

Probability equivalent level of Value at Risk and higher-order Expected Shortfalls [☆]

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ABSTRACT

We investigate the probability equivalent level of Value at Risk and n^{th} -order Expected Shortfall (called PELVE_n), which can be considered as a variant of the notion of the probability equivalent level of Value at Risk and Expected Shortfall (called PELVE) due to Li and Wang (2022). We study the finiteness, uniqueness and several properties of PELVE_n , we calculate PELVE_n of some notable distributions, PELVE_2 of a random variable having generalized Pareto excess distribution, and we describe the asymptotic behaviour of PELVE_2 of regularly varying distributions as the level tends to 0. Some properties of n^{th} -order Expected Shortfall are also investigated. Among others, it turns out that the Gini Shortfall at some level $p \in [0, 1)$ corresponding to a (loading) parameter $\lambda \geq 0$ is the linear combination of the Expected Shortfall at level p and the 2^{nd} -order Expected Shortfall at level p with coefficients $1 - 2\lambda$ and 2λ , respectively.

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1. Introduction

The Fundamental Review of the Trading Book (FRTB) was introduced by the Basel Committee on Banking Supervision in the years following the Global Financial Crisis of 2007–2009. FRTB is expected to make a complete revision of the approach to calculating risk-based capital requirements for investments. It was originally supposed to be implemented in January 2023, but a new starting date, January 2025, has been recently announced. Value at Risk (VaR , see Definition 1.1) and Expected Shortfall (ES , see Definition 1.2) are popular risk measures used to measure portfolio risk. According to FRTB, the banks are supposed to use ES at the level 0.975 instead of VaR at the level 0.99 for the bank-wide internal models to determine market risk capital requirements.

Motivated by the FRTB, Li and Wang (2022) have recently introduced the notion of probability equivalent level of VaR and ES (PELVE , see Definition 1.4). Roughly speaking, for an integrable random variable X and $\varepsilon \in (0, 1)$, the PELVE of X at the level ε is the infimum of those values $c \in [1, \frac{1}{\varepsilon}]$ for which the ES of X at level $1 - c\varepsilon$ is less than or equal to the VaR of X at the level $1 - \varepsilon$. One can see that the level $\varepsilon = 0.01$ corresponds to the replacement of VaR at the level 0.99 with the ES at some appropriate level, which has particular importance due to the FRTB.

Very recently, Fiori and Rosazza Gianin (2022) have proposed two generalizations of PELVE . As a first generalization, they have replaced the pair (VaR, ES) in the definition of PELVE with a general pair of monotone risk measures $(\varrho, \tilde{\varrho})$, where $\tilde{\varrho}$ is obtained from ϱ by integration similarly as ES can be obtained from VaR by integration, for more details, see Definition 1.5. As a special case of their

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generalization, Fiori and Rosazza Gianin (2022) have also considered the so-called conditional PELVE, where ϱ is chosen to be the ES. As a second generalization, Fiori and Rosazza Gianin (2022, Definition 2) have introduced the so-called distorted PELVE associated to a family of distortions, for more details, see Remark 1.7.

Both the PELVE due to Li and Wang (2022) and its generalizations due to Fiori and Rosazza Gianin (2022) are defined under the minimal assumption that the random variable representing the risk has a finite first moment. These risk measures enjoy satisfactory invariance and ordering properties, and they have interesting links to the tail index of a regularly varying random variable. Fiori and Rosazza Gianin (2022, Section 4) have also studied PELVE and conditional PELVE of a random variable having generalized Pareto excess distribution.

In the present paper we study the probability equivalent level of VaR and a higher-order ES (see Definition 1.2), i.e., we replace ES with a higher-order ES in the definition of PELVE due to Li and Wang (2022). We note that this variant of PELVE is a special case of the newly introduced notion, called distorted PELVE, in Fiori and Rosazza Gianin (2022, Definition 2) (for more details, see Remark 1.7). In Appendix B, we point out that the Gini Shortfall at some level $p \in [0, 1)$ corresponding to a (loading) parameter $\lambda \geq 0$, introduced in Furman et al. (2017, formula (4.1)), is the linear combination of the Expected Shortfall at level p and the 2nd-order Expected Shortfall at level p with coefficients $1 - 2\lambda$ and 2λ , respectively. This underlines the importance of studying the previously mentioned variant of PELVE in a more detailed way.

Let \mathbb{N} and \mathbb{R} denote the set of positive integers and real numbers, respectively. For a function $f : \mathbb{R} \rightarrow \mathbb{R}$, its range $\{f(x) : x \in \mathbb{R}\}$ is denoted by $\text{Range}(f)$. The random variables will be defined on an appropriate probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The distribution function of a random variable $X : \Omega \rightarrow \mathbb{R}$ is given by $F_X : \mathbb{R} \rightarrow [0, 1]$, $F_X(x) := \mathbb{P}(X \leq x)$, $x \in \mathbb{R}$. The set of random variables X satisfying $\mathbb{E}(|X|) < \infty$ is denoted by L^1 . Convergence in distribution is denoted by $\xrightarrow{\mathcal{D}}$.

First, we recall the notion of Value at Risk.

Definition 1.1. Let X be a random variable. The Value at Risk of X at a level $p \in [0, 1]$ is defined by

$$\text{VaR}_X(p) := \inf\{x \in \mathbb{R} : F_X(x) \geq p\},$$

with the convention $\inf \emptyset := \infty$.

Note that $\text{VaR}_X(p)$ is also called the (lower) quantile or a generalized inverse of X at a level $p \in (0, 1)$. One may think about X as the loss and profit of some financial position at a given time point, and, using actuarial notation, positive values of X represent losses, while negative values profit. For each $p \in (0, 1)$, $\text{VaR}_X(p)$ is the smallest value x such that the probability of a loss X greater than x is at most $1 - p$, or, roughly speaking, $\text{VaR}_X(p)$ is the loss that is likely to be exceeded only $(1 - p)100\%$ of the time. While VaR is widely used and easy to compute, it has no information on the magnitude of the biggest $(1 - p)100\%$ of the losses. Note also that $\text{VaR}_X(0) = -\infty$ for any random variable X .

Next, we recall the notion of n^{th} -order Expected Shortfall due to Fuchs et al. (2017, Example 2, part (4)).

Definition 1.2. Let X be a random variable such that $X \in L^1$, and let $n \in \mathbb{N}$. The n^{th} -order Expected Shortfall of X at a level $p \in [0, 1)$ is defined by

$$\text{ES}_{X,n}(p) := \frac{n}{1-p} \int_p^1 \left(\frac{s-p}{1-p}\right)^{n-1} \text{VaR}_X(s) \, ds.$$

Note that $\text{ES}_{X,n}(p)$ can be written in the form $\text{ES}_{X,n}(p) = \int_p^1 \frac{n(s-p)^{n-1}}{(1-p)^{n-1}} \text{VaR}_X(s) \, ds$, where the function $\frac{n(s-p)^{n-1}}{(1-p)^{n-1}}$, $s \in [p, 1]$, can be considered as a weight function with integral 1 on $[p, 1]$ such that higher losses are weighted higher.

In the next remark we recall some basic properties of higher-order Expected Shortfalls. For some further properties of higher-order Expected Shortfalls, see Appendix A.

Remark 1.3. (i). For $X \in L^1$, $n \in \mathbb{N}$, and $p \in [0, 1)$, we have $\text{ES}_{X,n}(p) \in \mathbb{R}$, see Lemma A.1. Note also that the first order Expected Shortfall coincides with the usual Expected Shortfall (also called Conditional Value at Risk), so $\text{ES}_{X,1}$ is simply denoted by ES_X .

(ii). By Lemma 2 in Fuchs et al. (2017), the n^{th} -order Expected Shortfall is monotone (in the sense that if $X \leq Y$, $X, Y \in L^1$, then $\text{ES}_{X,n}(p) \leq \text{ES}_{Y,n}(p)$, $p \in [0, 1)$), positive homogeneous and translation invariant. Further, using that the distortion function corresponding to the n^{th} -order Expected Shortfall (see the function h_p in the proof of Lemma A.1) is convex, we get that the n^{th} -order Expected Shortfall is subadditive, see Fuchs et al. (2017, Example 1/(4) and Theorem 4). The above mentioned properties of the n^{th} -order Expected Shortfall also follow from Proposition 2 and Theorem 3 in Wang et al. (2020). All in all, the n^{th} -order Expected Shortfall is a coherent risk measure on L^1 .

(iii). For $X \in L^1$ and $n_1, n_2 \in \mathbb{N}$ with $n_1 \leq n_2$, we have

$$\text{VaR}_X(p) \leq \text{ES}_X(p), \quad p \in [0, 1), \quad \text{and} \quad \text{ES}_{X,n_1}(p) \leq \text{ES}_{X,n_2}(p), \quad p \in [0, 1), \tag{1.1}$$

where the second inequality is a consequence of Fuchs et al. (2017, Corollary 4, part (1)). Indeed,

$$\text{ES}_{X,n}(p) = \int_0^1 \text{VaR}_X(s) \, dD_{n,p}(s), \quad n \in \mathbb{N}, \quad p \in [0, 1),$$

where $D_{n,p}(s) := \left(\frac{s-p}{1-p}\right)^n \mathbf{1}_{[p,1]}(s)$, $s \in [0, 1]$, and we have that $D_{n_2,p}(s) \leq D_{n_1,p}(s)$, $s \in [0, 1]$ for each $n_1, n_2 \in \mathbb{N}$ with $n_1 \leq n_2$. In particular, we have that $\text{ES}_X(p) \leq \text{ES}_{X,n}(p)$, $n \in \mathbb{N}$, $p \in [0, 1]$. The second inequality in (1.1) also follows by Wang et al. (2020, part (i) of Proposition 2). \square

In Appendix B, we point out the fact that the 2nd-order Expected Shortfall is nothing else but a special Gini Shortfall introduced in Furman et al. (2017, formula (4.1)), and a (general) Gini Shortfall is a linear combination of Expected Shortfall and 2nd-order Expected Shortfall. This observation could also underline the importance of studying properties of higher-order (especially, 2nd-order) Expected Shortfalls.

Next, we recall the notion of probability equivalent level of Value at Risk and Expected Shortfall (abbreviated as PELVE) due to Li and Wang (2022, formula (2)).

Definition 1.4. Let X be a random variable such that $X \in L^1$. The probability equivalent level of Value at Risk and Expected Shortfall (abbreviated as PELVE) of X at a level $\varepsilon \in (0, 1)$ is defined by

$$\Pi_\varepsilon(X) := \inf \left\{ c \in \left[1, \frac{1}{\varepsilon}\right] : \text{ES}_X(1 - c\varepsilon) \leq \text{VaR}_X(1 - \varepsilon) \right\},$$

where $\inf \emptyset = \infty$.

We give a motivation why the infimum in Definition 1.4 of $\Pi_\varepsilon(X)$ is taken over $\left[1, \frac{1}{\varepsilon}\right]$. The level $1 - c\varepsilon$ of ES_X should be non-negative yielding that $c \leq \frac{1}{\varepsilon}$; and, by (1.1), we have $\text{VaR}_X(1 - c\varepsilon) \leq \text{ES}_X(1 - c\varepsilon)$ for any $c \in (0, \frac{1}{\varepsilon}]$, which together with the requested inequality $\text{ES}_X(1 - c\varepsilon) \leq \text{VaR}_X(1 - \varepsilon)$ in the definition of $\Pi_\varepsilon(X)$ imply that $\text{VaR}_X(1 - c\varepsilon) \leq \text{VaR}_X(1 - \varepsilon)$. Provided that $[1 - \varepsilon, 1) \ni p \mapsto \text{VaR}_X(p)$ is strictly monotone increasing, this yields that $c \geq 1$, since otherwise $1 - c\varepsilon > 1 - \varepsilon$ implying that $\text{VaR}_X(1 - c\varepsilon) > \text{VaR}_X(1 - \varepsilon)$.

Next, we recall a generalization of PELVE due to Fiori and Rosazza Gianin (2022, Definition 8).

Definition 1.5. For each $\alpha \in (0, 1)$, let $\varrho_\alpha : L^1 \rightarrow \mathbb{R}$ be a risk measure such that the family $\{\varrho_\alpha : \alpha \in (0, 1)\}$ is monotone, i.e., if $0 < \alpha_1 \leq \alpha_2 < 1$, then $\varrho_{\alpha_1}(X) \leq \varrho_{\alpha_2}(X)$, $X \in L^1$. For each $p \in (0, 1)$, let us introduce the risk measure $\tilde{\varrho}_p : L^1 \rightarrow \mathbb{R} \cup \{\infty\}$,

$$\tilde{\varrho}_p(X) := \frac{1}{1-p} \int_p^1 \varrho_\alpha(X) d\alpha, \quad X \in L^1.$$

Given a random variable $X \in L^1$, the generalized PELVE of X at a level $\varepsilon \in (0, 1)$ corresponding to the pair $((\varrho_\alpha)_{\alpha \in (0,1)}, (\tilde{\varrho}_p)_{p \in (0,1)})$ is defined by

$$\Pi_\varepsilon^g(X) := \inf \left\{ c \in \left[1, \frac{1}{\varepsilon}\right] : \tilde{\varrho}_{1-c\varepsilon}(X) \leq \varrho_{1-\varepsilon}(X) \right\},$$

where $\inf \emptyset = \infty$. In the special case $\varrho_\alpha(X) = \text{ES}_X(\alpha)$, $\alpha \in (0, 1)$, $X \in L^1$, the corresponding generalized PELVE is called the conditional PELVE (abbreviated as c-PELVE).

By Proposition 7 in Fiori and Rosazza Gianin (2022), the family $\{\tilde{\varrho}_p : p \in (0, 1)\}$ is monotone, and $\tilde{\varrho}_p \geq \varrho_p$, $p \in (0, 1)$. Hence one can give a similar motivation why the infimum in the definition of $\Pi_\varepsilon^g(X)$ is taken over $\left[1, \frac{1}{\varepsilon}\right]$ just as we did in case of $\Pi_\varepsilon(X)$ (see the paragraph after Definition 1.4). Further, note that if one chooses $\varrho_\alpha(X) = \text{VaR}_X(\alpha)$, $\alpha \in (0, 1)$, $X \in L^1$ in Definition 1.5, then the corresponding generalized PELVE is nothing else but (usual) PELVE due to Li and Wang (2022) recalled in Definition 1.4.

Both Li and Wang (2022, Propositions 1-2 and Theorem 1) and Fiori and Rosazza Gianin (2022, Propositions 9-11) have studied finiteness, uniqueness, and some properties of PELVE and the generalized PELVE in Definition 1.5, respectively. The PELVE values of some notable distributions, such as uniform, exponential, normal, lognormal, t and Pareto distributions, have been calculated or approximated in Li and Wang (2022). The conditional PELVE values of uniform, normal and Pareto distributions have been also calculated in Fiori and Rosazza Gianin (2022), and it turned out that for uniform and Pareto distributions, the corresponding PELVE and conditional PELVE values coincide, see Fiori and Rosazza Gianin (2022, Subsection 3.2.1). Li and Wang (2022, Section 4.2) have described convergence of PELVE of regularly varying random variables as the level tends to 0, while Fiori and Rosazza Gianin (2022, Proposition 15) showed that PELVE and conditional PELVE of a random variable having generalized Pareto excess distribution coincide.

In the following definition we replace the Expected Shortfall in Definition 1.4 by the n^{th} -order Expected Shortfall, where $n \in \mathbb{N}$.

Definition 1.6. Let X be a random variable such that $X \in L^1$, and let $n \in \mathbb{N}$. The probability equivalent level of Value at Risk and n^{th} -order Expected Shortfall (abbreviated as PELVE _{n}) of X at a level $\varepsilon \in (0, 1)$ is defined by

$$\Pi_{\varepsilon,n}(X) := \inf \left\{ c \in \left[1, \frac{1}{\varepsilon}\right] : \text{ES}_{X,n}(1 - c\varepsilon) \leq \text{VaR}_X(1 - \varepsilon) \right\},$$

where $\inf \emptyset = \infty$.

First of all, we emphasize that the notion of PELVE _{n} given in Definition 1.6 is a special case of the so-called distorted PELVE introduced in Fiori and Rosazza Gianin (2022, Definition 2) that we recall below in Remark 1.7. The research on distorted PELVE in Fiori and Rosazza Gianin (2022, Section 3.1) and our research on PELVE _{n} have been carried out parallelly, and hence we decided to keep our original presentation as it is, but we always mention those results in Fiori and Rosazza Gianin (2022) which generalize our results.

