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Parisian ruin with random deficit-dependent delays for spectrally negative Lévy processes ☆

Duy Phat Nguyen, Konstantin Borovkov *

School of Mathematics and Statistics, The University of Melbourne, Parkville 3010, Australia

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ABSTRACT

We consider an interesting natural extension to the Parisian ruin problem under the assumption that the risk reserve dynamics are given by a spectrally negative Lévy process. The distinctive feature of this extension is that the distribution of the random implementation delay windows' lengths can depend on the deficit at the epochs when the risk reserve process turns negative, starting a new negative excursion. This includes the possibility of an immediate ruin when the deficit hits a certain subset. In this general setting, we derive a closed-form expression for the Parisian ruin probability and the joint Laplace transform of the Parisian ruin time and the deficit at ruin.

1. Introduction and main results

The concept of Parisian ruin was first introduced in actuarial risk theory by Dassios and Wu (2008) in 2008: "Parisian type ruin will occur if the surplus falls below zero and stays below zero for a continuous time interval of length *d*. In some respects, this might be a more appropriate measure of risk than classical ruin as it gives the office some time to put its finances in order." The time period during which the surplus is allowed to remain negative is called implementation delay (or grace) period, often referred to just as the delay period.

It is often noted that the idea (and the name as well) of such a concept goes back to the so-called Parisian options whose payoffs depend on the lengths of the excursions of the underlying asset prices above or below a flat barrier. For example, the owner of a Parisian down-and-out option will lose the option if the underlying asset price drops below a given level and stays constantly below that level for a time interval longer than a given quantity *d*. Stopping times of this kind were first considered by Chesney et al. (1997).

However, discussions of a delayed Parisian-type ruin appeared in the actuarial literature even earlier. In particular, it was pointed out in Dos Reis (1993) that when an insurance company has many portfolios and only one of them has got negative, the company can have enough funds (either from another line of business or as a loan from a bank) to support the affected portfolio for some time in the hope that it would recover in the near future thus allowing the company to keep its business alive (see also Cheung, 2012).

More recently, it was stated in several papers that the Parisian ruin could be viewed as a theoretical description of reorganization under Chapter 11 of the US Bankruptcy Code of a company in distress rather than its immediate liquidation under Chapter 7 of the code. Chapter 11 allows the company to remain in control of its operations with a bankruptcy court providing oversight. The court grants the company an observation period during which the company manager can restructure the debt. If the reorganization plan fails, the company may be forced to be liquidated (see e.g. Li et al., 2014). Modeling this situation, we must take into account the fact that the length of the observation period can depend on the debt size, so that we are dealing here with *deficit-dependent delay periods.* This and other aspects of better modeling the complex Chapter 11 reorganization were also discussed, for instance, in Galai et al. (2007), Makarov (2016), Corbae and D'Erasmo (2021) and Zhang et al. (2022).

Over the last decade, analysis of Parisian ruin probabilities and times in different settings has become a popular topic in the litera-

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* Corresponding author.
E-mail address: borovkov@unimelb.edu.au (K. Borovkov).

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ture. First we will mention papers where the delay period length d was assumed to be *deterministic and fixed* (i.e., depending neither of the deficit at the beginning of a negative excursion nor on any other quantity, and remaining the same for all negative excursions of the risk reserve process).

Dassios and Wu (2008) derived the Laplace transform of the time until the Parisian ruin and the probability thereof for the classical Cramér–Lundberg (CL) model. Loeffen et al. (2013) derived an elegant compact formula for the Parisian ruin probability in the case where the surplus process is modeled by a spectrally negative Lévy process (SNLP) $X = \{X_t\}_{t\ge 0}$ (whose trajectories may be of unbounded variation), the answer involving only the scale function of X and the distribution of X_d. Czarna (2016) studied, also in the SNLP framework, Parisian ruin probabilities with an "ultimate bankruptcy level", meaning that ruin will also occur whenever the deficit reaches a given deterministic negative level. Simpler proofs and further results for that setting were obtained in Czarna and Renaud (2016).

Li et al. (2018) and Lkabous (2019) studied the concept of Parisian ruin under the "hybrid observation scheme" for SNLPs, where the surplus process is monitored discretely at arrival epochs of an independent Poisson process (that can be interpreted as the observation times of the regulatory body), but is continuously observed once the process value drops below zero.

Lkabous et al. (2017) studied Parisian ruin for a refracted SNLP model assuming that the premium payment rate is higher when the process is below zero. In Czarna et al. (2017), the joint law of the Parisian ruin time and the number of claims until that time was derived for the CL model. A compact formula for the Parisian ruin probability for a spectrally negative Markov additive risk process was obtained by Zhao and Dong (2018). The probability was expressed in terms of the scale matrix and transition rate matrix of the process.

A more flexible (and more realistic) model with *random delays* was first considered in Landriault et al. (2014). In their setup, along with the risk reserve SNLP with trajectories of locally bounded variation, there is an independent of it sequence of i.i.d. random variables that serve as implementation delay times (so that for each new negative excursion of the process, there is a new independent delay time). The authors studied the Laplace transform of the Parisian ruin time when delays were exponentially distributed or followed Erlang mixture distributions (noting that switching from deterministic delays to stochastic ones with such distributions improves the tractability of the resulting expressions). They also studied a version of the two-sided exit problem "when the first passage time below level zero is substituted by the Parisian ruin time". Frostig and Keren-Pinhasik (2020) studied Parisian ruin with an ultimate bankruptcy barrier (as in Czarna (2016) in the case of deterministic delay) and i.i.d. exponentially (and then Erlang) distributed random delays. Baurdoux et al. (2016) studied the Gerber–Shiu distribution at Parisian ruin with exponential implementation delays in the SNLP setup. We will further comment on this paper below. Lkabous and Renaud (2019) considered Parisian time of ruin where each delay is the minimum of the deterministic and random exponential ones, and computed the joint Laplace transform of that Parisian ruin time and the deficit at ruin for a general spectrally negative Lévy process (independent of path variations).

