{2n}} = \sum_{l=1}^{n} \left(j_{2l} - j_{2l-1}\right)$ and dependence parameter satisfies $\alpha \in [-1, 1]$. For this example, we select the dependence parameter $\alpha = 0.5$. We aim to approximate the cdf of $S = X_1 + X_2 + X_3$ through discretization methods and using Algorithm 1.

Fig. 1 presents the cdf of $\tilde{S}^{(m,h)}$ for $m \in \{l, u\}$ and $h \in \{0.5, 1, 2\}$. The relationship in (34) is satisfied, it follows that the cdf of the continuous aggregate rv is between the green (lower method) and blue (upper method) curves. Table 1 presents the values of the TVaR risk measure at levels $\kappa \in \{0.9, 0.99, 0.999\}$ for the rvs $\tilde{S}^{(m,h)}$ for $m \in \{l, u\}$ and $h \in \{0.1, 0.5, 1, 2\}$. One can state, therefore, that $60.43 \leq \mathrm{TVaR}_{0.9}(S) \leq 60.73$, without ever knowing the true cdf of S. Also, one may decrease the range between the lower and upper bounds by selecting a smaller span h at the cost of more computations. For instance, selecting $h = 0.01$ yields an interval $60.56 \leq \mathrm{TVaR}_{0.9}(S) \leq 60.59$, but the computation time goes from 0.01 seconds for $h = 0.1$ to 64 seconds for $h = 0.01$.

5. Stochastic ordering of aggregate rvs

In this section, we will leverage the stochastic representation of FGM copulas to study the impact of dependence on the aggregate rv S. We briefly recall the notions required for this section. Let $\boldsymbol{X}$ and $\boldsymbol{X}'$ be two d-variate random vectors whose cdfs belong to the same Fréchet class. We aim to compare the rvs S and S', which respectively correspond to the sum of rvs from the random vectors $\boldsymbol{X}$ and $\boldsymbol{X}'$. An important stochastic order in actuarial science, which measures the variability of a rv, is the convex order.

Definition 3 *(Convex order).* Let Y and Y' be two rvs with finite expectations. We say that Y is smaller than Y' under the convex order if $E[\phi(Y)] \leq E[\phi(Y')]$ for every convex function ϕ, when the expectations exist. We denote two rvs ordered according to the convex order as $Y \preceq_{cx} Y'$.

Some relevant implications of the relation $S \preceq_{cx} S'$ are that $E[S] = E[S']$, $\mathrm{Var}(S) \leq \mathrm{Var}(S')$ (assuming that they exist), and $\mathrm{TVaR}_\kappa(S) \leq \mathrm{TVaR}_\kappa(S')$, for all $\kappa \in (0, 1)$, see Müller and Stoyan (2002); Denuit et al. (2006); Shaked and Shanthikumar (2007) for a more comprehensive list.

In our quest to compare the aggregate rvs according to the convex order, we will first need to compare vectors of rvs, $(V_1, \ldots, V_d)$ and $(V_1', \ldots, V_d')$, using dependence stochastic orders, where, for each $j \in \{1, \ldots, d\}$, V_j and V_j' have the same marginal distribution. In Sections 3.8 and 3.9 of Müller and Stoyan (2002), the authors present the supermodular order.

Definition 4 *(Supermodular order).* We say $\boldsymbol{V}$ is smaller than $\boldsymbol{V}'$ under the supermodular order, denoted $\boldsymbol{V} \preceq_{sm} \boldsymbol{V}'$, if $E[\phi(\boldsymbol{V})] \leq E\left[\phi(\boldsymbol{V}')\right]$ for all supermodular functions ϕ, given that the expectations exist. A function $\phi : \mathbb{R}^d \to \mathbb{R}$ is said to be supermodular if

$$\begin{aligned} &\phi(x_1, \ldots, x_i + \varepsilon, \ldots, x_j + \delta, \ldots, x_d) - \phi(x_1, \ldots, x_i + \varepsilon, \ldots, x_j, \ldots, x_d) \\ &\geq \phi(x_1, \ldots, x_i, \ldots, x_j + \delta, \ldots, x_d) - \phi(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_d) \end{aligned}$$

holds for all $(x_1, \ldots, x_d) \in \mathbb{R}^d$, $1 \leq i < j \leq d$ and all $\varepsilon, \delta > 0$.

The supermodular order satisfies the nine desired properties for dependence orders as mentioned in Section 3.8 of Müller and Stoyan (2002). See also Shaked and Shanthikumar (2007) and Denuit et al. (2006) for more details on the supermodular order. We recall the following lemma from Blier-Wong et al. (2022) which presents the general result for supermodular orders within the family of FGM copulas.

Lemma 3 *(Theorem 4.2 of Blier-Wong et al. (2022)). Let $\boldsymbol{I}$ and $\boldsymbol{I}'$ be random vectors with $F_{\boldsymbol{I}}, F_{\boldsymbol{I}'} \in \mathcal{B}_d$. Let $\boldsymbol{U}$ and $\boldsymbol{U}'$ be random vectors constructed using (13) and $\boldsymbol{X}$ and $\boldsymbol{X}'$ be random vectors constructed using (14). If $\boldsymbol{I} \preceq_{sm} \boldsymbol{I}'$, then $\boldsymbol{U} \preceq_{sm} \boldsymbol{U}'$ and $\boldsymbol{X} \preceq_{sm} \boldsymbol{X}'$.*

Establishing the supermodular order within a class of copulas has important consequences for risk aggregation, as the following proposition shows.

Proposition 1. *If $\boldsymbol{X} \preceq_{sm} \boldsymbol{X}'$ holds, then $\sum_{j=1}^{d} X_j \preceq_{cx} \sum_{j=1}^{d} X_j'$, where $\preceq_{cx}$ is the convex order.*

Proof. See Theorem 8.3.3 of Müller and Stoyan (2002) or Proposition 6.3.9 of Denuit et al. (2006). □

It follows from Proposition 1 and Lemma 3 that one may order the aggregate rvs S and S' within the context of the current paper if one first orders the random vectors $\boldsymbol{X}$ and $\boldsymbol{X}'$. In the remainder of this section, we investigate the implications of this observation.