Remark 1.7. Let $g : [0, 1] \rightarrow [0, 1]$ be an increasing and concave function that is continuous at 0 with $g(0) = 0$ and $g(1) = 1$. Let us consider the corresponding family of distortions $g_p : [0, 1] \rightarrow [0, 1]$, $p \in [0, 1)$, given by

$$g_p(x) := \begin{cases} g\left(\frac{x}{1-p}\right) & \text{if } x \in [0, 1-p), \\ 1 & \text{if } x \in [1-p, 1). \end{cases}$$

For each $p \in [0, 1)$ and $X \in L^1$, let us introduce the so-called Wang premia of X at level p given by

$$g\text{VaR}_X(p) := \int_0^1 \text{VaR}_X(1-s) dg_p(s),$$

provided that $\int_0^1 \text{VaR}_X(1-s) dg_p(s) \in \mathbb{R}$. Fiori and Rosazza Gianin (2022, Definition 2) have recently introduced the notion of distorted PELVE associated to the family of distortions g_p , $p \in [0, 1)$, by replacing $\text{ES}_{X,n}(1-c\varepsilon)$ with $g\text{VaR}_X(1-c\varepsilon)$ in Definition 1.6. For each $n \in \mathbb{N}$, one can check that $\text{ES}_{X,n}(p) = g\text{VaR}_X(p)$, $p \in [0, 1)$, with the function $g(x) := 1 - (1-x)^n$, $x \in [0, 1]$, which implies that PELVE_n is a distorted PELVE associated to the family of distortions g_p with the given function g . \square

Note that, just like the original PELVE, for each $n \in \mathbb{N}$, PELVE_n is defined under the minimal assumption that the random variable representing the risk has a finite first moment. Remark also that, since $\text{ES}_{X,n}(p) \geq \text{VaR}_X(p)$, $p \in [0, 1)$ (see (1.1)), we can give a similar motivation why the infimum in the definition of $\Pi_{\varepsilon,n}(X)$ is taken over $[1, \frac{1}{\varepsilon}]$ just as we did in case of $\Pi_\varepsilon(X)$ (see the paragraph after Definition 1.4). Note also that, since $\text{ES}_{X,n_1}(p) \leq \text{ES}_{X,n_2}(p)$ for $p \in [0, 1)$ and $n_1, n_2 \in \mathbb{N}$ with $n_1 \leq n_2$ (see (1.1)), we have $\Pi_{\varepsilon,n_1}(X) \leq \Pi_{\varepsilon,n_2}(X)$ for $\varepsilon \in (0, 1)$ and $n_1, n_2 \in \mathbb{N}$ with $n_1 \leq n_2$ (for more details, see Remark 2.2). In particular, $\Pi_\varepsilon(X) \leq \Pi_{\varepsilon,n}(X)$, $\varepsilon \in (0, 1)$, $n \in \mathbb{N}$.

We will prove results that can be considered as counterparts of the above mentioned results of Li and Wang (2022) and Fiori and Rosazza Gianin (2022). Our forthcoming Propositions 2.1, 2.3 and Theorem 2.4 are special cases of Propositions 3, 4, and 5 in Fiori and Rosazza Gianin (2022) for distorted PELVE, respectively. As we mentioned earlier, the two research works have been carried out parallelly, and hence we decided to present proofs of our Propositions 2.1, 2.3 and Theorem 2.4.

The paper is organized as follows. In Section 2 we study the finiteness, uniqueness and some basic properties of PELVE_n such as inequalities for PELVE_n of sum of comonotonic random variables (see Definition A.4), see Propositions 2.1, 2.3 and Theorem 2.4. In Section 3, under some appropriate conditions, we show that for each $n \in \mathbb{N}$, we have $\Pi_{\varepsilon,n}(X_m) \rightarrow \Pi_{\varepsilon,n}(X)$ as $m \rightarrow \infty$ whenever $X_m \xrightarrow{\mathcal{D}} X$ as $m \rightarrow \infty$. In Section 4, we calculate the PELVE_n -values of uniform, exponential and Pareto distributions for each $n \geq 2$, $n \in \mathbb{N}$, and we approximate the PELVE_2 -values of normal distributions. In particular, it turns out that, for a uniformly distributed random variable X , the PELVE_n value $\Pi_{\varepsilon,n}(X)$ of X equals $n+1$ for each $\varepsilon \in (0, \frac{1}{n+1}]$, i.e., it is the same constant for $\varepsilon \in (0, \frac{1}{n+1}]$. Similar phenomena occur in case of exponential and Pareto distributions, but not in case of normal distributions. In Section 5, we study PELVE_2 of a non-negative random variable having a generalized Pareto excess distribution function; and in Section 6 we describe the asymptotic behaviour of PELVE_2 of regularly varying distributions as the level tends to 0. In Sections 5 and 6, we consider the PELVE_2 -values, and not the PELVE_n -values of the random variables in question mainly due to the less complexity of computation in case of PELVE_2 . However, note that the 2nd-order Expected Shortfall plays a central role in the decomposition of Gini Shortfall presented in Appendix B.

Section 7 is devoted to presenting some simulations and real data analysis for PELVE_2 on S&P 500 daily returns. An interesting phenomenon occurs, PELVE_2 clearly shows the effect of the COVID-19 pandemic via analysing S&P 500 daily returns. We close the paper with three appendices. In Appendix A, we study some properties of higher-order Expected Shortfalls given in Definition 1.2 such as finiteness, continuity, monotonicity, additivity for comonotonic random variables and connection with weak convergence. Our results on finiteness and the connection with weak convergence are in fact consequences of recent results of Wang et al. (2020, Proposition 1 and Theorem 6) on so called distortion risk metrics. For completeness, we present independent proofs of these results as well. Appendix B is devoted to develop a connection between 2nd-order Expected Shortfall and Gini Shortfall. Finally, in Appendix C, we formulate a Karamata theorem for regularly varying functions at 0 with index $\kappa > -1$.

2. Finiteness, uniqueness and basic properties of PELVE_n

In what follows, when we write $\text{ES}_{X,n}$, $\Pi_{\varepsilon,n}(X)$ and PELVE_n we always mean that $n \in \mathbb{N}$ without mentioning it explicitly. The following result for PELVE_n can be considered as the counterpart of the corresponding result for PELVE due to Li and Wang (2022, Proposition 1). It is a special case of Proposition 3 for distorted PELVE in Fiori and Rosazza Gianin (2022).

Proposition 2.1. *Let X be a random variable such that $X \in L^1$, $\varepsilon \in (0, 1)$ and $n \in \mathbb{N}$. Then the following statements are equivalent:*

(i) *There exists $c_0 \in [1, 1/\varepsilon]$ such that*

$$\text{ES}_{X,n}(1 - c_0\varepsilon) = \text{VaR}_X(1 - \varepsilon). \tag{2.1}$$

(ii) *$\Pi_{\varepsilon,n}(X) \in [1, 1/\varepsilon]$ and (2.1) holds for $\Pi_{\varepsilon,n}(X)$, i.e., $\text{ES}_{X,n}(1 - \Pi_{\varepsilon,n}(X)\varepsilon) = \text{VaR}_X(1 - \varepsilon)$.*

(iii) *$\text{ES}_{X,n}(0) \leq \text{VaR}_X(1 - \varepsilon)$.*

(iv) *$\Pi_{\varepsilon,n}(X) < \infty$.*

Proof. (i) \Rightarrow (ii): By (i), the set $\{c \in [1, 1/\varepsilon] : \text{ES}_{X,n}(1 - c\varepsilon) \leq \text{VaR}_X(1 - \varepsilon)\}$ is nonempty yielding that $\Pi_{\varepsilon,n}(X) < \infty$ and $\Pi_{\varepsilon,n}(X) \in [1, 1/\varepsilon]$. By the definition of infimum, there exists a sequence $(c_m)_{m \in \mathbb{N}}$ in $[1, 1/\varepsilon]$ such that $c_m \downarrow \Pi_{\varepsilon,n}(X)$ as $m \rightarrow \infty$, and $\text{ES}_{X,n}(1 - c_m\varepsilon) \leq \text{VaR}_X(1 - \varepsilon)$, $m \in \mathbb{N}$. By Lemma A.2, the function $[0, 1) \ni p \mapsto \text{ES}_{X,n}(p)$ is continuous and monotone increasing, so it is continuous at the point $1 - \Pi_{\varepsilon,n}(X)\varepsilon \in [0, 1)$, and consequently, by taking the limit of both sides of $\text{ES}_{X,n}(1 - c_m\varepsilon) \leq \text{VaR}_X(1 - \varepsilon)$ as $m \rightarrow \infty$, we have

$$\mathbb{E}S_{X,n}(1 - \Pi_{\varepsilon,n}(X)\varepsilon) \leq \text{VaR}_X(1 - \varepsilon). \tag{2.2}$$

Further, using again (i), there exists $c_0 \in [1, 1/\varepsilon]$ such that $\mathbb{E}S_{X,n}(1 - c_0\varepsilon) = \text{VaR}_X(1 - \varepsilon)$, and hence, by the definition of infimum, $\Pi_{\varepsilon,n}(X) \leq c_0$. Since the function $[0, 1) \ni p \mapsto \mathbb{E}S_{X,n}(p)$ is continuous and monotone increasing,

$$\text{VaR}_X(1 - \varepsilon) = \mathbb{E}S_{X,n}(1 - c_0\varepsilon) \leq \mathbb{E}S_{X,n}(1 - \Pi_{\varepsilon,n}(X)\varepsilon). \tag{2.3}$$

Inequalities (2.2) and (2.3) yield (ii).

(ii) \Rightarrow (iii): Since the function $[0, 1) \ni p \mapsto \mathbb{E}S_{X,n}(p)$ is continuous and monotone increasing (see Lemma A.2), we have

$$\inf_{c \in [1, 1/\varepsilon]} \mathbb{E}S_{X,n}(1 - c\varepsilon) = \mathbb{E}S_{X,n}(0) = n \int_0^1 s^{n-1} \text{VaR}_X(s) \, ds.$$

On the contrary to (iii), let us suppose that $\mathbb{E}S_{X,n}(0) > \text{VaR}_X(1 - \varepsilon)$. Then $\inf_{c \in [1, 1/\varepsilon]} \mathbb{E}S_{X,n}(1 - c\varepsilon) > \text{VaR}_X(1 - \varepsilon)$, and hence $\inf\{c \in [1, 1/\varepsilon] : \mathbb{E}S_{X,n}(1 - c\varepsilon) \leq \text{VaR}_X(1 - \varepsilon)\} = \emptyset$. Consequently, by definition, $\Pi_{\varepsilon,n}(X) = \infty$, which leads us to a contradiction, since $\Pi_{\varepsilon,n}(X) < \infty$ (due to $\Pi_{\varepsilon,n}(X) \in [1, 1/\varepsilon]$).

(iii) \Rightarrow (iv): By (iii), choosing $c = 1/\varepsilon$, the set $\{c \in [1, 1/\varepsilon] : \mathbb{E}S_{X,n}(1 - c\varepsilon) \leq \text{VaR}_X(1 - \varepsilon)\}$ is nonempty, so $\Pi_{\varepsilon,n}(X) < \infty$, as desired.

(iv) \Rightarrow (i): Using (1.1), we have $\mathbb{E}S_{X,n}(p) \geq \text{VaR}_X(p)$, $p \in (0, 1)$. In fact, this inequality is a direct consequence of the fact that the function $(0, 1) \ni p \mapsto \text{VaR}_X(p)$ is monotone increasing:

$$\begin{aligned} \mathbb{E}S_{X,n}(p) &= \frac{n}{1-p} \int_p^1 \left(\frac{s-p}{1-p}\right)^{n-1} \text{VaR}_X(s) \, ds \geq \frac{n}{(1-p)^n} \int_p^1 (s-p)^{n-1} \text{VaR}_X(p) \, ds \\ &= \frac{n}{(1-p)^n} \text{VaR}_X(p) \frac{(1-p)^n}{n} = \text{VaR}_X(p). \end{aligned}$$

Further, $\mathbb{E}S_{X,n}(0) \leq \text{VaR}_X(1 - \varepsilon)$, since otherwise $\Pi_{\varepsilon,n}(X) = \infty$ would hold, which can be checked similarly as in the proof of part (ii) \Rightarrow (iii). Consequently, by the previous two inequalities, we have

$$\mathbb{E}S_{X,n}(1 - \varepsilon) \geq \text{VaR}_X(1 - \varepsilon) \geq \mathbb{E}S_{X,n}(0). \tag{2.4}$$

Since the function $[1, 1/\varepsilon] \ni c \mapsto \mathbb{E}S_{X,n}(1 - c\varepsilon)$ is continuous and monotone decreasing (see Lemma A.2), by Bolzano's intermediate value theorem and (2.4), there exists $c_0 \in [1, 1/\varepsilon]$ such that (2.1) holds, as desired. \square

In the next remark, we formulate a consequence of part (iii) of Proposition 2.1, and we also compare $\Pi_\varepsilon(X)$ and $\Pi_{\varepsilon,n}(X)$, where $\varepsilon \in (0, 1)$ and $X \in L^1$.

Remark 2.2. (i). For $X \in L^1$ and $\varepsilon \in (0, 1)$, the inequality $\mathbb{E}S_{X,n}(0) \leq \text{VaR}_X(1 - \varepsilon)$ in part (iii) in Proposition 2.1 implies $\mathbb{E}(X) \leq \text{VaR}_X(1 - \varepsilon)$ (since $\mathbb{E}(X) = \mathbb{E}S_X(0)$), which is nothing else but the inequality in part (iii) in Proposition 1 in Li and Wang (2022).

(ii). For $X \in L^1$, the inequality $\mathbb{E}S_{X,n_1}(p) \leq \mathbb{E}S_{X,n_2}(p)$ for $p \in [0, 1)$ and $n_1, n_2 \in \mathbb{N}$ with $n_1 \leq n_2$ (see (1.1)) yields $\Pi_{\varepsilon,n_1}(X) \leq \Pi_{\varepsilon,n_2}(X)$ for $\varepsilon \in (0, 1)$ and $n_1, n_2 \in \mathbb{N}$ with $n_1 \leq n_2$. In particular, $\Pi_\varepsilon(X) \leq \Pi_{\varepsilon,n}(X)$ for $\varepsilon \in (0, 1)$ and $n \in \mathbb{N}$. \square

The following result for PELVE_n can be considered as the counterpart of the corresponding result for PELVE due to Li and Wang (2022, Proposition 2). It is a special case of Proposition 4 for distorted PELVE in Fiori and Rosazza Gianin (2022).

Proposition 2.3. Let X be a random variable such that $X \in L^1$, $\varepsilon \in (0, 1)$, and $n \in \mathbb{N}$. Let us suppose that the function $(0, 1) \ni p \mapsto \text{VaR}_X(p)$ is not constant on the interval $[1 - \varepsilon, 1)$, and let us assume that $\mathbb{E}S_{X,n}(0) \leq \text{VaR}_X(1 - \varepsilon)$. Then there exists a unique $c_0 \in [1, 1/\varepsilon]$ such that (2.1) holds.

Proof. By part (iii) \Rightarrow (i) of Proposition 2.1, there exists $c_0 \in [1, 1/\varepsilon]$ such that (2.1) holds. Further, the function $[0, 1) \ni p \mapsto \mathbb{E}S_{X,n}(p)$ is continuous and it is strictly monotone increasing on $[0, 1 - \varepsilon]$ (see Lemmas A.2 and A.3), yielding that the function $[1, 1/\varepsilon] \ni c \mapsto \mathbb{E}S_{X,n}(1 - c\varepsilon)$ is continuous and strictly monotone decreasing. This together with the existence of $c_0 \in [1, 1/\varepsilon]$ satisfying (2.1) yields the uniqueness of such a c_0 , as desired. \square

Note that the assumption $\mathbb{E}S_{X,n}(0) \leq \text{VaR}_X(1 - \varepsilon)$ in Proposition 2.3 yields that $\mathbb{E}S_X(0) \leq \text{VaR}_X(1 - \varepsilon)$ (due to the second inequality in (1.1)), which is nothing else but the corresponding condition for PELVE in Proposition 2 in Li and Wang (2022). Consequently, under the assumptions of Proposition 2.3 we also have that $\Pi_\varepsilon(X) < \infty$ and $\mathbb{E}S_X(1 - \Pi_\varepsilon(X)\varepsilon) = \text{VaR}_X(1 - \varepsilon)$.

The following result for PELVE_n can be considered as the counterpart of the corresponding result for PELVE due to Li and Wang (2022, Theorem 1). It is a special case of Proposition 5 for distorted PELVE in Fiori and Rosazza Gianin (2022).

