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Portfolio choice with illiquid asset for a loss-averse pension fund investor

Zheng Chen [a], Zhongfei Li [b,*], Yan Zeng [c]

[a] School of Management, Guangdong University of Technology, Guangzhou 510520, PR China
[b] Department of Finance, Southern University of Science and Technology, Shenzhen 518055, PR China
[c] Lingnan College, Sun Yat-sen University, Guangzhou 510275, PR China

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A B S T R A C T

This paper explores the optimization of liquid and illiquid assets investment for a defined contribution (DC) pension plan and investigates the impact of illiquidity on portfolio choice. In addition to three kinds of liquid assets, there is an illiquid asset that can only be traded at time 0, and it provides returns at retirement. The investor exhibits both risk-seeking and loss-averse behaviors, with S-shaped utility from the return on investment at retirement. In the long run, the investor also faces the risks caused by the time-varying income and inflation. The martingale method is adopted first to analyze the characteristics of the optimal investment strategy in a complete market. Then the optimal illiquid asset trading strategy is identified and determined. The results are proven to be applicable in a variety of market model settings through some extended analyses. Finally, several numerical findings are illustrated.

1. Introduction

Illiquid asset trading has always been occupying an important position in the financial market. Illiquid assets mainly include long-term receivables, engineering materials, long-term equity investment, real estate investment, construction projects in progress, fixed assets, long-term deferred expenses, available-for-sale financial assets, intangible assets, and so on. According to the study of Ang et al. (2014), illiquid assets held by individuals usually accounted for over 80% of their total net assets and included bank deposits, private equity, housing, etc. Broeders et al. (2021) considered pension funds as an important way of investment in illiquid assets, such as real estate, private equity, mortgages, infrastructure, and hedge funds, which could provide benefits through a liquidity premium, portfolio diversification, and liability hedging. The Global Pension Assets Study 2022 reports that the ratio of pension funds to other assets (real estate and other alternatives, including illiquid assets) increased from 5% in 2001 to the estimated 19% at the end of 2021. In practice, the illiquid asset market is well-developed and provides multiple opportunities for pension funds.

In this paper, a finite-horizon continuous-time model is established to investigate the effect of illiquid assets on portfolio selection in a defined contribution (DC) pension plan when the investor shows a preference for the prospect theory. This study is aimed at explicitly analyzing the optimization problem under loss aversion and illiquid asset trading over a long-term horizon. The results show that when there were illiquid asset investment opportunities, the illiquid asset could provide the lowest protection of investment for the loss-averse investor. The main contributions of this paper are as follows: (1) A theoretical model is established to investigate the impact of illiquid asset trading on the investment behavior of a loss-averse pension investor. (2) The optimum of the illiquid asset trading strategy is obtained, the explicit investment strategy for liquid assets is derived, and the properties and implications of the results are clarified. (3) A revised approach based on that of Desmettre and Seifried (2016) is developed to solve the optimization problem. In addition, from a

* Corresponding author.
E-mail addresses: chenzheng@gdut.edu.cn (Z. Chen), lizf6@sustech.edu.cn (Z. Li), zengy36@mail.sysu.edu.cn (Y. Zeng).

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macro point of view, Broeders et al. (2021) empirically studied the influence of capital and liquidity requirements on the illiquid asset allocation of a pension fund. By contrast, this paper, from a micro point of view, provides some theoretical support and supplements through the study of the illiquidity asset allocation of the pension fund based on the portfolio theory.

The first key feature of the model is that it incorporates illiquid asset trading for a long-term pension plan investor. In the current literature on portfolio choice, the notion of illiquidity has been described in several ways. The first is to assume the trading time of illiquid assets to be discontinuous. For most assets such as long-term bank deposits or venture capital, it is usually hard to convert them into currencies and their trading is infrequent. According to Ang et al. (2014), private equity and venture capital portfolios are typically held between 3 and 10 years with a fixed investment horizon in advance. The second is to use the transaction cost theory to address the characteristics of illiquid assets: investment in stocks does not require transaction costs while illiquid asset trading is expensive. Following this view, investors can obtain the liquidity of illiquid assets at a certain cost. The third is to adopt the theory of search frictions to study the cost of waiting for illiquid assets' trading opportunities.[1] In our model, the first way is taken to describe the characteristics of illiquid assets. We also study the optimal trading decision of illiquid assets at the initial time. According to the descriptions by Schwartz and Tebaldi (2006) and Desmettre and Seifried (2016), it is assumed in this paper that illiquid assets can only be traded at the beginning of the investment period, and trading is not allowed until the terminal time. The deterministic trading time is simple yet meaningful in our model due to the following reason. In a pension plan, the investment horizon is finite. For a finite pension fund investor, illiquid assets with uncertain trading time turn out to be less attractive because the costs generated by illiquidity are magnified by finite horizons (Ang et al., 2014).[2] In solving the model, a baseline model where the market is complete is investigated to derive the analytical results that are favorable for explicitly illustrating the properties of model parameters. It is also extended to a general case where illiquid assets can be traded at the stochastic time. Besides, the impact of liquidity constraints on the investor's trading of illiquid assets is studied. Since these situations are too complicated to provide analytical results, a preliminary analysis is conducted through approximation methods. The main results are proven to be applicable to a variety of market settings.

Some scholars have studied this investment problem with illiquidity. Dai et al. (2016) obtained some interesting results of asset allocation strategies when market closures existed. Under the assumption that investors buy and sell stocks with transaction costs, they concluded that the existence of market closures greatly affected trading strategies and transaction costs had a first-order effect on the level of the liquidity premium. Then empirical findings were provided to verify the reliability of their theoretical results. Desmettre and Seifried (2016) proposed another framework to study illiquid asset investment problems. In their model, the investors were allowed to invest in a traditional liquidity market that contains one stock and one risk-free bond. In addition to these two kinds of liquid assets, the investors were allowed to invest in an illiquid asset that can only be traded at the beginning of the investment period and provide returns at the end of the investment period. Bichuch and Guasoni (2018) identified optimal trading strategies for investment, including safe assets, liquid risky assets, and illiquid risky assets with proportional transaction costs. In spite of the increasing popularity of these assets in pension funds, there is little literature on the dynamic allocation of illiquid assets in such a market. Recently only Broeders et al. (2021) found from an empirical perspective that illiquid asset allocations were becoming increasingly popular in pension fund management. Their results demonstrate that the allocation of illiquid assets is significantly affected by hedging needs.

The second key feature of the model is that we assume the pension investor is loss-averse. Different from almost all the above works based on the concave utility function[3] to identify investors' preferences, however, this study adopts the concept of loss aversion put forward by Kahneman and Tversky (1979) to describe the investor's preference. This assumption reveals two realistic behavioral features. The first one is that the investor focuses on the gains and losses associated with a predetermined reference point, namely a wealth target. The second one is that the investor exhibits higher sensitivity to losses, which usually leads to more discomfort, than gains. As a major feature proposed in the prospect theory, loss aversion has been supported by a large number of empirical findings. For instance, Benartzi and Thaler (1995) argued that loss aversion could be used for explaining the equity premium puzzle. According to Barberis et al. (2001), loss aversion resulted in the high mean and excess volatility of stock returns. Fortin and Hlouskova (2011) found that loss-averse portfolios clearly outperformed Mean-Variance and CVaR portfolios using 13 EU and US assets. Furthermore, Curatola (2015) combined loss aversion with habit formation to explain the high premium of equity, the counter-cyclical performance of stock returns, and the joint-term structures of interest rates and equity. Some scholars (Han and Hsu, 2004) have already found that institutional investors and traders show aversion to losses rather than risks. Evidence has also supported the loss aversion feature of pension fund management (Broadbent et al., 2006; Xie et al., 2018). In the relevant literature on pension fund investment, Blake et al. (2013), Guan and Liang (2016), Chen et al. (2017), and Dong and Zheng (2020) adopted the S-shape utility function to characterize the investors' preferences. However, they all considered the allocation of the pension fund during the investment period in a liquid market. Therefore, it is more meaningful to consider loss aversion when dealing with the investment decision of the DC pension fund.

This paper is related to the work of Desmettre and Seifried (2016). We develop a theoretical model to investigate the impact of illiquid assets on pension investment. However, our paper is different from theirs at least in two aspects. Firstly, in this paper, the investor is assumed to be averse to losses, which is not considered in their paper. The non-concave property of S-shaped utility leads to the inapplicability of Desmettre and Seifried's (2016) method for addressing the optimization problem to the resolution of the investment problem in this paper. Thus, a revised approach is developed to overcome this difficulty. Secondly, we propose an optimization problem from a DC pension plan investor's point of view. In the pension plan, the existence of pension contributions causes the income risk to become an important background risk, and the long period of pension investment makes the inflation risk not to be ignored. Therefore,

[1] In particular, for the first and second descriptions, Ang et al. (2014) explained both their advantages and disadvantages and supported the first description, indicating that the use of the transaction cost theory was inaccurate in some cases. For example, during the financial crisis, there will be liquidity freezes in many markets where investors cannot trade, even though they are willing to pay transaction costs. For the second and the third descriptions, Gârleanu (2009) pointed out that transaction costs generated trading losses while search frictions generated costs of waiting or “not trading”. It is also revealed that in the view of the transaction cost theory, investors could obtain the liquidity of assets at a certain cost. Therefore, Gârleanu (2009) took the third way to describe illiquid assets. In Gârleanu's paper, however, illiquid assets are assumed to be an endowment at the initial time (Schwartz and Tebaldi, 2006). While in our paper, the optimal trading decision of illiquid assets at the initial time is studied and determined.

[2] For example, an investor with a one-year horizon views the illiquid asset with a ten-year horizon as a very unattractive asset.

[3] For example, the constant relative risk aversion (CRRA) utility function and the constant absolute risk aversion (CRRA) utility function.

income and inflation risks are considered in this paper. We, therefore, obtain quite different results from those of Desmettre and Seifried (2016), and consider their results as a special case.

We make some technical contributions to the resolution of the optimization problem. It is worth noting that the introduction of illiquid assets has brought some new characterizations of the optimization problem. The main change in the model is reflected in the following fact. The wealth of the investor who conducts illiquid asset trading usually consists of two parts, namely the wealth gained from investment in stocks and bonds and the wealth gained from investment in illiquid assets. Therefore, in the optimization problem, the wealth process is different from the traditional case. The wealth process is divided into a liquid wealth process and a total wealth process. At the same time, the two-step method and the martingale approach are combined to address the compound optimization problem, and numerical illustrations are further provided to analyze the impacts of model parameters, such as illiquid assets, loss aversion preference, and income risk on the optimal investment strategy.

In the extension study, our results are further expanded to consider several scenarios by studying the effects of stochastic trading time and liquidity constraints on pension investment behavior. Following the assumptions by Ang et al. (2014), we show that when facing the stochastic trading time of illiquid assets, the investor must pay for this uncertainty by trading illiquid assets, and the existence of liquidity constraints could reduce the investor's motive for illiquid asset trading.

The remainder of this paper is organized as follows. Section 2 develops the portfolio selection model with an illiquid asset for a DC pension plan investor. Section 3 generalizes the martingale method to solve this investment problem in complete market. Section 4 extends the model considering stochastic trading times and liquidity constraints. Section 5 provides some numerical examples and economic implications. Section 6 concludes this paper. Appendices provide detailed mathematical derivations.

2. The model

In this section, an asset allocation model with an illiquid asset and the inflation risk is established for a DC pension investor. In this model, a finite time horizon $[0, T]$, in which T is the retirement time, is considered. Furthermore, there is a trading opportunity of an illiquid asset, which compared with the liquid asset such as a stock or a bond, can only be traded at the beginning of the investment period, and provides a stochastic pay-off at retirement. Generally speaking, the illiquid asset is a risky investment characterized by both marketed and non-marketed risks. The investor is assumed to have aversion to losses and exhibits higher sensitivity to losses than gains. As a result, she holds different attitudes towards risks in different scenarios and respect gains and losses with a reference point.

2.1. Investment opportunities

Consider a financial market where there are no transaction costs or taxes. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathcal{P})$ be a probability space supporting a two-dimensional Brownian motion $\mathbb{W}(t) := (W_1(t), W_2(t))'$ with an associated filtration $\{\mathcal{F}_t\}_{0\leq t\leq T}$ satisfying the usual conditions. $W_1(t)$ is independent of $W_2(t)$ and the prime ($'$) denotes transposition of a matrix throughout this paper.

The accumulation phase in a pension plan usually lasts long (20 to 40 years), and long-term cumulative inflation may seriously damage the real return rate of pension funds.[4] For a pension fund investor, neither the interest rate nor the equity yield can help avoid the deterioration of purchasing power as a result of inflation (Beletski, 2006). Therefore, the inflation risk should be evaded efficiently (Owadally et al., 2021). In order to improve the market's efficiency in hedging against the inflation risk, the inflation index bonds are introduced into financial markets in this paper based on the previous literature (Zhang and Ewald, 2010; Han and Hung, 2012; Chen et al., 2017; Baltas et al., 2022). The index bond is linked to the price level and acts as a long-term risk-free asset in the real economic environment. Moreover, the strong correlation between the return rate of index bonds and inflation stimulates investors to hedge against the inflation risk through investment in the index bond. In reality, the index bond is known as a Real Return Bond (RRB) in Canada, Treasury Inflation Protected Securities (TIPS) in the U.S., and an Index-linked Gilt in the UK. Campbell and Viceira (2001) who found a rapid development of the index bond market indicated that if investors were allowed to invest in index bonds, their utility would be improved.

In this model with an illiquid asset, the inflation risk is considered for another reason. Some new risks to the pay-off of the illiquid asset brought by the inflation may have significant impacts on the investor's decision about the purchase of the illiquid asset. To ensure that the pension fund investor is free from the inflation risk, the financial market is assumed to be composed of three tradable assets, namely an index bond, a risk-free bond, and a stock. Accordingly, a stochastic price level $P(t)$ is defined based on the geometric Brownian motion (GBM) to capture the inflation risk.

$$\frac{dP(t)}{P(t)} = idt + \sigma_I dW_1(t) = idt + \Sigma_I' d\mathbb{W}(t), \quad P(0) = p_0 > 0, \tag{1}$$

where the constant i is the expected rate of price, $\sigma_I > 0$ is the volatility of the price, and $\Sigma_I = (\sigma_I, 0)'$.

The index bond offers a constant rate of real return r and its price process follows[5]

$$\frac{dI(t)}{I(t)} = rdt + \frac{dP(t)}{P(t)} = (r+i)dt + \Sigma_I' d\mathbb{W}(t), \quad I(0) = i_0. \tag{2}$$

[4] Some earlier literature has investigated the problem of asset allocation for individual investors to deal with the inflation risk (Campbell and Viceira, 2001; Brennan and Xia, 2002; Munk et al., 2004; etc). In these studies, Battocchio and Menoncin (2004) examined a DC pension fund manager's optimal asset allocation strategy under the inflation risk. In their study, the interest rate, contribution, and inflation rate were all assumed to be stochastic, and the fund manager was aimed at maximizing the expected utility from the pension's real wealth at retirement. It was found that the fund manager could hedge against the inflation risk by establishing a cash-dominated portfolio. The authors also considered three types of risks, namely the interest, equity, and inflation risks, which can provide real returns in the financial market. Since there was no real riskless asset in their model, the manager could not adjust risk exposure based on his aversion to risks. In some other portfolio selection literature, stocks were regarded as an alternative for hedging against inflation. However, Brown et al. (2004) found that the role of stock in hedging against inflation was inefficient in practice after studying the US market.

[5] This price process refers to a study by Zhang and Ewald (2010). In Brennan and Xia (2002), they consider an investor's portfolio selection problem under inflation risk and assume there is no real asset in the market to deal with the inflation risk. They provide a portfolio that mimics as closely as possible a hypothetical indexed bond.

The price dynamics of the risk-free bond evolves according to equation

$$\frac{dB(t)}{B(t)} = Rdt, \quad B(0) = b_0 \tag{3}$$

with a constant rate of nominal return R, which is deterministic. In the presence of inflation risk, the price dynamics of the stock follows

$$\begin{aligned}\frac{dS(t)}{S(t)} &= \mu_S dt + \sigma_S(\rho_{IS} dW_1(t) + \sqrt{1-\rho_{IS}^2} dW_2(t)) \\ &= \mu_S dt + \Sigma_S' d\mathbb{W}(t), \quad S(0) = s_0,\end{aligned} \tag{4}$$

where μ_S is the appreciation rate and $\sigma_S > 0$ is the volatility, $\rho_{IS} \in [-1, 1]$ is a constant and $\Sigma_S = (\sigma_S \rho_{IS}, \sigma_S \sqrt{1-\rho_{IS}^2})'$. We assume that $|\rho_{IS}| < 1$ represents the correlation between the indexed bond and the stock. The existence of the index bond helps the investor to hedge the inflation risk and makes the market of liquid assets complete.[6] Thus, the unique stochastic discount factor $H(t)$ is given by

$$H(t) = e^{-Rt - \frac{1}{2}\|\Xi\|^2 t - \Xi' \mathbb{W}(t)}. \tag{5}$$

Alternatively, we rewrite it as

$$\frac{dH(t)}{H(t)} = -Rdt - \Xi' d\mathbb{W}(t), \; H(0) = 1. \tag{6}$$

In addition to these three liquid assets, the investor has access to buy an illiquid asset $F(t)$. For example, investment in the illiquid asset is always attracting particular interest (fixed-term bank deposits, private equity investments and housing, etc.). In contrast to the liquid assets, this illiquid asset can only be traded at some deterministic times. At initial time, the illiquid asset has a price $F(0)$; subsequently, its dynamics is given by

$$\begin{aligned}\frac{dF(t)}{F(t)} &= \mu_F dt + \sigma_F(\rho_{IF} dW_1(t) + \rho_{SF} dW_2(t) + \sqrt{1-\rho_{IF}^2-\rho_{SF}^2} dW_F(t)) \\ &= \mu_F dt + \Sigma_F' d\mathbb{W}(t) + \sigma_F \sqrt{1-\rho_{IF}^2-\rho_{SF}^2} dW_F(t),\end{aligned} \tag{7}$$

where μ_F is the compounded total expected rate of return on this illiquid asset. $W_F(t)$ is another standard Brownian motion which is independent of $W_1(t)$ and $W_2(t)$. ρ_{IF} and ρ_{SF} represent the correlation between the illiquid asset and the indexed bond and the stock, respectively. Here $\Sigma_F = (\sigma_F \rho_{IF}, \sigma_F \rho_{SF})'$. From Eq. (7), the price process contains both marketed and non-marketed risk drivers for this illiquid asset. The financial market is incomplete in general case.