While the upper bound under the supermodular order for multivariate Bernoulli random vectors is well-known (see the EPD FGM copula further in this section), its lower bound is still an open problem. For this reason, we will restrict our analysis to the class of exchangeable FGM copulas, studied in Blier-Wong et al. (2024), for which a lower bound exists. The lower and upper bounds of the supermodular order within the families of exchangeable FGM copulas, called the extreme negative dependence (END) and extreme positive dependence (EPD), respectively satisfy

$$\boldsymbol{U}^{END} \preceq_{sm} \boldsymbol{U} \preceq_{sm} \boldsymbol{U}^{EPD},$$

for all $\boldsymbol{U}$ with $F_{\boldsymbol{U}}$ being an exchangeable FGM copula as defined in Blier-Wong et al. (2024). Further, $\boldsymbol{U} \preceq_{sm} \boldsymbol{U}^{EPD}$ holds for all $\boldsymbol{U}$ with $F_{\boldsymbol{U}} \in \mathcal{C}^{FGM}$. We recall the definition of the EPD FGM copula from Blier-Wong et al. (2022).

Definition 5 *(Extreme positive dependent FGM copula).* The FGM copula associated with the random vector $\boldsymbol{I}$ whose components are comonotonic rvs is the EPD FGM copula, denoted by C^{EPD}. The expression of the EPD FGM copula is given by

$$C^{EPD}(\boldsymbol{u}) = \prod_{k=1}^{d} u_k \left(1 + \sum_{n=1}^{\left\lfloor \frac{d}{2} \right\rfloor} \sum_{1 \le j_1 < \cdots < j_{2n} \le d} \overline{u}_{j_1} \cdots \overline{u}_{j_{2n}}\right), \quad \boldsymbol{u} \in [0,1]^d, \tag{35}$$

where $\lfloor y \rfloor$ is the floor function returning the greatest integer smaller or equal to y. The n-dependence parameters are $\theta_n = (1 + (-1)^n)/2$, for $n \in \{2, \ldots, d\}$.

The END FGM copula is derived in Blier-Wong et al. (2024) and recalled below.

Definition 6 *(Extreme negative dependent FGM copula).* The expression of the FGM END copula, denoted by C^{END}, is given by

$$C^{END}(\boldsymbol{u}) = \prod_{k=1}^{d} u_k \left(1 + \sum_{n=1}^{\left\lfloor \frac{d}{2} \right\rfloor} \sum_{1 \le j_1 < \cdots < j_{2n} \le d} \frac{\Gamma(n+1)\Gamma\left(\frac{1}{2} - \left\lfloor \frac{d+1}{2} \right\rfloor\right)}{2^n \Gamma\left(\frac{n}{2}+1\right)\Gamma\left(\frac{n+1}{2} - \left\lfloor \frac{d+1}{2} \right\rfloor\right)} \overline{u}_{j_1} \cdots \overline{u}_{j_{2n}}\right), \quad \boldsymbol{u} \in [0,1]^d. \tag{36}$$

That is, the n-dependence parameters for the FGM END copula are given by

$$\theta_n = {}_2F_1\left(-\left\lfloor \frac{d+1}{2} \right\rfloor, -n, 2\left\lfloor \frac{d+1}{2} \right\rfloor, 2\right) = \frac{(1+(-1)^n)}{2} \frac{\Gamma(n+1)\Gamma\left(\frac{1}{2} - \left\lfloor \frac{d+1}{2} \right\rfloor\right)}{2^n \Gamma\left(\frac{n}{2}+1\right)\Gamma\left(\frac{n+1}{2} - \left\lfloor \frac{d+1}{2} \right\rfloor\right)}, \tag{37}$$

for $n \in \{2, \ldots, d\}$ and where ${}_2F_1$ is the ordinary hypergeometric function.

Example 5. Consider a vector $\boldsymbol{X}$ with joint cdf $F_{\boldsymbol{X}}(\boldsymbol{x}) = C(F(x_1), \ldots, F(x_d))$, $\boldsymbol{x} \in \mathbb{R}_+^d$, where $F(x) = 1 - e^{-\beta x}$ and C is an exchangeable FGM copula. We denote the aggregate rv for a portfolio of d risks as S_d, and omit the subscript when d is arbitrary. In this example, we study the special cases of S which lead to the lower bound and the upper bound under the convex order for exchangeable FGM copulas, respectively denoted S^{END} and S^{EPD}, along with the aggregate rv under the assumption of independence, denoted S^{Ind}. Note that $E[S^{END}] = E[S] = E[S^{EPD}]$, $\mathrm{Var}(S^{END}) \le \mathrm{Var}(S) \le \mathrm{Var}(S^{EPD})$ and $\mathrm{TVaR}_\kappa(S^{END}) \le \mathrm{TVaR}_\kappa(S) \le \mathrm{TVaR}_\kappa(S^{EPD})$, for all $\kappa \in (0,1)$ and for all S constructed within the setup of this example. By using the representation in (18) and Theorem 1 (see also Appendix A.1), the LST of S^{EPD} is

$$\mathcal{L}_{S_d^{EPD}}(t) = \frac{1}{2}\left(\frac{2\beta}{2\beta + t}\right)^d + \frac{1}{2}\left(\frac{\beta}{\beta + t}\frac{2\beta}{2\beta + t}\right)^d, \quad t \ge 0,$$

while the LST of S^{END} is

Table 2
VaR and TVaR of W_d with the END, independent and EPD copulas.