Theorem 2.4. Let X be a random variable such that $X \in L^1$, $\varepsilon \in (0, 1)$, and $n \in \mathbb{N}$. Let us suppose that $\mathbb{E}S_{X,n}(0) \leq \text{VaR}_X(1 - \varepsilon)$ holds. Then the following statements hold:

- (i) scale-location invariance: $\Pi_{\varepsilon,n}(\lambda X + a) = \Pi_{\varepsilon,n}(X)$ for each $\lambda > 0$ and $a \in \mathbb{R}$.
- (ii) $\Pi_{\varepsilon,n}(f(X)) \leq \Pi_{\varepsilon,n}(X)$ for each monotone increasing and concave function $f : \mathbb{R} \rightarrow \mathbb{R}$ with $f(X) \in L^1$.
- (iii) $\Pi_{\varepsilon,n}(X) \leq \Pi_{\varepsilon,n}(g(X))$ for each strictly monotone increasing and convex function $g : \mathbb{R} \rightarrow \mathbb{R}$ with $\text{Range}(g) = \mathbb{R}$ and $g(X) \in L^1$.

(iv) quasi-convexity and quasi-concavity for comonotonic random variables:

$$\min\{\Pi_{\varepsilon,n}(f(X)), \Pi_{\varepsilon,n}(g(X))\} \leq \Pi_{\varepsilon,n}(\lambda f(X) + (1 - \lambda)g(X)) \leq \max\{\Pi_{\varepsilon,n}(f(X)), \Pi_{\varepsilon,n}(g(X))\}$$

for each $\lambda \in [0, 1]$ and monotone increasing functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$ with $f(X), g(X) \in L^1$.

Proof. First, we prove that (ii) yields (i). Let $\lambda > 0, a \in \mathbb{R}, f : \mathbb{R} \rightarrow \mathbb{R}, f(x) := \lambda x + a, x \in \mathbb{R}$, and $g : \mathbb{R} \rightarrow \mathbb{R}, g(y) := \frac{1}{\lambda}(y - a), y \in \mathbb{R}$. Then f and g are monotone increasing, linear (hence convex and concave) functions, and they are inverses of each other. Further, $f(X), g(X) \in L^1$, since $\mathbb{E}(|f(X)|) \leq \lambda \mathbb{E}(|X|) + |a| < \infty$ and $\mathbb{E}(|g(X)|) \leq \frac{1}{\lambda} \mathbb{E}(|X|) + \frac{|a|}{\lambda} < \infty$ due to $X \in L^1$. Consequently, by (ii),

$$\Pi_{\varepsilon,n}(f(X)) \leq \Pi_{\varepsilon,n}(X) = \Pi_{\varepsilon,n}(g \circ f)(X) \leq \Pi_{\varepsilon,n}(f(X)),$$

yielding $\Pi_{\varepsilon,n}(f(X)) = \Pi_{\varepsilon,n}(X)$, i.e., (i), as desired.

Now we prove (ii). Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a monotone increasing and concave function with $f(X) \in L^1$. Since f is concave and defined on (the open interval) \mathbb{R} , we have f is continuous. Further, it is known that for any monotone increasing and continuous function $h : \mathbb{R} \rightarrow \mathbb{R}$, we have $\text{VaR}_{h(X)}(p) = h(\text{VaR}_X(p)), p \in (0, 1)$, see, e.g., Shorack and Wellner (2009, Exercise 3, page 9) or Dhaene et al. (2002, part (a) of Theorem 1). Consequently, we get

$$\text{VaR}_{f(X)}(p) = f(\text{VaR}_X(p)), \quad p \in (0, 1). \tag{2.5}$$

Let $c_1 := \Pi_{\varepsilon,n}(X)$ and $c_2 := \Pi_{\varepsilon,n}(f(X))$. Since, by assumption, $\text{ES}_{X,n}(0) \leq \text{VaR}_X(1 - \varepsilon)$, using the equivalence of (ii), (iii) and (iv) in Proposition 2.1, we have $c_1 < \infty$ and

$$\text{ES}_{X,n}(1 - c_1\varepsilon) = \text{VaR}_X(1 - \varepsilon). \tag{2.6}$$

Note that for each $p \in [0, 1)$, the function $\sigma_p : [0, 1) \rightarrow [0, \infty), \sigma_p(s) := \frac{n}{(1-p)^n}(s - p)^{n-1} \mathbf{1}_{[p,1]}(s), s \in [0, 1]$, is a distortion function in the sense of Definition 3.6 in Pflug and Pichler (2014), since it is non-negative, monotone increasing and $\int_0^1 \sigma_p(s) ds = 1$. Hence for each $p \in [0, 1)$ one can apply Corollary 3.19 in Pflug and Pichler (2014) with the distortion function σ_p , and we have

$$\text{ES}_{f(X),n}(p) = \sup_{U \text{ is uniformly distributed on } [0, 1]} \mathbb{E}(f(X)\sigma_p(U)), \quad p \in [0, 1).$$

Here we implicitly assumed that the underlying probability space is rich enough such that there exists a random variable on it with uniform distribution on $[0, 1]$. This assumption is not a restriction, see, e.g., Rachev (1991, Theorem 2.5.1). In particular, if the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless, then there exists a random variable $U : \Omega \rightarrow \mathbb{R}$ with uniform distribution on $[0, 1]$, see, e.g., Rachev (1991, Theorems 2.4.1 and 2.5.2). Further, one can check that for any random variable U which is uniformly distributed on $[0, 1]$, we have $\mathbb{Q}(A) := \int_A (\sigma_p(U))(\omega) \mathbb{P}(d\omega), A \in \mathcal{F}$, is a probability measure on (Ω, \mathcal{F}) such that \mathbb{Q} is absolutely continuous with respect to \mathbb{P} and $\mathbb{E}(f(X)\sigma_p(U)) = \mathbb{E}_{\mathbb{Q}}(f(X))$. Hence, for any random variable U which is uniformly distributed on $[0, 1]$, by Jensen's inequality, we get

$$\mathbb{E}(f(X)\sigma_p(U)) = \mathbb{E}_{\mathbb{Q}}(f(X)) \leq f(\mathbb{E}_{\mathbb{Q}}(X)) = f(\mathbb{E}(X\sigma_p(U))).$$

Consequently, using also that f is monotone increasing, we have

$$\begin{aligned} \text{ES}_{f(X),n}(p) &\leq \sup_{U \text{ is uniformly distributed on } [0, 1]} f(\mathbb{E}(X\sigma_p(U))) \leq f\left(\sup_{U \text{ is uniformly distributed on } [0, 1]} \mathbb{E}(X\sigma_p(U))\right) \\ &= f(\text{ES}_{X,n}(p)), \quad p \in [0, 1). \end{aligned} \tag{2.7}$$

Since the functions $f, [0, 1) \ni p \mapsto \text{ES}_{X,n}(p)$ and $[0, 1) \ni p \mapsto \text{ES}_{f(X),n}(p)$ are continuous (see Lemma A.2), by taking the limit of both sides of the inequality (2.7) as $p \downarrow 0$, we have

$$\text{ES}_{f(X),n}(0) \leq f(\text{ES}_{X,n}(0)) \leq f(\text{VaR}_X(1 - \varepsilon)) = \text{VaR}_{f(X)}(1 - \varepsilon),$$

where, for the second inequality, we used that f is monotone increasing and $\text{ES}_{X,n}(0) \leq \text{VaR}_X(1 - \varepsilon)$ (by assumption), and, for the equality, (2.5). So, by Proposition 2.1, we have $c_2 = \Pi_{\varepsilon,n}(f(X)) < \infty$ and $\text{ES}_{f(X),n}(1 - c_2\varepsilon) = \text{VaR}_{f(X)}(1 - \varepsilon)$. Using (2.7) with the choice of $p := 1 - c_1\varepsilon$, (2.6) and (2.5), we get

$$\text{ES}_{f(X),n}(1 - c_1\varepsilon) \leq f(\text{ES}_{X,n}(1 - c_1\varepsilon)) = f(\text{VaR}_X(1 - \varepsilon)) = \text{VaR}_{f(X)}(1 - \varepsilon),$$

and, by Definition 1.6 of PELVE_n , we have $\Pi_{\varepsilon,n}(f(X)) \leq c_1$, i.e., $\Pi_{\varepsilon,n}(f(X)) \leq \Pi_{\varepsilon,n}(X)$, as desired.

(iii). Since g is a convex function defined on \mathbb{R} , it is continuous. Further, due to our assumptions, $g^{-1} : \mathbb{R} \rightarrow \mathbb{R}$ is a strictly monotone increasing and concave function with $\text{Range}(g^{-1}) = \mathbb{R}$, and since $g(X) \in L^1$ and $g^{-1}(g(X)) = X \in L^1$, part (ii) yields that

$$\Pi_{\varepsilon,n}(X) = \Pi_{\varepsilon,n}(g^{-1}(g(X))) \leq \Pi_{\varepsilon,n}(g(X)),$$

as desired.

(iv). One can easily check that for any $c, d \in [1, \frac{1}{\varepsilon}]$ and $Y \in L^1$, we have

$$d < \Pi_{\varepsilon,n}(Y) \leq c \iff \text{ES}_{Y,n}(1 - c\varepsilon) \leq \text{ES}_{Y,n}(1 - \Pi_{\varepsilon,n}(Y)\varepsilon) = \text{VaR}_Y(1 - \varepsilon) < \text{ES}_{Y,n}(1 - d\varepsilon), \tag{2.8}$$

for a proof, see our ArXiv version Barczy et al. (2022, the proof of formula (2.8)).

Since $f, g : \mathbb{R} \rightarrow \mathbb{R}$ are monotone increasing and $\lambda \in [0, 1]$, we have $\lambda f(X)$ and $(1 - \lambda)g(X)$ are comonotonic random variables, so, by the additivity of VaR for comonotonic random variables (see, e.g., McNeil et al., 2015, Proposition 7.20 or Dhaene et al., 2006, Theorem 4.2.1), we get

$$\text{VaR}_{\lambda f(X)+(1-\lambda)g(X)}(1 - \varepsilon) = \text{VaR}_{\lambda f(X)}(1 - \varepsilon) + \text{VaR}_{(1-\lambda)g(X)}(1 - \varepsilon).$$

Let $c_1 := \Pi_{\varepsilon,n}(f(X))$ and $c_2 := \Pi_{\varepsilon,n}(g(X))$. Let us suppose that $c_1 = \infty$ and $c_2 = \infty$. In this case $\min(c_1, c_2) = \infty$ and $\max(c_1, c_2) = \infty$, so it is enough to check that $\Pi_{\varepsilon,n}(\lambda f(X) + (1 - \lambda)g(X)) = \infty$. By Proposition 2.1, the positive homogeneity and comonotonic additivity of Value at Risk and n^{th} -order Expected Shortfall (see Proposition A.5), we have

$$\begin{aligned} \Pi_{\varepsilon,n}(\lambda f(X) + (1 - \lambda)g(X)) = \infty &\Leftrightarrow \text{ES}_{\lambda f(X)+(1-\lambda)g(X),n}(0) > \text{VaR}_{\lambda f(X)+(1-\lambda)g(X)}(1 - \varepsilon) \\ &\Leftrightarrow \lambda \text{ES}_{f(X),n}(0) + (1 - \lambda) \text{ES}_{g(X),n}(0) > \lambda \text{VaR}_{f(X)}(1 - \varepsilon) + (1 - \lambda) \text{VaR}_{g(X)}(1 - \varepsilon). \end{aligned}$$

Here the last inequality is satisfied, since using again Proposition 2.1, $c_1 = \Pi_{\varepsilon,n}(f(X)) = \infty$, $c_2 = \Pi_{\varepsilon,n}(g(X)) = \infty$, and $f(X), g(X) \in L^1$, we have

$$\text{ES}_{f(X),n}(0) > \text{VaR}_{f(X)}(1 - \varepsilon) \quad \text{and} \quad \text{ES}_{g(X),n}(0) > \text{VaR}_{g(X)}(1 - \varepsilon).$$

Let us suppose now that at least one of c_1 and c_2 is finite. Then for each $d < \min\{c_1, c_2\}$ with $d \in [1, \frac{1}{\varepsilon}]$, using (2.8) and Proposition 2.1, we get

$$\text{VaR}_{f(X)}(1 - \varepsilon) < \text{ES}_{f(X),n}(1 - d\varepsilon) \quad \text{and} \quad \text{VaR}_{g(X)}(1 - \varepsilon) < \text{ES}_{g(X),n}(1 - d\varepsilon).$$

Indeed, if both c_1 and c_2 are finite, then it readily follows by (2.8); and if $c_1 < \infty$ and $c_2 = \infty$, then, by (2.8), $\text{VaR}_{f(X)}(1 - \varepsilon) < \text{ES}_{f(X),n}(1 - d\varepsilon)$, and, by Proposition 2.1 and the monotone increasing property of the function $[0, 1) \ni p \mapsto \text{ES}_{g(X),n}(p)$, we have $\text{VaR}_{g(X)}(1 - \varepsilon) < \text{ES}_{g(X),n}(0) \leq \text{ES}_{g(X),n}(1 - d\varepsilon)$, as desired. The case $c_1 = \infty$ and $c_2 < \infty$ can be handled similarly.

As a consequence, for each $\lambda \in [0, 1]$, we have

$$\lambda \text{VaR}_{f(X)}(1 - \varepsilon) + (1 - \lambda) \text{VaR}_{g(X)}(1 - \varepsilon) < \lambda \text{ES}_{f(X),n}(1 - d\varepsilon) + (1 - \lambda) \text{ES}_{g(X),n}(1 - d\varepsilon),$$

and using again the positive homogeneity and comonotonic additivity of Value at Risk and n^{th} -order Expected Shortfall, we have

$$\text{VaR}_{\lambda f(X)+(1-\lambda)g(X)}(1 - \varepsilon) < \text{ES}_{\lambda f(X)+(1-\lambda)g(X),n}(1 - d\varepsilon), \quad \lambda \in [0, 1].$$

Hence, by Definition 1.6 and the continuity and monotone increasing property of the function $[0, 1) \ni p \mapsto \text{ES}_{\lambda f(X)+(1-\lambda)g(X),n}(p)$, we have $d < \Pi_{\varepsilon,n}(\lambda f(X) + (1 - \lambda)g(X))$ for $\lambda \in [0, 1]$ and $d < \min\{c_1, c_2\}$. By taking the limit $d \uparrow \min\{c_1, c_2\}$, we have

$$\min\{c_1, c_2\} \leq \Pi_{\varepsilon,n}(\lambda f(X) + (1 - \lambda)g(X)),$$

as desired.

If $\max(c_1, c_2) = \infty$, then the second inequality in (iv) automatically holds. If $\max(c_1, c_2) < \infty$, i.e., both c_1 and c_2 are finite, then let $c := \max\{c_1, c_2\}$. Then $\Pi_{\varepsilon,n}(f(X)) = c_1 \leq c$ and $\Pi_{\varepsilon,n}(g(X)) = c_2 \leq c$, so, by (2.8),

$$\text{ES}_{f(X),n}(1 - c\varepsilon) \leq \text{VaR}_{f(X)}(1 - \varepsilon) \quad \text{and} \quad \text{ES}_{g(X),n}(1 - c\varepsilon) \leq \text{VaR}_{g(X)}(1 - \varepsilon).$$

Hence for each $\lambda \in [0, 1]$,

$$\lambda \text{ES}_{f(X),n}(1 - c\varepsilon) + (1 - \lambda) \text{ES}_{g(X),n}(1 - c\varepsilon) \leq \lambda \text{VaR}_{f(X)}(1 - \varepsilon) + (1 - \lambda) \text{VaR}_{g(X)}(1 - \varepsilon),$$

and then the positive homogeneity and comonotonic additivity of VaR and n^{th} -order Expected Shortfall yield that

$$\text{ES}_{\lambda f(X)+(1-\lambda)g(X),n}(1 - c\varepsilon) \leq \text{VaR}_{\lambda f(X)+(1-\lambda)g(X)}(1 - \varepsilon).$$

Consequently, using (2.8) with $Y := \lambda f(X) + (1 - \lambda)g(X)$, we have

$$\Pi_{\varepsilon,n}(\lambda f(X) + (1 - \lambda)g(X)) \leq c = \max\{c_1, c_2\},$$

as desired. \square

In the next remark, we point out that part (iii) of Theorem 2.4 does not hold for a general monotone increasing and convex function $g : \mathbb{R} \rightarrow \mathbb{R}$.