Here, we provide a complementary discussion of the restrictions on the trading times of this illiquid asset. In particular, we assume that the investor can buy this illiquid asset at price $F(0)$ at time 0; and she will obtain a stochastic payoff $F(T)$ at retirement. That is, we assume that the trading time of the illiquid asset is limited to the initial time and the retirement time, which is fixed. Thus, if the investor decides to buy λ positions of the illiquid asset at time 0, she receives a lump-sum payment $\lambda F(T)$ at time T. We further assume that shorting the illiquid asset is unfeasible.[7]

Remark 2.1. Our assumption of the trading times of the illiquid asset is consistent with that in the works by Kahl et al. (2003), Ilhan et al. (2005), Schwartz and Tebaldi (2006), Longstaff (2009), and Desmettre and Seifried (2016). But in some studies like those by Gârleanu (2009) and Schwartz and Tebaldi (2006), it was assumed that the investor had an endowment of the illiquid asset at the initial time and then considered the impact of the illiquid asset on the investor's investment decisions. As a result, they handled the illiquid asset in a way similar to that for dealing with human capital, which is, therefore, commonly considered as a form of illiquid asset in most literature. In this paper, the investor's trading decisions about the illiquid asset at the initial time are considered, and the illiquid asset is distinguished from human capital. More importantly, a new non-concave utility function is adopted.

Remark 2.2. In another work, Ang et al. (2014) investigated the impact of stochastic trading times of illiquid assets on the investor's investment strategies based on an infinite horizon model. However, in a pension plan, the investor always faces a finite retirement time.

[6] For simplicity, we denote the volatility matrix

$$\Sigma = \begin{bmatrix} \sigma_I & 0 \\ \sigma_S \rho_{IS} & \sigma_S \sqrt{1-\rho_{IS}^2} \end{bmatrix},$$

which is nonsingular. Then, there exists a unique market price of risk Ξ satisfying

$$\Xi = \Sigma^{-1} \begin{pmatrix} r+i-R \\ \mu_S - R \end{pmatrix} \equiv \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = \begin{pmatrix} \frac{r+i-R}{\sigma_I} \\ \frac{1}{\sqrt{1-\rho_{IS}^2}} \left(\frac{\mu_S - R}{\sigma_S} - \frac{\rho_{IS}(r+i-R)}{\sigma_I} \right) \end{pmatrix}.$$

We assume that $\|\Xi\| > 0$, where $\|\cdot\|$ denotes the module of vector in this paper.

[7] These conditions guarantee no arbitrage opportunities in the model. For more details, see Desmettre and Seifried (2016). In the presence of salary, the solvency requirement $X_t^{\psi,\pi} \geq 0$ (a.e.) in their paper is changed into a new form in our model (see it later), showing the significant role played by the contributor's human capital in pension investment. Except for these two conditions, in reality the payment of the illiquid asset should satisfy some basic properties. For example, $F(T) \geq F(0)e^{RT}$ (a.e.), otherwise the illiquid asset is redundant. In this model, however, we adopt a general model which may include these cases in every situation, hence we omit the discussions for the constraint of specific model parameters.

The illiquidity faced by the pension fund at retirement is thus very costly, which may lead to the investor's abandonment of the illiquid asset. This result is also supported by the magnified impact of illiquidity with a finite horizon in the study by Ang et al. (2014). Therefore, a finite horizon model with deterministic trading times of an illiquid asset is developed in this paper. At the same time, as a comparison, a preliminary analysis is conducted to study the stochastic trading times later. In particular, to alleviate the magnified impact of illiquidity with a finite horizon, the model setting in Ang et al. (2014) is modified in this paper, and some simple assumptions are made, which are further discussed in Section 4.

2.2. Labor income and the investor's preference

During the working period, the pension fund receives contributions continuously over time 0 to T. Suppose that the contribution rate is a percentage c of income. The pension investor's objective is to maximize the expected *S-shaped* utility from the total terminal wealth until retirement.

Suppose that the income is stochastic and follows

$$\begin{aligned}\frac{dY(t)}{Y(t)} &= \mu_Y dt + \sigma_Y(\rho_{IY} dW_1(t) + \sqrt{1-\rho_{IY}^2}\, dW_2(t)) \\ &= \mu_Y dt + \Sigma_D' d\mathbb{W}(t), \quad Y(0) = y_0 > 0,\end{aligned} \tag{8}$$

where μ_Y is the appreciation rate, $\sigma_Y > 0$ is the volatility, $\rho_{IY} \in [-1, 1]$ is a constant and $\Sigma_D = (\sigma_Y \rho_{IY}, \sigma_Y \sqrt{1-\rho_{IY}^2})'$.

In previous works such as El Karoui and Jeanblanc-Picque (1998), Bodie et al. (2004) and Dybvig and Liu (2010), all of these authors assume that the salary process is spanned by the market assets. Since we consider the inflation risk, it is reasonable to allow the income to be correlated with both the inflation risk and the stock market risk. In this case, we assume $\rho_{IY} > 0$, which means that the income has a positive correlation with the inflation. Intuitively, when the inflation becomes larger, the pension investor wishes to have a higher income.[8]

At time 0, the investor's initial total wealth is x_0, and she decides to buy λ positions of the illiquid asset, which costs her $\lambda F(0)$. Then, she engages in the liquid market. Let $\Pi(t) = (\pi_1(t), \pi_2(t))'$, where $\pi_1(t)$, $\pi_2(t)$ are the proportions of liquid wealth invested in the indexed bond and the stock respectively. Denote $\pi_0(t) := 1 - \pi_1(t) - \pi_2(t)$ be the proportion of liquid wealth invested in the risk-free bond. We call the wealth gained from investing in the liquid assets the liquid wealth, and distinguish it from the total wealth which consists of wealth gained from both liquid and illiquid assets investment. Then, the liquid wealth process $X^{\Pi,\lambda}(t)$ follows

$$dX^{\Pi,\lambda}(t) = X^{\Pi,\lambda}(t)[Rdt + \Pi'(t)\Sigma(\Xi dt + d\mathbb{W}(t))] + cY(t)dt. \tag{9}$$

Obviously, the initial liquid wealth is $X^{\Pi,\lambda}(0) = x_0 - \lambda F(0)$. At retirement, the illiquid asset offers the investor a payoff of $\lambda F(T)$, and her terminal total wealth is $X_T = X^{\Pi,\lambda}(T) + \lambda F(T)$.

Remark 2.3. Given a fixed illiquid asset trading rule λ, we find that, in the liquid wealth process, $cY(t)$ is the contribution provided by the income at time t. In this sense, the human capital plays a similar role to the illiquid asset. Unlike Gârleanu (2009) and Schwartz and Tebaldi (2006), at the initial time of investment period, we do not assume that the investor has an exogenous endowment of the illiquid asset, but instead assume the investor decides what positions to buy for the illiquid asset. Decision variable λ reveals the fact that we distinguish between the illiquid asset and the human capital in our model. This assumption makes our model more complicated yet meaningful than theirs.

Almost all the works related to illiquid asset trading have adopted standard utility functions (CRRA type or CARA type) to characterize investors' investment preferences. Thus, the investor is always assumed to have a strong aversion to the risks to the terminal wealth. On the one hand, the CRRA type preference, which means a large proportion of investment in stocks, could be too risky for pension funds. On the other hand, investors who have the CARA type preference exhibit stable and absolute aversion to risks. That is to say, their attitudes towards the risk are independent of the quantity of their wealth. Therefore, in this paper, an S-shaped utility proposed by Kahneman and Tversky (1979) is adopted and shown by

$$U(X(T)) = \begin{cases} -A(\theta - X(T))^{\gamma_1}, & \text{if} \quad X(T) \le \theta, \\ B(X(T) - \theta)^{\gamma_2}, & \text{if} \quad X(T) > \theta, \end{cases} \tag{10}$$

where A and B are positive constants; $A > B$ holds for loss aversion, which means that losses cause more uncomfortableness[9]; γ_1 and γ_2 characterize the degree of loss aversion and risk aversion, respectively; $0 < \gamma_1 < 1$ and $0 < \gamma_2 < 1$ hold for a convex-concave shape, which shows the investor's distinct attitudes toward risk in different situations; and θ is the reference point. The reference point θ plays a key role and a loss-averse investor's behavior is sensitive to the choice of the reference point (Chen et al., 2017). Some papers focusing on DC pension investment problems, such as those by Blake et al. (2013), Guan and Liang (2016), Chen et al. (2017) and Dong and Zheng (2020), already apply this utility function to describe the preferences of the pension investors. Therefore, we adopt the utility function with an illiquid asset.

[8] According to many studies including this paper, the income risk is insurable in the financial market. The introduction of an independent Brownian motion into the stochastic salary would be more realistic, and the uninsurable income risk generates frictions in the financial market. In this case, therefore, deriving closed-form solutions would be quite difficult. More details can be found in the study by Koo (1995, 1998).

[9] Many empirical findings show that loss aversion corresponds better with an investor's behavior than traditional utility maximization theory in real-world risk situations. Such as Kahneman and Tversky (1992), Benartzi and Thaler (1995), Barberis et al. (2001), Curatola (2015).

Now, the investor's objective is to maximize the expected S-shaped utility from her terminal wealth until retirement.[10] In other words, the investor aims to maximize $\mathbf{E}[U(X^{\pi,\lambda}(T)+\lambda F(T))]$ for some feasible trading strategies. Before we define the admissible strategies, we conduct some necessary calculations. We follow Chen et al. (2017), first define $D(t):=c\mathbf{E}_t[\int_t^T \frac{H(s)}{H(t)}Y(s)ds]$, where $\mathbf{E}_t[\int_t^T \frac{H(s)}{H(t)}Y(s)ds]$ is called human capital in Bodie et al. (1992). The human capital plays a significant role in pension plan: the higher the income is, the larger the pension fund at retirement is. We calculate it as $D(t)=\frac{1}{\beta}(e^{\beta(T-t)}-1)cY(t)$ where $\beta=\mu_Y-R-\sigma_Y\rho_{IY}\xi_1-\sigma_Y\sqrt{1-\rho_{IY}^2}\xi_2$, and we denote $d_0\equiv D(0)$ for simplicity.

Definition 2.4. A portfolio strategy $\{(\Pi(t),\lambda):t\in[0,T]\}$ is said to be admissible if, for all $t\in[0,T]$, $\Pi(t)\in\mathcal{F}_t$, $\int_0^T[(X^{\Pi(t),\lambda}(t)\pi_1(t))^2+(X^{\Pi(t),\lambda}(t)\pi_2(t))^2]dt<+\infty$, $\lambda\geq 0$ and

$$X^{\Pi,\lambda}(t)+D(t)\geq 0 \tag{11}$$

holds. We denote the set of all admissible portfolio strategies by $\mathcal{A}$.

From the definition, we have the constraint imposed on λ as follows:

$$X^{\Pi,\lambda}(0)+D(0)\geq 0, \tag{12}$$

which leads the following constraint to hold: $\lambda\leq\frac{x_0+d_0}{F(0)}$. The admissible strategy reveals that in the market, the investor makes her investment decision not only on the current liquid wealth, but also on her human capital.

Given the initial total wealth x_0, the investor's optimization problem is how to choose the investment $\{\Pi,\lambda\}$ in admissible strategy set to maximize the expected utility from the terminal total wealth $X^{\Pi,\lambda}(T)+\lambda F(T)$. In the next section, we dedicate to solving this problem.

3. Optimal investment strategies

In this section, analytical solutions are derived for a simple case when illiquid asset investment is spanned by the market of these liquid assets.[11] Based on an assumed spanned illiquid asset, the market is assumed to be complete. This assumption simplifies the analysis in this paper but deserves some discussions. In incomplete markets, it is not feasible to obtain closed-form solutions to the optimization problems through the martingale approach, which would significantly complicate the analysis of the problems. Such complications are unnecessary for the current purpose.

3.1. Main result

We assume that the illiquid asset's price dynamics is given by

$$\frac{dF(t)}{F(t)}=\mu_F dt+\sigma_F(\rho_{IF}dW_1(t)+\rho_{SF}dW_2(t))=\mu_F dt+\Sigma_F' d\mathbb{W}(t). \tag{13}$$

In this case, $\rho_{SF}=\sqrt{1-\rho_{IF}^2}$ and $\Sigma_F=(\sigma_F\rho_{IF},\sigma_F\sqrt{1-\rho_{IF}^2})'$.

Define a state variable $Z^{\Pi,\lambda}(t)=X^{\Pi,\lambda}(t)+D(t)$ which corresponds to the total present value of the future liquid wealth, comprising financial wealth and a share of human capital. We call $Z^{\Pi,\lambda}(t)$ liquid surplus process. It is easy to have that $D(T)=0$ and $Z^{\Pi,\lambda}(T)=X^{\Pi,\lambda}(T)$. By calculating $dD(t)$ and $dZ^{\Pi,\lambda}(t)$ using Itô's formula, we show that the surplus process $Z^{\Pi,\lambda}(t)$ satisfies the following process

$$\begin{aligned} dZ^{\Pi,\lambda}(t)=&Z^{\Pi,\lambda}(t)Rdt+X^{\Pi,\lambda}(t)\Pi'\Sigma\Xi dt+D(t)\Sigma_D'\Xi dt\\ &+(X^{\Pi,\lambda}(t)\Pi'\Sigma+D(t)\Sigma_D')d\mathbb{W}(t), \end{aligned} \tag{14}$$

which is self-financing. In particular, $Z^{\Pi,\lambda}(0)=x_0+d_0-\lambda F(0)$. We formulate the optimization problem shown by:

$$\begin{cases} \max\limits_{(\Pi,\lambda)\in\mathcal{A}} & \mathbf{E}[U(Z^{\Pi,\lambda}(T)+\lambda F(T))]\\ \text{s.t.} & (\Pi,\lambda) \text{ satisfies (8), (9), (13) and (14).} \end{cases} \tag{15}$$

We consider the case $0\leq\lambda\leq\lambda_{max}\equiv\frac{x_0+d_0}{F(0)}$. To solve this equivalent problem, we take a two-step approach as is shown in Desmettre and Seifried (2016).[12] First, for a given illiquid asset investment λ, we solve the optimal investment problem in the liquid market consisting of the index bond, the stock and the risk-free bond. Second, we look for an optimal illiquid asset investment strategy of λ^*, and the key point is to guarantee that the maximum of λ is attained in our model.

[10] It has been shown in the appendix that, when the index bond is a traded asset, maximizing expected utility of real terminal wealth is equivalent to maximizing expected utility of nominal terminal wealth. As a result, we use this objective function for simplicity.

[11] As shown in the study by Desmettre and Seifried (2016) and earlier literature, when the illiquid asset contained non-marketed risk, there would be no analytical expression of the optimal investment strategy for the illiquid asset.

[12] Although the approach similar to those adopted in previous literature is adopted in this paper, there are some differences. First, the optimization of investment in a DC pension plan is studied by considering both the income and inflation risks, which play a significant role in the pension investor's investment decision-making. Second, the S-shaped utility is adopted to describe the pension investor's investment preferences, and loss aversion is used to characterize the investor's behavior more accurately. The non-concave utility, which is used by Desmettre and Seifried (2016) and depends on the concave structure of the utility function, does not work here. A revised proof is provided in this paper, which includes theirs as a special case. What remains to be noted is that the use of the two-step approach to this type of problem is quite natural.

To show more, we first consider the following problem under a fixed investment λ,

$$\max \quad \mathbf{E}[U(Z^{\Pi,\lambda}(T)+\lambda F(T))]=\mathbf{E}[\bar{U}(\bar{Z}^{\Pi}(T))] \quad \text{over} \quad \Pi \quad \text{with} \quad (\Pi,\lambda)\in\mathcal{A},$$

where the utility function $\bar{U}$ is defined by $\bar{U}(\bar{z})\equiv U(\bar{z}+\lambda F(T))$ for $\bar{z}\in(0,+\infty)$, and the notation $\bar{Z}^{\Pi}(t)\equiv Z^{\Pi,\lambda}(t)$ stands for the liquid surplus process for a given illiquid asset investment λ. Similarly, we take $\bar{X}^{\Pi}(t)\equiv X^{\Pi,\lambda}(t)$ which represents the liquid wealth process for the given illiquid asset investment λ. Notice that on the first-step, the investor no longer controls the illiquid asset investment, so we omit the superscript λ now. As before, $\bar{Z}^{\Pi}(t)$ satisfies

$$\begin{aligned} d\bar{Z}^{\Pi}(t) &= \bar{Z}^{\Pi}(t)Rdt+\bar{X}^{\Pi}(t)\Pi'\Sigma\Xi dt+D(t)\Sigma_D'\Xi dt \\ &\quad +(\bar{X}^{\Pi}(t)\Pi'\Sigma+D(t)\Sigma_D')d\mathbb{W}(t), \end{aligned} \tag{16}$$

and $\bar{Z}^{\Pi}(0)=x_0+d_0-\lambda F(0)$.

Then we apply the martingale method to solve this problem. We first solve a terminal static optimization problem to obtain the optimal terminal wealth. In this step, we formulate the equivalent static problem

$$\max_{\bar{Z}^{\Pi}(T)} \quad \mathbf{E}[\bar{U}(\bar{Z}^{\Pi}(T))] \tag{17}$$

$$\text{s.t.} \quad \mathbf{E}[H(T)\bar{Z}^{\Pi}(T)]\le x_0+d_0-\lambda F(0), \tag{18}$$

$$\bar{Z}^{\Pi}(T)\ge 0. \tag{19}$$

To solve this problem, we consider two cases: $F(T)$ is a fixed constant or is stochastic. In a particular baseline case, when the terminal payoff of the illiquid asset $F(T)$ is a fixed constant, we take the Legendre-Fenchel transform and solve the pointwise maximization problem, see Chen et al. (2017) for more details. Now we state the result of this problem. In general case, when a stochastic payoff $F(T)$ at retirement is studied, the resolving thought is similar but more complicated. The main difference is that in process of obtaining the uniqueness of the optimal solution, $F(T)$, as a random variable, has more values and makes the derivation quite hard.