In the present paper, we consider a natural interesting extension to the Parisian ruin problem with a risk reserve SNLP. The distinctive feature of this extension is that the distribution of the random delay windows' lengths can depend on the deficit at the epochs when the risk reserve process turns negative, starting a new negative excursion. This includes the possibility of an immediate ruin when the deficit hits a certain subset. The presented extension was motivated, in particular, by the above-mentioned practical considerations related to Chapter 11 of the US Bankruptcy Code.

In this general setting, we derive closed-form expressions for the Parisian ruin probability and the joint Laplace transform of the Parisian ruin time and the deficit at ruin. Our results are illustrated by examples where the risk reserve follows the classical CL dynamics with claim distributions being exponential or hyperexponential, whereas the delay period distributions are finite mixtures of Erlang distributions with parameters depending on the deficit value at the beginning of the respective negative excursion.

More formally, we assume in this paper that $X := \{X_t\}_{t\ge 0}$ is an SNLP with càdlàg trajectories starting at $X_0 = u \in \mathbb{R}$. To indicate this for different values of u, the respective probability and expectation symbols will be endowed with subscript u, as in $\mathbf{E}_u$. The cumulant generating function $\psi(\theta) := \ln \mathbf{E}_0 e^{\theta X_1}$ of such a process X is clearly finite for all $\theta \ge 0$ and, for some constants $a, \sigma \in \mathbb{R}$, has the form

$$\psi(\theta) = a\theta + \frac{1}{2}\sigma^2\theta^2 + \int\limits_{(-\infty,0)} (e^{\theta x} - 1 - \theta x \mathbf{1}(x > -1))\Pi(dx), \quad \theta \ge 0, \tag{1}$$

where the jump measure Π is such that $\int_{(-\infty,0)}(1\wedge x^2)\Pi(dx) < \infty$. We also assume satisfied the standard safety loading condition

$$\mathbf{E}_0 X_1 > 0 \tag{2}$$

(clearly, $\mathbf{E}_0|X_1| < \infty$ under the above condition as X is spectrally negative).

To formally describe the Parisian ruin mechanism with deficit-dependent random delay windows, we will first assume that the trajectories of X are of *locally bounded variation.* It is well-known that, in this case, $\sigma = 0$ and $\int_{(-1,0)} |x|\Pi(dx) < \infty$ and so (1) simplifies to

$$\psi(\theta) = a_1\theta + \int\limits_{(-\infty,0)} (e^{\theta x} - 1)\Pi(dx), \quad \theta \ge 0. \tag{3}$$

This means that our process is just a linear drift minus a pure jump subordinator (see, e.g., Section 8.1 in Kyprianou, 2014).

Denote by $\mathbb{F} := \{\mathcal{F}_t\}_{t\ge 0}$ the natural filtration for X. For $x, y \in \mathbb{R}$, introduce the first hitting times

$$\tau_x^- := \inf\{t > 0 : X_t < x\} \quad \text{and} \quad \tau_y^+ := \inf\{t > 0 : X_t > y\}.$$

In view of (2), τ_x^- is an improper random variable when $x \le X_0$.

Setting $\tau_{0,0}^+ := 0$, we further define recursively for $k = 1, 2, \dots$ the following (improper, due to (2)) $\mathbb{F}$-stopping times:

$$\tau_{0,k}^- := \inf\{t > \tau_{0,k-1}^+ : X_t < 0\} \quad \text{and} \quad \tau_{0,k}^+ := \inf\{t > \tau_{0,k}^- : X_t > 0\}.$$

Note that, due to (2), the time $\tau_{0,k}^+$ is always finite on the event $\{\tau_{0,k}^- < \infty\}$.

In words, $\tau_{0,k}^-$ is the time when the kth negative excursion of the process X starts and $\tau_{0,k}^+$ is the time when that excursion ends. If $\tau_{0,k-1}^- < \infty$ but $\tau_{0,k}^- = \infty$ for some $k \ge 1$, then there are only $k-1$ negative excursions of the risk reserve process.

To formally construct random delay times, suppose that $P_x(B)$ is a stochastic kernel on $(-\infty, 0) \times \mathcal{B}([0,\infty))$. That is, for any fixed $B \in \mathcal{B}([0,\infty))$, $P_x(B)$ is a measurable function of x and, for any fixed $x < 0$, $P_x(B)$ is a probability measure in $B \in \mathcal{B}([0,\infty))$. Further, let $F_x(s) := P_x((-\infty, s])$, $s \ge 0$, be the distribution function of P_x, $\overline{F}_x(s) := 1 - F_x(s)$ its right tail. Denote by

$$F_x^{\leftarrow}(y) := \inf\{s \ge 0 : F_x(s) \ge y\}, \quad y \in (0,1),$$

the generalized inverse of F_x. Note that $F_x^{\leftarrow}(y)$, $(x, y) \in D := (-\infty, 0) \times (0, 1)$, is a measurable function. Indeed, since $F_x(y)$ is right-continuous and non-decreasing in y, for any $z \geq 0$ one has $\{(x, y) \in D : F_x^{\leftarrow}(y) \leq z\} = \{(x, y) \in D : F_x(z) - y \geq 0\}$, which is a measurable set on the plane as both $F_x(z)$ and y are measurable functions of (x, y).