Remark 2.5. Let $g : \mathbb{R} \rightarrow \mathbb{R}$, $g(x) = A$, $x \in \mathbb{R}$, with some $A \in \mathbb{R}$. Then for any random variable X , $n \in \mathbb{N}$, and $p \in (0, 1)$, we have $\text{VaR}_{g(X)}(p) = A$ and

$$\text{ES}_{g(X),n}(p) = \frac{n}{1-p} \int_p^1 \left(\frac{s-p}{1-p}\right)^{n-1} \text{VaR}_{g(X)}(s) ds = \frac{nA}{(1-p)^n} \int_p^1 (s-p)^{n-1} ds = A,$$

and hence for each $\varepsilon \in (0, 1)$,

$$\Pi_{\varepsilon,n}(g(X)) = \inf \left\{ c \in \left[1, \frac{1}{\varepsilon}\right] : \text{ES}_{g(X),n}(1 - c\varepsilon) \leq \text{VaR}_{g(X)}(1 - \varepsilon) \right\} = \inf \left[1, \frac{1}{\varepsilon}\right] = 1.$$

Consequently, if X is random variable such that $\Pi_{\varepsilon,n}(X) > 1$, then $\Pi_{\varepsilon,n}(g(X)) \geq \Pi_{\varepsilon,n}(X)$ cannot hold. Note that g is not strictly increasing and $\text{Range}(g) = \{A\} \neq \mathbb{R}$. All in all, part (iii) of Theorem 2.4 does not hold for a general monotone increasing and convex function $g : \mathbb{R} \rightarrow \mathbb{R}$. \square

3. Convergence properties of PELVE_n

The following result for PELVE_n can be considered as the counterpart of the corresponding result for PELVE due to Li and Wang (2022, Theorem 2).

Theorem 3.1. *Let $X_m \in L^1$, $m \in \mathbb{N}$, and $X \in L^1$ be random variables. Let $\varepsilon \in (0, 1)$ and $n \in \mathbb{N}$. If*

- (i) $\text{ES}_{X,n}(0) < \text{VaR}_X(1 - \varepsilon)$,
- (ii) *the function $(0, 1) \ni p \mapsto \text{VaR}_X(p)$ is not constant on the interval $[1 - \varepsilon, 1)$ and it is continuous at $1 - \varepsilon$,*
- (iii) $X_m \xrightarrow{D} X$ as $m \rightarrow \infty$,
- (iv) $\{X_m : m \in \mathbb{N}\}$ *is uniformly integrable,*

then $\Pi_{\varepsilon,n}(X_m) \rightarrow \Pi_{\varepsilon,n}(X)$ as $m \rightarrow \infty$.

Proof. By the second part of (ii) and (iii), the quantile convergence theorem (see, e.g., Shorack and Wellner, 2009, Exercise 5, page 10) yields that

$$\text{VaR}_{X_m}(1 - \varepsilon) \rightarrow \text{VaR}_X(1 - \varepsilon) \quad \text{as } m \rightarrow \infty. \tag{3.1}$$

Using (iii), (iv) and Lemma A.6, we have

$$\text{ES}_{X_m,n}(t) \rightarrow \text{ES}_{X,n}(t) \quad \text{as } m \rightarrow \infty \text{ for each } t \in [0, 1). \tag{3.2}$$

Let us introduce the functions $f_m : [0, 1) \rightarrow \mathbb{R}$, $m \in \mathbb{N}$, and $f : [0, 1) \rightarrow \mathbb{R}$, given by

$$f_m(t) := \text{ES}_{X_m,n}(t) - \text{VaR}_{X_m}(1 - \varepsilon), \quad t \in [0, 1),$$

and

$$f(t) := \text{ES}_{X,n}(t) - \text{VaR}_X(1 - \varepsilon), \quad t \in [0, 1).$$

By (3.1) and (3.2), we have f_m converges pointwise to f on $[0, 1)$ as $m \rightarrow \infty$, and, using also Lemma A.2, we get that f_m , $m \in \mathbb{N}$, and f are continuous and monotone increasing functions. Further, the first part of (ii) and Lemma A.3 yield that f is strictly monotone increasing on the interval $[0, 1 - \varepsilon]$. Then, by a result from calculus, we have that f_m converges uniformly on any interval $[0, 1 - \delta]$ to f as $m \rightarrow \infty$, where $\delta \in (0, 1)$.

Let us consider the reparametrizations $g_m : [1, \frac{1}{\varepsilon}] \rightarrow \mathbb{R}$, $m \in \mathbb{N}$, and $g : [1, \frac{1}{\varepsilon}] \rightarrow \mathbb{R}$ of f_m , $m \in \mathbb{N}$, and f , respectively, given by

$$g_m(c) := f_m(1 - c\varepsilon) = \text{ES}_{X_m,n}(1 - c\varepsilon) - \text{VaR}_{X_m}(1 - \varepsilon), \quad c \in \left[1, \frac{1}{\varepsilon}\right],$$

and

$$g(c) := f(1 - c\varepsilon) = \text{ES}_{X,n}(1 - c\varepsilon) - \text{VaR}_X(1 - \varepsilon), \quad c \in \left[1, \frac{1}{\varepsilon}\right].$$

Then g_m , $m \in \mathbb{N}$, and g are continuous and monotone decreasing functions, and g is strictly monotone decreasing as well. Further, g_m converges uniformly on $[1, \frac{1}{\varepsilon}]$ to g as $m \rightarrow \infty$.

Using (i), (3.1) and (3.2) with $t = 0$, we have $\text{ES}_{X_m,n}(0) < \text{VaR}_{X_m}(1 - \varepsilon)$ for large enough $m \in \mathbb{N}$. Hence, by Proposition 2.1, for large enough $m \in \mathbb{N}$, we have $\Pi_{\varepsilon,n}(X_m) \in [1, \frac{1}{\varepsilon}]$ and $\Pi_{\varepsilon,n}(X_m)$ solves the equation $g_m(c) = 0$, $c \in [1, \frac{1}{\varepsilon}]$, and we also have $\Pi_{\varepsilon,n}(X) \in [1, \frac{1}{\varepsilon}]$ and $\Pi_{\varepsilon,n}(X)$ solves the equation $g(c) = 0$, $c \in [1, \frac{1}{\varepsilon}]$.

Then for each $m \in \mathbb{N}$,

$$|\Pi_{\varepsilon,n}(X_m)| = |g(\Pi_{\varepsilon,n}(X_m)) - g_m(\Pi_{\varepsilon,n}(X_m))| \leq \sup_{c \in [1, \frac{1}{\varepsilon}]} |g_m(c) - g(c)| \rightarrow 0 \quad \text{as } m \rightarrow \infty,$$

since g_m converges uniformly on $[1, \frac{1}{\varepsilon}]$ to g as $m \rightarrow \infty$. So $\lim_{m \rightarrow \infty} g(\Pi_{\varepsilon,n}(X_m)) = 0$. Further, if $\tilde{c} \in [1, \frac{1}{\varepsilon}]$ is a limit point of the sequence $(\Pi_{\varepsilon,n}(X_m))_{m \in \mathbb{N}}$, then there exists a subsequence $(\Pi_{\varepsilon,n}(X_{m_k}))_{k \in \mathbb{N}}$ in $[1, \frac{1}{\varepsilon}]$ such that $\Pi_{\varepsilon,n}(X_{m_k}) \neq \tilde{c}$, $k \in \mathbb{N}$, and $\Pi_{\varepsilon,n}(X_{m_k}) \rightarrow \tilde{c}$ as $k \rightarrow \infty$. Since g is continuous, we have $g(\Pi_{\varepsilon,n}(X_{m_k})) \rightarrow g(\tilde{c})$ as $k \rightarrow \infty$, where $g(\tilde{c}) = 0$ due to $\lim_{m \rightarrow \infty} g(\Pi_{\varepsilon,n}(X_m)) = 0$. Hence $\tilde{c} \in [1, \frac{1}{\varepsilon}]$ is a root of g , and using that g has a unique root $\Pi_{\varepsilon,n}(X)$ on $[1, \frac{1}{\varepsilon}]$ (since we already checked that $\Pi_{\varepsilon,n}(X)$ is a root of g and g is continuous and strictly monotone decreasing), we get $\tilde{c} = \Pi_{\varepsilon,n}(X)$. All in all, for any limit point \tilde{c} of $(\Pi_{\varepsilon,n}(X_m))_{m \in \mathbb{N}}$, we have $\tilde{c} = \Pi_{\varepsilon,n}(X)$. Since $(\Pi_{\varepsilon,n}(X_m))_{m \in \mathbb{N}}$ is a bounded sequence in $[1, \frac{1}{\varepsilon}]$, it has a limit point, and taking into account our previous considerations, $\Pi_{\varepsilon,n}(X_m)$ converges to its unique limit point $\Pi_{\varepsilon,n}(X)$ as $m \rightarrow \infty$. \square

4. PELVE_n of some notable distributions

In this section, we calculate the PELVE_n-values of uniform, exponential and Pareto distributions for each $n \geq 2$, $n \in \mathbb{N}$, and we approximate the PELVE₂-values of normal distributions. Some direct computations are omitted in this section, for the details, see our ArXiv version Barczy et al. (2022, Section 4).

Example 4.1 (Uniform distribution). Let X be a random variable with uniform distribution on the interval $[0, 1]$, and let $n \in \mathbb{N}$. Then $\text{VaR}_X(p) = p$, $p \in (0, 1)$, and

$$\text{ES}_{X,n}(p) = \frac{p}{n+1} + \frac{n}{n+1}, \quad p \in [0, 1).$$

So for each $\varepsilon \in (0, 1)$, the inequality $\text{ES}_{X,n}(0) \leq \text{VaR}_X(1 - \varepsilon)$ is equivalent to $\frac{n}{n+1} \leq 1 - \varepsilon$, and hence, by Proposition 2.1, if $\varepsilon \in (0, \frac{1}{n+1}]$, then $\Pi_{\varepsilon,n}(X)$ is a solution of the equation $\text{ES}_{X,n}(1 - c\varepsilon) = \text{VaR}_X(1 - \varepsilon)$, $c \in [1, \frac{1}{\varepsilon}]$, taking the form

$$\frac{1 - c\varepsilon}{n+1} + \frac{n}{n+1} = 1 - \varepsilon, \quad c \in \left[1, \frac{1}{\varepsilon}\right].$$

Hence $\Pi_{\varepsilon,n}(X) = n + 1$ for $\varepsilon \in (0, \frac{1}{n+1}]$. If $\varepsilon \in (\frac{1}{n+1}, 1)$, then, by Definition 1.6, $\Pi_{\varepsilon,n}(X) = \infty$. Note that, by Li and Wang (2022, part (i) of Example 5), $\Pi_\varepsilon(X) = 2$ for $\varepsilon \in (0, \frac{1}{2}]$, and hence $\Pi_{\varepsilon,n}(X) > \Pi_\varepsilon(X)$ for $\varepsilon \in (0, \frac{1}{n+1}]$ (as it is expected, see part (ii) of Remark 2.2).

Let Y be a random variable with uniform distribution on the interval $[a, b]$, where $a < b$, $a, b \in \mathbb{R}$. Using that the distribution of Y coincides with that of $(b - a)X + a$, part (i) of Theorem 2.4 yields $\Pi_{\varepsilon,n}(Y) = \Pi_{\varepsilon,n}((b - a)X + a) = \Pi_{\varepsilon,n}(X)$, so

$$\Pi_{\varepsilon,n}(Y) = \begin{cases} n + 1 & \text{if } \varepsilon \in (0, \frac{1}{n+1}], \\ \infty & \text{if } \varepsilon \in (\frac{1}{n+1}, 1). \end{cases} \quad \square$$

Example 4.2 (Exponential distribution). Let X be an exponentially distributed random variable with parameter 1, and let $n \in \mathbb{N}$. Then $\text{VaR}_X(p) = -\ln(1 - p)$, $p \in (0, 1)$, and

$$\text{ES}_{X,n}(p) = -\frac{n}{(1-p)^n} \int_0^{1-p} (1-p-s)^{n-1} \ln(s) \, ds, \quad p \in [0, 1).$$

Using formula 2.725/2 in Gradshteyn and Ryzhik (2007) with $a := 1 - p$, $b := -1$ and $m := n - 1$, we have

$$\int_0^{1-p} (1-p-s)^{n-1} \ln(s) \, ds = -\frac{1}{n} \left((1-p-s)^n - (1-p)^n \right) \ln(s) - \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{(-1)^k}{(k+1)^2} (1-p)^{n-1-k} s^{k+1} + C,$$

where $C \in \mathbb{R}$. It yields that

$$\text{ES}_{X,n}(p) = -\ln(1-p) + n \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{(-1)^k}{(k+1)^2} = H_n - \ln(1-p), \quad p \in [0, 1),$$

where $H_n := \sum_{k=1}^n \frac{1}{k}$ denotes the n -th harmonic number. In particular, we have

$$\text{ES}_{X,2}(p) = -\ln(1-p) + \frac{3}{2}, \quad p \in [0, 1).$$

So for each $\varepsilon \in (0, 1)$, the inequality $\text{ES}_{X,n}(0) \leq \text{VaR}_X(1 - \varepsilon)$ is equivalent to $H_n \leq -\ln(\varepsilon)$, i.e., $\varepsilon \in (0, e^{-H_n}]$. Hence, by Proposition 2.1, if $\varepsilon \in (0, e^{-H_n}]$, then $\Pi_{\varepsilon,n}(X)$ is a solution of the equation $\text{ES}_{X,n}(1 - c\varepsilon) = \text{VaR}_X(1 - \varepsilon)$, $c \in [1, \frac{1}{\varepsilon}]$, taking the form

$$-\ln(c\varepsilon) + H_n = -\ln(\varepsilon), \quad c \in \left[1, \frac{1}{\varepsilon}\right].$$

Hence

$$\Pi_{\varepsilon,n}(X) = e^{H_n} \quad \text{for } \varepsilon \in (0, e^{-H_n}].$$

If $\varepsilon \in (e^{-H_n}, 1)$, then, by Definition 1.6, $\Pi_{\varepsilon,n}(X) = \infty$

In particular, we have $\Pi_{\varepsilon,2}(X) = e^{\frac{3}{2}} \approx 4.482$ for $\varepsilon \in (0, e^{-3/2}]$, and $\Pi_{\varepsilon,2}(X) = \infty$ for $\varepsilon \in (e^{-3/2}, 1)$.

Note that, by Li and Wang (2022, Example 5, part (ii)), $\Pi_\varepsilon(X) = e$ for $\varepsilon \in (0, e^{-1}]$, so $\Pi_{\varepsilon,n}(X) > \Pi_\varepsilon(X)$ for $\varepsilon \in (0, e^{-H_n}]$ (as it is expected, see part (ii) of Remark 2.2).

Let Y be an exponentially distributed random variable with parameter $\lambda > 0$. Using that the distribution of Y coincides with that of $\frac{1}{\lambda}X$, part (i) of Theorem 2.4 yields $\Pi_{\varepsilon,n}(Y) = \Pi_{\varepsilon,n}(\frac{1}{\lambda}X) = \Pi_{\varepsilon,n}(X)$, so

$$\Pi_{\varepsilon,n}(Y) = \begin{cases} e^{H_n} & \text{if } \varepsilon \in (0, e^{-H_n}], \\ \infty & \text{if } \varepsilon \in (e^{-H_n}, 1). \end{cases} \quad \square$$

Example 4.3 (Normal distribution). Let X be a standard normally distributed random variable. Let Φ and φ denote the distribution function and density function of X , respectively. Then $\text{VaR}_X(p) = \Phi^{-1}(p)$, $p \in (0, 1)$, and, by substitution $s = \Phi(x)$, we have

$$\text{ES}_{X,2}(p) = \frac{2}{(1-p)^2} \left(\int_{\Phi^{-1}(p)}^{\infty} x\Phi(x)\varphi(x) \, dx - p \int_{\Phi^{-1}(p)}^{\infty} x\varphi(x) \, dx \right), \quad p \in [0, 1),$$

where $\Phi^{-1}(0)$ is defined to be $-\infty$. Here, since $\varphi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$, $x \in \mathbb{R}$, we have

$$-p \int_{\Phi^{-1}(p)}^{\infty} x\varphi(x) \, dx = -p\varphi(\Phi^{-1}(p)), \quad p \in [0, 1).$$

Further, by partial integration and then substitution $x = \frac{y}{\sqrt{2}}$, we get

$$\int_{\Phi^{-1}(p)}^{\infty} x\Phi(x)\varphi(x) \, dx = p\varphi(\Phi^{-1}(p)) + \frac{1}{2\sqrt{\pi}}(1 - \Phi(\sqrt{2}\Phi^{-1}(p))), \quad p \in [0, 1).$$

Hence

$$\text{ES}_{X,2}(p) = \frac{1}{\sqrt{\pi}(1-p)^2}(1 - \Phi(\sqrt{2}\Phi^{-1}(p))), \quad p \in [0, 1).$$

So for each $\varepsilon \in (0, 1)$, the inequality $\text{ES}_{X,2}(0) \leq \text{VaR}_X(1 - \varepsilon)$ is equivalent to $\frac{1}{\sqrt{\pi}} \leq \Phi^{-1}(1 - \varepsilon)$, i.e., $\varepsilon \in (0, 1 - \Phi(1/\sqrt{\pi})]$, and hence, by Proposition 2.1, if $\varepsilon \in (0, 1 - \Phi(1/\sqrt{\pi})] \approx (0, 0.286]$, then $\Pi_{\varepsilon,2}(X)$ is a solution of the equation $\text{ES}_{X,2}(1 - c\varepsilon) = \text{VaR}_X(1 - \varepsilon)$, $c \in [1, \frac{1}{\varepsilon}]$, taking the form

$$\frac{1}{\sqrt{\pi}c^2\varepsilon^2}(1 - \Phi(\sqrt{2}\Phi^{-1}(1 - c\varepsilon))) = \Phi^{-1}(1 - \varepsilon), \quad c \in \left[1, \frac{1}{\varepsilon}\right]. \tag{4.1}$$

If $\varepsilon \in (1 - \Phi(1/\sqrt{\pi}), 1)$, then, by Definition 1.6, $\Pi_{\varepsilon,2}(X) = \infty$.