Proposition 3.1. *The optimal terminal liquid surplus (or wealth) of the preceding problem for a loss-averse investor under a fixed illiquid asset investment $\lambda\in[0,\lambda_{max}]$ is, when $\theta\ge\lambda F(T)$ holds,*

$$\bar{Z}^{\Pi^*}(T)=\bar{X}^{\Pi^*}(T)=\begin{cases}\theta+(\frac{B\gamma_2}{y(\lambda)H(T)})^{\frac{1}{1-\gamma_2}}-\lambda F(T), & \text{if} \quad H(T)<\bar{H}(\lambda),\\ 0, & \text{if} \quad H(T)\ge\bar{H}(\lambda),\end{cases} \tag{20}$$

when $\theta<\lambda F(T)$ holds,

$$\bar{Z}^{\Pi^*}(T)=\bar{X}^{\Pi^*}(T)=\left(\theta+(\frac{B\gamma_2}{y(\lambda)H(T)})^{\frac{1}{1-\gamma_2}}-\lambda F(T)\right)^+, \tag{21}$$

where $\bar{H}(\lambda)$ satisfies $f(\bar{H}(\lambda))=0$ with

$$f(x)=\frac{1-\gamma_2}{\gamma_2}(\frac{1}{y(\lambda)x})^{\frac{\gamma_2}{1-\gamma_2}}(B\gamma_2)^{\frac{1}{1-\gamma_2}}-\theta y(\lambda)x+A\theta^{\gamma_1} \tag{22}$$

and $y(\lambda)>0$, which is the corresponding Lagrange multiplier, satisfies

$$\mathbf{E}[H(T)\bar{Z}^{\Pi^*}(T)]=x_0+d_0-\lambda F(0).$$

Proof. See Appendix A. □

The optimal terminal liquid wealth for a loss-averse investor (under fixed illiquid asset investment λ) is divided into three parts. The reference point plays a crucial role in determining the optimal terminal wealth. On the one hand, when $\theta\ge\lambda F(T)$ holds, the investor faces a reference point which is higher than the payoff paid by the illiquid asset. The investor will invest in the liquid market to achieve the target. The state of the market has a big impact on the terminal wealth available to the investor. In particular, the investor's terminal total wealth achieves $\theta+(\frac{B\gamma_2}{y(\lambda)H(T)})^{\frac{1}{1-\gamma_2}}-\lambda F(T)+\lambda F(T)=\theta+(\frac{B\gamma_2}{y(\lambda)H(T)})^{\frac{1}{1-\gamma_2}}$ if $H(T)<\bar{H}(\lambda)$, or $\lambda F(T)$ if $H(T)\ge\bar{H}(\lambda)$. As a special case, when $\lambda=0$, $\theta\ge\lambda F(T)$ holds. In this case, the investor does not buy any illiquid asset. This model degenerates to the model studied by Chen et al. (2017). On the other hand, when $\theta<\lambda F(T)$ holds, the illiquid asset provides a sufficient high payoff for the investor, and the investor will always achieve the target. In this case, the investor's nonnegative surplus process constraint always holds, which means that the investor's S-shaped utility degenerates to CRRA type utility. Similar to Desmettre and Seifried (2016), the optimal terminal wealth of the investor achieves $\left[\theta+(\frac{B\gamma_2}{y(\lambda)H(T)})^{\frac{1}{1-\gamma_2}}\right]\vee\lambda F(T)$. As a special case, when $\theta=0$, $\theta\le\lambda F(T)$ holds. Our model degenerates to the model studied by Desmettre and Seifried (2016). The optimal terminal wealth $(\frac{B\gamma_2}{y(\lambda)H(T)})^{\frac{1}{1-\gamma_2}}\vee\lambda F(T)$ coincides with theirs.

In our model, the separation of liquid and illiquid wealth makes our results different from the previous results in Berkelaar et al. (2004), Guan and Liang (2016) and Chen et al. (2017). In those papers, they all find that the terminal wealth reaches above the investor's reference point θ in good economic states, while this property does not hold any more in the presence of the illiquid asset. Although the wealth decreases to 0 as economic conditions deteriorate, the existence of the illiquid asset trading opportunities provides additional chance to gain benefit. In our model, particularly, there exists two situations where the investor's liquid final wealth becomes 0. First, as

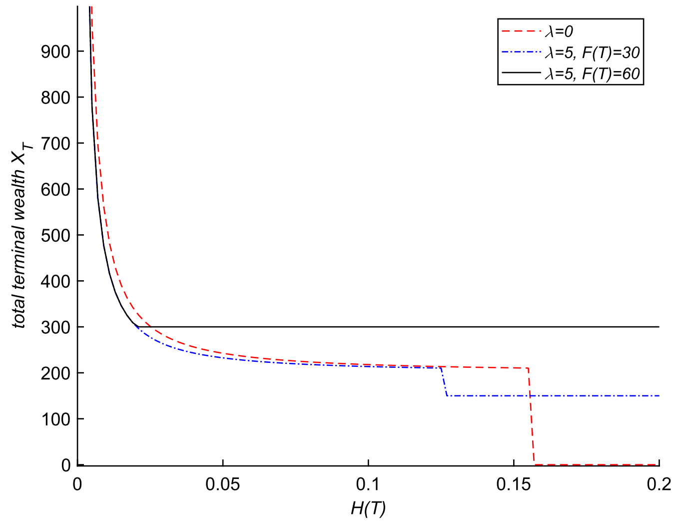


Fig. 1. Effect of illiquid asset trading on the terminal total wealth. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

shown in (20), the investor's liquid wealth target is adjusted by the payoff of the illiquid asset $\lambda F(T)$, and when $\theta - \lambda F(T) < 0$, there is a possibility of event that "the investor who even owns a negative liquid wealth may fulfill the aim". At this time, the definition of admissible strategy precludes this occurrence of negative liquid wealth. As a result, the investor's liquid wealth is always 0. This statement means that even in good states in the market, the investor may obtain a final liquid wealth of 0. This happens naturally for high payoffs of the illiquid asset. Second, in bad states, when facing large losses compared with the reference point, the loss-averse investor now becomes risk-seeking and will maximize the probability of beating the goal, which also leads her final liquid wealth to be 0. In particular, from the view of total wealth, the illiquid asset always provides a guaranteed return as $\lambda F(T)$. In Chen et al. (2017), the minimum performance constraint provides a manner to protect the pension investor's wealth from becoming 0 at retirement. In this paper, we find that investing in the illiquid asset helps the pension investor to hedge the portfolio risk.

As shown in Fig. 1, investing in the illiquid asset helps the pension investor to hedge the liquid assets portfolio risk efficiently and provides a minimum guaranteed wealth $\lambda F(T)$ even in bad states. When the pricing kernel is low, the investor faces a nice environment and obtains high benefits mainly by investing in the liquid assets. While as the pricing kernel grows, the environment deteriorates in liquid market. The investor will suffer loss investing in the liquid assets, and can only obtain a minimum guaranteed wealth $\lambda F(T)$ by buying the illiquid asset. When the payoff is high, the investor is more motivated to buy the illiquid asset, and earns more.

After obtaining the optimal terminal wealth at retirement time, we need to guarantee that the optimal illiquid asset investment λ^* is attained before deriving the explicit expressions for the optimal investment strategy Π^*.

In the second-step, we define a new function of $\lambda: \quad [0, \lambda_{max}] \to \mathbb{R}$,

$$V(\lambda) \equiv \mathbf{E}[\bar{U}(\bar{Z}^{\Pi^*}(T))].$$

Then after direct calculations, we have

$$V(\lambda) = \mathbf{E}[U(Z^{\Pi^*,\lambda}(T) + \lambda F(T))] = \max_{\Pi:\ (\Pi,\lambda)\in\mathcal{A}} \mathbf{E}[U(X^{\Pi,\lambda}(T) + \lambda F(T))], \tag{23}$$

according to utility function (10) and terminal liquid wealth (20) and (21), we have the following result. When $\theta \geq \lambda F(T)$ holds,

$$U(X^{\Pi^*,\lambda}(T) + \lambda F(T)) = \begin{cases} B(\frac{B\gamma_2}{y(\lambda)H(T)})^{\frac{\gamma_2}{1-\gamma_2}}, & \text{if} \quad H(T) < \bar{H}(\lambda), \\ -A(\theta - \lambda F(T))^{\gamma_1}, & \text{if} \quad H(T) \geq \bar{H}(\lambda), \end{cases} \tag{24}$$

when $\theta < \lambda F(T)$ holds,

$$U(X^{\Pi^*,\lambda}(T) + \lambda F(T)) = B\left([\theta + (\frac{B\gamma_2}{y(\lambda)H(T)})^{\frac{1}{1-\gamma_2}}] \vee \lambda F(T) - \theta\right)^{\gamma_2}, \tag{25}$$

where $\bar{H}(\lambda)$ and $y(\lambda) > 0$ are defined as before. In order to obtain the optimal investment strategy Π^* to the original problem of the investor, we must determine the optimal illiquid asset investment λ^*. Unfortunately, it is not easy to derive the explicit form of λ^* (we will show it later), we therefore aim to guarantee the existence and uniqueness of λ^*.

When $\theta < \lambda F(T)$, the S-shaped utility degenerates into CRRA type utility. In this case the investor always is able to achieve her goal. According to Desmettre and Seifried (2016), there will always be an unique λ^{**} that maximizes the $V_2(\lambda) \triangleq \mathbf{E}[B([\theta + (\frac{B\gamma_2}{y(\lambda)H(T)})^{\frac{1}{1-\gamma_2}}] \vee \lambda F(T) - \theta)^{\gamma_2}]$.

When $\theta \geq \lambda F(T)$, we calculate $V_1(\lambda) \triangleq \mathbf{E}\left[B(\frac{B\gamma_2}{y(\lambda)H(T)})^{\frac{\gamma_2}{1-\gamma_2}}\mathbb{I}_{\{H(T)<\bar{H}(\lambda)\}} - A(\theta - \lambda F(T))^{\gamma_1}\mathbb{I}_{\{H(T)\geq\bar{H}(\lambda)\}}\right]$, leading to

$$V_1(\lambda) = \Phi(d_2(\bar{H}(\lambda)))B(\frac{B\gamma_2}{y(\lambda)})^{\frac{1}{1-\gamma_2}}e^{\Gamma} - A(\theta - \lambda F(T))^{\gamma_1}(1 - \Phi(d_1(\bar{H}(\lambda))) + \|\Xi\|\sqrt{T}),$$

where

$$d_1(x) = \frac{\ln x + (R - \frac{1}{2}\|\Xi\|^2)T}{\|\Xi\|\sqrt{T}},$$
$$d_2(x) = d_1(x) + \frac{\|\Xi\|\sqrt{T}}{1-\gamma_2},$$
$$\Gamma = \frac{\gamma_2}{1-\gamma_2}(R + \frac{1}{2}\|\Xi\|^2)T + \frac{1}{2}(\frac{\gamma_2}{1-\gamma_2})^2\|\Xi\|^2 T,$$

$\Phi(\cdot)$ denotes the cumulative standard normal distribution function and $\phi(\cdot)$ denotes the standard normal density function. We can see that the first part is decreasing with respect to λ while the second part is increasing w.r.t. λ.

Overall, in general case, when a stochastic payoff $F(T)$ at retirement is considered, the function $V(\lambda)$ has a non-trivial interior solution in general, and it is not clear how to differentiate the function $V(\lambda)$ with respect to (w.r.t.) λ. We will follow these steps for analysis.

In a particular baseline case, when the terminal payoff of the illiquid asset $F(T)$ is a fixed constant, we will notice the following situations. In particular, we recall the function $V_2(\lambda)$, which is easily verified to be a concave function.[13] When $\lambda \in [0, \lambda_{max}]$, there is a unique maximum value of function $V_2(\lambda)$, and we denote $\lambda^{**} \equiv \arg\max_{\lambda\in[0,\lambda_{max}]} \mathbf{E}\left[B\left([\theta + (\frac{B\gamma_2}{y(\lambda)H(T)})^{\frac{1}{1-\gamma_2}}] \vee \lambda F(T) - \theta\right)^{\gamma_2}\right]$. According to the optimal terminal wealth shown in Eqs. (24)–(25), we have the following conclusions. First, when $H(T) < \bar{H}^{**}$ ($\bar{H}^{**}$ represents the value $\bar{H}(\lambda)$ for fixed illiquid asset position λ^{**}, similarly hereinafter), the particular part $\Phi(d_2(\bar{H}(\lambda)))B(\frac{B\gamma_2}{y(\lambda)})^{\frac{1}{1-\gamma_2}}e^{\Gamma}$ is decreasing w.r.t. λ. We also have $H(T) < \bar{H}^{**} < \bar{H}(0)$, where $\bar{H}(0)$ represents the value of $\bar{H}$ when the trading positions of the illiquid asset is 0. At the same time $\theta > 0$ always holds. It concludes that the optimal investment strategy of the illiquid asset is $\lambda^* = 0$. Second, when $H(T) \geq \bar{H}^{**}$, we will show that a high or low reference point for the investor would determine her different illiquid asset investment strategies. As $\theta > \lambda_{max}F(T)$, together with the relationship $H(T) \geq \bar{H}^{**} \geq \bar{H}(\lambda_{max})$, and the fact that the particular part $-A(\theta - \lambda F(T))^{\gamma_1}(1 - \Phi(d_1(\bar{H}(\lambda))) + \|\Xi\|\sqrt{T})$ is increasing w.r.t. λ, the optimal investment strategy of the illiquid asset is $\lambda^* = \lambda_{max}$. As $\theta \leq \lambda^{**}F(T)$, together with the condition $H(T) \geq \bar{H}^{**}$, the optimal investment strategy of the illiquid asset is $\lambda^* = \lambda^{**}$. As $\lambda^{**}F(T) < \theta \leq \lambda_{max}F(T)$, according to the property of continuous function and similar monotonicity analysis, there exists a unique $\lambda^{***} \in [\lambda^{**}, \lambda_{max}]$, and it is the optimal investment strategy of the illiquid asset. Because the terminal payoff of the illiquid asset $F(T)$ is a fixed constant, only one of the above three relationships is true, we denote in the corresponding case, the optimal strategy is $\lambda^{\diamond}$ (it represents λ^{**}, λ^{***} or λ_{max} in above three different situations). As a result, the optimal illiquid asset investment λ^* is given by[14]

$$\lambda^* = \begin{cases} 0, & \text{if} \quad H(T) < \bar{H}^{**}, \\ \lambda^{\diamond}, & \text{if} \quad H(T) \geq \bar{H}^{**}. \end{cases} \tag{26}$$

In general case, when we consider a stochastic payoff $F(T)$ at retirement, the resolving thought is similar. The difference is that all three of these scenarios are happening in a probabilistic way. Therefore, the optimal solution is similar in expression form. Unfortunately, there is still no perfect method to calculate the analytical expression of the optimal investment strategy of the illiquid asset, but numerically, the main idea is to solve the optimal investment strategy through statistical average on the basis of Monte Carlo simulation. The relevant part will be further analyzed in the numerical part. Here we further analyze the problem inspired by the baseline case.

Based on the previous analysis and combined with Proposition 3.1, we have the following findings. Under the S-shaped utility framework, the major role played by the illiquid asset is to provide the investor with a guarantee of return when the market is in bad state. However, when the market is in good condition, that is, $H(T) < \bar{H}^{**}$, the investor can obtain greater returns by investing in liquid stock markets. Therefore, the investor does not choose to buy any amount of the illiquid asset at this time, that is, $\lambda^* = 0$. However, when the market condition gradually deteriorates, that is, $H(T) \geq \bar{H}^{**}$, or the reference point (target) of the investor is too high, that is, $\theta > \lambda_{max}F(T)$, the liquid stock market cannot provide enough returns for her at this time, and the optimal position of the illiquid asset for the investor to hold is no longer zero, that is, $\lambda^* = \lambda^{\diamond}$. And at this time, the investor's target level has an important impact on her position of the illiquid asset. Specifically, the investor's illiquid asset position increases as the reference point increases, reflecting the maximum effort of a risk-seeking investor to achieve her goal.

In conclusion, we present the following proportion to show that the maximum of the function $V(\lambda)$ is attainable and to give the optimal terminal wealth of the investor under the optimal illiquid asset investment strategy. The proof process is similar to Proposition 3.1, which is omitted here.

Proposition 3.2. *Suppose that $\mathbf{E}[H(T)X^{\Pi,\lambda}(T)] < \infty$ and $\mathbf{E}[U(X^{\Pi,\lambda}(T))] < \infty$ for any Π with $(\Pi, \lambda) \in \mathcal{A}$ under a given λ hold by the investor, then a unique static investment position λ^* exists which maximizes the function $V(\lambda)$. Moreover, there is an optimal solution (Π^*, λ^*) of the investment problem with a fixed position of illiquid asset. As a result, under the optimal illiquid asset investment λ^*, the pension investor's optimal terminal total wealth is given by*

$$X^{\Pi^*,\lambda^*}(T) + \lambda^* F(T), \tag{27}$$

[13] Please refer to the appendix B for the certification process.