Further, let $\{U_n\}_{n\geq 1}$ be a sequence of i.i.d. random variables uniformly distributed on $(0, 1)$ that is independent of the process X. The length η_k of the k-th delay window, $k = 1, 2, \ldots$, is then defined on the event $\{\tau_{0,k}^- < \infty\}$ as

$$\eta_k := F_{\chi_k}^{\leftarrow}(U_k), \quad \text{where } \chi_k := X_{\tau_{0,k}^-} \tag{4}$$

(on $\{\tau_{0,k}^- = \infty\}$ we can leave both χ_k and η_k undefined). Note that this allows one to model situations where $\eta_k = 0$ for some values of χ_k. This happens, for instance, in cases where delay is only granted when the deficit χ_k is above a certain negative threshold.

We say that Parisian ruin occurs in our model if

$$N := \inf\{k \geq 1 : \tau_{0,k}^- < \infty, \tau_{0,k}^- + \eta_k < \tau_{0,k}^+\} < \infty,$$

and define on the event $\{N < \infty\}$ the Parisian ruin time as

$$T := \tau_{0,N}^- + \eta_N. \tag{5}$$

To state our results, we have to recall the definition of the scale functions. For $q \geq 0$, the q-scale function $W^{(q)}$ for the process X is defined as a function on $\mathbb{R}$ such that (i) $W^{(q)}(x) = 0$ for $x < 0$ and (ii) $W^{(q)}(x)$ is continuous on $[0, \infty)$ and

$$\int_{[0,\infty)} e^{-\beta x} W^{(q)}(x)dx = \frac{1}{\psi(\beta) - q}, \quad \beta > \Phi(q), \tag{6}$$

where $\Phi(q) := \sup\{\theta \geq 0 : \psi(\theta) = q\}$, $q \geq 0$ (see, e.g., Section 8.2 in Kyprianou, 2014). One refers to $W := W^{(0)}$ as just the scale function for X. Note that the q-scale functions can be obtained as the scale functions for SNLPs with "tilted distributions": for $q \geq 0$,

$$W^{(q)}(x) = e^{\Phi(q)x} W_{\Phi(q)}(x), \quad x \in \mathbb{R}, \tag{7}$$

where $W_\nu(x)$ is the scale function for the Lévy process with the cumulant function $\psi_\nu(\theta) := \psi(\theta + \nu) - \psi(\nu)$ (Proposition 2 in Surya, 2008).

Several important characteristics of and fluctuation identities for SNLPs can be expressed in terms of their scale functions. In particular, the distribution $\mathbf{P}_u(\chi_1 \in \cdot\,, \tau_0^- < \infty)$ of the first negative value χ_1 of the process given $X_0 = u > 0$ has (defective) density $h_u(x)$ that can be written as

$$h_u(x) = \int_{(0-,u]} \Pi(x + z - u)dW(z), \quad x < 0,$$
$$\text{where } \Pi(y) := \Pi((-\infty, y]),\ y < 0, \tag{8}$$

see, e.g., p. 277 in Kyprianou (2014) (note that the formula for that distribution on that page in Kyprianou (2014) contains a typo: instead of Π there must be the Lévy measure for the spectrally positive process $-X$). Another formula we will use below provides an expression for the "incomplete Laplace transform" for τ_y^+: for $q \geq 0$ and $t, y > 0$,

$$\mathbf{E}_0(e^{-q\tau_y^+}; \tau_y^+ \leq t) = e^{-qt}\Lambda^{(q)}(-y, t), \tag{9}$$

where

$$\Lambda^{(q)}(x, t) := \int_0^\infty W^{(q)}(x + z)\frac{z}{t}\mathbf{P}_0(X_t \in dz), \quad x \in \mathbb{R},\ t > 0$$

(see, e.g., Lemma 4.2 in Loeffen et al., 2018); one could also compute the left-hand side of (9) using the expression for the distribution function of τ_y^+ provided in (10).

One must note here that closed-form expressions for the scale function are only available for processes from several special classes of SNLPs (in particular, in the cases considered in Section 3 below). Hubalek and Kyprianou (2010) presented several examples where closed-form expressions for the scale function are available and described a methodology for finding such expressions. In the general case, one can compute approximations to the scale functions. A "robust" numerical method for computing $W^{(q)}$ based on (6) and numerical inversion of (7) for $W_{\Phi(q)}$ was described in Surya (2008), whereas Egami and Yamazaki (2014) presented a possible "phase-type-fitting approach" to approximating the scale functions.

Finally, we set

$$G_y(t) := \mathbf{P}_0(\tau_y^+ \leq t) = -y\frac{\partial}{\partial y}\int_0^t \mathbf{P}_0(X_s > y)\frac{ds}{s}, \quad y, t > 0, \tag{10}$$

where the expression on the right-hand side comes from the celebrated Kendall's formula (see, e.g., Borovkov and Burq, 2001 or p. 725 in Bingham, 1975, Zolotarev, 1964) and let

$$K(x) := \mathbf{E}_0\overline{F}_x(\tau_{|x|}^+) = \int_0^\infty \overline{F}_x(t)dG_{|x|}(t), \quad x < 0, \tag{11}$$

$$H(v) := \int_{-\infty}^0 K(x)h_v(x)dx, \quad v \geq 0. \tag{12}$$

Our first result is stated in the following assertion.