Let Y be a normally distributed random variable with mean m and variance σ^2 , where $m \in \mathbb{R}$ and $\sigma > 0$. Using that the distribution of Y coincides with that of $\sigma X + m$, part (i) of Theorem 2.4 yields $\Pi_{\varepsilon,2}(Y) = \Pi_{\varepsilon,2}(\sigma X + m) = \Pi_{\varepsilon,2}(X)$.

Using the software R, for levels $\varepsilon \in \{0.1, 0.05, 0.01, 0.005\}$ we calculate an approximated value of the unique root $\Pi_{\varepsilon,2}(X)$ of the equation (4.1), see Table 1. \square

Table 1
Approximations of PELVE_2 -values of normal distribution $\mathcal{N}(m, \sigma^2)$ ($m \in \mathbb{R}$, $\sigma > 0$).

ε	PELVE_2 of $\mathcal{N}(m, \sigma^2)$
0.100	3.92217
0.050	4.04082
0.010	4.18527
0.005	4.22188

Example 4.4 (Pareto distribution). Let X be a random variable with Pareto distribution having parameters $k > 0$ and $\alpha > 0$, i.e., the distribution function of X takes the form $F_X : \mathbb{R} \rightarrow [0, 1]$,

$$F_X(x) := \begin{cases} 1 - \left(\frac{k}{x}\right)^\alpha & \text{if } x \geq k, \\ 0 & \text{if } x < k. \end{cases}$$

Further, let $n \in \mathbb{N}$. Then

$$\text{VaR}_X(p) = k(1-p)^{-\frac{1}{\alpha}}, \quad p \in (0, 1).$$

In what follows we suppose that $\alpha > 1$, yielding that $X \in L^1$. In this case we have

$$\text{ES}_{X,n}(p) = kn(1-p)^{-\frac{1}{\alpha}} \sum_{j=0}^{n-1} \binom{n-1}{j} \frac{(-1)^j}{j - \frac{1}{\alpha} + 1}, \quad p \in [0, 1).$$

In particular, we have

$$\text{ES}_{X,2}(p) = \frac{2k\alpha^2}{(\alpha-1)(2\alpha-1)}(1-p)^{-\frac{1}{\alpha}}, \quad p \in [0, 1).$$

So for each $\varepsilon \in (0, 1)$, the inequality $\text{ES}_{X,n}(0) \leq \text{VaR}_X(1 - \varepsilon)$ is equivalent to

$$kn \sum_{j=0}^{n-1} \binom{n-1}{j} \frac{(-1)^j}{j - \frac{1}{\alpha} + 1} \leq k\varepsilon^{-\frac{1}{\alpha}}, \quad \text{i.e.,} \quad \varepsilon \in \left(0, \left(n \sum_{j=0}^{n-1} \binom{n-1}{j} \frac{(-1)^j}{j - \frac{1}{\alpha} + 1} \right)^{-\alpha} \right],$$

and hence, by Proposition 2.1, if $\varepsilon \in \left(0, \left(n \sum_{j=0}^{n-1} \binom{n-1}{j} \frac{(-1)^j}{j - \frac{1}{\alpha} + 1} \right)^{-\alpha} \right]$, then $\Pi_{\varepsilon,n}(X)$ is a solution of the equation $\text{ES}_{X,n}(1 - c\varepsilon) = \text{VaR}_X(1 - \varepsilon)$, $c \in [1, \frac{1}{\varepsilon}]$, taking the form

$$kn(c\varepsilon)^{-\frac{1}{\alpha}} \sum_{j=0}^{n-1} \binom{n-1}{j} \frac{(-1)^j}{j - \frac{1}{\alpha} + 1} = k\varepsilon^{-\frac{1}{\alpha}}, \quad c \in \left[1, \frac{1}{\varepsilon} \right].$$

Hence

$$\Pi_{\varepsilon,n}(X) = \left(n \sum_{j=0}^{n-1} \binom{n-1}{j} \frac{(-1)^j}{j - \frac{1}{\alpha} + 1} \right)^{\alpha} \quad \text{for } \varepsilon \in \left(0, \left(n \sum_{j=0}^{n-1} \binom{n-1}{j} \frac{(-1)^j}{j - \frac{1}{\alpha} + 1} \right)^{-\alpha} \right].$$

If $\varepsilon \in \left(\left(n \sum_{j=0}^{n-1} \binom{n-1}{j} \frac{(-1)^j}{j - \frac{1}{\alpha} + 1} \right)^{-\alpha}, 1 \right)$, then, by Definition 1.6, $\Pi_{\varepsilon,n}(X) = \infty$.

In particular, we have

$$\Pi_{\varepsilon,2}(X) = \left(\frac{2\alpha^2}{(\alpha - 1)(2\alpha - 1)} \right)^{\alpha} \quad \text{for } \varepsilon \in \left(0, \left(\frac{(\alpha - 1)(2\alpha - 1)}{2\alpha^2} \right)^{\alpha} \right], \tag{4.2}$$

and if $\varepsilon \in \left(\left(\frac{(\alpha - 1)(2\alpha - 1)}{2\alpha^2} \right)^{\alpha}, 1 \right)$, then $\Pi_{\varepsilon,2}(X) = \infty$. Note that

$$\lim_{x \rightarrow \infty} \left(\frac{(x - 1)(2x - 1)}{2x^2} \right)^x = \lim_{x \rightarrow \infty} \left(1 - \frac{1}{x} \right)^x \left[\left(1 - \frac{1}{2x} \right)^{2x} \right]^{\frac{1}{2}} = e^{-1} e^{-\frac{1}{2}} = e^{-\frac{3}{2}},$$

and the function $(1, \infty) \ni x \mapsto \left(\frac{2x^2}{(x - 1)(2x - 1)} \right)^x$ is monotone decreasing. Indeed,

$$\left(\frac{2x^2}{(x - 1)(2x - 1)} \right)^x = \left(\frac{x}{x - 1} \right)^x \left(\left(\frac{2x}{2x - 1} \right)^{2x} \right)^{\frac{1}{2}}, \quad x \in (1, \infty),$$

and the logarithm of the function $(1, \infty) \ni x \mapsto (x/(x - 1))^x$, i.e., the function $(1, \infty) \ni x \mapsto x \ln(x/(x - 1))$, is monotone decreasing. Hence, by (4.2), for each $\varepsilon \in \left(0, \left(\frac{(\alpha - 1)(2\alpha - 1)}{2\alpha^2} \right)^{\alpha} \right]$, we have

$$\Pi_{\varepsilon,2}(X) \geq \lim_{x \rightarrow \infty} \left(\frac{2x^2}{(x - 1)(2x - 1)} \right)^x = e^{\frac{3}{2}} \approx 4,482, \tag{4.3}$$

where, by Example 4.2, the limit $e^{\frac{3}{2}}$ is nothing else but the PELVE_2 -value (at the given level ε) of an exponentially distributed random variable. The inequality (4.3) for PELVE_2 can be considered as the counterpart of the corresponding inequality for PELVE of X due to Li and Wang (2022, inequality (8)). For the parameters $\alpha \in \{2, 10, 30\}$, we calculate $\Pi_{\varepsilon,2}(X)$, see Table 2. \square

Table 2
 PELVE_2 -values of Pareto distribution $\text{Pareto}(k, \alpha)$ ($k > 0, \alpha \in \{2, 10, 30\}$), rounded up to 3 decimal places.

PELVE_2	$\varepsilon \in \left(0, \frac{9}{64} \right]$	$\varepsilon \in \left(\frac{9}{64}, \left(\frac{171}{200} \right)^{10} \right]$	$\varepsilon \in \left(\left(\frac{171}{200} \right)^{10}, \left(\frac{1711}{1800} \right)^{30} \right]$	$\varepsilon \in \left(\left(\frac{1711}{1800} \right)^{30}, 1 \right)$
Pareto(k,2)	7.112	∞	∞	∞
Pareto(k,10)	4.791	4.791	∞	∞
Pareto(k,30)	4.578	4.578	4.578	∞

5. Generalized Pareto distributions and PELVE_2

In this section, we calculate the PELVE_2 -values of random variables with some generalized Pareto distribution and random variables of which the excess distribution function over a threshold is given by the distribution function of a generalized Pareto distribution with tail index less than 1. It will turn out that the PELVE_2 -value at a level ε of such random variables depends on the tail index but not on ε (below some threshold). Such a result was already established for the PELVE -values and conditional PELVE -values of the random variables in question by Fiori and Rosazza Gianin (2022, Proposition 15). Both results might be useful for estimating the tail index of the random variables in question.

First, we recall the notion of a generalized Pareto distribution.

Definition 5.1. Let $\kappa \in \mathbb{R}$ and $\beta > 0$. We say that a random variable X has a generalized Pareto distribution with parameters κ and β if its distribution function F_X takes the form:

- in case of $\kappa > 0$,

$$F_X(x) := \begin{cases} 1 - \left(1 + \frac{\kappa}{\beta}x\right)^{-\frac{1}{\kappa}} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0, \end{cases}$$

- in case of $\kappa = 0$,

$$F_X(x) := \begin{cases} 1 - e^{-\frac{x}{\beta}} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0, \end{cases}$$

- in case of $\kappa < 0$,

$$F_X(x) := \begin{cases} 1 & \text{if } x > -\frac{\beta}{\kappa}, \\ 1 - \left(1 + \frac{\kappa}{\beta}x\right)^{-\frac{1}{\kappa}} & \text{if } x \in [0, -\frac{\beta}{\kappa}], \\ 0 & \text{if } x < 0. \end{cases}$$

The parameter κ is sometimes called the tail index of X . The distribution function of a random variable having generalized Pareto distribution with parameters $\kappa \in \mathbb{R}$ and $\beta > 0$ is denoted by $G_{\kappa, \beta}$.

Remark 5.2. If X is a random variable having a generalized Pareto distribution with parameters $\kappa > 0$ and $\beta > 0$, then $X + \frac{\beta}{\kappa}$ has a (usual) Pareto distribution with parameters $\frac{\beta}{\kappa}$ and $\frac{1}{\kappa}$ (recalled in Example 4.4). Further, if X is a random variable having a generalized Pareto distribution with parameters $\kappa = 0$ and $\beta > 0$, then X is in fact exponentially distributed with parameter $\frac{1}{\beta}$. \square

Next, to give an application of Theorem 2.4, we calculate the PELVE_2 -value of a generalized Pareto distribution with parameters $\kappa \in (0, 1)$ and $\beta := 1$.

Example 5.3. Let $\kappa \in (0, 1)$ and let X be a random variable with distribution function $F_X : \mathbb{R} \rightarrow [0, 1]$,

$$F_X(x) := \begin{cases} 1 - (1 + \kappa x)^{-\frac{1}{\kappa}} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Then X has a generalized Pareto distribution with parameters κ and $\beta := 1$, and X has the unbounded support $[0, \infty)$. Consequently, $X + \frac{1}{\kappa}$ has a (usual) Pareto distribution with parameters $\frac{1}{\kappa}$ and $\frac{1}{\kappa}$. This yields that the calculations of VaR , 2nd-order ES and PELVE_2 of X can be traced back to those of $X + \frac{1}{\kappa}$, for which we can use Example 4.4.

Using Example 4.4 and the translation invariance of VaR , we have

$$\text{VaR}_X(p) = \text{VaR}_{X + \frac{1}{\kappa}}(p) - \frac{1}{\kappa} = \frac{1}{\kappa}(1 - p)^{-\kappa} - \frac{1}{\kappa} = \frac{-1 + (1 - p)^{-\kappa}}{\kappa}, \quad p \in (0, 1).$$

Since $\kappa \in (0, 1)$, we have $X + \frac{1}{\kappa} \in L^1$, and hence $X \in L^1$. Consequently, using again Example 4.4 and the translation invariance of the 2nd-order ES (being a coherent risk measure on L^1 , see part (ii) of Remark 1.3), we have

$$\text{ES}_{X,2}(p) = \text{ES}_{X + \frac{1}{\kappa},2}(p) - \frac{1}{\kappa} = -\frac{1}{\kappa} + \frac{2(1 - p)^{-\kappa}}{\kappa(1 - \kappa)(2 - \kappa)}, \quad p \in [0, 1).$$

Consequently, for each $\varepsilon \in (0, 1)$, the inequality $\text{ES}_{X,2}(0) \leq \text{VaR}_X(1 - \varepsilon)$ is equivalent to

$$-\frac{1}{\kappa} + \frac{2}{\kappa(1 - \kappa)(2 - \kappa)} \leq \frac{-1 + \varepsilon^{-\kappa}}{\kappa}, \quad \text{i.e.,} \quad \varepsilon \in \left(0, \left(\frac{(1 - \kappa)(2 - \kappa)}{2}\right)^{\frac{1}{\kappa}}\right].$$

Using part (i) of Theorem 2.4 and (4.2) yield that

$$\Pi_{\varepsilon,2}(X) = \Pi_{\varepsilon,2}\left(X + \frac{1}{\kappa}\right) = \left(\frac{2}{(1 - \kappa)(2 - \kappa)}\right)^{\frac{1}{\kappa}} \quad \text{for } \varepsilon \in \left(0, \left(\frac{(1 - \kappa)(2 - \kappa)}{2}\right)^{\frac{1}{\kappa}}\right].$$

If $\varepsilon \in \left(\left(\frac{(1 - \kappa)(2 - \kappa)}{2}\right)^{\frac{1}{\kappa}}, 1\right)$, then, by Definition 1.6, $\Pi_{\varepsilon,2}(X) = \infty$.

Finally, note that, by L'Hospital's rule, we get

$$\lim_{\kappa \downarrow 0} \left(-\frac{1}{\kappa} + \frac{2(1 - p)^{-\kappa}}{\kappa(1 - \kappa)(2 - \kappa)}\right) = -\ln(1 - p) + \frac{3}{2}, \quad p \in [0, 1),$$

where $-\ln(1 - p) + \frac{3}{2}$ is nothing else but the 2nd-order Expected Shortfall of an exponentially distributed random variable at a level p , see Example 4.2. This is in accordance with in accordance with Lemma A.6, since a generalized Pareto distribution having parameters $\kappa > 0$ and 1 converges in distribution to the exponential distribution with parameter 1 as $\kappa \downarrow 0$. \square

For a random variable X with distribution function F_X , let x_{F_X} denote the right endpoint of F_X , i.e., $x_{F_X} := \sup\{x \in \mathbb{R} : F_X(x) < 1\}$. If X has a generalized Pareto distribution with parameters $\kappa \in \mathbb{R}$ and $\beta > 0$, then

$$x_{F_X} = \begin{cases} \infty & \text{if } \kappa \geq 0, \\ -\frac{\beta}{\kappa} & \text{if } \kappa < 0. \end{cases}$$

Definition 5.4. Let X be a non-negative random variable with distribution function F_X . The excess distribution function corresponding to F_X over a threshold $u \in [0, x_{F_X})$ is given by $F_{X,u} : [0, \infty) \rightarrow \mathbb{R}$,

$$F_{X,u}(x) := \mathbb{P}(X - u \leq x | X > u) = \frac{F_X(x + u) - F_X(u)}{1 - F_X(u)}, \quad x \geq 0.$$

Note that if $x_{F_X} < \infty$, then $F_{X,u}(x) = 1$ for $x \geq x_{F_X} - u$.

For the forthcoming Propositions 5.5, 5.6 and 5.7, one can refer to Example 5.19, Lemma 5.22, formulae (5.18), (5.19) and (5.20) in McNeil et al. (2015).