[14] Another way to determine the optimal illiquid asset investment strategy is, more intuitively, that when $\frac{\theta}{F(T)} < \lambda \leq \lambda_{max}$, there exists an unique solution λ^{**} for function $V_2(\lambda)$. When $0 \leq \lambda \leq \frac{\theta}{F(T)}$, the continuous function $V_1(\lambda)$ must have a maximum on a closed interval λ^*. However, the maximum value may not be unique and we need to compare these local maximum points to obtain the optimal solution. This line of argument is also valid, while it does not shed much light on the characteristics of the optimal illiquid asset investment strategy.

where $X^{\Pi^*,\lambda^*}(T)$ is the investor's terminal liquid wealth, following

$$X^{\Pi^*,\lambda^*}(T)=\begin{cases}\theta+(\frac{B\gamma_2}{y^*H(T)})^{\frac{1}{1-\gamma_2}}-\lambda^*F(T), & \text{if} \quad H(T)<\bar{H}^*,\\ \left(\theta+(\frac{B\gamma_2}{y^*H(T)})^{\frac{1}{1-\gamma_2}}-\lambda^*F(T)\right)^+\mathbb{I}_{\{\theta<\lambda^*F(T)\}}, & \text{if} \quad H(T)\geq\bar{H}^*,\end{cases} \tag{28}$$

where $\bar{H}^*$ satisfies $f(\bar{H}^*)=0$ with

$$f(x)=\frac{1-\gamma_2}{\gamma_2}(\frac{1}{y^*x})^{\frac{\gamma_2}{1-\gamma_2}}(B\gamma_2)^{\frac{1}{1-\gamma_2}}-\theta y^*x+A\theta^{\gamma_1} \tag{29}$$

and $y^*\geq 0$, which is the corresponding Lagrange multiplier, satisfies

$$\mathbf{E}[H(T)X^{\Pi^*,\lambda^*}(T)]=x_0+d_0-\lambda^*F(0).$$

From this proposition, we find that when $H(T)\geq\bar{H}^*$ a.e., at initial time the investor will spend some or all of her wealth buying the illiquid asset; at retirement she will get the total wealth $\lambda^*F(T)$ provided by the illiquid asset. In this environment, the illiquid will protect her pension wealth from falling to zero. In fact, $H(T)\geq\bar{H}^*$ a.e. reveals that the economic state is quite terrible, it is therefore reasonable to take the opportunity of illiquid asset trading. When $H(T)<\bar{H}^*$ a.e., the investor can obtain greater returns by investing in liquid stock markets. Therefore, the investor does not choose to buy any amount of the illiquid asset at this time, that is, $\lambda^*=0$. As a result, even if the investor does not put any of her wealth ($\lambda^*=0$) in the illiquid asset, that is, she invests her whole wealth in the liquid market, and the favorable market conditions allow the investor to earn sufficient returns.

Now we provide a characterization of the investor's optimal investment strategy in the liquid assets in the liquid market. That is, given the optimal illiquid asset investment λ^*, we derive the closed-form expressions for the optimal liquid assets investment strategy Π^* as follows. In particular, we present the result under the case where $\theta\geq\lambda^*F(T)$, and the result under the case $\theta<\lambda^*F(T)$ is shown in the subsection 3.3.

Proposition 3.3. *(i) The optimal wealth of the investor at time $0\leq t<T$ is given by*

$$\begin{aligned}X^{\Pi^*,\lambda^*}(t)=&\theta e^{-R(T-t)}\Phi(d_1(\bar{H}^*))+(\frac{B\gamma_2}{y^*H(t)})^{\frac{1}{1-\gamma_2}}e^{\Gamma(t)}\Phi(d_2(\bar{H}^*))\\&-\lambda^*F(T)e^{-R(T-t)}\Phi(d_1(\bar{H}^*))-D(t),\end{aligned} \tag{30}$$

where

$$d_1(x)=\frac{\ln(\frac{x}{H(t)})+(R-\frac{1}{2}\|\Xi\|^2)(T-t)}{\|\Xi\|\sqrt{T-t}}, \tag{31}$$

$$d_2(x)=d_1(x)+\frac{\|\Xi\|\sqrt{T-t}}{1-\gamma_2}, \tag{32}$$

$$\Gamma(t)=\frac{\gamma_2}{1-\gamma_2}(R+\frac{1}{2}\|\Xi\|^2)(T-t)+\frac{1}{2}(\frac{\gamma_2}{1-\gamma_2})^2\|\Xi\|^2(T-t), \tag{33}$$

and $\bar{H}^*$ and y^* are given in Proposition 3.2.

(ii) Let $\Lambda(t)=\frac{\theta e^{-R(T-t)}-\lambda^*F(T)e^{-R(T-t)}}{\|\Xi\|\sqrt{T-t}}\phi(d_1(\bar{H}^*))+(\frac{B\gamma_2}{y^*H(t)})^{\frac{1}{1-\gamma_2}}e^{\Gamma(t)}(\frac{\phi(d_2(\bar{H}^*))}{\|\Xi\|\sqrt{T-t}}+\frac{\Phi(d_2(\bar{H}^*))}{1-\gamma_2})$. Then, the percentage of wealth invested in the liquid assets is

$$\Pi^*(t)=\underbrace{\frac{\Sigma^{-1}\Xi}{X^{\Pi^*,\lambda^*}(t)}\Lambda(t)}_{\text{Part 1}}-\underbrace{\frac{\Sigma^{-1}\Sigma_D}{X^{\Pi^*,\lambda^*}(t)}D(t)}_{\text{Part 2}}, \tag{34}$$

in particular,

$$\pi_1^*(t)=\frac{\sigma_I^{-1}\xi_1}{X^{\Pi^*,\lambda^*}(t)}\Lambda(t)-\frac{\sigma_I^{-1}\sigma_Y\rho_{IY}}{X^{\Pi^*,\lambda^*}(t)}D(t), \tag{35}$$

$$\begin{aligned}\pi_2^*(t)=&\frac{1}{\sqrt{1-\rho_{IS}^2}}\frac{\sigma_S^{-1}\xi_2-\sigma_I^{-1}\rho_{IS}\xi_1}{X^{\Pi^*,\lambda^*}(t)}\Lambda(t)\\&-\frac{1}{\sqrt{1-\rho_{IS}^2}}\frac{\sigma_S^{-1}\sigma_Y\sqrt{1-\rho_{IY}^2}-\sigma_I^{-1}\rho_{IS}\sigma_Y\rho_{IY}}{X^{\Pi^*,\lambda^*}(t)}D(t),\end{aligned} \tag{36}$$

where $\Phi(\cdot)$ denotes the cumulative standard normal distribution function and $\phi(\cdot)$ denotes the standard normal density function.

Proof. See Chen et al. (2017). It is straightforward and we omit it here. □

In the optimal liquid wealth, we find that $X^{\Pi^*,\lambda^*}(t)$ is affected by the presence of the illiquid asset trading. When the terminal liquid wealth $X^{\Pi^*,\lambda^*}(T)$ is larger than 0, the investor invests in the liquid market and makes profits, and these gains are bigger than buying the illiquid asset. However, these gains are greatly affected by the state of the liquid market. In general, if the illiquid asset trading opportunities are not available, the investor tends to receive a zero return when the market state is bad. However, when the illiquid asset is allowed to be traded, there are conditions under which the investor can still earn a positive return by investing in a portion of the illiquid asset, even if the market state is bad. At this time, the investor has a strong desire to buy the illiquid asset.

In the optimal investment strategy in Eq. (34), on the one hand, part 1 illustrates her optimal investment strategy in liquid market in the presence of the illiquid asset and in absence of income risk. It is adjusted by the condition $X^{\Pi^*,\lambda^*}(T) > 0$, determined by the specific role of the illiquid asset in the model.[15] On the other hand, for the remaining component part 2, we find that the income exists in the portfolio and generates a new hedging component regarding income risk. In particular, when $\sigma_Y > 0$, the investor invests less aggressively in the risky assets as a result of substitution effect between the risky assets and income. That is, the percentage of wealth invested in the risky assets decreases w.r.t. $D(t)$. As a result, the investor will invest more wealth in the liquid assets as a result of beneficial diversified portfolio.

3.2. Time-varying reference point

In this subsection we study the effect of dynamic choice of reference point on the investment strategy. We assume that the reference point θ is no longer a constant, but time-varying during the working period. In particular, at initial time the investor is assumed to have an initial reference point $\theta(0)$; then the investor will update her reference point continuously, depending on the development of her wealth. To be more specific, the investor adjusts her initial reference point $\theta(0)$ with the constant riskless rate R weighted by $1-\alpha$, and with the change of her wealth $dX^{\Pi,\lambda}(t)$ weighted by α, where $\alpha \in [0, 1)$. As a result, the investor's reference point evolves dynamically according to the process as follows:

$$d\theta(t) = (1-\alpha)\theta(0)Rdt + \alpha dX^{\Pi,\lambda}(t). \tag{37}$$

Integrating two sides of previous equation from 0 to T, we obtain

$$\theta(T) = \theta(0) + (1-\alpha)\theta(0)RT + \alpha[X^{\Pi,\lambda}(T) - x_0]. \tag{38}$$

Recall that the investor's S-shaped utility is now shown by

$$U(X(T)) = \begin{cases} -A(\theta(T) - X(T))^{\gamma_1}, & \text{if} \quad X(T) \le \theta(T), \\ B(X(T) - \theta(T))^{\gamma_2}, & \text{if} \quad X(T) > \theta(T). \end{cases} \tag{39}$$

To solve the portfolio choice problem with a stochastic reference point (37), we first show that it is equivalent to a portfolio choice problem of a loss-averse investor with a non-stochastic reference point. Notice that $X(T) \le \theta(T)$ means that $X^{\Pi,\lambda}(T) - \theta(T) = (1-\alpha)X^{\Pi,\lambda}(T) - \theta(0) - (1-\alpha)\theta(0)RT + \alpha x_0 \le 0$, that is $X^{\Pi,\lambda}(T) \le \frac{1}{1-\alpha}\{\theta(0) + (1-\alpha)\theta(0)RT - \alpha x_0\}$, we denote

$$\theta_T^*(\alpha) \equiv \frac{1}{1-\alpha}\{\theta(0) + (1-\alpha)\theta(0)RT - \alpha x_0\} \tag{40}$$

for simplicity, and obviously the new reference point $\theta_T^*(\alpha)$ is nonstochastic. It depends on α, $\theta(0)$, R, T and x_0. Given these variables, it's a constant. Then we can rewrite the utility function as follows:

$$U(X(T)) = \begin{cases} -A(1-\alpha)^{\gamma_1}(\theta_T^*(\alpha) - X(T))^{\gamma_1}, & \text{if} \quad X(T) \le \theta_T^*(\alpha), \\ B(1-\alpha)^{\gamma_2}(X(T) - \theta_T^*(\alpha))^{\gamma_2}, & \text{if} \quad X(T) > \theta_T^*(\alpha). \end{cases} \tag{41}$$

The new form shows that the utility over final wealth of a loss-averse investor who dynamically updates her reference point with the rule (39) is identical to the utility over final wealth of a loss-averse investor with a constant reference point of $\theta_T^*(\alpha)$ and loss aversion level $(A/B)(1-\alpha)^{\gamma_1-\gamma_2}$. The result is also consistent with that in Berkelaar et al. (2004).

There are two special cases which should be explained. First, an intuitive case is that the investor chooses her initial endowment x_0 as the initial reference point $\theta(0)$. At this time we derive a more simple equation for $\theta_T^*(\alpha)$ in the following:

$$\theta_T^* \equiv x_0(1+RT), \tag{42}$$

which shows that θ_T^* now only depends on x_0, R and T. That is, the weight α does not affect the new reference point, however, it still affects the loss aversion level $(A/B)(1-\alpha)^{\gamma_1-\gamma_2}$. Second, we consider the case of $\alpha = 1$. At this time the investor's reference point evolves completely according to the changes of her wealth, that is, $d\theta(t) = dX^{\Pi,\lambda}(t)$. Consequently we have the relationship $X^{\Pi,\lambda}(T) - \theta(T) = x_0 - \theta(0)$. The result derived here is also interesting: when the investor chooses her reference point only from the change of wealth, whether her investment goal can be achieved is completely determined by the comparison between initial wealth and initial reference point.

[15] As shown before, in the presence of illiquid asset, the value of its payoff determines the features of the terminal wealth. This condition affects the investor's investment strategy significantly.

3.3. Comparison of S-shaped utility and concave utility

To show the differences in the investment strategies between a loss-averse investor and a risk-averse investor, we first show that our model can be simplified to a special case studied in Desmettre and Seifried (2016). For simplicity, we assume that the risk-averse investor receives no salary. In order to inherit the attitude of risk aversion in our model, we choose the reference point θ to be 0. According to the definition of admissible strategies, liquid surplus process $Z^{\pi,\lambda}(t)$ is always nonnegative. Thus the utility function degenerates to $U(Z^{\pi,\lambda}(T)) = BZ^{\pi,\lambda}(T)^{\gamma_2}$, which is just the CRRA type. When $\theta = 0$, we have $\bar{H}(\lambda) = +\infty$ for admissible λ by simple calculations, therefore $\Phi(d_1(\bar{H}(\lambda))) = \Phi(d_2(\bar{H}(\lambda))) = 1$ and $\phi(d_1(\bar{H}(\lambda))) = \phi(d_2(\bar{H}(\lambda))) = 0$, we derive the optimal terminal wealth as follows

$$\bar{Z}^{\pi^*}(T) = \bar{X}^{\pi^*}(T) = \left((\frac{B\gamma_2}{y(\lambda)H(T)})^{\frac{1}{1-\gamma_2}} - \lambda F(T) \right)^+ .$$

As a result, the terminal total wealth achieves $\left((\frac{B\gamma_2}{y(\lambda)H(T)})^{\frac{1}{1-\gamma_2}} - \lambda F(T) \right)^+ + \lambda F(T) = [(\frac{B\gamma_2}{y(\lambda)H(T)})^{\frac{1}{1-\gamma_2}}] \vee \lambda F(T)$.

In addition, we study another case where risk-averse preference emerges under the availability of illiquid asset trading. We refer to the optimal terminal liquid surplus in Proposition 3.1. When $\theta < \lambda F(T)$ holds, the illiquid asset provides a sufficient high payoff for the investor, and the investor will always achieve the target. In this case, the investor's nonnegative surplus process constraint always holds, which means that the investor's S-shaped utility also degenerates to CRRA type utility. Similar to the first case where θ is chosen to be 0, the optimal terminal wealth of the investor achieves $\left[\theta + (\frac{B\gamma_2}{y(\lambda)H(T)})^{\frac{1}{1-\gamma_2}}\right] \vee \lambda F(T)$. The optimal terminal wealth $(\frac{B\gamma_2}{y(\lambda)H(T)})^{\frac{1}{1-\gamma_2}} \vee \lambda F(T)$ coincides with the result of Desmettre and Seifried (2016).

The optimal wealth of the investor is given by

$$\bar{X}^{\pi^*}(t) = \mathbf{E}_t \left\{ \left[\frac{H(T)}{H(t)} \left((\frac{B\gamma_2}{y(\lambda)H(T)})^{\frac{1}{1-\gamma_2}} - \lambda F(T) \right) \right] \mathbb{I}_{\{\bar{X}^{\pi^*}(T)>0\}} \right\}, \tag{43}$$

which is calculated by applying the method in Proposition 3.3, and is shown by

$$\bar{X}^{\pi^*}(t) = (\frac{B\gamma_2}{y(\lambda)H(t)})^{\frac{1}{1-\gamma_2}} e^{\Gamma(t)} \Phi(d_2(G(\lambda))) - \lambda F(T) e^{-R(T-t)} \Phi(d_1(G(\lambda))), \tag{44}$$

where $G(\lambda) = \frac{B\gamma_2}{y(\lambda)(\lambda F(T))^{1-\gamma_2}}$, and $d_1(x)$, $d_2(x)$, $\Gamma(t)$ are defined in Proposition 3.3. The wealth invested in the stock is given by

$$\bar{X}^{\pi^*}(t)\pi^*(t) = \frac{\xi/\sigma_S}{1-\gamma_2} \left(\bar{X}^{\pi^*}(t) + \lambda F(T) e^{-R(T-t)} \Phi(d_1(G(\lambda))) \right). \tag{45}$$

We find that the strategy (45) is the same as the optimal investment strategy for CRRA investors in theorem 4.3 in Desmettre and Seifried (2016).

Second, we show the difference between these two results. On the one hand, for investment strategy, we find that for a risk-averse investor, the optimal trading strategy is the Merton strategy that the investor would implement if her total wealth were given as the sum of the market prices of her liquid securities, and the value of her holdings in the illiquid asset. While for a loss-averse investor in our model, it is optimal to hold a time-varying portfolio weight, even in absence of income risk. Merton's constant proportion strategy is no longer optimal due to the fact that states of the market and stochastic pricing kernel can significantly affect the risk attitude of the investor, thus choosing different investment strategies. On the other hand, for optimal terminal wealth, we find that for a risk-averse investor, the optimal terminal wealth is always $(\frac{B\gamma_2}{y(\lambda)H(T)})^{\frac{1}{1-\gamma_2}} \vee \lambda F(T)$. While for a loss-averse investor, if the state of the market is good, the investor accumulates more wealth and exceeds her reference point. However, even when the state of the market is bad, the investor can lock in a guaranteed return as low as $\lambda F(T)$. Thus, the terminal wealth of a loss-averse investor is higher than that of a CRRA type investor in good economic states. As economic conditions deteriorate, however, the terminal wealth drops to $\lambda F(T)$ provided by the illiquid asset trading.

Based on the above comparison, another finding of this paper is that the illiquid asset can provide more adequate protection under the loss-averse utility function than the traditional utility function. This is because, in poor market conditions, the risk-seeking investor will adopt very aggressive investment strategies to maximize the realization of their investment goals. However, this behavior always leads to the dissolution of the investor's wealth. As a result, the presence of the illiquid asset will make the investor in the market of a poor state consider investment opportunities outside the stock market. The investor will estimate the gains from illiquid asset investments, measuring the gap between the gains and his target gains. Compared with the traditional model, a less radical investment strategy will be employed by the investor so that the illiquid asset investment can play a protective role, which is reflected in real estate investment, regular savings, and infrastructure investment behavior. Although investors cannot achieve high returns from these illiquid assets as they achieve from stocks in a bull market, they can obtain basic protection in a bear market and be compensated for their losses in the stock market.

4. Stochastic trading times and liquidity constraints

In this section, the impacts of stochastic trading time and liquidity constraints on pension investment behavior are studied. It is assumed that illiquid assets are traded at the stochastic time and borrowing the future income is not applicable. Under these cases, some preliminary analyses are conducted to verify the applicability of the main results in this paper.

4.1. Stochastic trading times

In the study by Ang et al. (2014), the trading time of illiquid assets followed a Poisson process with intensity ν, and thus the expected interval between each transaction of illiquid assets was $\frac{1}{\nu}$. It can be concluded that Ang et al. (2014) developed an infinite horizon model, which demonstrates the magnified impacts of illiquidity with finite horizon. However, in a pension plan, the investor always faces a finite retirement time. According to Ang et al. (2014), the illiquidity faced by the pension fund at retirement is very costly, which may lead to the investor to abandon illiquid assets. In this subsection, this phenomenon is further explained, and some new conclusions are drawn. For example, it is found that the magnified effects of illiquidity with a finite horizon are alleviated under some simple assumptions and modifications. In a pension plan, since the event that 'illiquid assets are traded at retirement time T' happens with a zero probability, the investor's wealth at retirement no longer contains the return on illiquid asset investment, and the investor chooses the optimal strategy $\{\Pi, \lambda\}$ to maximize the expected utility of terminal wealth $X^{\Pi,\lambda}(T)$. This problem is addressed in the following three cases.