Theorem 1. *Assume that the reserve process X is an SNLP that has trajectories of bounded variation and satisfies condition (2). In the Parisian ruin scheme with random deficit-dependent delays specified by (4), the probability of no Parisian ruin when the initial reserve is $X_0 = u \geq 0$ is equal to*

$$\mathbf{P}_u(N = \infty) = \mathbf{E}_0 X_1\left(W(u) + \frac{W(0)}{1 - H(0)}H(u)\right). \tag{13}$$

One can also compute the joint Laplace transform for the Parisian ruin time and the deficit at the time of that ruin. To state our result, we need to introduce further notations. For $v, w \geq 0$ and $x < 0$, set

$$M_1(v, w, x) := \int_0^1 \left[e^{(\psi(w)-v)F_x^{\leftarrow}(s)+wx} - e^{-vF_x^{\leftarrow}(s)}\Lambda^{(\psi(w))}(x, F_x^{\leftarrow}(s))\right]ds, \tag{14}$$

$$M_2(v, x) := \mathbf{E}_0 e^{-v\tau_{|x|}^+}\mathbf{1}(\tau_{|x|}^+ \leq F_x^{\leftarrow}(U_1)) = \int_0^1 e^{-vF_x^{\leftarrow}(s)}\Lambda^{(v)}(x, F_x^{\leftarrow}(s))\,ds, \tag{15}$$

where the last equality holds true since $\mathbf{E}_0 e^{-v\tau_{|x|}^+}\mathbf{1}(\tau_{|x|}^+ \leq r) = e^{-vr}\Lambda^{(v)}(x, r)$ by Lemma 4.2 in Loeffen et al. (2018) and U_1 is independent of $\tau_{|x|}^+$.

Finally, assuming in addition that $u \in [0, b]$, $b > 0$, we set

$$
\begin{aligned}
&Q_{1,b}(u, v, w) \\
&:= \mathbf{E}_u e^{-v\tau_0^-} \mathbf{1}(\tau_0^- < \tau_b^+) M_1(v, w, \chi_1) \\
&= \int_0^b \int_{(-\infty,-y)} M_1(v, w, y+\theta)\Big(\frac{W^{(v)}(u)W^{(v)}(b-y)}{W^{(v)}(b)} \\
&\quad - W^{(v)}(u-y)\Big)\Pi(d\theta)dy, \\
&Q_{2,b}(u, v) \\
&:= \mathbf{E}_u e^{-v\tau_0^-} \mathbf{1}(\tau_0^- < \tau_b^+) M_2(v, \chi_1) \\
&= \int_0^b \int_{(-\infty,-y)} M_2(v, y+\theta)\Big(\frac{W^{(v)}(u)W^{(v)}(b-y)}{W^{(v)}(b)} \\
&\quad - W^{(v)}(u-y)\Big)\Pi(d\theta)dy
\end{aligned}
$$

(the second equalities in both formulae follow from the result of Exercise 10.6 on p. 303 in Kyprianou, 2014).

Theorem 2. *Under the assumptions of Theorem 1, for $b, v, w \geq 0$ and $u \in [0, b]$, the joint Laplace transform of the Parisian ruin time and the deficit at the time of that ruin on the event $\{T < \tau_b^+\}$ is equal to*

$$
\mathbf{E}_u(e^{-vT+wX_T}; T < \tau_b^+) = Q_{1,b}(u, v, w) + \frac{Q_{1,b}(0, v, w)Q_{2,b}(u, v)}{1 - Q_{2,b}(0, v)}. \tag{16}
$$

Corollary 1. *Under the assumptions of Theorem 1, the joint Laplace transform of the Parisian ruin time and the deficit at the time of that ruin is given by*

$$
\mathbf{E}_u(e^{-vT+wX_T}; T < \infty) = Q_{1,\infty}(u, v, w) + \frac{Q_{1,\infty}(0, v, w)Q_{2,\infty}(u, v)}{1 - Q_{2,\infty}(0, v)}, \tag{17}
$$

where

$$
\begin{aligned}
&Q_{1,\infty}(u, v, w) \\
&= \int_0^\infty \int_{(-\infty,-y)} M_1(v, w, y+\theta)\big(W^{(v)}(u)e^{-\Phi(v)y} \\
&\quad - W^{(v)}(u-y)\big)\Pi(d\theta)dy, \\
&Q_{2,\infty}(u, v) \\
&= \int_0^\infty \int_{(-\infty,-y)} M_2(v, y+\theta)\big(W^{(v)}(u)e^{-\Phi(v)y} \\
&\quad - W^{(v)}(u-y)\big)\Pi(d\theta)dy.
\end{aligned}
$$

Now we will turn to the case of SNLP X with *trajectories of unbounded variation.* For such processes, the recursive procedure we used above to introduce Parisian ruin does not work. Instead, in the case of non-random delays of a fixed length $d > 0$, the Parisian ruin time was defined in Dassios and Wu (2008); Loeffen et al. (2013) as

$$
T_d := \inf\{t > 0 : t - g_t > d\}, \quad \text{where } g_t := \sup\{s \in [0, t] : X_s \geq 0\},
$$

using the convention that $\inf \varnothing = \infty$, $\sup \varnothing = 0$. In the case of bounded variation trajectories and a common degenerate distribution $F_x(t) = \mathbf{1}(d \leq t)$, $x < 0$, for delay windows, the thus defined T_d clearly coincides with our T defined in (5).