Proposition 5.5. Let X be a random variable having a generalized Pareto distribution with parameters $\kappa \in \mathbb{R}$ and $\beta > 0$. For the excess distribution function $F_{X,u}$ corresponding to F_X over a threshold u , we have $F_{X,u}(x) = G_{\kappa, \beta + \kappa u}(x)$ for $x \geq 0$ and $u \geq 0$ in case of $\kappa \geq 0$; and for $x \geq 0$ and $u < -\frac{\beta}{\kappa}$ in case of $\kappa < 0$.

Proposition 5.6. Let X be a non-negative random variable, and assume that there exist $u \in [0, x_{F_X})$, $\kappa \in \mathbb{R}$ and $\beta > 0$ such that $F_{X,u}(x) = G_{\kappa, \beta}(x)$ for $x \in [0, x_{F_X} - u)$. Then $F_{X,v}(x) = G_{\kappa, \beta + \kappa(v-u)}(x)$ for $x \geq 0$ and $v \geq u$ in case of $\kappa \geq 0$; and for $x \in [0, -\frac{\beta}{\kappa} - (v - u))$ and $v \in [u, u - \frac{\beta}{\kappa})$ in case of $\kappa < 0$.

Proposition 5.7. Let X be a non-negative random variable, and assume that there exist $u \in [0, x_{F_X})$, $\kappa \in \mathbb{R}$ and $\beta > 0$ such that $F_{X,u}(x) = G_{\kappa, \beta}(x)$ for $x \in [0, x_{F_X} - u)$. Then for each $p \in (F_X(u), 1)$, we have

$$\text{VaR}_X(p) = \begin{cases} u + \frac{\beta}{\kappa} \left(\left(\frac{1-p}{1-F_X(u)} \right)^{-\kappa} - 1 \right) & \text{in case of } \kappa \neq 0, \\ u - \beta \ln \left(\frac{1-p}{1-F_X(u)} \right) & \text{in case of } \kappa = 0. \end{cases} \tag{5.1}$$

If, in addition $\kappa < 1$, then we have

$$\text{ES}_X(p) = \frac{1}{1-\kappa} \text{VaR}_X(p) + \frac{\beta - \kappa u}{1-\kappa}, \quad p \in (F_X(u), 1),$$

and

$$\lim_{p \uparrow 1} \frac{\text{ES}_X(p)}{\text{VaR}_X(p)} = \begin{cases} \frac{1}{1-\kappa} & \text{in case of } \kappa \in [0, 1), \\ 1 & \text{in case of } \kappa < 0. \end{cases} \tag{5.2}$$

In the next proposition, we calculate the PELVE₂ of a non-negative random variable having a generalized Pareto excess distribution function. This result can be considered as a counterpart of the corresponding results for PELVE and conditional PELVE (see Definition 1.5) in Proposition 15 in Fiori and Rosazza Gianin (2022).

Proposition 5.8. Let X be a non-negative random variable, and assume that there exist $u \in [0, x_{F_X})$, $\kappa < 1$ and $\beta > 0$ such that $F_{X,u}(x) = G_{\kappa, \beta}(x)$ for $x \in [0, x_{F_X} - u)$. In case of $\kappa < 1$ with $\kappa \neq 0$, we have

$$\Pi_{\varepsilon, 2}(X) = \left(\frac{2}{(1-\kappa)(2-\kappa)} \right)^{\frac{1}{\kappa}} \quad \text{for } 0 < \varepsilon < (1 - F_X(u)) \left(\frac{(1-\kappa)(2-\kappa)}{2} \right)^{\frac{1}{\kappa}},$$

and in case of $\kappa = 0$, we have

$$\Pi_{\varepsilon, 2}(X) = e^{\frac{3}{2}} \quad \text{for } 0 < \varepsilon < (1 - F_X(u))e^{-\frac{3}{2}}.$$

Further,

$$\lim_{p \uparrow 1} \frac{\text{ES}_{X, 2}(p)}{\text{VaR}_X(p)} = \begin{cases} \frac{2}{(1-\kappa)(2-\kappa)} & \text{if } \kappa \in [0, 1), \\ 1 & \text{if } \kappa < 0. \end{cases} \tag{5.3}$$

Proof. Recall that $\text{VaR}_X(p)$, $p \in (F_X(u), 1)$, is given in (5.1). Next, we calculate $\text{ES}_{X,2}(p)$ for $p \in (F_X(u), 1)$.

First, we consider the case of $\kappa = 0$. For $p \in (F_X(u), 1)$ we have

$$\begin{aligned} \text{ES}_{X,2}(p) &= \frac{2}{(1-p)^2} \int_p^1 (s-p) \left(u - \beta \ln \left(\frac{1-s}{1-F_X(u)} \right) \right) ds \\ &= \frac{2}{(1-p)^2} \left(u + \beta \ln(1-F_X(u)) \right) \int_p^1 (s-p) ds - \frac{2\beta}{(1-p)^2} \int_p^1 s \ln(1-s) ds + \frac{2\beta p}{(1-p)^2} \int_p^1 \ln(1-s) ds. \end{aligned}$$

By partial integration, one can check that for $p \in (F_X(u), 1)$, we get

$$\text{ES}_{X,2}(p) = u + \beta \ln(1-F_X(u)) - \beta \ln(1-p) + \frac{3}{2}\beta = \frac{3}{2}\beta + \text{VaR}_X(p),$$

for the omitted details, see our ArXiv version Barczy et al. (2022, proof of Proposition 5.8).

Note that if $u = 0$, $\kappa = 0$ and $\beta = 1$, then $\text{ES}_{X,2}(p) = -\ln(1-p) + \frac{3}{2}$, which is nothing else but the 2nd-order Expected Shortfall at the level p of an exponentially distributed random variable (see Example 4.2). This is expected, since if $u = 0$, $\kappa = 0$ and $\beta = 1$, then $F_X(x) = F_{X,u}(x) = G_{0,1}(x)$, $0 \leq x < x_{F_X}$ with $x_{F_X} = \infty$, yielding that $F_X(x) = G_{0,1}(x)$, $x \geq 0$, where $G_{0,1}$ is the distribution function of an exponentially distributed random variable with parameter 1.

Let $\varepsilon \in (0, 1 - F_X(u))$. Then the inequality

$$\text{ES}_{X,2}(1 - c\varepsilon) \leq \text{VaR}_X(1 - \varepsilon), \quad c \in \left[1, \frac{1 - F_X(u)}{\varepsilon} \right)$$

takes the form

$$u + \beta \ln(1 - F_X(u)) - \beta \ln(c\varepsilon) + \frac{3}{2}\beta \leq u - \beta \ln \left(\frac{\varepsilon}{1 - F_X(u)} \right), \quad c \in \left[1, \frac{1 - F_X(u)}{\varepsilon} \right).$$

This inequality holds if and only if $c \geq e^{\frac{3}{2}}$ and $c \in \left[1, \frac{1 - F_X(u)}{\varepsilon} \right)$. Consequently, using that the function $\left[1, \frac{1}{\varepsilon} \right] \ni c \mapsto \text{ES}_{X,2}(1 - c\varepsilon)$ is monotone decreasing (see Lemma A.2), for $0 < \varepsilon < (1 - F_X(u))e^{-\frac{3}{2}}$ we have

$$\Pi_{\varepsilon,2}(X) = \inf \left\{ c \in \left[1, \frac{1}{\varepsilon} \right] : \text{ES}_{X,2}(1 - c\varepsilon) \leq \text{VaR}_X(1 - \varepsilon) \right\} = \inf \left\{ c \in \left[1, \frac{1 - F_X(u)}{\varepsilon} \right) : c \geq e^{\frac{3}{2}} \right\} = e^{\frac{3}{2}},$$

as desired.

Next, we consider the case of $\kappa < 1$ with $\kappa \neq 0$. For $p \in (F_X(u), 1)$, we have

$$\begin{aligned} \text{ES}_{X,2}(p) &= \frac{2}{(1-p)^2} \int_p^1 (s-p) \left(u + \frac{\beta}{\kappa} \left(\left(\frac{1-s}{1-F_X(u)} \right)^{-\kappa} - 1 \right) \right) ds \\ &= \frac{2}{(1-p)^2} \left(u - \frac{\beta}{\kappa} \right) \int_p^1 (s-p) ds + \frac{2\beta(1-F_X(u))^\kappa}{(1-p)^{2\kappa}} \int_p^1 s(1-s)^{-\kappa} ds - \frac{2\beta p(1-F_X(u))^\kappa}{(1-p)^{2\kappa}} \int_p^1 (1-s)^{-\kappa} ds. \end{aligned}$$

By partial integration, one can check that for $p \in (F_X(u), 1)$, we get that

$$\text{ES}_{X,2}(p) = u - \frac{\beta}{\kappa} + \frac{2\beta(1-F_X(u))^\kappa}{\kappa(1-\kappa)(2-\kappa)}(1-p)^{-\kappa} = \text{VaR}_X(p) + \frac{\beta(3-\kappa)}{(1-\kappa)(2-\kappa)} \left(\frac{1-p}{1-F_X(u)} \right)^{-\kappa},$$

for the omitted details, see our ArXiv version Barczy et al. (2022, proof of Proposition 5.8). Let $\varepsilon \in (0, 1 - F_X(u))$. Then the inequality

$$\text{ES}_{X,2}(1 - c\varepsilon) \leq \text{VaR}_X(1 - \varepsilon), \quad c \in \left[1, \frac{1 - F_X(u)}{\varepsilon} \right)$$

takes the form

$$u - \frac{\beta}{\kappa} + \frac{2\beta(1-F_X(u))^\kappa}{\kappa(1-\kappa)(2-\kappa)}(c\varepsilon)^{-\kappa} \leq u + \frac{\beta}{\kappa} \left(\left(\frac{\varepsilon}{1-F_X(u)} \right)^{-\kappa} - 1 \right), \quad c \in \left[1, \frac{1 - F_X(u)}{\varepsilon} \right).$$

This inequality holds if and only if

$$c \geq \left(\frac{2}{(1-\kappa)(2-\kappa)} \right)^{\frac{1}{\kappa}} \quad \text{and} \quad c \in \left[1, \frac{1 - F_X(u)}{\varepsilon} \right).$$

Consequently, using that the function $\left[1, \frac{1}{\varepsilon} \right] \ni c \mapsto \text{ES}_{X,2}(1 - c\varepsilon)$ is monotone decreasing (see Lemma A.2), for $0 < \varepsilon < (1 - F_X(u)) \left(\frac{(1-\kappa)(2-\kappa)}{2} \right)^{\frac{1}{\kappa}}$ we have

$$\begin{aligned} \Pi_{\varepsilon,2}(X) &= \inf \left\{ c \in \left[1, \frac{1}{\varepsilon} \right] : \text{ES}_{X,2}(1 - c\varepsilon) \leq \text{VaR}_X(1 - \varepsilon) \right\} \\ &= \inf \left\{ c \in \left[1, \frac{1 - F_X(u)}{\varepsilon} \right] : c \geq \left(\frac{2}{(1 - \kappa)(2 - \kappa)} \right)^{\frac{1}{\kappa}} \right\} = \left(\frac{2}{(1 - \kappa)(2 - \kappa)} \right)^{\frac{1}{\kappa}}, \end{aligned}$$

as desired.

Now, we turn to prove (5.3). In case of $\kappa = 0$, we have $\lim_{p \uparrow 1} \text{VaR}_X(p) = \infty$ and

$$\lim_{p \uparrow 1} \frac{\text{ES}_{X,2}(p)}{\text{VaR}_X(p)} = \lim_{p \uparrow 1} \frac{\text{VaR}_X(p) + \frac{3}{2}\beta}{\text{VaR}_X(p)} = 1.$$

In case of $\kappa \in (0, 1)$, we have $\lim_{p \uparrow 1} \text{VaR}_X(p) = \infty$ and

$$\lim_{p \uparrow 1} \frac{\text{ES}_{X,2}(p)}{\text{VaR}_X(p)} = 1 + \lim_{p \uparrow 1} \frac{\frac{\beta(3-\kappa)}{(1-\kappa)(2-\kappa)} \left(\frac{1-p}{1-F_X(u)} \right)^{-\kappa}}{u + \frac{\beta}{\kappa} \left(\left(\frac{1-p}{1-F_X(u)} \right)^{-\kappa} - 1 \right)} = \frac{2}{(1-\kappa)(2-\kappa)}.$$

In case of $\kappa < 0$, we have $\lim_{p \uparrow 1} \text{VaR}_X(p) = u - \frac{\beta}{\kappa}$ and

$$\lim_{p \uparrow 1} \frac{\text{ES}_{X,2}(p)}{\text{VaR}_X(p)} = 1 + \lim_{p \uparrow 1} \frac{\frac{\beta(3-\kappa)}{(1-\kappa)(2-\kappa)} \left(\frac{1-p}{1-F_X(u)} \right)^{-\kappa}}{\text{VaR}_X(p)} = 1,$$

as desired. \square

6. PELVE₂ of regularly varying distributions

First, we recall the notion of regularly varying functions.

Definition 6.1. A measurable function $U : (x_0, \infty) \rightarrow (0, \infty)$ (where $x_0 \geq 0$) is called regularly varying at infinity with index $\rho \in \mathbb{R}$ if for all $q > 0$, we have

$$\lim_{x \rightarrow \infty} \frac{U(qx)}{U(x)} = q^\rho.$$

In case of $\rho = 0$, we call U slowly varying at infinity. A measurable function $V : (0, x_0) \rightarrow (0, \infty)$ (where $x_0 > 0$) is called regularly varying at 0 with index $\kappa \in \mathbb{R}$ if for all $q > 0$, we have

$$\lim_{x \downarrow 0} \frac{V(qx)}{V(x)} = q^\kappa.$$

Next, we recall the notion of regularly varying non-negative random variables.

Definition 6.2. A non-negative random variable X is called regularly varying with index $\alpha \geq 0$ if $\bar{F}_X(x) := \mathbb{P}(X > x) > 0$ for all $x > 0$, and \bar{F}_X is regularly varying at infinity with index $-\alpha$.

Next, we study the asymptotic behaviour of the PELVE₂ of X at a level ε tending to 0. Recall that if X is a regularly varying non-negative random variable with index $\alpha > 1$, then $\mathbb{E}(X) < \infty$, i.e., $X \in L^1$, and hence in this case PELVE₂ of X at any level $\varepsilon \in (0, 1)$ is well-defined.

Theorem 6.3. Let X be a regularly varying non-negative random variable with index $\alpha > 1$ such that $(0, 1) \ni p \mapsto \text{VaR}_X(p)$ is continuous. Then

$$\lim_{\varepsilon \downarrow 0} \Pi_{\varepsilon,2}(X) = \left(\frac{2\alpha^2}{(\alpha - 1)(2\alpha - 1)} \right)^\alpha.$$

Proof. Recall that

$$\lim_{\varepsilon \downarrow 0} \frac{\text{VaR}_X(1 - t\varepsilon)}{\text{VaR}_X(1 - \varepsilon)} = t^{-\frac{1}{\alpha}}, \quad t > 0, \tag{6.1}$$

i.e., the function $(0, 1) \ni p \mapsto \text{VaR}_X(p)$ is regularly varying at 0 with index $-\frac{1}{\alpha}$, see, e.g., Li and Wang (2022, formula (A.9)) or our ArXiv version Barczy et al. (2022, formula (6.1)). Recall also that

$$\lim_{\varepsilon \downarrow 0} \frac{\text{ES}_X(1 - \varepsilon)}{\text{VaR}_X(1 - \varepsilon)} = \frac{\alpha}{\alpha - 1}, \quad t > 0, \tag{6.2}$$

see, e.g., Li and Wang (2022, formula (A.10)) or our ArXiv version Barczy et al. (2022, formula (6.2)).