Case A. $\nu = 0$. The investor intuitively will not buy assets as the expected interval between each transaction of illiquid assets is infinite. In this paper, this problem is attributed to the model without illiquid asset trading. Among numerous studies on this problem, those by Guan and Liang (2016) and Chen et al. (2017) are referred to for more discussions.

Case B. $\nu = \infty$. Illiquid assets can be seen as a kind of liquid assets, which can be traded continuously during the investor's investment period. In the financial market, this kind of assets is favorable for completing the market without being distinguished from liquid wealth. We call this kind of illiquid assets security F. At this time, the problem becomes another familiar one, that is, in the complete market, how does the investor allocate his wealth into four liquid assets to maximize the expected utility from the terminal wealth at retirement? For the comparison, the results are presented without proof in this paper, and the calculations are quite straightforward.

Proposition 4.1. *(i) Let $\pi(t) = (\pi_1(t), \pi_2(t), \pi_3(t))'$, where $\pi_1(t)$, $\pi_2(t)$, $\pi_3(t)$ (instead of λ, here the trading of the illiquid asset occurs continuously) and $1 - \pi_1(t) - \pi_2(t) - \pi_3(t)$ are the proportions invested in the index bond, the stock, the security F and the risk-free bond, respectively. The optimal wealth of the investor at time $0 \le t < T$ is given by*

$$X^{\pi^*}(t) = \theta e^{-R(T-t)}\Phi(d_1(\bar{H})) + (\frac{B\gamma_2}{yH(t)})^{\frac{1}{1-\gamma_2}} e^{\Gamma(t)}\Phi(d_2(\bar{H})) - D(t), \tag{46}$$

where

$$\sigma = \begin{pmatrix} \sigma_I & 0 & 0 \\ \sigma_S\rho_{IS} & \sigma_S\sqrt{1-\rho_{IS}^2} & 0 \\ \sigma_F\rho_{IF} & \sigma_F\rho_{SF} & \sigma_F\sqrt{1-\rho_{IF}^2-\rho_{SF}^2} \end{pmatrix}, \tag{47}$$

$$\xi = \sigma^{-1}\begin{pmatrix} r+i-R \\ \mu_S - R \\ \mu_F - \delta - R \end{pmatrix} \equiv \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}, \tag{48}$$

assume a three-dimensional Brownian motion $W(t) := (W_1(t), W_2(t), W_F(t))'$ and hence

$$H(t) = e^{-Rt-\frac{1}{2}\|\xi\|^2 t - \xi' W(t)} \tag{49}$$

is the stochastic discount factor;

$$d_1(x) = \frac{\ln(\frac{x}{H(t)}) + (R - \frac{1}{2}\|\xi\|^2)(T-t)}{\|\xi\|\sqrt{T-t}}, \tag{50}$$

$$d_2(x) = d_1(x) + \frac{\|\xi\|\sqrt{T-t}}{1-\gamma_2}, \tag{51}$$

$$\Gamma(t) = \frac{\gamma_2}{1-\gamma_2}(R + \frac{1}{2}\|\xi\|^2)(T-t) + \frac{1}{2}(\frac{\gamma_2}{1-\gamma_2})^2\|\xi\|^2(T-t), \tag{52}$$

where $\bar{H}$ satisfies $f(\bar{H}) = 0$ with

$$f(x) = \frac{1-\gamma_2}{\gamma_2}(\frac{1}{yx})^{\frac{\gamma_2}{1-\gamma_2}}(B\gamma_2)^{\frac{1}{1-\gamma_2}} - \theta yx + A\theta^{\gamma_1}, \tag{53}$$

and $y \ge 0$, the corresponding Lagrange multiplier, satisfies

$$\mathbf{E}[H(T)X^{\pi^*}(T)] = x_0 + d_0.$$

(ii) Let $\Lambda(t) = \frac{\theta e^{-R(T-t)}}{\|\xi\|\sqrt{T-t}}\phi(d_1(\bar{H})) + (\frac{B\gamma_2}{yH(t)})^{\frac{1}{1-\gamma_2}} e^{\Gamma(t)}(\frac{\phi(d_2(\bar{H}))}{\|\xi\|\sqrt{T-t}} + \frac{\Phi(d_2(\bar{H}))}{1-\gamma_2})$. Then, the percentage of wealth invested in the risky assets is

$$\pi^*(t) = \frac{\sigma^{-1}\xi}{X^{\pi^*}(t)}\Lambda(t) - \frac{\sigma^{-1}\sigma_D}{X^{\pi^*}(t)}D(t), \tag{54}$$

where $\Phi(\cdot)$ denotes the cumulative standard normal distribution function and $\phi(\cdot)$ denotes the standard normal density function.

Case C. $0 < \nu < \infty$. We first show that when $0 < \nu < \infty$, the value function is bounded below by the problem studied in case A, where illiquid assets are unavailable; and is bounded above by the problem studied in case B, where illiquid assets can be traded continuously. The reason is obvious. When it is optimal to hold a positive position λ^* in illiquid assets, the case A would not provide the investor with the opportunity to do it. When considering case B, we show that each possible optimal strategy under condition $0 < \nu < \infty$ can be replicated by a strategy chosen by the investor in case B. For example, when $\lambda^* = \lambda_{max}$, we already show that it is an optimal choice for the investor to employ all her initial surpluses to buy illiquid assets. In case B, this strategy is implemented simply by choosing $\pi_3(t) = 1$. And if the optimal trading position of illiquid assets is 0, the investor will invest no money in illiquid asset F. We consider a general case where the investor has an optimal position λ^* in illiquid assets. Since the investor could select her portfolio continuously, she can always adjust her investment proposition $\pi_3(T)$ at retirement to achieve the same return of $\lambda^* F(T)$ on illiquid asset F. Furthermore, as she is able to allocate her wealth without the constraint on investment propositions, she will take advantage of this mechanism to gain profits during her working period. All these strategies just mentioned will provide gains no fewer than those in the case under condition $0 < \nu < \infty$.

The case of $0 < \nu < \infty$ is studied by Ang et al. (2014). In their model, however, the investor faces an infinite horizon. In a pension plan, we show that the situation is something different. Given that time T is unlikely to be the trading time of illiquid assets, the investor's wealth at retirement no longer contains returns on illiquid assets. The problem is still how to measure the returns on illiquid assets at retirement reasonably since the investor can obtain utility from it while facing difficulty in trading it without delay. If utility is not derived from the wealth accumulated by illiquid assets, then the optimal choice is to avoid purchasing this illiquid asset. In fact, there is no essential difference between it and case A.

Without loss of generality, we assume there exist suitable wealth weights on liquid wealth and illiquid wealth, respectively. That is, the investor aims to maximize the expected utility from the weighted terminal total wealth $\hat{X}^{\hat{\Pi},\lambda}(T) + \omega \cdot \lambda \hat{F}(T)$, where ω represents the weighting endowed to the illiquid asset by the investor.

Next, we pay attention to how to determine a weighting coefficient ω. At this time the terminal total wealth which brings the investor the utility consists of $X^{\Pi,\lambda}(T)$, which represents the liquid final wealth, and $\omega \cdot \lambda F(T)$, which represents the benefit of the illiquid asset. We just modify the condition in the Proposition 3.1 to

$$\bar{U}(\bar{z}) = \begin{cases} -A\,[(\theta - \omega \cdot \lambda F(T)) - \bar{z}]^{\gamma_1}\,, & \text{if} \quad \bar{z} \le \theta - \omega \cdot \lambda F(T), \\ B\,[\bar{z} - (\theta - \omega \cdot \lambda F(T))]^{\gamma_2}\,, & \text{if} \quad \bar{z} > \theta - \omega \cdot \lambda F(T), \end{cases} \tag{55}$$

as a result, the difference lies in the form of terminal total wealth between this case and the baseline model.

Adopting a similar procedure in Section 3, we are able to derive optimal terminal wealth, the existence of optimal trading rules of the illiquid asset, and the optimal investment strategy. In particular, we find the terminal total wealth achieves $\left(\theta + (\frac{B\gamma_2}{y(\lambda)H(T)})^{\frac{1}{1-\gamma_2}} - \omega \cdot \lambda F(T)\right)^{+} + \omega \cdot \lambda F(T) = \left[\theta + (\frac{B\gamma_2}{y(\lambda)H(T)})^{\frac{1}{1-\gamma_2}}\right] \vee \omega \cdot \lambda F(T)$ if $H(T) < \bar{H}(\lambda)$, or $\omega \cdot \lambda F(T)$ if $H(T) \ge \bar{H}(\lambda)$. The terminal total wealth is no more than the case in section 3, which shows that in an environment with stochastic trading times for the illiquid asset, this uncertainty always generates cost in trading the illiquid asset. As a result, we obtain a concluding result.

Proposition 4.2. *(1) When the weighting rate ω is small, the investor has little motive to buy the illiquid asset; especially, when $\omega = 0$, the investor will buy no illiquid asset.*

(2) As the weighting rate ω grows, the probability $\mathbb{P}\left(\theta + (\frac{B\gamma_2}{y(\lambda)H(T)})^{\frac{1}{1-\gamma_2}} \le \omega \cdot \lambda F(T)\right)$ grows, and the investor is more willing to buy the illiquid asset.

(3) When facing stochastic trading times of the illiquid asset, the investor must pay a cost for this uncertainty in trading the illiquid asset. When $\omega \to 1$, the solutions gradually converges to that of the baseline model in Section 3.

4.2. Liquidity constraints

Except for the illiquidity caused by illiquid assets, there is another kind of illiquidity faced by investors in virtually all real financial markets. It is usually called liquidity constraints or borrowing constraints, which refer to the limitations on the investors' opportunities to borrow their future salaries in this model.

In previous works, He and Pages (1993) and El Karoui and Jeanblanc-Picque (1998) conducted a formal analysis of liquidity constraints using standard models. Detemple and Serrat (2003) considered a dynamic equilibrium problem with liquidity constraints. They found that, in the presence of liquidity constraints, the solution to the new problem was similar to the standard solution. The only difference lies in the different forms of state price densities (pricing kernels). In the presence of liquidity constraints, the adjusted state price density is lower than that in the standard model. Dybvig and Liu (2010) obtained a similar result when considering constrained borrowing.

In the presence of illiquid assets, this model is aimed at addressing the following two problems: (i) what is the impact of liquid constraints on the investor's decision on illiquid asset investment? This problem is relatively new and was not studied in previous works. (ii) what is the impact of liquid constraints on the investor's investment strategies for liquid assets?

Mathematically, this inability to borrow implies that the investor's investments are restricted by a constraint which mandates the liquid wealth. Specifically, the liquid wealth must be nonnegative during the investment period, i.e., for all $t \in [0, T]$, satisfies

$$X^{\Pi,\lambda}(t) \ge 0. \tag{56}$$

We find that in this case the new admissible set $\mathcal{A}^*$ is restricted to a smaller range in $\mathcal{A}$.

In baseline model, we define a state variable $Z^{\Pi,\lambda}(t) = X^{\Pi,\lambda}(t) + D(t)$ which must be nonnegative during the investment period, as a result, $X^{\Pi,\lambda}(t) \ge -D(t)$. In order to solve the liquid constraint problem, we add another condition $X^{\Pi,\lambda}(t) \ge 0$ into the optimization model,

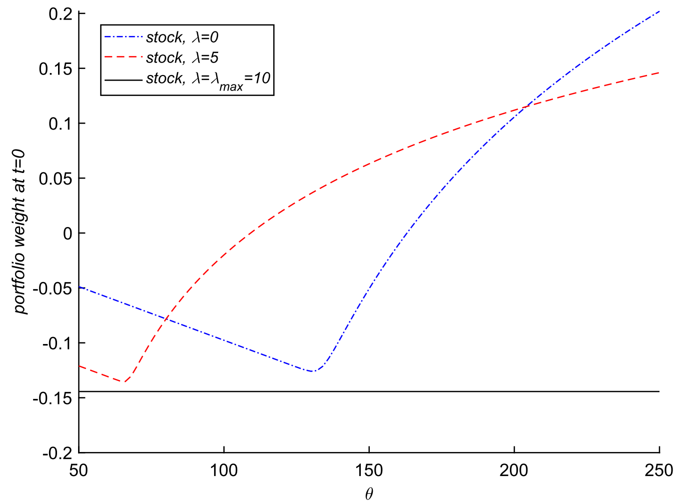


Fig. 2. Effect of reference points under different illiquid trading. The basic parameters are: at $t=0$, $H(0)=1$, $x_0=10$, $d_0=10$, $F(0)=2$ and $\lambda_{max}=\frac{x_0+d_0}{F(0)}=10$. In the S-shaped utility, we assume $A=2.25$ and $B=1$ as estimated by Kahneman and Tversky (1992), $\gamma_1=0.15$ and $\gamma_2=0.2$ for the curvature parameters for losses and gains. Investment horizon is $T=40$ years. For simplicity, we consider the stock investment. And the result of index bond is similar.

and use the Lagrange multiplier method to deal with this constraint. In particular, we introduce the multiplier $\chi(t)$ which represents the shadow price of this constraint. The complementary slackness condition tells that

$$X^{\Pi,\lambda}(t)d\chi(t)=0. \tag{57}$$

We first answer question (ii). For a given illiquid asset trading rule λ, we are able to derive the optimal terminal liquid wealth and investment strategies in a similar way. After obtaining the optimal wealth of the investor $X^{\Pi^*,\lambda}(t)$, we will make sure whether the condition $X^{\Pi^*,\lambda}(t)\geq 0$ holds. When it holds, the complementary slackness condition shows that $\chi^*(t)=0$; when it does not hold, we need $X^{\Pi,\lambda}(t)=0$, at this time $\chi^*(t)>0$. We adopt the multiplier $\chi^*(t)$ to adjust the pricing kernel $H(t)$, in fact, we obtain an adjusted pricing kernel as $(1-\chi^*(t))H(t)\equiv H_{LC}(t)$.[16] We replace $H_{LC}(t)$ with $H(t)$, we rewrite the solutions easily and we omit it.

We next turn to question (i). The effect of the liquidity constraint on the illiquid asset trading comes from two paths. First, in the presence of liquidity constraint, the investor is unable to borrow and has less wealth to buy the illiquid asset at initial time. As a result, the static position λ now satisfies $\lambda\leq\frac{x_0}{F(0)}\equiv\lambda_{max,LC}$. This is the direct impact of liquidity constraints on the trading of the illiquid asset. Due to the liquid constraint, the investor can buy only $\lambda_{max,LC}$ positions of the illiquid asset at most. The second way becomes effective through the role of adjusted pricing kernel. As for the adjusted pricing kernel $H_{LC}(t)$, in the optimal investment strategies, the probability $\mathbb{P}\left(\theta+(\frac{B\gamma_2}{y(\lambda)H_{LC}(T)})^{\frac{1}{1-\gamma_2}}\leq\lambda F(T)\right)$ is smaller than that in the baseline model under the same market environment, the investor is therefore more likely to invest in the liquid market.

Integrating (i) and (ii), we also notice a situation happens that even the investor buys $\lambda_{max,LC}$ positions of the illiquid asset at time 0, there are also opportunities for the investor to invest in the liquid market. Under liquidity constraints, however, the liquid wealth must be nonnegative, therefore the liquid wealth at each time t is given by

$$\begin{aligned}X^{\Pi^*,\lambda^*}(t)=&\theta e^{-r(T-t)}\Phi(d_1(\bar{H}^*))+(\frac{B\gamma_2}{y^*H_{LC}(t)})^{\frac{1}{1-\gamma_2}}e^{\Gamma(t)}\Phi(d_2(\bar{H}^*))\\&-\lambda^*F(T)e^{-r(T-t)}\Phi(d_1(\bar{H}^*))-D_{LC}(t),\end{aligned} \tag{58}$$

where the new state variables $D_{LC}(t)$ is calculated by the adjusted pricing kernel $H_{LC}(t)$, and we find that the present value of the future salary in the presence of liquidity constraints is less than that without liquidity constraints. This result comes from the cost of liquidity constraints and is reasonable.

In conclusion, it is found from this study that the existence of liquidity constraints reduces the investor's motive for illiquid asset trading. Compared with the findings in prior literature, the findings in this study show that in the presence of trading opportunities for illiquid assets, liquidity constraints have a similar impact on the investment strategies of liquid assets while having a more significant impact on illiquid asset trading. Facing liquidity constraints, the investor is more willing to invest in liquid assets, and the potential benefits from illiquid asset investment are reduced.

5. A sensitivity analysis

When we consider the trading of the illiquid asset, the impact of the reference point on investment strategy is shown in Fig. 2. It can be seen from the figure that, the size of the reference point for the pension investor with loss aversion preference plays a key role in her

[16] In this study, the relative results from the studies by He and Pages (1993), El Karoui and Jeanblanc-Picque (1998), and Bodie et al. (2004) are used without providing more detailed derivations. The calculations are not important (necessary) for the main purpose of this study. Thus, only a preliminary analysis is conducted to investigate the impact of liquidity constraints on illiquid asset trading. For more mathematical backgrounds and details, there is prior literature.