Extending this definition to the case of random delay windows is a non-trivial task. An approach to doing this in the simple situation where all the delay windows are independent and follow a common exponential distribution was suggested in Baurdoux et al. (2016). In that paper, the authors denoted by G the set of all left-end points of the negative excursions of the process X and then, "for each $g \in G$," considered "an independent, exponentially distributed random variable $\mathbf{e}_q^g$, also independent of X" (q represents here the rate of the exponential distribution). The time of the Parisian ruin with i.i.d. exponentially distributed delay windows was defined as

$$
\inf\{t > 0 : X_t < 0 \text{ and } t - g_t > \mathbf{e}_q^{g_t}\}.
$$

This description seems to be lacking detail and requires more clarification with regard to exactly how these random times $\mathbf{e}_q^g$ are to be constructed. For instance, if one just starts with an "exponential white noise" $\{\mathbf{e}_q(t)\}_{t>0}$ and then takes $\mathbf{e}_q^{g_t} := \mathbf{e}_q(t)$, one will encounter measurability issues.

Moreover, from the practical viewpoint, this problem setup is hardly meaningful as the suggested mechanism is not feasible. For instance, if, say, $X_t = X_0 + ct + \sigma B_t$, $t \geq 0$, where B is the standard Brownian motion process then, immediately after the start of the first negative excursion at time τ_0^-, one would have to generate infinitely many independent exponential random times as the process X will have infinitely many negative excursions in any right neighborhood of that time τ_0^-.

To avoid the above-mentioned complications and end up with an implementable Parisian-type ruin scheme, one can consider "ε-Parisian ruin times" T^ε constructed for $\varepsilon > 0$ by "activating the clock" for random delay windows at the times when the value of X_t drops below the level $-\varepsilon$ (such ruin times were considered in Loeffen et al. (2013); Baurdoux et al. (2016) as well). This makes it possible to use the recursive procedure we employed in the case of processes with trajectories of bounded variation. More precisely, for a fixed $\varepsilon > 0$, starting with $\tau_{0,0}^{+,\varepsilon} := 0$, we introduce recursively for $k = 1, 2, \ldots$ the following (improper) $\mathbb{F}$-stopping times:

$$
\tau_{-\varepsilon,k}^- := \inf\{t > \tau_{0,k-1}^{+,\varepsilon} : X_t < -\varepsilon\} \quad \text{and}
$$
$$
\tau_{0,k}^{+,\varepsilon} := \inf\{t > \tau_{-\varepsilon,k}^- : X_t > 0\}.
$$

Note that, due to (2), the time $\tau_{0,k}^{+,\varepsilon}$ is always finite on the event $\{\tau_{-\varepsilon,k}^- < \infty\}$. Then, similarly to (4), we set $\chi_k^\varepsilon := X_{\tau_{-\varepsilon,k}^-}$ and $\eta_k^\varepsilon := F_{\chi_k^\varepsilon}^{\leftarrow}(U_k)$. We say that ε-Parisian ruin occurs if

$$
N^\varepsilon := \inf\{k \geq 1 : \tau_{-\varepsilon,k}^- < \infty, \tau_{-\varepsilon,k}^- + \eta_k^\varepsilon < \tau_{0,k}^{+,\varepsilon}\} < \infty
$$

and define on the event $\{N^\varepsilon < \infty\}$ the ε-Parisian ruin time as $T^\varepsilon := \tau_{-\varepsilon,N^\varepsilon}^- + \eta_{N^\varepsilon}^\varepsilon$.

Theorem 3. *Assume that the reserve process X is a general SNLP that satisfies condition (2). Then, for any $\varepsilon > 0$, the probability of no ε-Parisian ruin with random deficit-dependent delays when the initial reserve is $X_0 = u \geq 0$ is equal to*

$$
\mathbf{P}_u(N^\varepsilon = \infty) = \mathbf{E}_0 X_1 \left[W(u+\varepsilon) + \frac{W(\varepsilon)\mathbf{E}_{u+\varepsilon}(K(\chi_1 - \varepsilon); \tau_0^- < \infty)}{1 - \mathbf{E}_\varepsilon(K(\chi_1 - \varepsilon); \tau_0^- < \infty)} \right]. \tag{18}
$$

2. Proofs

We will start with the following simple auxiliary assertions that may be well-known.

Lemma 1. *Let ξ and ζ be random variables on a common probability space, $\mathcal{G}$ be a sub-σ-algebra on that space.*

(i) *If $\mathbf{E}(|\xi|+|\zeta|)<\infty$ and ξ is independent of the pair $(\zeta,\mathcal{G})$ then*

$$\mathbf{E}(\xi\zeta|\mathcal{G})=\mathbf{E}\xi\cdot\mathbf{E}(\zeta|\mathcal{G}).$$