	$\kappa = 0.9$			$\kappa = 0.99$			$\kappa = 0.999$		
e	END	Ind	EPD	END	Ind	EPD	END	Ind	EPD
$\mathrm{VaR}(W_1^e)$	23.03	23.03	23.03	46.05	46.05	46.05	69.08	69.08	69.08
$\mathrm{TVaR}(W_1^e)$	33.03	33.03	33.03	56.05	56.05	56.05	79.08	79.08	79.08
$\mathrm{VaR}(W_2^e)$	18.09	19.45	20.90	29.91	33.19	35.55	41.46	46.17	48.86
$\mathrm{TVaR}(W_2^e)$	23.25	25.47	27.37	34.93	38.85	41.36	46.47	51.66	54.43
$\mathrm{VaR}(W_{10}^e)$	13.63	14.21	17.85	17.58	18.78	23.19	20.95	22.66	27.40
$\mathrm{TVaR}(W_{10}^e)$	15.38	16.24	20.26	19.06	20.48	25.05	22.31	24.20	29.04
$\mathrm{VaR}(W_{100}^e)$	11.13	11.30	15.93	12.14	12.47	17.39	12.92	13.38	18.44
$\mathrm{TVaR}(W_{100}^e)$	11.58	11.83	16.60	12.48	12.87	17.86	13.21	13.72	18.82
$\mathrm{VaR}(W_{1000}^e)$	10.35	10.41	15.30	10.65	10.75	15.74	10.87	11.01	16.04
$\mathrm{TVaR}(W_{1000}^e)$	10.49	10.56	15.50	10.75	10.86	15.87	10.95	11.10	16.15

$$\mathcal{L}_{S_d^{END}}(t) = \begin{cases} \left(\frac{2\beta}{2\beta+t}\right)^d \left(\frac{\beta}{\beta+t}\right)^{d/2}, & d \text{ is even} \\ \left(\frac{2\beta}{2\beta+t}\right)^d \left(\frac{1}{2}\left(\frac{\beta}{\beta+t}\right)^{(d-1)/2} + \frac{1}{2}\left(\frac{\beta}{\beta+t}\right)^{(d+1)/2}\right), & d \text{ is odd} \end{cases}.$$

Both LSTs correspond to the LST of mixed Erlang distributions. Using an optimization tool, we obtain the values for the VaR by inverting the cdf of S, then we compute the TVaR. To simplify comparisons, we introduce the rv $W_d^e = S_d^e/d$, for $e \in \{END, Ind, EPD\}$ and omit the superscript for arbitrary dependence structures. We present the values of VaR and TVaR for W_d in Table 2. We present the results for $d \in \{1, 2, 10, 100, 1000\}$ and $\kappa \in \{0.9, 0.99, 0.999\}$. We compute every risk measure with $\beta = 0.1$, that is, $E[X] = 10$, $E[S_d] = 10 \times d$ and $E[W_d] = 10$. Let us examine the effect of dependence on the risk measures for $\kappa = 0.9$. We aim to compute the relative effect of dependence for different portfolio sizes. For $d = 2$, we have

$$\frac{\mathrm{TVaR}_{0.9}(W_2^{END}) - \mathrm{TVaR}_{0.9}(W_2^{Ind})}{\mathrm{TVaR}_{0.9}(W_2^{Ind})} = -0.0870; \quad \frac{\mathrm{TVaR}_{0.9}(W_2^{EPD}) - \mathrm{TVaR}_{0.9}(W_2^{Ind})}{\mathrm{TVaR}_{0.9}(W_2^{Ind})} = 0.0744,$$

while, for $d = 1000$, we have

$$\frac{\mathrm{TVaR}_{0.9}(W_{1000}^{END}) - \mathrm{TVaR}_{0.9}(W_{1000}^{Ind})}{\mathrm{TVaR}_{0.9}(W_{1000}^{Ind})} = -0.0072; \quad \frac{\mathrm{TVaR}_{0.9}(W_{1000}^{EPD}) - \mathrm{TVaR}_{0.9}(W_{1000}^{Ind})}{\mathrm{TVaR}_{0.9}(W_{1000}^{Ind})} = 0.4671.$$

The most negative relative effect of dependence (-0.0870) appears for $d = 2$ with the END copula and decreases as the portfolio size d increases. This isn't surprising, as the impact of the negative dependence structure decreases when the dimension d increases, see Blier-Wong et al. (2024). The most positive relative effect of dependence (0.4671) appears for the EPD copula and is an increasing function of d, that is, increasing the portfolio size with the EPD copula increases the relative effect of dependence on the TVaR.

6. Risk allocation and risk sharing

It is natural, in the context of risk management, to study the impact of aggregating risks in an insurance portfolio or pool. To do so, we will study allocation rules, a problem related to the aggregation of rvs. Allocations have actuarial applications in peer-to-peer insurance and regulatory capital allocation. Throughout this section, we consider a portfolio of d risks, each of which follows mixed Erlang distributions with a common rate parameter. The dependence structure is once again induced by a FGM copula. From Section 4.1.1, we know that the aggregate rv is also mixed Erlang distributed.

6.1. Conditional mean risk sharing

The rise of peer-to-peer insurance has ignited a lot of interest in risk allocation and risk-sharing rules. A participant to a pool of insurance risk should contribute relative to the risk he contributes to the pool hence one seeks fair risk sharing rules to determine this value; see, for instance, Denuit (2019), Denuit (2020) or Denuit et al. (2022) for discussions. The conditional mean risk sharing is one such rule, where the participant pays his expected contribution, given the total realized losses (denoted s, with $s \geq 0$) in the pool, that is,

$$E[X_k | S = s] = \frac{E\left[X_k \times 1_{\{S=s\}}\right]}{f_S(s)}, \tag{38}$$

see Denuit and Dhaene (2012), Denuit and Robert (2021), Denuit et al. (2022) and Jiao et al. (2022) for details and properties of the conditional mean risk-sharing rule. When each risk is mixed Erlang distributed and the dependence structure is induced by a FGM copula, we have the following result.