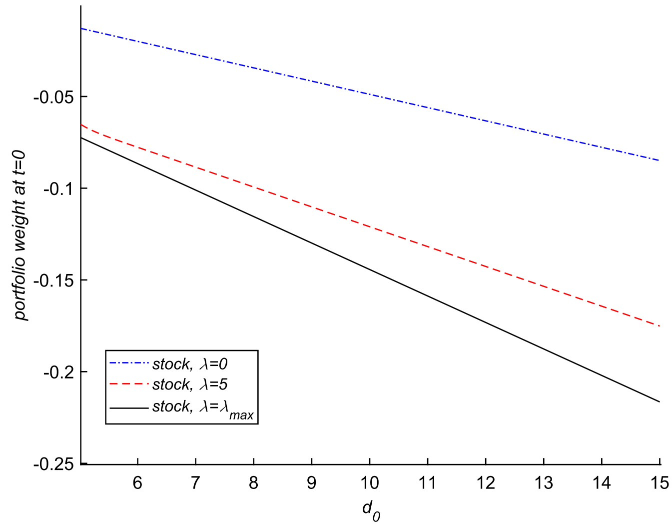


Fig. 3. Effect of salary levels under different illiquid trading. The basic parameters remains the same. We fix the reference point $\theta = 60$.

optimal asset allocation, and different reference point levels lead to distinct results. When in the absence of the illiquid asset, the optimal wealth propositions of risky assets (stocks) show a V-shaped curve as the reference point level grows. However, comparing the dashed and dotted-dashed lines in the figure, we find that when there is illiquid asset trading (from the dotted-dashed line to the dashed line), the stock's investment ratio is reducing for low or high references, that is, the introduction of the illiquid asset provides more trading opportunities, which reduces the risk exposure of the investor. When the reference point is very high, it is hardly to be reached at this time, so the investor undertakes too much risk. Investing in the illiquid asset can help to reduce risk. For moderate reference points, the proportion of wealth invested in the stock is increased along with the reference level and the trading volume of the illiquid asset. At this time, when the investor buys the illiquid asset, the wealth invested in the liquid market is reduced. The lower level of wealth leads the investor to be risk seeking due to loss aversion. Therefore, the investor increases the proportion of wealth invested in the stock.

At the same time, for each curve, when the level of the reference point is above a certain threshold, the pension investor will be in the area of losses, implying risk seeking. Therefore, as the reference point level further increases, her investment in the risky assets also increases. As can be seen from the solid line in the figure, when the investor spends all her wealth in purchasing the illiquid asset, the size of the reference point does not affect the investment strategy of the stock.

Fig. 3 shows the impact of different contributions on the optimal stock investment strategy for the pension investor in the presence of different illiquid asset investment decisions at zero time. Comparing the dashed and dotted-dashed lines in the figure, we find that when there exists the trading of the illiquid asset (from the dotted-dashed line to the dashed line), it causes the investor to invest less in the stock and reduces the portfolio risk as the initial investment in the liquidity market declines. When we further investigate each curve separately, we find that increasing the expected discount for future contribution levels will reduce the proportion of wealth invested in the stock. From a view of economic, when d_0 is low, the investor will be in the area of losses. Under the influence of S-type utility, she is risk seeking and invests more in the stock. With d_0 increasing, her pension account will receive more expected contribution, so her motivation of risk seeking is getting weaker, and the proportion of wealth invested in the stock is getting lower. We also find that when the investor spends all her initial wealth in purchasing of the illiquid asset, the part 1 in the optimal stock investment at this time is disappearing, and it can be clearly seen that the proportion of stock investment decreases as d_0 increases. This is also verified in the figure.

Fig. 4 shows the impact of different initial wealth levels on the optimal stock investment strategy for the pension investor in the presence of different illiquid asset investment decisions at zero time. Comparing the dashed and dashed-dotted lines in the figure, we find that when there are illiquid asset investments (from the dotted-dashed line to the dashed line), as the initial wealth rises, the declining rate for investor's investing in the stock slows down. When the investor buys no illiquid asset, the situation at this time is the same as the case in Chen et al. (2017), shown by the dotted-dashed line. At this time we find that increasing the initial wealth will reduce the proportion of wealth invested in the stock. Because with the increase of initial wealth, the pension investor will gradually approach the target reference point, she will gradually reduce the proportion of wealth invested in the stock in order to control the portfolio risk. When the investor buys some illiquid asset, the proportion of wealth invested in the stock still decreases with the increase in initial wealth, but at this time, because she invests some of her initial wealth in the illiquid asset, she therefore cannot approach the target reference point easily. As a result, the declining rate for investor's investing in the stock at this time is relatively modest. What is more, we find that when the investor puts all her initial wealth in purchasing of the illiquid asset, the proportion of wealth invested in the stock increases as initial wealth grows, as shown by the solid line.

Fig. 5 shows the effects of the loss aversion ratio A/B and the curvature parameters γ_1 and γ_2 on the optimal investment proportion for the stock. It should be noted that we need to establish the experiment in the absence of the trading for the illiquid asset. Otherwise, when the investor puts all her initial wealth in purchasing of the illiquid asset, the optimal investment strategy does not depend on the parameters of the S-type utility. In the figure, the effects of S-type utility parameters on investor stock investment strategies are intuitive. As the degree of loss aversion increases, the investor is more loss averse. To reduce the portfolio risk, the proportion of wealth invested in the stock is reduced. The investor with a lower degree of loss aversion γ_1 would invest more in the stock because she is more risk-seeking and aims to eliminate losses in the loss domain; the investor with a lower γ_2 would invest less in the stock since she is more risk averse and aims to protect her wealth in the gain domain.

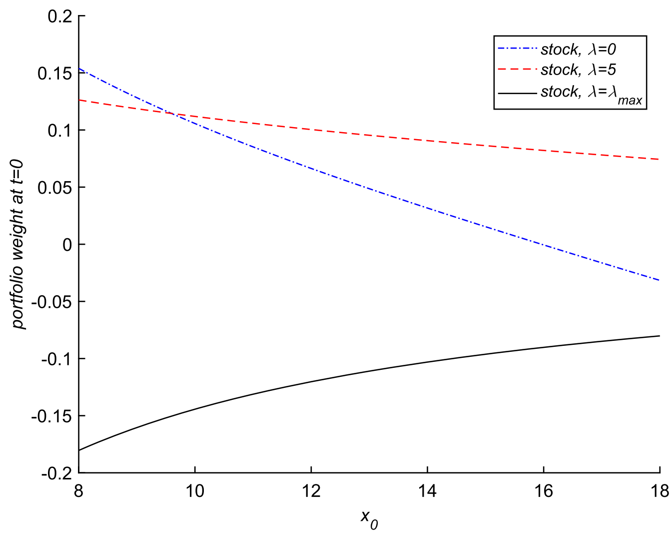


Fig. 4. Effect of initial wealth under different illiquid trading. The basic parameters remains the same. We fix the reference point $\theta = 200$.

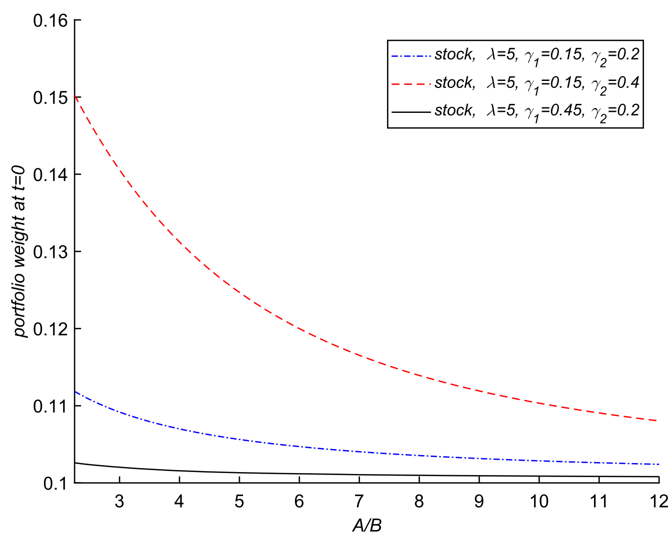


Fig. 5. Effect of S-shaped parameters under different illiquid trading. The basic parameters remains the same. We fix the reference point $\theta = 200$.

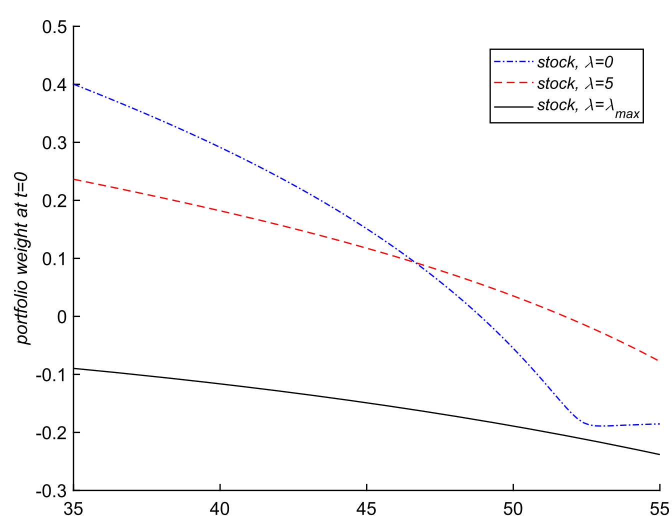


Fig. 6. Effect of investment horizons under different illiquid trading. The basic parameters remains the same. We fix the reference point $\theta = 300$.

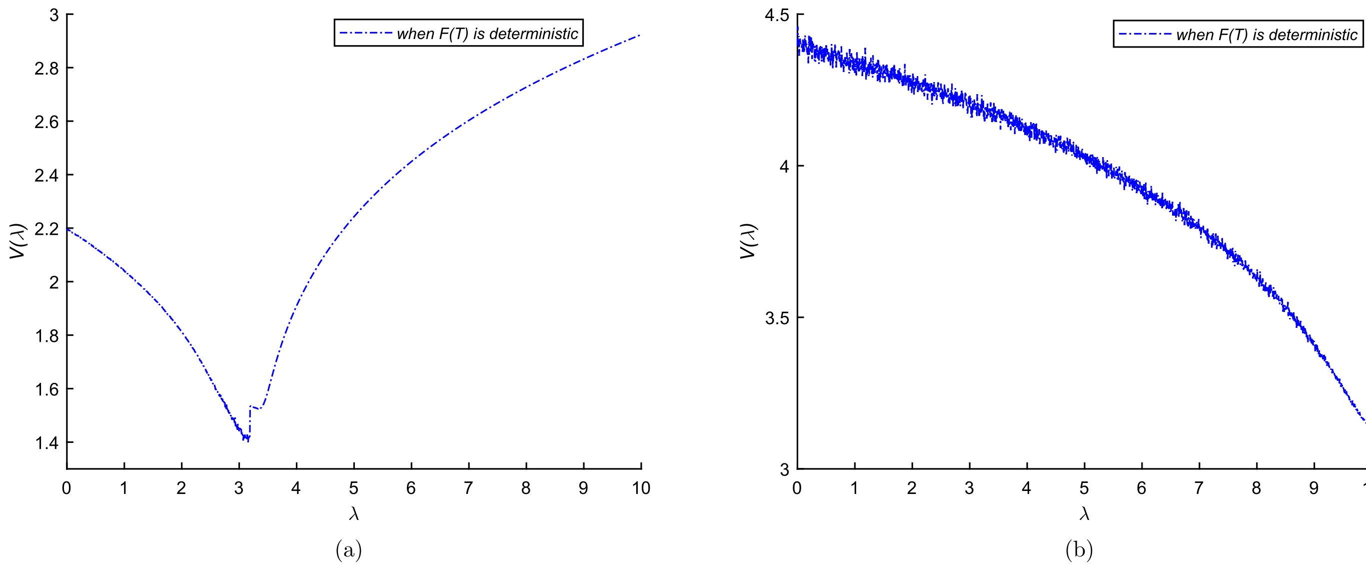


Fig. 7. Monte Carlo simulations for $V(\lambda)$ when payoff $F(T)$ is deterministic.

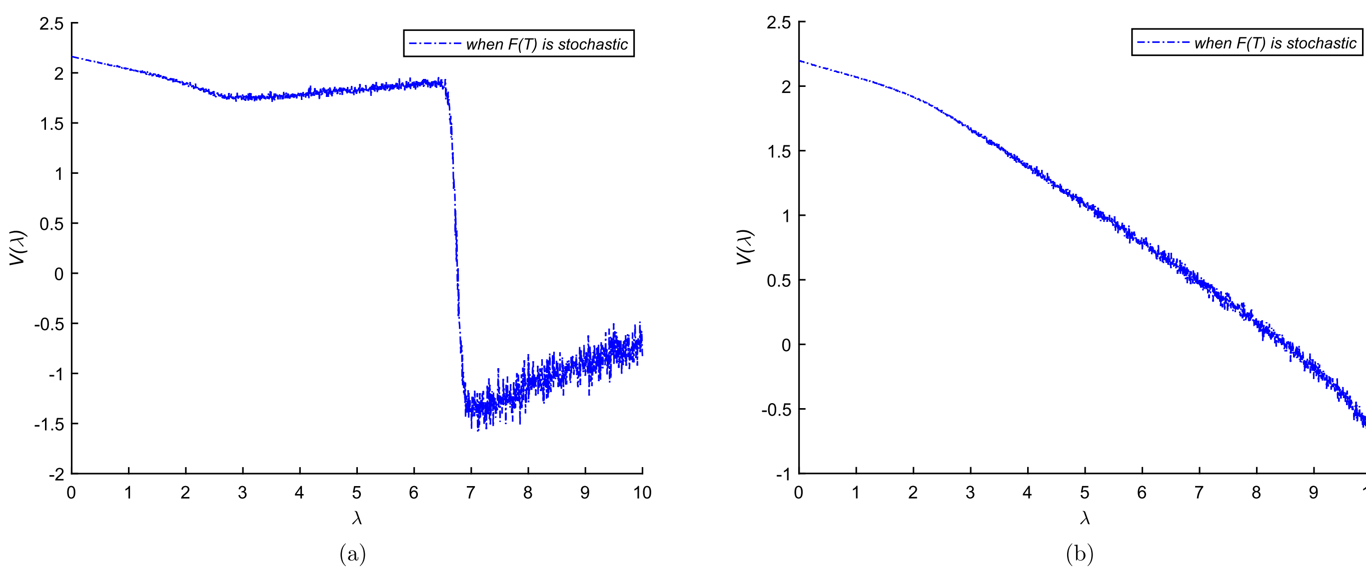
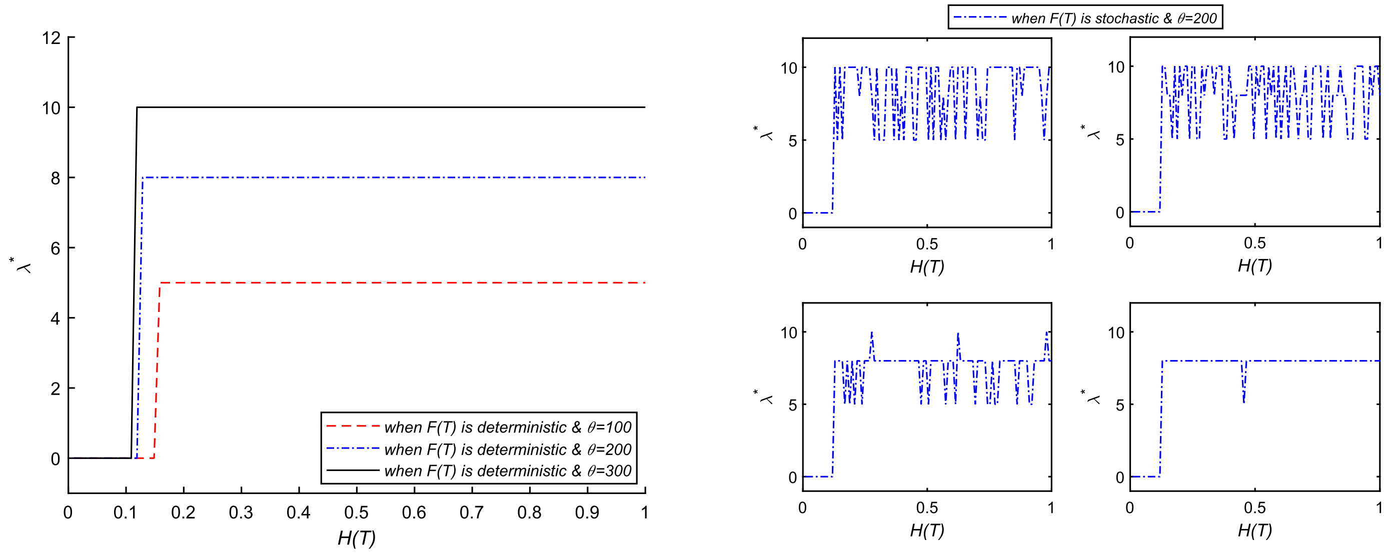


Fig. 8. Monte Carlo simulations for $V(\lambda)$ when payoff $F(T)$ is stochastic.

Fig. 6 shows the impact of changes in investment horizons on the optimal stock investment strategy for the pension investor. In Chen et al. (2017), when salary risk is considered under the pension investment framework, the human capital gradually increases with the prolongation of the investment period, optimal portfolio weight for the stock decreases along with the time horizon T. We show that when there is illiquid asset trading, the proportion of optimal stock investment still decreases along with the time horizon T. When the investor buys no illiquid asset, we find that the declining rate for investor's investing in the stock is more fast. As the investment horizon increases, she will gain more human capital accumulation, she will gradually reduce the risk exposure for the stock. We also notice that at this time, when the investment period is very long, the investor will increase the proportion of stock investment. This is due to the fact that the accumulation of human capital caused by the increase in the investment period is high enough for the investor to reach the target reference point level. Therefore, at this time, the investor prefers to take more risk to earn more than she would invest her wealth only in the risk-free asset. When the investor puts all her initial wealth in purchasing of the illiquid asset, the situation is similar to the other two cases. Finally, we conclude that the optimal stock investment decreases along with the investment horizon.

For the existence of the optimal illiquid asset investment strategy, we already provide the corresponding theorem. In order to represent it more intuitively, we present the numerical description of the optimal illiquid asset investment strategy below. Correspondingly, we illustrate an image of this function.

Figs. 7 and 8 show the existence of the maximum value of the function with help of Monte Carlo simulations. Fig. 7 presents the situation when the payoff of the illiquid asset at retirement $F(T)$ is deterministic for two types of model parameters. From previous analysis, we show that when pricing kernel $H(T)$ is lower than some threshold value, it is optimal to pursue none of the illiquid asset. For other cases, it is optimal to buy a positive position of the illiquid asset. According to Fig. 7(a), both two curves are divided into two parts: the left part is decreasing with λ, while the right part is concave. For the left part, the maximum value is obtained at $\lambda^* = 0$. For the left part, we show that the payoff may grow with the positions of the illiquid asset, it is optimal for the investor to purchase nonzero



(a)   (b)

Fig. 9. Monte Carlo simulations for λ^* when payoff $F(T)$ is deterministic or stochastic.

positions of the illiquid asset at the beginning of the investment period, and her utility at this time is maximum.[17] Fig. 8 presents the situation when the payoff of the illiquid asset at retirement $F(T)$ is stochastic for two types of model parameters. We find that the results are similar but complicated than the deterministic case. We carry Monte Carlo simulations by 10000 times, and we find that there is a positive probability that the optimal positions of the investor's buying and holding the illiquid asset is zero. It is due to the fact that when $F(T)$ is stochastic, it is risky for the investor to trade in the illiquid asset. As a result, she does not choose to buy any amount of the illiquid asset. However, there is also other possibilities in which the optimal strategy for the investor is to buy non-negative positions of the illiquid asset.