(ii) *If ζ is $\mathcal{G}$-measurable, ξ is independent of $\mathcal{G}$ and has distribution function G, then*

$$\mathbf{E}(\mathbf{1}(\xi>\zeta)|\mathcal{G})=1-G(\zeta).$$

Proof. Both statements can be verified by straightforward computations. First observe that the right-hand sides in the above relations are clearly $\mathcal{G}$-measurable. Second, for an arbitrary $A\in\mathcal{G}$, in case (i) by independence one has $\mathbf{E}\xi\zeta\mathbf{1}_A=\mathbf{E}\xi\mathbf{E}\zeta\mathbf{1}_A=\mathbf{E}\xi\mathbf{E}(\mathbf{E}(\zeta|\mathcal{G})\mathbf{1}_A)$, yielding the desired relation, whereas in case (ii) one has

$$\begin{aligned}\mathbf{E}\mathbf{1}(\xi>\zeta)\mathbf{1}_A&=\int\mathbf{E}(\mathbf{1}(\xi>\zeta)\mathbf{1}_A|\zeta=y)\mathbf{P}(\zeta\in dy)\\&=\int\mathbf{E}\mathbf{1}(\xi>y)\mathbf{E}(\mathbf{1}_A|\zeta=y)\mathbf{P}(\zeta\in dy)\\&=\int(1-G(y))\mathbf{E}(\mathbf{1}_A|\zeta=y)\mathbf{P}(\zeta\in dy)\\&=\mathbf{E}(1-G(\zeta))\mathbf{1}_A,\end{aligned}$$

which establishes the second desired relation. □

Proof of Theorem 1. Our initial step is similar to the one from Loeffen et al. (2013). The probability of no Parisian ruin when the initial reserve is $u>0$ equals

$$\begin{aligned}&\mathbf{P}_u(N=\infty)\\&\quad=\mathbf{P}_u(\tau_0^-=\infty)+\mathbf{P}_u(\tau_0^-<\infty,N=\infty)\\&\quad=\mathbf{P}_u(\tau_0^-=\infty)+\mathbf{E}_u\mathbf{E}_u\big(\mathbf{1}(\tau_0^-<\infty)\mathbf{1}(N=\infty)|\mathcal{F}_{\tau_0^-}\big)\\&\quad=\mathbf{P}_u(\tau_0^-=\infty)+\mathbf{E}_u\big[\mathbf{1}(\tau_0^-<\infty)\mathbf{E}_u\big(\mathbf{1}(N=\infty)|\mathcal{F}_{\tau_0^-}\big)\big].\end{aligned}\tag{19}$$

Observe that, by the strong Markov property and the absence of positive jumps, on the event $\{\tau_0^-<\infty\}$ the process

$$\widetilde{X}:=\{\widetilde{X}_t:=X_{\tau_{0,1}^++t}\}_{t\ge0}\tag{20}$$

is an independent of $\mathcal{F}_{\tau_{0,1}^+}$ Lévy process with cumulant (3) and initial value $\widetilde{X}_0=0$ (see, e.g., Theorem 3.1 in Kyprianou, 2014). We will keep all the notations we introduced for the functionals of the process X for the respective functionals of $\widetilde{X}$, endowing them with a tilde, so that, say, $\widetilde{N}$ denotes the total number of negative excursions in $\widetilde{X}$ needed for the Parisian ruin when the risk reserve dynamics are represented by that process ($\widetilde{N}=\infty$ if there is no such ruin).

Now we can write that, on the event $\{\tau_0^-<\infty\}$, one has

$$\begin{aligned}&\mathbf{E}_u\big(\mathbf{1}(N=\infty)|\mathcal{F}_{\tau_0^-}\big)\\&\quad=\mathbf{E}_u\big(\mathbf{1}(\tau_0^-+\eta_1\ge\tau_{0,1}^+)\mathbf{1}(\widetilde{N}=\infty)|\mathcal{F}_{\tau_0^-}\big)\\&\quad=\mathbf{E}_u\big[\mathbf{E}_u\big(\mathbf{1}(\tau_0^-+\eta_1\ge\tau_{0,1}^+)\mathbf{1}(\widetilde{N}=\infty)|\mathcal{F}_{\tau_{0,1}^+}\big)|\mathcal{F}_{\tau_0^-}\big]\\&\quad=\mathbf{P}_0(N=\infty)\mathbf{E}_u\big[\mathbf{E}_u\big(\mathbf{1}(\eta_1\ge\tau_{0,1}^+-\tau_0^-)|\mathcal{F}_{\tau_{0,1}^+}\big)|\mathcal{F}_{\tau_0^-}\big],\end{aligned}\tag{21}$$

where, to get the third equality, we used Lemma 1(i) with $\xi=\mathbf{1}(\widetilde{N}=\infty)$ to re-express the inner conditional expectation in the second line. As

$$\{\eta_1\ge\tau_{0,1}^+-\tau_0^-\}=\{F_{\chi_1}^{\leftarrow}(U_1)\ge\tau_{0,1}^+-\tau_0^-\}=\{U_1\ge F_{\chi_1}(\tau_{0,1}^+-\tau_0^-)\}$$

and $F_{\chi_1}(\tau_{0,1}^+-\tau_0^-)$ is $\mathcal{F}_{\tau_{0,1}^+}$-measurable, we conclude from Lemma 1(ii) that

$$\begin{aligned}\mathbf{E}_u\big(\mathbf{1}(\eta_1\ge\tau_{0,1}^+-\tau_0^-)|\mathcal{F}_{\tau_{0,1}^+}\big)&=\mathbf{E}_u\big[\mathbf{1}(U_1\ge F_{\chi_1}(\tau_{0,1}^+-\tau_0^-))|\mathcal{F}_{\tau_{0,1}^+}\big]\\&=\overline{F}_{\chi_1}(\tau_{0,1}^+-\tau_0^-).\end{aligned}$$