Next, we prove that

$$\lim_{\varepsilon \downarrow 0} \frac{\text{ES}_{X,2}(1 - \varepsilon)}{\text{VaR}_X(1 - \varepsilon)} = \frac{2\alpha^2}{(\alpha - 1)(2\alpha - 1)}.$$

For each $\varepsilon \in (0, 1)$, we have

$$\begin{aligned} \frac{\text{ES}_{X,2}(1 - \varepsilon)}{\text{VaR}_X(1 - \varepsilon)} &= \frac{\frac{2}{\varepsilon^2} \int_{1-\varepsilon}^1 (u - (1 - \varepsilon)) \text{VaR}_X(u) \, du}{\text{VaR}_X(1 - \varepsilon)} = \frac{2 \int_0^\varepsilon (\varepsilon - v) \text{VaR}_X(1 - v) \, dv}{\varepsilon^2 \text{VaR}_X(1 - \varepsilon)} \\ &= 2 \frac{\int_0^\varepsilon \text{VaR}_X(1 - v) \, dv}{\varepsilon \text{VaR}_X(1 - \varepsilon)} - 2 \frac{\int_0^\varepsilon v \text{VaR}_X(1 - v) \, dv}{\varepsilon \cdot \varepsilon \text{VaR}_X(1 - \varepsilon)} \\ &= 2 \frac{\text{ES}_X(1 - \varepsilon)}{\text{VaR}_X(1 - \varepsilon)} - 2 \frac{\int_0^\varepsilon v \text{VaR}_X(1 - v) \, dv}{\varepsilon \cdot \varepsilon \text{VaR}_X(1 - \varepsilon)}. \end{aligned}$$

Here the function $(0, 1) \ni v \mapsto v \text{VaR}_X(1 - v)$ is regularly varying at 0 with index $\frac{\alpha-1}{\alpha} > -1$, since, using (6.1), we have

$$\lim_{v \downarrow 0} \frac{tv \text{VaR}_X(1 - v)}{v \text{VaR}_X(1 - v)} = t \cdot t^{-\frac{1}{\alpha}} = t^{\frac{\alpha-1}{\alpha}}, \quad t > 0.$$

So, by Karamata theorem for regularly varying functions at zero (see Lemma C.1), we have

$$\lim_{\varepsilon \downarrow 0} \frac{\int_0^\varepsilon v \text{VaR}_X(1 - v) \, dv}{\varepsilon \cdot \varepsilon \text{VaR}_X(1 - \varepsilon)} = \frac{1}{\frac{\alpha-1}{\alpha} + 1} = \frac{\alpha}{2\alpha - 1}.$$

Hence, using (6.2), we get

$$\lim_{\varepsilon \downarrow 0} \frac{\text{ES}_{X,2}(1 - \varepsilon)}{\text{VaR}_X(1 - \varepsilon)} = \frac{2\alpha}{\alpha - 1} - \frac{2\alpha}{2\alpha - 1} = \frac{2\alpha^2}{(\alpha - 1)(2\alpha - 1)}.$$

Consequently, using (6.1), we have

$$\lim_{\varepsilon \downarrow 0} \frac{\text{ES}_{X,2}(1 - t\varepsilon)}{\text{VaR}_X(1 - \varepsilon)} = \lim_{\varepsilon \downarrow 0} \frac{\text{ES}_{X,2}(1 - t\varepsilon)}{\text{VaR}_X(1 - t\varepsilon)} \cdot \frac{\text{VaR}_X(1 - t\varepsilon)}{\text{VaR}_X(1 - \varepsilon)} = \frac{2\alpha^2}{(\alpha - 1)(2\alpha - 1)} t^{-\frac{1}{\alpha}}, \quad t > 0. \tag{6.3}$$

Note that for each $\alpha > 1$, the function $(0, \infty) \ni t \mapsto \frac{2\alpha^2}{(\alpha-1)(2\alpha-1)} t^{-\frac{1}{\alpha}}$ is strictly monotone decreasing, and it takes value 1 if and only if

$$t = \left(\frac{2\alpha^2}{(\alpha - 1)(2\alpha - 1)} \right)^\alpha.$$

Let t_1 and t_2 be such that

$$0 < t_1 < \left(\frac{2\alpha^2}{(\alpha - 1)(2\alpha - 1)} \right)^\alpha < t_2,$$

yielding

$$\frac{2\alpha^2}{(\alpha - 1)(2\alpha - 1)} t_2^{-\frac{1}{\alpha}} < 1 < \frac{2\alpha^2}{(\alpha - 1)(2\alpha - 1)} t_1^{-\frac{1}{\alpha}}.$$

Hence, using (6.3), for sufficiently small $\varepsilon > 0$ (which may depend on t_1 and t_2), we have

$$\frac{\text{ES}_{X,2}(1 - t_2\varepsilon)}{\text{VaR}_X(1 - \varepsilon)} < 1 \quad \text{and} \quad \frac{\text{ES}_{X,2}(1 - t_1\varepsilon)}{\text{VaR}_X(1 - \varepsilon)} > 1.$$

Since X is non-negative and regularly varying, we have $\text{VaR}_X(1 - v) \rightarrow \infty$ as $v \downarrow 0$, and hence for sufficiently small $\varepsilon > 0$, we get $\text{VaR}_X(1 - \varepsilon) > 0$ and

$$\text{ES}_{X,2}(1 - t_2\varepsilon) < \text{VaR}_X(1 - \varepsilon) < \text{ES}_{X,2}(1 - t_1\varepsilon). \tag{6.4}$$

Using (again) $\text{VaR}_X(1 - v) \rightarrow \infty$ as $v \downarrow 0$, and $\text{ES}_{X,2}(0) < \infty$, we have $\text{ES}_{X,2}(0) < \text{VaR}_X(1 - \varepsilon)$ for sufficiently small $\varepsilon > 0$. Hence, using Proposition 2.1, for sufficiently small $\varepsilon > 0$, we have $\Pi_{\varepsilon,2}(X) \in [1, \frac{1}{\varepsilon}]$ is a solution of the equation $\text{ES}_{X,2}(1 - c\varepsilon) = \text{VaR}_X(1 - \varepsilon)$, $c \in [1, \frac{1}{\varepsilon}]$. Consequently, using (6.4), the definition of PELVE_2 , and the continuity and monotone increasing property of $\text{ES}_{X,2}$ (see Lemma A.2), we get $\Pi_{\varepsilon,2}(X) \in (t_1, t_2]$ for sufficiently small $\varepsilon > 0$. Since $(t_1, t_2]$ can be chosen as a neighbourhood of $(\frac{2\alpha^2}{(\alpha - 1)(2\alpha - 1)})^\alpha$ with arbitrarily small length, the statement follows. \square

Remark 6.4. In Theorem 6.3, the limit is nothing else but the PELVE_2 of a Pareto distributed random variable with parameters $k > 0$ and $\alpha > 1$ at any level less than or equal to $(\frac{(\alpha-1)(2\alpha-1)}{2\alpha^2})^\alpha$, see Example 4.4. So Theorem 6.3 is in accordance with Theorem 3 in Li and Wang (2022). Note that a Pareto distributed random variable with parameters $k > 0$ and $\alpha > 0$ is regularly varying with index $-\alpha$, and a random variable with generalized Pareto distribution having parameters $\kappa > 0$ and $\beta > 0$ (see, Definition 5.1) is regularly varying with index $-\frac{1}{\kappa}$. \square

7. Simulations and real data analysis for PELVE₂

First, we present an empirical estimator of PELVE₂ of a random variable. Let $X \in L^1$, $m \in \mathbb{N}$, and let X_1, \dots, X_m be independent and identically distributed random variables such that their common distribution coincides with that of X , i.e., X_1, \dots, X_m is a sample of length m for X . Let $X_1^* \leq X_2^* \leq \dots \leq X_m^*$ be the corresponding ordered sample. Given $p \in (0, 1)$, an empirical estimator of $\text{VaR}_X(p)$ based on X_1, \dots, X_m is given by

$$\widehat{\text{VaR}}_X(p) := X_i^* \quad \text{if } p \in \left(\frac{i-1}{m}, \frac{i}{m} \right], \quad i = 1, \dots, m.$$

Following Acerbi (2007, Section 7) (where one can find a construction of empirical estimators of spectral risk measures), given $p \in [0, 1)$, an empirical estimator of the 2nd-order Expected Shortfall $\text{ES}_{X,2}(p)$ based on X_1, \dots, X_m is given by a weighted sum of X_1^*, \dots, X_m^* :

$$\widehat{\text{ES}}_{X,2}(p) := \sum_{i=1}^m w_i X_i^*,$$

where

$$w_i := \int_{(i-1)/m}^{i/m} \frac{2(s-p)}{(1-p)^2} \mathbb{1}_{[p,1)}(s) \, ds, \quad i = 1, \dots, m.$$

We check that if $p \in [0, \frac{m-1}{m})$, then

$$w_i = \begin{cases} 0 & \text{if } i \leq \lfloor mp \rfloor, \\ \frac{1}{(1-p)^2} \left(\frac{i}{m} - p \right)^2 & \text{if } i = \lfloor mp \rfloor + 1, \\ \frac{1}{(1-p)^2} \left(\left(\frac{i}{m} - p \right)^2 - \left(\frac{i-1}{m} - p \right)^2 \right) & \text{if } \lfloor mp \rfloor + 2 \leq i \leq m, \end{cases} \quad (7.1)$$

and if $p \in [\frac{m-1}{m}, 1)$, then

$$w_i = \begin{cases} 0 & \text{if } i = 1, \dots, m-1, \\ 1 & \text{if } i = m. \end{cases} \quad (7.2)$$

If $p \in [0, \frac{1}{m})$, then $\lfloor mp \rfloor = 0$ and

$$w_1 = \int_p^{1/m} \frac{2(s-p)}{(1-p)^2} \, ds = \frac{1}{(1-p)^2} \left(\frac{1}{m} - p \right)^2,$$

and

$$w_i = \int_{(i-1)/m}^{i/m} \frac{2(s-p)}{(1-p)^2} \, ds = \frac{1}{(1-p)^2} \left(\left(\frac{i}{m} - p \right)^2 - \left(\frac{i-1}{m} - p \right)^2 \right), \quad i = 2, \dots, m,$$

yielding (7.1) in case of $p \in [0, \frac{1}{m})$. If $p \in [\frac{1}{m}, \frac{2}{m})$, then $\lfloor mp \rfloor = 1$, $w_1 = \int_0^{1/m} 0 \, ds = 0$, and

$$w_2 = \int_p^{2/m} \frac{2(s-p)}{(1-p)^2} \, ds = \frac{1}{(1-p)^2} \left(\frac{2}{m} - p \right)^2,$$

and

$$w_i = \int_{(i-1)/m}^{i/m} \frac{2(s-p)}{(1-p)^2} \, ds = \frac{1}{(1-p)^2} \left(\left(\frac{i}{m} - p \right)^2 - \left(\frac{i-1}{m} - p \right)^2 \right), \quad i = 3, \dots, m,$$

yielding (7.1) in case of $p \in [\frac{1}{m}, \frac{2}{m})$. The case $p \in [\frac{2}{m}, \frac{m-1}{m})$ can be handled similarly. If $p \in [\frac{m-1}{m}, 1)$, then $\lfloor mp \rfloor = m-1$, $w_i = 0$, $i = 1, \dots, m-1$, and

$$w_m = \int_p^1 \frac{2(s-p)}{(1-p)^2} \, ds = 1,$$

yielding (7.1) in case of $p \in [\frac{m-1}{m}, 1)$, as desired.

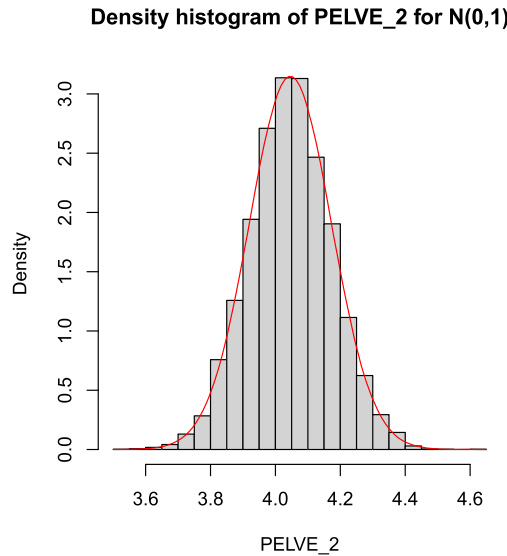


Fig. 1. Density histogram of empirical $PELVE_2$ values of a standard normal distribution at level $\varepsilon = 0.05$ based on 10000 samples of length $m = 5000$. The curve is the density function of the fitted normal distribution.

Consequently, if $p \in [0, \frac{m-1}{m})$, then

$$\widehat{ES}_{X,2}(p) = \frac{1}{(1-p)^2} \left[\left(\frac{\lfloor mp \rfloor + 1}{m} - p \right)^2 X_{\lfloor mp \rfloor + 1}^* + \sum_{i=\lfloor mp \rfloor + 2}^m \left(\left(\frac{i}{m} - p \right)^2 - \left(\frac{i-1}{m} - p \right)^2 \right) X_i^* \right],$$

and if $p \in [\frac{m-1}{m}, 1)$, then

$$\widehat{ES}_{X,2}(p) = X_m^*.$$

Given $\varepsilon \in (0, 1)$, an empirical estimator of the $PELVE_2$ value $\Pi_{\varepsilon,2}(X)$ based on X_1, \dots, X_m can be defined as

$$\widehat{\Pi}_{\varepsilon,2}(X) := \inf \left\{ c \in \left[1, \frac{1}{\varepsilon} \right] : \widehat{ES}_{X,2}(1 - c\varepsilon) \leq \widehat{\text{VaR}}_X(1 - \varepsilon) \right\}, \tag{7.3}$$

where $\inf \emptyset = \infty$.

In principle, the empirical estimator (7.3) of the $PELVE_2$ value $\Pi_{\varepsilon,2}(X)$ can be used even if the random variables X_1, \dots, X_m are not independent or identically distributed. We will do so in analyzing real data.

For illustrative purposes, we present a simulation result for calculating the $PELVE_2$ value of a standard normally distributed random variable at the level $\varepsilon = 0.05$. We generated 10000 samples of length $m = 5000$ for a standard normally distributed random variable X . For each generated sample, we calculated the empirical estimator $\widehat{\Pi}_{\varepsilon,2}(X)$ (given in (7.3)) of the $PELVE_2$ value of X at the given level ε . Then we made a density histogram based on the 10000 estimated $PELVE_2$ values, see Fig. 1. On this figure, we also plotted the density function of the fitted normal distribution in red. The theoretical $PELVE_2$ value of a standard normally distributed random variable at the level $\varepsilon = 0.05$ is approximately 4.040815. The sample mean of the 10000 estimated $PELVE_2$ values is 4.046066, which is quite close to the theoretical value. Based on Fig. 1, one could conjecture that a kind of central limit theorem might hold for $PELVE_2$ in case of a standard normal distribution. We do not study this question here. We only note that in case of $PELVE$, such a result is available due to Li and Wang (2022, Theorem 4).

As real data applications, we calculate the empirical estimator of $PELVE_2$ for S&P 500 daily returns based on two data sets: (i) ranging from 4th January 2020 to 4th January 2022, and (ii) ranging from 6th April 2020 to 4th January 2022. The S&P 500 historical data sets were downloaded from Investing.com. Note that in case (i) the data set contains approximately four months before the COVID-19 crisis started in Europe (i.e., before April 2020), and in case (ii) the data set just starts when the COVID-19 crisis started in Europe. Recall that, given some asset prices $S_t, t = 0, 1, \dots, N$, where $N \in \mathbb{N}$, the one-period (linear) return at time $t = 1, \dots, N$ is defined by $S_t/S_{t-1} - 1$. In the used data sets, the daily returns are rounded off to two decimal places. For both data sets in question, we calculated the empirical $PELVE_2$ estimator (7.3) and the empirical $PELVE$ estimator at levels ranging from 0.001 to 0.56, see Fig. 2. The empirical $PELVE$ estimator is not presented in the present paper, we used the same empirical estimators as Li and Wang (2022, Section 5) and Fiori and Rosazza Gianin (2022, Section 5).