Theorem 3. *Let $\boldsymbol{X}$ be a random vector with cdf $F_{\boldsymbol{X}}(\boldsymbol{x}) = C(F_{X_1}(x_1), \ldots, F_{X_d}(x_d))$, where F_{X_k} is the cdf of a mixed Erlang distributed rv, for $k = 1, \ldots, d$. Further, assume that $F_{X_1}, \ldots, F_{X_k}$ share the same rate parameter, and that $C \in \mathcal{C}_d^{FGM}$. For $k \in \{1, \ldots, d\}$, the conditional mean is given by*

$$E\left[X_k|S=s\right]=\frac{E[X_k\times 1_{\{S=s\}}]}{\sum_{j=1}^{\infty}q_{S,j}h(s;j,2\beta)},\tag{39}$$

where $h(s;\alpha,\beta)$ is the pdf associated with an Erlang distribution, for $s\geq 0$,

$$E[X_k\times 1_{\{S=s\}}]=\sum_{\boldsymbol{i}\in\{0,1\}^d}f_{\boldsymbol{I}}(\boldsymbol{i})\left[\sum_{l=2}^{\infty}\sum_{\ell=1}^{l-1}\Pr\left(L_{k,\{i_k+1\}}=\ell\right)\times \Pr\left(\sum_{\nu=1,\nu\neq k}^{d}L_{\nu,\{i_\nu+1\}}=l-\ell\right)\frac{\ell}{2\beta}h(s;l+1,2\beta)\right]\tag{40}$$

and $q_{S,j}$ for $j\in\mathbb{N}_1$ is defined as in (25).

Proof. The denominator in (38) follows from differentiating the cdf in (25). It remains to provide an expression for $E\left[X_k\times 1_{\{S=s\}}\right]$. From the joint LST in (17), we condition on $\boldsymbol{I}$ to notice that the distribution of the bivariate random vector (X_k,S_{-k}) is expressed as a mixture of independent bivariate distributions. The bivariate LST of (X_k,S_{-k}) is

$$\begin{aligned}\mathcal{L}_{X_k,S_{-k}}(t_1,t_2)&=E_{\boldsymbol{I}}\left[\mathcal{P}_{L_{k,\{1+I_k\}}}\left(\frac{2\beta}{2\beta+t_1}\right)\prod_{\nu=1,\nu\neq k}^{d}\mathcal{P}_{L_{\nu,\{1+I_\nu\}}}\left(\frac{2\beta}{2\beta+t_2}\right)\right]\\&=\sum_{\boldsymbol{i}\in\{0,1\}^d}f_{\boldsymbol{I}}(\boldsymbol{i})\mathcal{P}_{L_{k,\{1+i_k\}}}\left(\frac{2\beta}{2\beta+t_1}\right)\prod_{\nu=1,\nu\neq k}^{d}\mathcal{P}_{L_{\nu,\{1+i_\nu\}}}\left(\frac{2\beta}{2\beta+t_2}\right),\end{aligned}$$

for $(t_1,t_2)\in\mathbb{R}_+^2$. Then, the expected allocation is

$$\begin{aligned}E\left[X_k\times 1_{\{S=s\}}\right]&=\int_0^s xf_{X_k,S_{-k}}(x,s-x)\,\mathrm{d}x\\&=\int_0^s x\left[\sum_{\boldsymbol{i}\in\{0,1\}^d}f_{\boldsymbol{I}}(\boldsymbol{i})f_{X_{k,[1+i_k]}}(x)f_{\sum_{\nu=1,\nu\neq k}^{d}X_{\nu,[1+i_\nu]}}(s-x)\right]\mathrm{d}x\\&=\sum_{\boldsymbol{i}\in\{0,1\}^d}f_{\boldsymbol{I}}(\boldsymbol{i})\left[\int_0^s xf_{X_{k,[1+i_k]}}(x)f_{\sum_{\nu=1,\nu\neq k}^{d}X_{\nu,[1+i_\nu]}}(s-x)\,\mathrm{d}x\right].\end{aligned}$$

The result in (40) follows from Propositions 4 and 5 of Cossette et al. (2012) since each integral is an expectation from a pair of independent mixed Erlang rvs. □

Example 6. We consider a portfolio of six risks where each follows a mixed Erlang distribution with common rate parameter 1/2, that is,

$$F_{X_k}(s)=\Pr(X_k\leq s)=\sum_{j=1}^{\infty}q_{k,j}H(s;j,1/2),\quad s>0,$$

for $k\in\{1,\ldots,6\}$, and

$$\begin{cases}q_{1,j}=1\times 1_{\{j=1\}};\\q_{2,j}=\frac{1}{2}^j\times 1_{\{j\in\mathbb{N}_1\}};\\q_{3,j}=5^{j-1}e^{-5}/(j-1)!\times 1_{\{j\in\mathbb{N}_1\}};\\q_{4,j}=\Gamma(j-1+2)/\Gamma(2)/(j-1)!0.25^2(0.75)^{j-1}\times 1_{\{j\in\mathbb{N}_1\}};\\q_{5,j}=10^{j-1}e^{-10}/(j-1)!\times 1_{\{j\in\mathbb{N}_1\}};\\q_{6,j}=\Gamma(j-1+3)/\Gamma(3)/(j-1)!0.2^3(0.8)^{j-1}\times 1_{\{j\in\mathbb{N}_1\}}.\end{cases}$$

For convenience, we artificially construct vectors of probabilities whose masses correspond to known discrete distributions (Dirac, geometric, Poisson, negative binomial); this will help us control the shape of the marginal distributions. Notice that the risks are highly heterogeneous since $\{q_{6,j},j\in\mathbb{N}_1\}$ comes from a distribution with a heavier tail than $\{q_{1,j},j\in\mathbb{N}_1\}$. Also note that $E[X_4]<E[X_5]$, but $\mathrm{Var}(X_4)>\mathrm{Var}(X_5)$. In Table 3, we present the values of the expectation, the variance, the VaR and the TVaR for each rv.