Now setting, for $a>0$,

$$\widehat{X}:=\{\widehat{X}_t:=X_{\tau_0^-+t}-\chi_1\}_{t\ge0},\quad\widehat{\tau}_a^+:=\inf\{t>0:\widehat{X}_t>a\},\tag{22}$$

we obtain that, on the event $\{\tau_0^-<\infty\}$, the second factor in the last line of (21) equals

$$\begin{aligned}\mathbf{E}_u\big(\overline{F}_{\chi_1}(\tau_{0,1}^+-\tau_0^-)\big|\mathcal{F}_{\tau_0^-}\big)&=\mathbf{E}_u\big(\overline{F}_{\chi_1}(\widehat{\tau}_{|\chi_1|}^+)\big|\mathcal{F}_{\tau_0^-}\big)\\&=\mathbf{E}_u\big(\overline{F}_{\chi_1}(\widehat{\tau}_{|\chi_1|}^+)\big|\chi_1\big)=K(\chi_1),\end{aligned}$$

where, to get the last two equalities, we used the observation that, on that event, by the strong Markov property, the process $\widehat{X}$ is an independent of $\mathcal{F}_{\tau_0^-}$ (and hence of χ_1) Lévy process with cumulant (3) and initial value $\widehat{X}_0=0$ (recall that the function K was defined in (11)). From this and (19), (21) we derive that

$$\mathbf{P}_u(N=\infty)=\mathbf{P}_u(\tau_0^-=\infty)+\mathbf{P}_0(N=\infty)\mathbf{E}_u(K(\chi_1);\,\tau_0^-<\infty).$$

Setting now $u=0$ yields

$$\mathbf{P}_0(N=\infty)=\frac{\mathbf{P}_0(\tau_0^-=\infty)}{1-\mathbf{E}_0(K(\chi_1);\,\tau_0^-<\infty)}.$$

Recalling that, in the case of an SNLP with positive drift, one has

$$\mathbf{P}_u(\tau_0^-=\infty)=\mathbf{E}_0X_1W(u)\tag{23}$$

(see, e.g., Theorem 8.1(ii) in Kyprianou, 2014), we obtain that

$$\mathbf{P}_u(N=\infty)=\mathbf{E}_0X_1\left[W(u)+W(0)\frac{\mathbf{E}_u(K(\chi_1);\,\tau_0^-<\infty)}{1-\mathbf{E}_0(K(\chi_1);\,\tau_0^-<\infty)}\right].$$

Expressing the expectations inside the square brackets in terms of the function H defined in (12) yields representation (13). This completes the proof of Theorem 1. □

In the proof of Theorem 2 we will use the following observation that may be well-known, but for which we could not find a suitable reference.

Lemma 2. *Assume that τ and σ are stopping times relative to a filtration $\{\mathcal{H}_t\}_{t\ge0}$. Then $\{\tau<\sigma\}\in\mathcal{H}_\tau$.*

Proof. For $t\ge0$, we have

$$\begin{aligned}\{\tau<\sigma\}\cap\{\tau\le t\}&=\{\tau<\sigma,\tau\le t,\sigma>t\}\cup\{\tau<\sigma,\tau\le t,\sigma\le t\}\\&=\{\tau\le t,\sigma>t\}\cup\{\tau<\sigma,\sigma\le t\}.\end{aligned}$$

The first event in the union in the second line is clearly in $\mathcal{H}_t$, whereas for the second one we have

$$\{\tau<\sigma,\sigma\le t\}=\bigcup_{r\in\mathbb{Q},\,r<t}(\{\tau\le r\}\cap\{r<\sigma\le t\}),$$

where obviously $\{\tau\le r\}\in\mathcal{H}_r\subseteq\mathcal{H}_t$ and $\{r<\sigma\le t\}\in\mathcal{H}_t$ when $r<t$. Lemma 2 is proved. □

Proof of Theorem 2. Our starting point is to observe that, for $u, v, w \geq 0$, one has

$$\begin{aligned}
&\mathbf{E}_u(e^{-vT+wX_T}; T<\tau_b^+)\\
&\quad=\mathbf{E}_u\mathbf{E}_u(e^{-vT+wX_T}\mathbf{1}(T<\tau_b^+)\mathbf{1}(\tau_0^-<\tau_b^+)|\mathcal{F}_{\tau_0^-})\\
&\quad=\mathbf{E}_u\big[e^{-v\tau_0^-}\mathbf{1}(\tau_0^-<\tau_b^+)\mathbf{E}_u(e^{-v(T-\tau_0^-)+wX_T}\mathbf{1}(T<\tau_b^+)|\mathcal{F}_{\tau_0^-})\big],
\end{aligned} \tag{24}$$

where the second equality follows from Lemma 2. The conditional expectation in the second line is equal to $E_1 + E_2$, where

$$E_1 := \mathbf{E}_u(e^{-v(T-\tau_0^-)+wX_T}\mathbf{1}(T<\tau_b^+)\mathbf{1}(N=1)|\mathcal{F}_{\tau_0^-}),$$

$$E_2 := \mathbf{E}_u(e^{-v(T-\tau_0^-)+wX_T}\mathbf{1}(T<\tau_b^+)\mathbf{1}(N>1)|\mathcal{F}_{\tau_0^-}).$$