On Fig. 2, the red horizontal lines correspond to the $PELVE_2$ value of an exponential distribution, i.e., $e^{3/2} \approx 4.482$ (see Example 4.2); and the green horizontal lines correspond to the $PELVE$ value of an exponential distribution, i.e., $e \approx 2.718$ (see Li and Wang (2022, Example 5, part (ii))). On Fig. 2, one can see that there are levels for which no empirical $PELVE_2$ or $PELVE$ values are plotted. It just means that the corresponding empirical $PELVE_2$ or $PELVE$ values are infinity. For example, on the right subfigure of Fig. 2, no $PELVE_2$ values are plotted at levels greater than (approximately) 0.27. As shown in the left subfigure of Fig. 2, most of the empirical $PELVE_2$ values are above $e^{3/2}$, but it is not the case for the right subfigure of Fig. 2. A possible explanation for this phenomenon is that the data set used for the left subfigure of Fig. 2 contains four months daily returns of S&P 500 before the start of the COVID-19 crisis in Europe (i.e., before April 2020), while the data set used for the right subfigure of Fig. 2 does not include these four months, it just starts at April

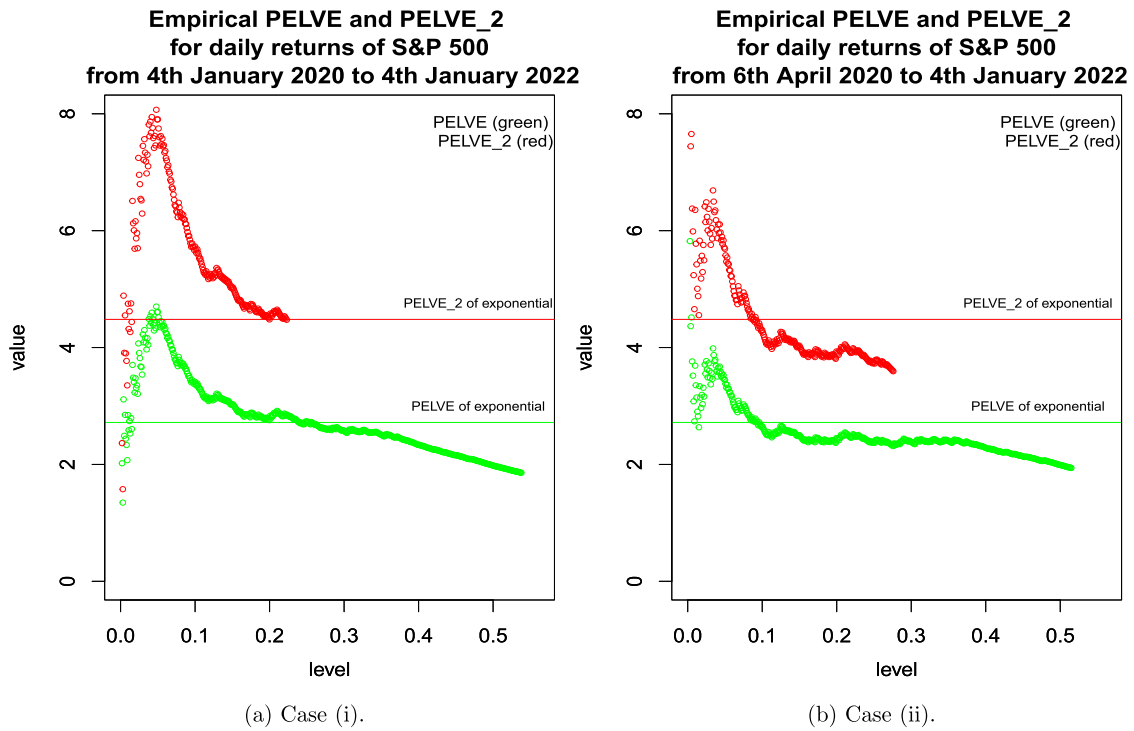


Fig. 2. Empirical PELVE and PELVE₂ values for daily returns of S&P 500. (In both subfigures, the green curve is the lower one and the red one is the upper one.)

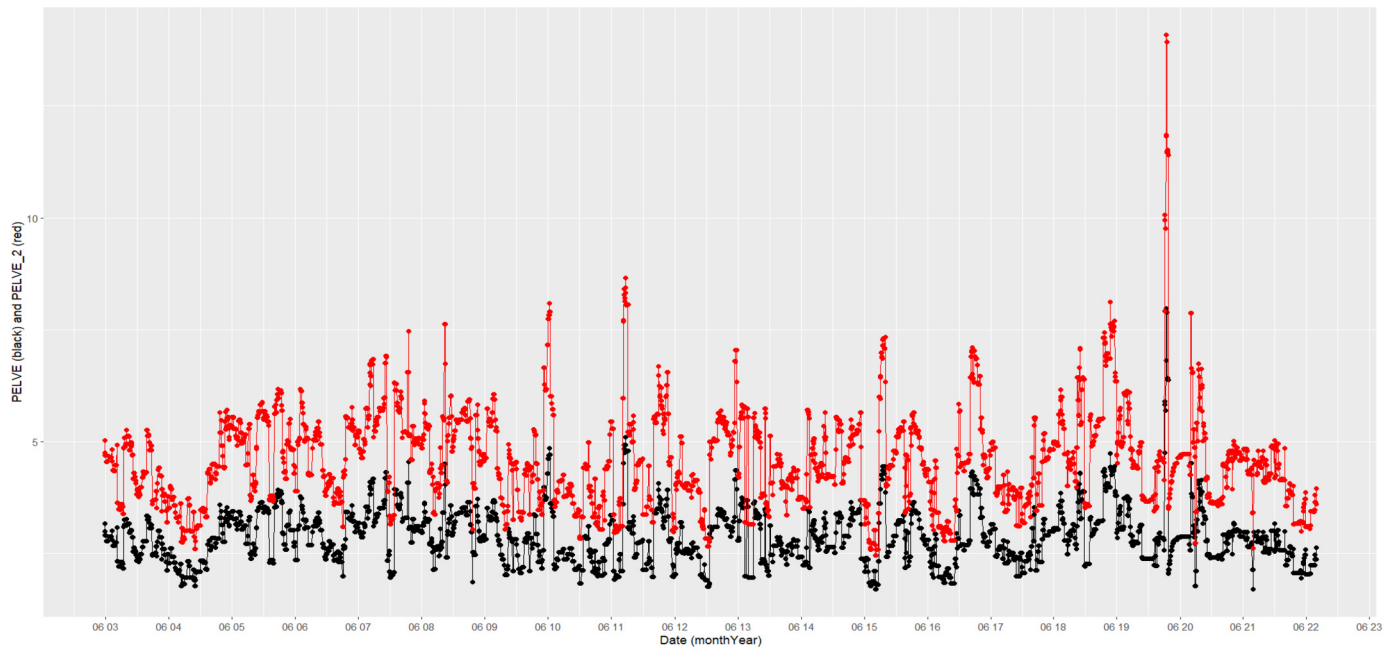


Fig. 3. PELVE (black) and PELVE₂ (red) values. (The black is the lower and the red is the upper one. For the coloured figure, the reader is referred to the web version of this article.)

2020. Our real data applications may suggest that PELVE₂ might be an indicator for structural changes in stock prices. Note also that the empirical PELVE₂ values on the left subfigure of Fig. 2 are greater than the corresponding empirical PELVE₂ values on the right subfigure of Fig. 2.

Finally, we present another approach to illustrate the changes in the PELVE and PELVE₂ values caused by COVID-19 based on a S&P 500 historical data set. Fixing the level $\varepsilon = 0.05$, for each day starting from May 27, 2003 and ending at July 28, 2022, we calculate a corresponding PELVE and PELVE₂ value based on the previous 99 days and the day in question itself (altogether 100 days). The graphs are shown in Fig. 3, where it is visible that both risk measures had a high peak at around March 2020 (the approximate starting date of COVID-19 crisis in Europe) as the effect of the pandemic. Note also that the PELVE₂ shows this effect more significantly.

We used the open software R for making the simulations and real data analysis.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

We used some S&P 500 historical data sets that were downloaded from Investing.com.

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Appendix A. Some properties of higher-order Expected Shortfalls

This appendix is devoted to study some properties of higher-order Expected Shortfalls given in Definition 1.2 such as finiteness, continuity, monotonicity, additivity for comonotonic random variables and connection with weak convergence. These results generalize the corresponding known properties for (the usual, i.e., first order) Expected Shortfall.

Our first result states the finiteness of higher-order Expected Shortfalls of random variables in L^1 .

Lemma A.1. For each $X \in L^1$, $n \in \mathbb{N}$, and $p \in [0, 1)$, we have $ES_{X,n}(p) \in \mathbb{R}$.

Lemma A.1 is an immediate consequence of Proposition 1 in Wang et al. (2020), for more details and an independent proof, see our ArXiv version Barczy et al. (2022, Lemma A.1 and Remark A.2).

Lemma A.2. Let X be a random variable such that $X \in L^1$, and let $n \in \mathbb{N}$. Then the function $[0, 1) \ni p \mapsto ES_{X,n}(p)$ is continuous and monotone increasing.

For a proof of Lemma A.2, see our ArXiv version Barczy et al. (2022, the proof of Lemma A.3).

Next, we provide some sufficient conditions under which $[0, 1 - \varepsilon) \ni p \mapsto ES_{X,n}(p)$ is strictly monotone increasing, where $\varepsilon \in (0, 1)$ and $n \in \mathbb{N}$. Such a result is known in case of $n = 1$, see, e.g., the proof of Proposition 2 in Li and Wang (2022).

Lemma A.3. Let X be a random variable such that $X \in L^1$, let $\varepsilon \in (0, 1)$, $n \in \mathbb{N}$, and let us suppose that the function $(0, 1) \ni p \mapsto VaR_X(p)$ is not constant on the interval $[1 - \varepsilon, 1)$. Then the function $[0, 1 - \varepsilon) \ni p \mapsto ES_{X,n}(p)$ is strictly monotone increasing.

For a proof of Lemma A.3, see our ArXiv version Barczy et al. (2022, the proof of Lemma A.4).

Definition A.4. The random variables X and Y are called comonotonic if there exist a random variable Z and monotone increasing functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$ such that $X = f(Z)$ and $Y = g(Z)$.

Proposition A.5. Let X and Y be comonotonic random variables such that $X, Y \in L^1$, and let $n \in \mathbb{N}$. Then

$$ES_{X+Y,n}(p) = ES_{X,n}(p) + ES_{Y,n}(p), \quad p \in [0, 1),$$

that is, the n^{th} -order Expected Shortfall is additive for comonotonic random variables belonging to L^1 .

Proposition A.5 is the direct consequence of the additivity of VaR for comonotonic random variables (see, e.g., McNeil et al., 2015, Proposition 7.20) and the additivity of Lebesgue integral.

Next, we investigate the connection between weak convergence of random variables and the convergence of their higher-order Expected Shortfalls.

Lemma A.6. Let $n \in \mathbb{N}$, $X_m, m \in \mathbb{N}$, and X be random variables such that $X_m \in L^1$, $m \in \mathbb{N}$, and $X \in L^1$. If $X_m \xrightarrow{\mathcal{D}} X$ as $m \rightarrow \infty$, and $\{X_m : m \in \mathbb{N}\}$ is uniformly integrable, then $ES_{X_m,n}(p) \rightarrow ES_{X,n}(p)$ as $m \rightarrow \infty$ for each $p \in [0, 1)$.

Lemma A.6 is a consequence of Theorem 6 in Wang et al. (2020), for more details and an independent proof, see our ArXiv version Barczy et al. (2022, Lemma A.7 and Remark A.8).

Appendix B. Second-order Expected Shortfall and Gini Shortfall

Let X be a random variable such that $X \in L^1$, and let $p \in [0, 1)$. Let U_p be a random variable uniformly distributed on the interval $(p, 1)$, and let $F_{X,p}$ be the distribution function of the random variable $VaR_X(U_p)$. The tail-Gini functional of X at a level $p \in [0, 1)$ is defined by

$$TGini_X(p) := \mathbb{E}(|X_p^* - X_p^{**}|),$$

where X_p^* and X_p^{**} are two independent, identically distributed random variables with a distribution function $F_{X,p}$, see Furman et al. (2017, formula (3.6)). Note that if $p = 0$, then $\text{VaR}_X(U_0)$ has the same distribution as X (see, e.g., Embrechts and Hofert, 2013, Proposition 2), and hence $\text{TGini}_X(0)$ is nothing else but the Gini variability measure of X given by $\mathbb{E}(|X^* - X^{**}|)$, where X^* and X^{**} are two independent copies of X . By Furman et al. (2017, (3.3) and Proposition 3.2), we have

$$\text{TGini}_X(p) = \frac{2}{(1-p)^2} \int_p^1 (2s - 1 - p) \text{VaR}_X(s) \, ds, \quad p \in [0, 1).$$

For $\lambda \geq 0$, the Gini Shortfall of X at a level $p \in [0, 1)$ corresponding to the (loading) parameter λ is defined by

$$\text{GS}_X(p, \lambda) := \text{ES}_X(p) + \lambda \text{TGini}_X(p),$$

see Furman et al. (2017, formula (4.1)). By Theorem 4.1 in Furman et al. (2017),

$$\text{GS}_X(p, \lambda) = \frac{1}{(1-p)^2} \int_p^1 \left(1 - p + 4\lambda \left(s - \frac{1+p}{2}\right)\right) \text{VaR}_X(s) \, ds, \quad p \in [0, 1), \lambda \geq 0, \tag{B.1}$$

and, using also Lemma 4.2 in Furman et al. (2017), the Gini Shortfall at a level p corresponding to the parameter λ is a coherent risk measure on L^1 if and only if $\lambda \in [0, \frac{1}{2}]$.

By (B.1), we have

$$\begin{aligned} \text{GS}_X\left(p, \frac{1}{2}\right) &= \frac{1}{(1-p)^2} \int_p^1 \left(1 - p + 2\left(s - \frac{1+p}{2}\right)\right) \text{VaR}_X(s) \, ds \\ &= \frac{2}{(1-p)^2} \int_p^1 (s - p) \text{VaR}_X(s) \, ds = \text{ES}_{X,2}(p), \quad p \in [0, 1), \end{aligned}$$

so for $p \in [0, 1)$, the 2nd-order Expected Shortfall of X at a level p is nothing else but the Gini Shortfall of X at a level p corresponding to the parameter $\frac{1}{2}$. In particular, we have that the 2nd-order Expected Shortfall (at any level $p \in [0, 1)$) is a coherent risk measure on L^1 (which is in accordance with part (ii) of Remark 1.3).

Further, using again (B.1), for each $p \in [0, 1)$ and $\lambda \geq 0$, we have

$$\begin{aligned} \text{GS}_X(p, \lambda) &= \frac{1}{(1-p)^2} \int_p^1 \left(1 - p + 4\lambda \left(s - p + \frac{p-1}{2}\right)\right) \text{VaR}_X(s) \, ds \\ &= (1 - 2\lambda) \frac{1}{1-p} \int_p^1 \text{VaR}_X(s) \, ds + 2\lambda \frac{2}{(1-p)^2} \int_p^1 (s - p) \text{VaR}_X(s) \, ds \\ &= (1 - 2\lambda) \text{ES}_X(p) + 2\lambda \text{ES}_{X,2}(p), \end{aligned}$$

yielding that the Gini Shortfall at a level $p \in [0, 1)$ corresponding to a parameter $\lambda \geq 0$ is the linear combination of the Expected Shortfall at level p and the 2nd-order Expected Shortfall at level p with coefficients $1 - 2\lambda$ and 2λ , respectively.

Appendix C. Karamata theorem for regularly varying functions at zero

For the notions of a regularly varying function at infinity and at 0, resp., see Definition 6.1.

We formulate a Karamata theorem for regularly varying functions at 0 with index $\kappa > -1$, which is used in the proof of Theorem 6.3. We could not address any reference for it, and hence, for completeness, we provide a proof as well.

Lemma C.1. *Let $x_0 > 0$ and $f : (0, x_0) \rightarrow (0, \infty)$ be a regularly varying function at 0 with index $\kappa > -1$. Then*

$$\lim_{\varepsilon \downarrow 0} \frac{\int_0^\varepsilon f(v) \, dv}{\varepsilon f(\varepsilon)} = \frac{1}{\kappa + 1}.$$

Proof. For each $\varepsilon \in (0, x_0)$, by the substitution $v = \frac{1}{u}$, we have

$$\int_0^\varepsilon f(v) \, dv = \int_{\frac{1}{\varepsilon}}^\infty u^{-2} f\left(\frac{1}{u}\right) \, du. \tag{C.1}$$

Here the function $(\frac{1}{x_0}, \infty) \ni u \mapsto u^{-2} f\left(\frac{1}{u}\right)$ is regularly varying at infinity with index $-\kappa - 2 < -1$ (see Definition 6.1), since it is measurable and

$$\lim_{u \rightarrow \infty} \frac{(qu)^{-2} f\left(\frac{1}{qu}\right)}{u^{-2} f\left(\frac{1}{u}\right)} = q^{-2} \lim_{\varepsilon \downarrow 0} \frac{f\left(\frac{1}{q\varepsilon}\right)}{f(\varepsilon)} = q^{-2} \left(\frac{1}{q}\right)^\kappa = q^{-\kappa-2} \quad \text{for each } q > 0.$$

Consequently, by Karamata theorem for regularly varying functions at infinity with index strictly less than -1 (see, e.g., Resnick, 1987, Theorem 0.6), we get

$$\lim_{\varepsilon \downarrow 0} \frac{\int_{\frac{1}{\varepsilon}}^{\infty} u^{-2} f\left(\frac{1}{u}\right) du}{\frac{1}{\varepsilon} \cdot \left(\frac{1}{\varepsilon}\right)^{-2} f(\varepsilon)} = \frac{1}{\kappa + 2 - 1} = \frac{1}{\kappa + 1}.$$

Hence, by (C.1), the assertion follows. \square

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