For every dependence structure, we have $E[S]=80$. In Table 4, we present the outcomes of random vectors under the conditional mean risk sharing rule, when the aggregate rv S takes either the value of $E[S]/2$, $E[S]$, or $2\times E[S]$. The rv X_1 (X_6) is the safest (riskiest), having the smallest (largest) mean, variance, VaR and TVaR at level 0.99. For $s=40$, we observe that *increasing* the dependence (according to the supermodular order) results in a decrease (increase) of the conditional mean for the rv X_1 (X_6). For $s=160$, we observe the opposite pattern: *increasing* the dependence (according to the supermodular order) results in an increase (decrease) of the conditional

Table 3
Summary description for marginal rvs.

k	1	2	3	4	5	6
$E[X_k]$	2	4	12	14	22	26
$\mathrm{Var}(X_k)$	4	16	44	124	84	292
$\mathrm{VaR}_{0.99}(X_k)$	9.21	18.42	31.44	50.86	47.45	79.72
$\mathrm{TVaR}_{0.99}(X_k)$	11.21	22.42	35.40	59.90	52.30	92.03

Table 4
Outcomes for risk premiums under the conditional mean risk sharing rule.

s	e	$E[X_k\|S^e=s]$					
		$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
40	END	1.928175	2.987516	7.996234	5.766606	13.401958	7.919511
	IND	1.575428	2.551020	7.668274	5.699930	13.761121	8.744228
	EPD	0.941819	1.757806	7.136790	5.658961	14.296102	10.208524
80	END	2.030938	4.123420	12.407195	13.910778	22.776325	24.751343
	IND	2.042401	4.106984	12.392149	13.867892	22.741896	24.848677
	EPD	2.205948	4.149484	12.499946	13.398856	22.671998	25.073768
160	END	1.721004	4.234335	13.912178	30.207898	27.145704	82.778881
	IND	2.330977	5.554256	15.892004	31.485783	29.453031	75.283950
	EPD	3.347377	7.541924	18.660443	32.720014	32.458935	65.271307

mean for the rv X_1 (X_6). Further, observe that when $S = 80$, the smallest conditional mean for X_2 and X_3 occurs when the dependence structure is independence. Note that $s \mapsto E[X_1|S^{END} = s]$ is not increasing, hence the conditional mean risk sharing rule, in this case, is not comonotonic, which is a desirable property of risk-sharing allocation principles.

6.2. Risk allocation based on the Euler allocation principle

For regulatory and capital requirement purposes, one must often decompose aggregate risk measures to the individual risks that contributed to it. The TVaR is a popular risk measure since it is coherent. The TVaR of a continuous rv is also called the expected shortfall, see, for instance, Artzner (1999), Artzner et al. (1999), Acerbi et al. (2001), Acerbi and Tasche (2002) for motivations and properties of the expected shortfall for risk management. When one establishes global capital with the TVaR, one may deconstruct this risk measure to TVaR-based allocations with the help of the Euler's risk allocation principle (Tasche (1999), Denault (2001)). Assuming that $E[S] < \infty$ and for some $\kappa \in (0,1)$, the contributions to the TVaR under the Euler risk allocation principle for continuous rvs is given by

$$\mathrm{TVaR}_\kappa(X_k; S) = E[X_k \times 1_{\{S>\mathrm{VaR}_\kappa(S)\}}]/(1-\kappa), \tag{41}$$

for $k \in \{1, \ldots, d\}$. Within the context of this paper, we have the following expression.

Theorem 4. *Let* $\boldsymbol{X}$ *be a random vector as described in Theorem 3. For some confidence level* $\kappa \in (0,1)$, *the contribution to the TVaR under Euler's allocation principle is*

$$\mathrm{TVaR}_\kappa(X_k; S) = \frac{1}{1-\kappa}\left\{\sum_{\boldsymbol{i}\in\{0,1\}^d} f_{\boldsymbol{I}}(\boldsymbol{i})\left[\sum_{l=2}^{\infty}\sum_{n=1}^{l-1}\Pr\left(L_{k,\{i_k+1\}}=n\right)\times \Pr\left(\sum_{\nu=1,\nu\neq k}^{d} L_{\nu,\{i_\nu+1\}} = l-n\right)\frac{n}{2\beta}\overline{H}(\mathrm{VaR}_\kappa(S); l+1, 2\beta)\right]\right\}, \tag{42}$$

for $k \in \{1, \ldots, n\}$, *where we compute* $\mathrm{VaR}_\kappa(S)$ *with numerical inversion of* (25).

Proof. The numerator in (41) is obtained by the relationship $E[X_k \times 1_{\{S>s\}}] = \int_s^\infty E[X_k \times 1_{\{S=x\}}]\,\mathrm{d}x$ and replacing $s = \mathrm{VaR}_\kappa(S)$. Inserting (40) into the latter and evaluating the integral, we have the desired result. □

Remark 3. Note that (42) has a similar form as other applications of risk aggregation with mixed Erlang distributions. See, e.g., Proposition 5 of Cossette et al. (2012), Proposition 4.2 of Cossette et al. (2013) and Theorem 1 of Willmot and Woo (2015). In particular, the result in (42) was also developed in equation (35) of Cossette et al. (2013), but the formula is very tedious. By studying multivariate mixed Erlang distributions from the order statistic perspective, and the FGM copula from its stochastic representation, one has an intuitive understanding of the underlying stochastic phenomenon and obtains straightforward expressions for the TVaR and the contributions to the TVaR under Euler's allocation principle. Also, (42) uses the stochastic formulation of the FGM copula (based on the symmetric multivariate Bernoulli random vector $\boldsymbol{I}$), which is more convenient in higher dimensions since most cases of interest (for instance, minimal and maximal dependence under the supermodular order for exchangeable FGM copulas) are easier to formulate with the stochastic representation. Also, the outer sum in (42) is a sum over 2^d values, which could be computationally prohibitive, but for most special cases, including minimal and maximal dependence under the supermodular order for exchangeable FGM copulas, the pmf is non-zero for few vectors of $\boldsymbol{i} \in \{0,1\}^d$.