According to the previous analysis ahead of Eq. (26), we now provide an image of this optimal illiquid asset holding position λ^*. The basic parameters of the illiquid asset price are $F(0) = 2$, $x_0 = 10$, $d_0 = 10$, then $\lambda_{max} \equiv \frac{x_0+d_0}{F(0)} = 10$. Recall that the concavity of the function $V_2(\lambda)$ guarantees the existence of a unique optimal solution λ^{**}. Consider a particular situation where the optimal solution is $\lambda^{**} = 5$. When $F(T)$ is deterministic, we follow the argument shown in section 3, and calculate the optimal illiquid asset investment λ^*, which is shown by the Fig. 9 (a). We find that in the deterministic case, the investment strategy of illiquid assets is closely related to the market status and the value of the investor's reference point. When the market environment is good, that is, $H(T) < \bar{H}^{**}$, the investor can obtain greater returns by investing in liquid stock markets. Therefore, the investor does not choose to buy any amount of the illiquid asset at this time, therefore $\lambda^* = 0$. However, when the market condition gradually deteriorates, at this point, the level of the reference point plays a crucial role in the investment strategy of the investor's illiquid asset. When reference point is high ($\theta = 300$), that is, $\theta > \lambda_{max} F(T)$, as shown in the figure, the optimal investment strategy of the illiquid asset is $\lambda^* = \lambda_{max} = 10$. When reference point is low ($\theta = 100$), that is, $\theta \le \lambda^{**} F(T)$, the optimal investment strategy of the illiquid asset is $\lambda^* = \lambda^{**} = 5$. When reference point ($\theta = 200$) satisfies $\lambda^{**} F(T) < \theta \le \lambda_{max} F(T)$, there exists a unique $\lambda^{***} = 8 \in [\lambda^{**}, \lambda_{max}]$, and it is the optimal investment strategy of the illiquid asset. The simulation results are in agreement with the theoretical results.

When $F(T)$ is stochastic, the difference is that all three of these scenarios are happening in a probabilistic way. Numerically, the main idea is to solve the optimal investment strategy through statistical average on the basis of Monte Carlo simulation. We consider the case $\theta = 200$. We carry on 10 times, 100 times, 1000 times and 10000 times simulations of the optimal illiquid asset investments. We find that when the investor faces with a very uncertain situation, the risk of $F(T)$ leads her to choose the illiquid asset with an optimal non-zero position, which will change rapidly with the fluctuation of the illiquid asset's price at retirement $F(T)$. However, when the number of simulations increases, the probability that investor will choose the optimal illiquid asset investment decision $\lambda^{***} = 8$ will gradually increase when the reference point is 200. When the number of simulations reaches a certain level, on average, the investor will choose the investment strategy consistent with the certainty case. The results in Fig. 9 (b) confirm these conclusions.

6. Conclusion

This paper investigates the influence of the introduction of illiquid assets to the optimal strategy for a pension plan investment with the S-shaped utility function. The work of Desmettre and Seifried (2016) is extended to address pension investment issues. Firstly, the investor's investment preference is described through the S-shaped utility function instead of the concave utility function. Secondly, both inflation and income risks are considered. In addition to the index bond, risk-free bond, and stock in the liquid market, the investor could also invest in illiquid assets, which can only be traded at the initial time and offer returns at the investor's retirement. When illiquid asset

[17] In this case, the investor will face a complicated situation. We have derived in the previous theoretical analysis that the function at this time is a concave function and therefore has a unique solution in the closed interval. However, the function $V(\lambda)$ has a non-trivial interior solution in general, while it is not difficult to differentiate $V(\lambda)$ with respect to (w.r.t.) λ. At this time we cannot obtain the analytical expression of the optimal illiquid asset investment strategy. But according to Fig. 7(b), it is a concave function, and there is a non-trivial solution to the optimal illiquid asset investment strategy, which is consistent with the conclusion obtained by Desmettre and Seifried (2016). We provide these results by simulations.

trading is added to the investor's investment opportunity set, the difference in the optimization model and solving procedure between this study and traditional works is mainly reflected in the two types of wealth processes: liquid wealth and total wealth. When a fixed investment decision is made for illiquid assets, the problem could be reduced to the asset allocation problem in the traditional liquidity market. Thus, the complex optimization problem is solved in this way with two steps, namely the use of the martingale approach to obtain the optimal investment strategy for liquid assets and the provision of the mathematical proof to guarantee the existence of the optimal investment strategy for illiquid assets. The findings obtained from the model in this paper demonstrate that illiquid assets could provide minimum income protection for the pension investor, especially when the liquid market performs badly.

However, this model can still be extended in some aspects. For example, the case when the trading time of illiquid assets follows a certain distribution can be considered. In addition, from the perspective of the personal life cycle, the trade-off between the introduction of illiquid assets and the stimulation of consumption smoothing or risk diversification can be studied.

Declaration of competing interest

The authors declared no potential conflicts of interest with respect to the research, authorship, and publication of this article.

Data availability

No data was used for the research described in the article.

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Appendix A. Proof of Proposition 3.1

We mainly follow Chen et al. (2017) and provide detailed calculation process. The difference lies in the new form of function $\bar{U}(\bar{z}) \equiv U(\bar{z}+\lambda F(T))$, showing

$$\bar{U}(\bar{z}) = \begin{cases} -A\,[(\theta-\lambda F(T))-\bar{z}]^{\gamma_1}\,, & \text{if} \quad \bar{z} \le \theta-\lambda F(T), \\ B\,[\bar{z}-(\theta-\lambda F(T))]^{\gamma_2}\,, & \text{if} \quad \bar{z} > \theta-\lambda F(T); \end{cases} \tag{59}$$

we denote $U_1(\bar{z}) = -A(\theta-\lambda F(T)-\bar{z})^{\gamma_1}$ and $U_2(\bar{z}) = B(\bar{z}-(\theta-\lambda F(T)))^{\gamma_2}$ for simplicity. As a result, the part of $\bar{z}+\lambda F(T)$ can be seen as 'Z' in Eq. (41) in Chen et al. (2017), in other words, the part of $\theta-\lambda F(T)$ now substitutes the θ in Chen et al. (2017). As a result, we consider two cases under the conditions $\theta \ge \lambda F(T)$ and $\theta < \lambda F(T)$, respectively. Furthermore, the constraint in admissible strategy (11) reveals that $\bar{Z}^{\Pi^*}(T)$ must be nonnegative. The detailed derivation details are provided in the following.

On the one hand, when $\theta \ge \lambda F(T)$ holds, we consider two cases. First if $\bar{z} \le \theta-\lambda F(T)$, the $U_1(\bar{z})$ is convex, and the Weirestrass theorem implies that the maximum $\bar{z}_1^*$ must lie on one of the boundaries $Z_1^*=0$ or $Z_1^*=\theta-\lambda F(T)$. While if $\bar{z} > \theta-\lambda F(T)$, the utility function $U_2(\bar{z})$ is concave and maximum $\bar{z}_2^*$ satisfies the KKT conditions: $L=\bar{U}(\bar{z})-yH(T)\bar{z}+\zeta\bar{z}$ is the Lagrange function,

$$\begin{cases} \bar{U}'(\bar{z}_2^*)-yH(T)+\zeta=0, & \bar{z}_2^* \ge 0, \\ \zeta\bar{z}_2^*=0, & \zeta \ge 0, \end{cases} \tag{60}$$

where ζ denotes another Lagrange multiplier associated with the nonnegativity constraint on wealth. After solving the KKT conditions, we have

$$\bar{z}_2^* = \Big(\frac{B\gamma_2}{yH(T)}\Big)^{\frac{1}{1-\gamma_2}} + \theta - \lambda F(T). \tag{61}$$

We find that under the condition $\theta \ge \lambda F(T)$, the surplus process $\bar{z}_2^*$ is always non-negative. In the following we compare the local maxima $\bar{z}_1^*$ and $\bar{z}_2^*$ to determine the global maximum. If

$$f(H(T)) \equiv \bar{U}(\bar{z}_2^*) - yH(T)\bar{z}_2^* - [\bar{U}(\bar{z}_1^*) - yH(T)\bar{z}_1^*] \ge 0, \tag{62}$$

then the optimal solution is $\bar{z}_2^*$.

Firstly, comparing $\bar{z}_2^*$ with $\bar{z}_1^* = \theta-\lambda F(T)$, we have

$$\begin{aligned} f(H(T)) &= \Big(\frac{B\gamma_2}{yH(T)}\Big)^{\frac{\gamma_2}{1-\gamma_2}}\Big[B - yH(T)\frac{B\gamma_2}{yH(T)}\Big] \\ &= \Big(\frac{B\gamma_2}{yH(T)}\Big)^{\frac{\gamma_2}{1-\gamma_2}} B(1-\gamma_2) > 0. \end{aligned} \tag{63}$$

As such, $f(H(T)) > 0$ holds for all $H(T)$, and $\bar{z}_1^* = \theta-\lambda F(T)$ is never the optimal level of wealth.

Secondly, comparing $\bar{z}_2^*$ with $\bar{z}_1^*=0$, we have

$$f(H(T)) = \Big(\frac{B\gamma_2}{yH(T)}\Big)^{\frac{\gamma_2}{1-\gamma_2}} B(1-\gamma_2) - yH(T)(\theta-\lambda F(T)) + A(\theta-\lambda F(T))^{\gamma_1}, \tag{64}$$

when $H(T) \le \frac{A}{y}(\theta - \lambda F(T))^{\gamma_1 - 1}$, $-yH(T)(\theta - \lambda F(T)) + A(\theta - \lambda F(T))^{\gamma_1} \ge 0$, so $f(H(T)) > 0$. Furthermore, $\lim_{H(T)\to\infty} f(H(T)) = -\infty$, $f'(H(T)) < 0$, so f is strictly decreasing. $f(H(T))$ has exactly one zero in the interval $(\frac{A}{y}(\theta - \lambda F(T))^{\gamma_1 - 1}, +\infty)$ which we denote by $\bar{H}$ with $f(\bar{H}) = 0$.

Therefore, we have $f(H(T)) > 0$, when $H(T) < \bar{H}$; and $f(H(T)) \le 0$, when $H(T) \ge \bar{H}$. That is, $\bar{z}_2^*$ is optimal for $H(T) < \bar{H}$ and $\bar{z}_1^* = 0$ is optimal for $H(T) \ge \bar{H}$. Then, we denote the optimal solution by $\bar{Z}^*(T)$. We can use similar steps to prove the uniqueness of the optimal solution. We have completed the argument in the case $\theta \ge \lambda F(T)$.

On the other hand, when $\theta < \lambda F(T)$ holds, the inequality $\bar{z} > \theta - \lambda F(T)$ always holds. As a result, we just focus on the optimal solution of $U_2(\bar{z}) = B(\bar{z} - (\theta - \lambda F(T)))^{\gamma_2}$, in the same way, we have $\bar{z}_2^* = (\frac{B\gamma_2}{yH(T)})^{\frac{1}{1-\gamma_2}} + \theta - \lambda F(T)$. Together with the condition $\bar{z}_2^* \ge 0$, we obtain the optimal solution as $\bar{Z}^*(T) = \bar{X}^{\Pi^*}(T) = \left(\theta + (\frac{B\gamma_2}{y(\lambda)H(T)})^{\frac{1}{1-\gamma_2}} - \lambda F(T)\right)^+$. □

Appendix B. The concavity of the function $V_2(\lambda)$

We have to show that, for $\lambda \in [0, \lambda_{max}]$, there exists a unique static illiquid asset position λ^* which maximizes the function $V(\lambda)$. Remember that $\bar{H}(\lambda)$ and $y(\lambda)$ are functions of λ, that is, for every $\lambda \in [0, \lambda_{max}]$, the values of $\bar{H}(\lambda)$ and $y(\lambda)$ are determined.

Because the optimal terminal liquid wealth under loss aversion is divided into two parts, as a result λ^* is studied in two cases. In advance, we rewrite $U(X^{\Pi^*,\lambda}(T) + \lambda F(T))$ as

$$U(X^{\Pi^*,\lambda}(T) + \lambda F(T)) = \begin{cases} U_1\left([\theta + (\frac{B\gamma_2}{y(\lambda)H(T)})^{\frac{1}{1-\gamma_2}}] \vee \lambda F(T) - \theta\right), & \text{if} \quad \theta < \lambda F(T), \\ B(\frac{B\gamma_2}{y(\lambda)H(T)})^{\frac{\gamma_2}{1-\gamma_2}}, & \text{if} \quad H(T) < \bar{H}(\lambda), \theta \ge \lambda F(T), \\ -A(\theta - \lambda F(T))^{\gamma_1}, & \text{if} \quad H(T) \ge \bar{H}(\lambda), \theta \ge \lambda F(T), \end{cases} \tag{65}$$

where $U_1(\cdot) = B(\cdot)^{\gamma_2}$ is a concave function.

When $\theta < \lambda F(T)$ holds, then $X^{\Pi^*,\lambda}(T) = \left(\theta + (\frac{B\gamma_2}{y(\lambda)H(T)})^{\frac{1}{1-\gamma_2}} - \lambda F(T)\right)^+$ and function $V_2(\lambda)$ is

$$V_2(\lambda) = \mathbf{E}\left[U_1\left([\theta + (\frac{B\gamma_2}{y(\lambda)H(T)})^{\frac{1}{1-\gamma_2}}] \vee \lambda F(T) - \theta\right)\right]. \tag{66}$$

We next verify that the function (66) is concave and then it attains its unique maximum on $[0, \lambda_{max}]$. On one hand, let $\lambda,\ \bar{\lambda} \in [0, \lambda_{max}]$ and $k \in [0, 1]$. We have

$$\begin{aligned}
&(1-k)V_2(\lambda) + kV_2(\bar{\lambda}) = (1-k)\max_{\Pi:(\Pi,\lambda)\in\mathcal{A}} \mathbf{E}[U_1(X^{\Pi,\lambda}(T) - \theta + \lambda F(T))] \\
&+k \max_{\bar{\Pi}:(\bar{\Pi},\bar{\lambda})\in\mathcal{A}} \mathbf{E}[U_1(X^{\bar{\Pi},\bar{\lambda}}(T) - \theta + \bar{\lambda} F(T))] \\
&= \max_{\Pi,\bar{\Pi}:\ (\Pi,\lambda),(\bar{\Pi},\bar{\lambda})\in\mathcal{A}} (1-k)\mathbf{E}[U_1(X^{\Pi,\lambda}(T) - \theta + \lambda F(T))] + k\mathbf{E}[U_1(X^{\bar{\Pi},\bar{\lambda}}(T) - \theta + \bar{\lambda} F(T))] \\
&\le \max_{\Pi,\bar{\Pi}:\ (\Pi,\lambda),(\bar{\Pi},\bar{\lambda})\in\mathcal{A}} \mathbf{E}[U_1((1-k)X^{\Pi,\lambda}(T) + kX^{\bar{\Pi},\bar{\lambda}}(T) - \theta + ((1-k)\lambda + k\bar{\lambda})F(T))],
\end{aligned} \tag{67}$$

where the last inequality holds due to the concavity of function U_1. On the other hand, we have

$$\begin{aligned}
V_2((1-k)\lambda + k\bar{\lambda}) &= \max_{\tilde{\Pi}:(\tilde{\Pi},(1-k)\lambda+k\bar{\lambda})\in\mathcal{A}} \mathbf{E}[U(X^{\tilde{\Pi},(1-k)\lambda+k\bar{\lambda}}(T) + ((1-k)\lambda + k\bar{\lambda})F(T))] \\
&= \mathbf{E}[U_1(X^{\tilde{\Pi},(1-k)\lambda+k\bar{\lambda}}(T) - \theta + ((1-k)\lambda + k\bar{\lambda})F(T))],
\end{aligned} \tag{68}$$

under the condition $H(T) < \bar{H}((1-k)\lambda + k\bar{\lambda})$.[18] According to the budget constraint (18), for strategies (Π, λ) and $(\bar{\Pi}, \bar{\lambda})$, we obtain

$$\begin{aligned}
\mathbf{E}[H(T)((1-k)X^{\Pi,\lambda}(T) + kX^{\bar{\Pi},\bar{\lambda}}(T))] &= (1-k)(x_0 + d_0 - \lambda F(0)) + k(x_0 + d_0 - \bar{\lambda} F(0)) \\
&= x_0 + d_0 - [(1-k)\lambda + k\bar{\lambda}]F(0),
\end{aligned} \tag{69}$$

let $\tilde{\lambda} = (1-k)\lambda + k\bar{\lambda}$, it is easy to see that $\tilde{\lambda} \in [0, \lambda_{max}]$, and we have

$$\mathbf{E}[H(T)X^{\tilde{\Pi},\tilde{\lambda}}(T)] = x_0 + d_0 - \tilde{\lambda} F(0), \tag{70}$$

as a result, for the illiquid asset investment $\tilde{\lambda}$, there is a strategy $\tilde{\Pi}$ such that $(\tilde{\Pi}, \tilde{\lambda}) \in \mathcal{A}$, and $(1-k)X^{\Pi,\lambda}(T) + kX^{\bar{\Pi},\bar{\lambda}}(T) = X^{\tilde{\Pi},\tilde{\lambda}}(T)$ holds. Thus we obtain the concavity of function (66). □

[18] In this case, we consider the case where both $H(T) < \bar{H}(\lambda)$ and $H(T) < \bar{H}(\bar{\lambda})$ hold. Due to the monotonicity of function $\bar{H}(\lambda)$, $H(T) < \bar{H}(\tilde{\lambda})$ still establishes for $\tilde{\lambda} = (1-k)\lambda + k\bar{\lambda}$ below.