Table 5
Values of $\text{TVaR}_{0.99}(X_k; S^e)$, for $k \in \{1, \ldots, 6\}$ for different $C \in \mathcal{C}_d^{FGM}$.

e	$\text{Var}(S^e)$	$\text{VaR}_{0.99}(S^e)$	$\text{TVaR}_{0.99}(S^e)$	$\text{TVaR}_{0.99}(X_k; S^e)$					
				$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
END	452.45	140.58	153.41	1.74	4.26	13.91	29.10	27.06	77.35
Ind	564	146.71	160.14	2.33	5.55	15.87	31.44	29.41	75.54
EPD	1121.77	163.57	177.24	3.39	7.79	19.08	36.48	33.23	77.25

Example 7. Consider the portfolio of six risks introduced in Example 6. In Table 5, we provide values of the TVaR-based risk allocation from the expression in (42).

As shown in Section 5, we have $\boldsymbol{U}^{END} \preceq_{sm} \boldsymbol{U}^{Ind} \preceq_{sm} \boldsymbol{U}^{EPD}$, hence $S^{END} \preceq_{cx} S^{Ind} \preceq_{cx} S^{EPD}$. This fact is verified from the size of the variance and the TVaR at level 0.99. Further, we observe for $k \in \{1, \ldots, 5\}$ that $\text{TVaR}_{0.99}(X_k; S)$ is smallest for the END FGM copula and largest for the EPD FGM copula. However, this is not the case for X_6, which is the riskiest in the portfolio. The authors were surprised to observe, for the rv X_6, that the smallest risk contribution occurs when the dependence structure is independence, while the largest risk contribution occurs with negative dependence. Investigating why this is the case represents an interesting avenue for future research.

7. Discussions

In this paper, we revisit risk aggregation and risk allocation with the FGM copula. By studying the problem using the stochastic representation of the FGM copula, we develop convenient representations for the cdf or moments of aggregate rvs when a FGM copula induces the dependence structure. One significant contribution of this work to the existing literature is our ability to order aggregate rvs according to stochastic orders.

In Section 4.1, we have provided convenient closed-form expressions for cdfs and moments of the aggregate rv S for positive and continuous distributions. Other closed-form expressions are possible for continuous distributions. For instance, if X has a cdf that is symmetric about $x = \mu$, we have $f_{X_{[1]}}(\mu + x) = f_{X_{[2]}}(\mu - x)$ and $\mu_{X_{[1]}}^{(m)} = (-1)^m \mu_{X_{[2]}}^{(m)}$. For $\mu = 0$, we have $X_{[1]} \stackrel{\mathcal{D}}{=} -X_{[2]}$. It follows that

$$E\left[S^m\right] = \sum_{j_1+\cdots+j_d=m} \frac{m!}{j_1! \ldots j_d!} E_{\boldsymbol{I}}\left[\prod_{k=1}^{d} (-1)^{j_k I_k} \mu_{X_{[1]}}^{(j_k)}\right].$$

We leave the study of risk aggregation under FGM dependence of rvs whose support is on $\mathbb{R}$ as future research.

In Section 6, we presented numerical illustrations of conditional mean risk sharing and risk allocation based on Euler's rule for mixed Erlang marginals. Since the results of the current paper allow for exact expressions, and the FGM copula admits multiple shapes of dependence (including negative dependence), we are in a position to investigate examples that provide apparent counter-intuitive results that were previously unknown (to the best of our knowledge) in the literature on risk sharing. Such developments open questions regarding the stochastic orderings of risk-sharing rules or ordering contributions based on Euler's rule or any other capital allocation rule.

Declaration of competing interest

There is no competing interest.

Data availability

No data was used for the research described in the article.

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Appendix A. Further results for continuous rvs

A.1. Special cases with exponential rvs

We have seen in Example 1 that when X is exponentially distributed, then $X_{[1]}$ and $X_{[2]}$ have convenient stochastic forms. It isn't surprising, given the link between the FGM copula and order statistics, that FGM copulas are based on exponential FGM distributions, first studied in their namesake papers, Eyraud (1936), Farlie (1960), Gumbel (1960) and Morgenstern (1956). When $\boldsymbol{X}$ has cdf $F_{\boldsymbol{X}}(\boldsymbol{x}) = C(F_{X_1}(x_1), \ldots, F_{X_d}(x_d))$ with $C \in \mathcal{C}_d^{FGM}$ and $F_{X_k}(x) = 1 - \exp\{-\beta_k x\}$, for $k \in \{1, \ldots, d\}$, then S will also have a convenient stochastic form.

We have seen in Section 4.1.1 that multivariate mixed Erlang distributions constructed with Sklar's theorem and FGM copulas lead to an aggregate rv that is mixed Erlang distributed. This is also the case for exponentially distributed rvs since exponential distributions are special cases of mixed Erlang distributions. In this section, we provide a simpler proof.

A.1.1. Case with exponential marginals with identical parameters

We now study the special case where $F_{X_k}(x) = F(x) = 1 - \exp\{-\beta x\}$, for $x \geq 0$ and $k = 1, \ldots, d$. For notational purposes, we introduce the rv N_d which corresponds to the sum of the components from $\boldsymbol{I}$, that is, $N_d = \sum_{k=1}^{d} I_k$. It follows from Theorem 1 that

$$\begin{aligned}\mathcal{L}_S(t) &= E\left[\left(\frac{2\beta}{2\beta+t}\frac{\beta}{\beta+t}\right)^{N_d}\left(\frac{2\beta}{2\beta+t}\right)^{d-N_d}\right] \\ &= \left(\frac{2\beta}{2\beta+t}\right)^{d} E\left[\left(\frac{\beta}{\beta+t}\right)^{N_d}\right] = \left(\frac{2\beta}{2\beta+t}\right)^{d} \mathcal{P}_{N_d}\left(\frac{\beta}{\beta+t}\right),\end{aligned} \tag{43}$$

for $t \geq 0$, where $\mathcal{P}_J(t)$ is the pgf of a discrete rv J. From the form of $\mathcal{L}_S$ in (43), one recognizes that S is the sum of two independent rvs Y_1 and Y_2, where $Y_1 \sim \mathit{Erlang}(d, 2\beta)$ and Y_2 follows a compound distribution with cdf

$$F_{Y_2}(x) = E\left[H(x, N_d, \beta)\right] = \sum_{j=0}^{d} \Pr(N_d = j) H(x, j, \beta), \quad x \geq 0,$$

with $H(x, 0, \beta) = 1$. We conclude that Y_2 follows a finite mixture of Erlang distributions with probabilities given by the pmf of N_d and rate parameter β.

Further, one can show that S follows a mixed Erlang distribution. Following Willmot and Woo (2007), we write the LST of Y_1 and Y_2 under the same rate parameter using the identity

$$\frac{\beta_1}{\beta_1+t} = \frac{\beta_2}{\beta_2+t}\left\{\frac{\beta_1/\beta_2}{1-(1-\beta_1/\beta_2)\frac{\beta_2}{\beta_2+t}}\right\}, \tag{44}$$

for $0 < \beta_1 \leq \beta_2 < \infty$ and $t \geq 0$. Specifically, combining (43) and (44), we obtain

$$\mathcal{L}_S(t) = \left(\frac{2\beta}{2\beta+t}\right)^{d} \mathcal{P}_{N_d}\left(\frac{2\beta}{2\beta+t}\left\{\frac{0.5}{1-0.5\frac{2\beta}{2\beta+t}}\right\}\right) = \mathcal{P}_M\left(\frac{2\beta}{2\beta+t}\right), \tag{45}$$

where

$$\mathcal{P}_M(t) = t^d \mathcal{P}_{N_d}\left(\frac{0.5t}{1-0.5t}\right), \quad t \geq 0.$$

From the expression in (45), we deduce that S follows a mixed Erlang distribution with rate 2β and parameters $q_j = \Pr(M = j)$, for $j \in \mathbb{N}_1$.

A.1.2. Case with exponential marginals with different parameters

We now consider the case where $F_{X_k}(x) = 1 - \exp\{-\beta_k x\}, x \geq 0, k = 1, \ldots, d$, and where $\beta_1 \neq \cdots \neq \beta_d$. Applying Theorem 1 to the order statistic representation of exponentially distributed rvs provided in Example 1, the LST of S is

$$\mathcal{L}_S(t) = E\left[\prod_{k=1}^{d}\left(\frac{2\beta_k}{2\beta_k+t}\right)\left(\frac{\beta_k}{\beta_k+t}\right)^{I_k}\right] = \left\{\prod_{k=1}^{d}\left(\frac{2\beta_k}{2\beta_k+t}\right)\right\} \times E\left[\prod_{k=1}^{d}\left(\frac{\beta_k}{\beta_k+t}\right)^{I_k}\right], \quad t \geq 0. \tag{46}$$

One may decompose the LST in (46) as the product of two LSTs; hence S is the sum of two independent rvs that we denote Y_1 and Y_2. One observes that Y_1 follows a generalized Erlang distribution with cdf

$$F_{Y_1}(x) = \sum_{k=1}^{d}\left(\prod_{j=1, j\neq k}^{d}\frac{\beta_j}{\beta_j-\beta_k}\right)\left(1-e^{-2\beta_j x}\right), \quad x \geq 0.$$

Since the distribution of $(Y_2|\boldsymbol{I} = \boldsymbol{i})$, for $\boldsymbol{i} \in \{0, 1\}^d$, is also generalized Erlang, Y_2 follows a finite mixture of generalized Erlang distributions with cdf

$$\begin{aligned}F_{Y_2}(x) &= E_{\boldsymbol{I}}\left[F_{Y_2|\boldsymbol{I}}(x)\right] = \sum_{\boldsymbol{i}\in\{0,1\}^d} f_{\boldsymbol{I}}(\boldsymbol{i}) F_{Y_2|\boldsymbol{I}=\boldsymbol{i}}(x) \\ &= \sum_{\boldsymbol{i}\in\{0,1\}^d} f_{\boldsymbol{I}}(\boldsymbol{i}) \sum_{\{k\in\{1,\ldots,d\}|i_k=1\}}\left(\prod_{\{j\in\{1,\ldots,d\}|i_j=1, j\neq k\}}\frac{\beta_j}{\beta_j-\beta_k}\right)\left(1-e^{-\beta_j x}\right), \quad x \geq 0.\end{aligned}$$

Once again, we can show that S follows a mixed Erlang distribution and use (44) to set all cdfs under the same rate parameter.