Appendix C. The equivalence of the optimization problems under nominal and real terms

Denote $\hat{X}^{\Pi,\lambda}(t) := \frac{X^{\Pi,\lambda}(t)}{P(t)}$, $\hat{Y}(t) := \frac{Y(t)}{P(t)}$ and $\hat{F}(t) := \frac{F(t)}{P(t)}$. Using Itô's formula, we have the real liquid wealth process as

$$\begin{aligned} d\hat{X}^{\Pi,\lambda}(t) &= \hat{X}^{\Pi,\lambda}(t)[Rdt + \Pi'\Sigma(\Xi - \Sigma_I)dt - (i - \sigma_I^2)dt] \\ &\quad + \hat{X}^{\Pi,\lambda}(t)[\Pi'\Sigma - \Sigma_I{}']d\mathbb{W}(t) + c\hat{Y}(t)dt + \lambda\delta\hat{F}(t)dt,\ \hat{X}^{\Pi,\lambda}(0) = \frac{x_0}{p_0}, \end{aligned} \tag{71}$$

and real income process as

$$d\hat{Y}(t) = \hat{Y}(t)[(\mu_Y - i + \sigma_I^2 - \sigma_Y\rho_{IY}\sigma_I)dt + (\Sigma_D - \Sigma_I)'d\mathbb{W}(t)],\ \hat{Y}(0) = \frac{y_0}{p_0}. \tag{72}$$

The real illiquid asset process is given by

$$\begin{aligned} d\hat{F}(t) &= \hat{F}(t)[(\mu_F - \delta - i + \sigma_I^2 - \sigma_F\rho_{IF}\sigma_I)dt + (\Sigma_F - \Sigma_I)'d\mathbb{W}(t)] \\ &\quad + \sigma_F\sqrt{1 - \rho_{IF}^2 - \rho_{SF}^2}dW_F(t),\ \hat{F}(0) = \frac{F(0)}{p_0}. \end{aligned} \tag{73}$$

At this time the investor's objective is to maximize the expected S-shaped utility from **real** wealth at retirement T. That is, the investor aims to maximize $\mathbf{E}[U\left(\frac{X^{\pi,\lambda}(T)+\lambda F(T)}{P(T)}\right)]$.

In parallel, the new market price of risk is $\hat{\Xi}$, we deduce the new state price density process as

$$\frac{d\hat{H}(t)}{\hat{H}(t)} = -rdt - \hat{\Xi}'d\mathbb{W}(t). \tag{74}$$

This new state price density process is adjusted by price (inflation), and has a different initial value for $\hat{H}(t)$. We determine it as p_0. In fact, the new state price density process can be constructed by $\hat{H}(t) = H(t)P(t)$. Eq. (74) and $\hat{H}(t) = H(t)P(t)$ are actually the same thing after straightforward calculations. As a result, the initial value of $\hat{H}(t)$ is p_0. It restates that the state price density process is inflation-adjusted.

For $\hat{X}^{\Pi,\lambda}(t) := \frac{X^{\Pi,\lambda}(t)}{P(t)}$, $\hat{Y}(t) := \frac{Y(t)}{P(t)}$ and $\hat{F}(t) := \frac{F(t)}{P(t)}$, we recall that the real liquid wealth process is

$$\begin{aligned} d\hat{X}^{\Pi,\lambda}(t) &= \hat{X}^{\Pi,\lambda}(t)[Rdt + \Pi'\Sigma(\Xi - \Sigma_I)dt - (i - \sigma_I^2)dt] \\ &\quad + \hat{X}^{\Pi,\lambda}(t)[\Pi'\Sigma - \Sigma_I{}']d\mathbb{W}(t) + c\hat{Y}(t)dt + \lambda\delta\hat{F}(t)dt. \end{aligned} \tag{75}$$

We calculate that

$$\begin{aligned} \Pi'\Sigma - \Sigma_I{}' &= (\pi_1(t),\ \pi_2(t))\begin{bmatrix} \sigma_I & 0 \\ \sigma_S\rho_{IS} & \sigma_S\sqrt{1-\rho_{IS}^2} \end{bmatrix} - (\sigma_I,\ 0) \\ &= \left(\pi_2(t)\sigma_S\rho_{IS} - (1-\pi_1(t))\sigma_I,\ \pi_2(t)\sigma_S\sqrt{1-\rho_{IS}^2}\right). \end{aligned} \tag{76}$$

By $\hat{\pi}_1(t) := \pi_0(t)$, $\hat{\pi}_2(t) := \pi_2(t)$, and $\hat{\Sigma} := \begin{bmatrix} -\sigma_I & 0 \\ \sigma_S\rho_{IS} - \sigma_I & \sigma_S\sqrt{1-\rho_{IS}^2} \end{bmatrix}$, we have

$$\begin{aligned} \Pi'\Sigma - \Sigma_I{}' &= \left(\pi_2(t)\sigma_S\rho_{IS} - (\pi_0(t)+\pi_2(t))\sigma_I,\ \pi_2(t)\sigma_S\sqrt{1-\rho_{IS}^2}\right) \\ &= (\pi_0(t),\ \pi_2(t))\begin{bmatrix} -\sigma_I & 0 \\ \sigma_S\rho_{IS} - \sigma_I & \sigma_S\sqrt{1-\rho_{IS}^2} \end{bmatrix} \\ &= \hat{\Pi}'\hat{\Sigma}. \end{aligned} \tag{77}$$

On the other hand, as $\hat{\Xi} := \Xi - \Sigma_I$, we have

$$\begin{aligned} R + \Pi'\Sigma(\Xi - \Sigma_I) - (i - \sigma_I^2) &= R - i + \sigma_I^2 + (\pi_1(t),\ \pi_2(t))\begin{bmatrix} \sigma_I & 0 \\ \sigma_S\rho_{IS} & \sigma_S\sqrt{1-\rho_{IS}^2} \end{bmatrix}\begin{pmatrix} \xi_1 - \sigma_I \\ \xi_2 \end{pmatrix} \\ &= R - i + \sigma_I^2 \\ &\quad + (1 - \pi_0(t) - \pi_2(t),\ \pi_2(t))\begin{pmatrix} \sigma_I(\xi_1 - \sigma_I) \\ \sigma_S\rho_{IS}(\xi_1 - \sigma_I) + \sigma_S\sqrt{1-\rho_{IS}^2}\xi_2 \end{pmatrix} \\ &= R - i + \sigma_I^2 + \sigma_I\xi_1 - \sigma_I^2 \\ &\quad + \left(\hat{\pi}_1(t),\ \hat{\pi}_2(t)\right)\begin{bmatrix} -\sigma_I & 0 \\ \sigma_S\rho_{IS} - \sigma_I & \sigma_S\sqrt{1-\rho_{IS}^2} \end{bmatrix}\begin{pmatrix} \xi_1 - \sigma_I \\ \xi_2 \end{pmatrix} \\ &= r + \hat{\Pi}'\hat{\Sigma}\hat{\Xi}, \end{aligned} \tag{78}$$

as a result, we obtain the real liquid wealth process as

$$d\hat{X}^{\hat{\Pi},\lambda}(t) = \hat{X}^{\hat{\Pi},\lambda}(t)[rdt + \hat{\Pi}'\hat{\Sigma}(\hat{\Xi}dt + d\mathbb{W}(t))] + c\hat{Y}(t)dt + \lambda\delta\hat{F}(t)dt. \tag{79}$$

The real salary process $d\hat{Y}(t)$ is straightforward calculated.

Comparing real liquid wealth process (79) with former liquid wealth process (9), we find that under these new notations, the two liquid wealth processes have a similar form. In the **real** environment, the investor faces a risk-free rate of r, and the indexed bond protects the investor against the inflation risk and provides a return rate r.

In the next, we pay attention to the assertion that given the new state price density process $\frac{d\hat{H}(t)}{\hat{H}(t)} = -rdt - \hat{\Xi}'d\mathbb{W}(t)$, the present value of expected future real contribution process $\hat{D}(t) = c\mathbf{E}_t[\int_t^T \frac{\hat{H}(s)}{\hat{H}(t)}\hat{Y}(s)ds]$, the process $\hat{X}^{\hat{\Pi},\lambda}(t) + \hat{D}(t)$ is self-financing, and $\hat{H}(t)[\hat{X}^{\hat{\Pi},\lambda}(t) + \hat{D}(t)]$ is a martingale.[19] Because the related processes are all given the explicit formulas, after a series of simple but tedious calculations we obtain the result.

Through this appendix, we pay attention to the real variables for the investor, and transform the original investment problem into a new investment problem. That is, we construct a new market where the variables are divided by price, in real sense, we show that the maximization problem is equivalent to the original case. In the case of complete market, we also prove that the related process $\hat{X}^{\hat{\Pi},\lambda}(t) + \hat{D}(t)$ is self-financing, and $\hat{H}(t)[\hat{X}^{\hat{\Pi},\lambda}(t) + \hat{D}(t)]$ is a martingale. These results guarantee the validity of the solution to maximize $\mathbf{E}[U(\frac{X^{\pi,\lambda}(T)+\lambda F(T)}{P(T)})]$.

Note that in real environment, the index bond is actually riskless and therefore we always focus on the investment proportions for the risky 'risk-free bond' and the stock. That is the reason why we derive the explicit strategies shown as $\hat{\pi}_1^*(t) = \pi_0(t)$ and $\hat{\pi}_2^*(t) = \pi_2(t)$. As a result, the percentages of wealth invested in the index bond and the stock are easily obtained as $\pi_1^*(t) \equiv 1 - \hat{\pi}_1^*(t) - \hat{\pi}_2^*(t)$ and $\pi_2^*(t) \equiv \hat{\pi}_2^*(t)$ in the nominal market, respectively.

As a result, in the paper nominal variables are used to calculate the results for some reasons as follows. Firstly, a relationship between a nominal market and a real market is identified as above. It is found from the analysis that in a new market where variables are real, the maximization problem is equivalent to the original one under a series of parallel notations. Secondly, the nominal variable is used to derive a relatively simple result, which excludes a lot of unnecessary troubles caused by the complex correlations between these standard Brownian motions. □

References

Ang, A., Papanikolaou, D., Westerfield, M.M., 2014. Portfolio choice with illiquid assets. Management Science 60 (11), 2737–2761.

Baltas, I., Dopierala, L., Kolodziejczyk, K., Szczepański, M., Yannacopoulos, A.N., 2022. Optimal management of defined contribution pension funds under the effect of inflation, mortality and uncertainty. European Journal of Operational Research 298 (3), 1162–1174. https://doi.org/10.1016/j.ejor.2021.08.038.

Barberis, N., Huang, M., Santos, T., 2001. Prospect theory and asset prices. The Quarterly Journal of Economics 116 (1), 1–53.

Battocchio, P., Menoncin, F., 2004. Optimal pension management in a stochastic framework. Insurance. Mathematics & Economics 34, 79–95.

Beletski, T., 2006. Inflation-linked products and optimal investment with macro derivatives. Unpublished thesis. Available at: http://kluedo.ub.uni-kl.de/volltexte/2006/2045/index.html.

Benartzi, S., Thaler, R., 1995. Myopic loss aversion and the equity premium puzzle. The Quarterly Journal of Economics 110 (1), 73–92.

Berkelaar, A.B., Kouwenberg, R., Post, T., 2004. Optimal portfolio choice under loss aversion. Review of Economics and Statistics 86 (4), 973–987.

Bichuch, M., Guasoni, P., 2018. Investing with liquid and illiquid assets. Mathematical Finance 28 (1), 119–152.

Blake, D., Wright, D., Zhang, Y.M., 2013. Target-driven investing: optimal investment strategies in defined contribution pension plans under loss aversion. Journal of Economic Dynamics and Control 37 (1), 195–209.

Bodie, Z., Merton, R., Samelson, W.F., 1992. Labor supply flexibility and portfolio choice in a life cycle model. Journal of Economic Dynamics and Control 16 (3), 427–449.

Bodie, Z., Detemple, J.B., Otruba, S., Walter, S., 2004. Optimal consumption-portfolio choices and retirement planning. Journal of Economic Dynamics and Control 28 (6), 1115–1148.

Brennan, M.J., Xia, Y., 2002. Dynamic asset allocation under inflation. The Journal of Finance 57 (3), 1201–1238.

Broadbent, J., Palumbo, M., Woodman, E., 2006. The Shift from Defined Benefit to Defined Contribution Pension Plans–Implications for Asset Allocation and Risk Management. Reserve Bank of Australia, Board of Governors of the Federal Reserve System and Bank of Canada, pp. 1–54.

Broeders, D., Jansen, K., Werker, B., 2021. Pension fund's illiquid assets allocation under liquidity and capital requirements. Journal of Pension Economics & Finance 20 (1), 102–124.

Brown, J., Mitchell, O., Poterba, J., 2004. The role of real annuities and indexed bonds in an individual accounts retirement program. In: Campbell, J.Y., Feldstein, M. (Eds.), Risk Aspects of Investment-Based Social Security Reform. NBER, pp. 321–370.

Chen, Z., Li, Z.F., Zeng, Y., Sun, J.Y., 2017. Asset allocation under loss aversion and minimum performance constraint in a DC pension plan with inflation risk. Insurance. Mathematics & Economics 75, 137–150.

Campbell, J.Y., Viceira, L.M., 2001. Who should buy long-term bonds? The American Economic Review 91 (1), 99–127.

Curatola, G., 2015. Loss aversion, habit formation and the term structures of equity and interest rates. Journal of Economic Dynamics and Control 53, 103–122.

Dai, M., Li, P., Liu, H., Wang, Y., 2016. Portfolio choice with market closure and implications for liquidity premia. Management Science 62 (2), 368–386.

Desmettre, S., Seifried, F.T., 2016. Optimal asset allocation with fixed-term securities. Journal of Economic Dynamics and Control 66, 1–19.

Detemple, J., Serrat, A., 2003. Dynamic equilibrium with liquidity constraints. The Review of Financial Studies 16 (2), 597–629.

Dong, Y.H., Zheng, H., 2020. Optimal investment with S-shaped utility and trading and value at risk constraints: an application to defined contribution pension plan. European Journal of Operational Research 281, 341–356.

Dybvig, P.H., Liu, H., 2010. Lifetime consumption and investment: retirement and constrained borrowing. Journal of Economic Theory 145 (3), 885–907.

El Karoui, N., Jeanblanc-Picque, M., 1998. Optimization of consumption with labor income. Finance and Stochastics 2 (4), 409–440.

Fortin, I., Hlouskova, J., 2011. Optimal asset allocation under linear loss aversion. Journal of Banking & Finance 35 (11), 2974–2990.

Gârleanu, N., 2009. Portfolio choice and pricing in illiquid markets. Journal of Economic Theory 144, 532–564.

Guan, G.H., Liang, Z.X., 2016. Optimal management of DC pension plan under loss aversion and value-at-risk constraints. Insurance. Mathematics & Economics 69, 224–237.

Han, N., Hung, M., 2012. Optimal asset allocation for the DC pension plans under inflation. Insurance. Mathematics & Economics 51, 172–181.

[19] In addition, we give a simple proof here. We already show the related process $\hat{X}^{\hat{\Pi},\lambda}(t) + \hat{D}(t)$ is self-financing. To apply the martingale approach directly, we need to guarantee $\hat{H}(t)[\hat{X}^{\hat{\Pi},\lambda}(t) + \hat{D}(t)]$ is a martingale. We are able to obtain this result by a series of simple but tedious calculations, using the new form of $\hat{H}(t)$ as (74). Otherwise, if we develop $\hat{H}(t)$ by $H(t)P(t)$, we can give a simple proof for that $\hat{H}(t)[\hat{X}^{\hat{\Pi},\lambda}(t) + \hat{D}(t)]$ is a martingale. That is, $\mathbf{E}[\hat{H}(t)[\hat{X}^{\hat{\Pi},\lambda}(t) + \hat{D}(t)]] = \mathbf{E}[H(t)[X^{\Pi,\lambda}(t) + D(t)]]$, where the latter is proved easily to be a martingale as in a similar way in Zhang and Ewald (2010) and Chen et al. (2017).

Han, B., Hsu, J., 2004. Prospect theory and its applications in finance. Working Paper.
He, H., Pages, H.F., 1993. Labor income, borrowing constraints, and equilibrium asset prices: a duality approach. Economic Theory 3, 663–696.
Ilhan, A., Jonsson, M., Sircar, R., 2005. Optimal investment with derivative securities. Finance and Stochastics 9 (4), 585–595.
Kahl, M., Liu, J., Longstaff, F.A., 2003. Paper millionaires: how valuable is stock to a stockholder who is restricted from selling it? Journal of Financial Economics 67 (3), 385–410.
Kahneman, D., Tversky, A., 1979. Prospect theory: an analysis of decision under risk. Econometrica 47 (2), 263–292.
Kahneman, D., Tversky, A., 1992. Advances in prospect theory: cumulative representation of uncertainty. Journal of Risk and Uncertainty 5 (4), 297–323.
Koo, H.K., 1995. Consumption and portfolio choice with labor income: evaluation of human capital. Mimeo. Washington University in St. Louis.
Koo, H.K., 1998. Consumption and portfolio selection with labor income: a continuous time approach. Mathematical Finance 8 (1), 49–65.
Longstaff, F.A., 2009. Portfolio claustrophobia: asset pricing in markets with illiquid assets. The American Economic Review 99, 1119–1144.
Owadally, I., Jang, C., Clare, A., 2021. Optimal investment for a retirement plan with deferred annuities allowing for inflation and labour income risk. European Journal of Operational Research 295, 1132–1146.
Schwartz, E.S., Tebaldi, C., 2006. Illiquid assets and optimal portfolio choice. NBER Working Paper 12633. National Bureau of Economic Research, Cambridge, MA.
Munk, C., Sorensen, C., Vinther, T.N., 2004. Dynamic asset allocation under meanreverting returns, stochastic interest rates and inflation uncertainty are popular recommendations consistent with rational behavior? International Review of Economics & Finance 13, 141–166.
Xie, Y., Hwang, S., Pantelous, A.A., 2018. Loss aversion around the world: empirical evidence from pension funds. Journal of Banking & Finance 88, 52–62.
Zhang, A.H., Ewald, C.O., 2010. Optimal investment for a pension fund under inflation risk. Mathematical Methods of Operations Research 71 (2), 353–369.