A.2. Special case with Pareto rvs

First assume that the rv X follows a Pareto type IV distribution, denoted Pareto(IV), with survival function

$$\overline{F}_X(x) = \left[1 + \left(\frac{x-\mu}{\sigma}\right)^{1/\gamma}\right]^{-\alpha}, \quad x > \mu,$$

with $\mu \in \mathbb{R}$ and $\sigma, \gamma, \alpha > 0$. One obtains the Lomax distribution, popular in actuarial science by setting $\mu = 0$ and $\gamma = 1$. The case $\mu = 0$ simplifies to a Burr type XII distribution, while $\alpha = 1$, simplifies to the log-logistic distribution. See Arnold (2015) for more details. When X follows a Pareto(IV), its mth moment exists for $-\gamma^{-1} < m < \alpha/\gamma$ and is given by

$$E[X^m] = \sigma^m \frac{\Gamma(\alpha - \gamma m)\Gamma(1+\gamma m)}{\Gamma(\alpha)}. \tag{47}$$

The survival function for $X_{[1]}$ when X follows a Pareto(IV) distribution is

$$\overline{F}_{X_{[1]}}(x) = \left[1 + \left(\frac{x-\mu}{\sigma}\right)^{1/\gamma}\right]^{-2\alpha}, \quad x > \mu,$$

which is the survival function of a Pareto(IV) distribution but with parameter 2α. Therefore, for $-\gamma^{-1} < m < 2\alpha/\gamma$, we have

$$\mu^{(m)}_{X_{[1]}} = \sigma^m \frac{\Gamma(2\alpha - \gamma m)\Gamma(1+\gamma m)}{\Gamma(2\alpha)}. \tag{48}$$

Inserting (47) and (48) within (21), we have, for $-\gamma^{-1} < m < \alpha/\gamma$, that

$$E\left[S^m\right] = \sum_{j_1+\cdots+j_d=m} \frac{m!}{j_1!\cdots j_d!} \left\{\prod_{k=1}^{d} \sigma^{j_k} \frac{\Gamma(\alpha - \gamma j_k)\Gamma(1+\gamma j_k)}{\Gamma(\alpha)}\right\} \times E_{\boldsymbol{I}}\left[\prod_{k=1}^{d}\left\{1 + (-1)^{I_k}\left(\frac{\Gamma(2\alpha - \gamma j_k)}{\Gamma(\alpha - \gamma j_k)} \frac{2^{1-2\alpha}\sqrt{\pi}}{\Gamma(\alpha + 1/2)} - 1\right)\right\}\right].$$

Alternatively from (22), we have

$$E\left[S^m\right] = \sum_{j_1+\cdots+j_d=m} \frac{m!}{j_1!\cdots j_d!} \left\{\prod_{k=1}^{d} \sigma^{m} \frac{\Gamma(\alpha - \gamma j_k)\Gamma(1+\gamma j_k)}{\Gamma(\alpha)}\right\} \times \left(1 + \sum_{n=2}^{d} \sum_{1\le \ell_1 < \cdots < \ell_n \le d} \theta_{\ell_1\cdots\ell_n} \left(\frac{\Gamma(2\alpha - \gamma j_{\ell_1})}{\Gamma(\alpha - \gamma j_{\ell_1})} \frac{2^{1-2\alpha}\sqrt{\pi}}{\Gamma(\alpha + 1/2)} - 1\right) \times \cdots \times \left(\frac{\Gamma(2\alpha - \gamma j_{\ell_n})}{\Gamma(\alpha - \gamma j_{\ell_n})} \frac{2^{1-2\alpha}\sqrt{\pi}}{\Gamma(\alpha + 1/2)} - 1\right)\right).$$

A.3. Special case with Weibull rvs

Next, assume that X follows a Weibull distribution with pdf

$$f_X(x) = \beta\tau(\beta x)^{\tau-1} e^{-(\beta x)^\tau}, \quad x \ge 0,$$

and survival function

$$\overline{F}_X(x) = e^{-(\beta x)^\tau}, \quad x \ge 0,$$

where $\beta, \tau > 0$, with moments given by $E[X^m] = \beta^{-m}\Gamma(1 + m/\tau)$. One computes

$$\mu^{(m)}_{X_{[1]}} = 2\int_0^\infty x^m (1 - e^{-(\beta x)^\tau})\beta\tau(\beta x)^{\tau-1} e^{-(\beta x)^\tau}\,\mathrm{d}x = 2E[X^m] - 2\int_0^\infty \beta^\tau \tau x^{\tau+m-1} e^{-2\beta^\tau x^\tau}\,\mathrm{d}x.$$

Letting $u = x^\tau$, we find

$$\mu^{(m)}_{X_{[1]}} = 2E[X^m] - \int_0^\infty 2\beta^\tau u^{m/\tau} e^{-2\beta^\tau u}\,\mathrm{d}u = \frac{1}{\beta^m}\Gamma\left(1 + \frac{m}{\tau}\right)\left(2 - 2^{-m/\tau}\right). \tag{49}$$

Inserting (49) within (21) or (22), we have respectively

$$E\left[S^m\right] = \sum_{j_1+\cdots+j_d=m} \frac{m!}{j_1!\cdots j_d!} \left\{\prod_{k=1}^{d} \frac{1}{\beta^{j_k}} \Gamma\left(1+\frac{j_k}{\tau}\right)\right\} E_{\boldsymbol{I}}\left[\prod_{k=1}^{d} \left\{1+(-1)^{I_k}\left(1-2^{-j_k/\tau}\right)\right\}\right] \tag{50}$$

$$= \sum_{j_1+\cdots+j_d=m} \frac{m!}{j_1!\cdots j_d!} \left\{\prod_{k=1}^{d} \frac{1}{\beta^{j_k}} \Gamma\left(1+\frac{j_k}{\tau}\right)\right\}$$

$$\times \left(1+\sum_{n=2}^{d} \sum_{1\le \ell_1<\cdots<\ell_n\le d} \theta_{\ell_1\cdots\ell_n}\left(1-2^{-j_{\ell_1}/\tau}\right)\cdots\left(1-2^{-j_{\ell_n}/\tau}\right)\right).$$

Note that in Section 5.6.3 of Kotz and Drouet (2001), the authors develop an expression similar to (50) for the product moments of $\boldsymbol{X}$ under the natural representation of the FGM copula for Weibull marginals. The advantage of the approach we take in the current paper is that one obtains the result by directly applying Corollary 1, and this corollary holds for any combination of marginal distributions.

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