First we will evaluate E_1. On the event $\{\tau_0^- < \tau_b^+, N=1\} = \{\tau_0^- < \tau_b^+, \eta_1 < \widehat{\tau}^+_{|\chi_1|}\}$ (see (22)), one has $T = \tau_0^- + \eta_1$, $X_T = X_{\tau_0^-+\eta_1} = \chi_1 + \widehat{X}_{\eta_1}$ and automatically $T < \tau_b^+$ (as the Parisian ruin occurs during the first negative excursion and that excursion started prior to time τ_b^+). Therefore, on the event $\{\tau_0^- < \tau_b^+\} \in \mathcal{F}_{\tau_0^-}$, one has

$$\begin{aligned}
E_1&=\mathbf{E}_u(e^{-v\eta_1+w(\widehat{X}_{\eta_1}+\chi_1)}\mathbf{1}(\tau_0^-<\tau_b^+)\mathbf{1}(N=1)|\mathcal{F}_{\tau_0^-})\\
&=e^{w\chi_1}\mathbf{E}_u(e^{-v\eta_1+w\widehat{X}_{\eta_1}}\mathbf{1}(\eta_1<\widehat{\tau}^+_{|\chi_1|})|\mathcal{F}_{\tau_0^-}).
\end{aligned}$$

On the event $\{\tau_0^- < \tau_b^+\}$ the process $\widehat{X}$ is an independent of $\mathcal{F}_{\tau_0^-}$ distributional copy of X (cf. our comment after (22)), so that the only random component inside the conditional expectation in the second line that is not independent of $\mathcal{F}_{\tau_0^-}$ is χ_1 (it participates in both η_1 and $\widehat{\tau}^+_{|\chi_1|}$). We conclude that, on the event $\{\tau_0^- < \tau_b^+\}$, that conditional expectation equals

$$\begin{aligned}
&\mathbf{E}_u(e^{-v\eta_1+w\widehat{X}_{\eta_1}}\mathbf{1}(\eta_1<\widehat{\tau}^+_{|\chi_1|})|\chi_1)\\
&\quad=\mathbf{E}_u\big[e^{-v\eta_1}\mathbf{E}_u(e^{w\widehat{X}_{\eta_1}}\mathbf{1}(\eta_1<\widehat{\tau}^+_{|\chi_1|})|\chi_1,\eta_1)\big|\chi_1\big].
\end{aligned}$$

Given that $\chi_1 = x < 0$, $\eta_1 = t > 0$, the inner conditional expectation on the right-hand side of the above formula is equal to $\mathbf{E}_0 e^{wX_t}\mathbf{1}(t < \tau^+_{|x|})$. This expression can be computed similarly to the argument used in the proof of Lemma 4.3 in Loeffen et al. (2018):

$$\begin{aligned}
&\mathbf{E}_0e^{wX_t}\mathbf{1}(\tau^+_{|x|}>t)\\
&\quad=\mathbf{E}_0e^{wX_t}-\mathbf{E}_0e^{wX_t}\mathbf{1}(\tau^+_{|x|}\le t)\\
&\quad=e^{t\psi(w)}-\int_{(0,t]}\mathbf{E}_0(e^{wX_t}|\tau^+_{|x|}=s)\mathbf{P}_0(\tau^+_{|x|}\in ds)\\
&\quad=e^{t\psi(w)}-e^{w|x|}\int_{(0,t]}\mathbf{E}_0(e^{w(X_t-X_s)}|\tau^+_{|x|}=s)\mathbf{P}_0(\tau^+_{|x|}\in ds)\\
&\quad=e^{t\psi(w)}-e^{-wx+t\psi(w)}\int_{(0,t]}e^{-s\psi(w)}\mathbf{P}_0(\tau^+_{|x|}\in ds),\\
&\quad=e^{t\psi(w)}-e^{-wx}\Lambda^{(\psi(w))}(x,t),
\end{aligned}$$

where we used the spectral negativity of X and the strong Markov property to get the third and fourth equalities and representation (9) to get the fifth one. Combining these computations, we obtain that, on the event $\{\tau_0^- < \tau_b^+\}$, one has

$$\begin{aligned}
E_1&=e^{w\chi_1}\mathbf{E}_u[e^{-v\eta_1}(e^{\psi(w)\eta_1}-e^{-w\chi_1}\Lambda^{(\psi(w))}(\chi_1,\eta_1))|\chi_1]\\
&=\mathbf{E}_u(e^{(\psi(w)-v)\eta_1+w\chi_1}-e^{-v\eta_1}\Lambda^{(\psi(w))}(\chi_1,\eta_1)|\chi_1)\\
&=M_1(v,w,\chi_1),
\end{aligned}$$

where $M_1(v, w, x)$ was introduced in (14).

Now we will turn to the term E_2. On the event $\{T < \tau_b^+\}$, relation $N > 1$ is equivalent to $\tau_0^- + \eta_1 \geq \tau^+_{0,1}$, so that on the event $\{\tau_0^- < \tau_b^+\}$ one has

$$\begin{aligned}
E_2&=\mathbf{E}_u(e^{-v(T-\tau_0^-)+wX_T}\mathbf{1}(T<\tau_b^+)\mathbf{1}(\tau_0^-+\eta_1\ge\tau^+_{0,1})|\mathcal{F}_{\tau_0^-})\\
&=\mathbf{E}_u\big[e^{-v(\tau^+_{0,1}-\tau_0^-)}\mathbf{1}(\tau_0^-+\eta_1\ge\tau^+_{0,1})\\
&\quad\times\mathbf{E}_u(e^{-v(T-\tau^+_{0,1})+wX_T}\mathbf{1}(T<\tau_b^+)|\mathcal{F}_{\tau^+_{0,1}},\eta_1)\big|\mathcal{F}_{\tau_0^-}\big]\\
&=\mathbf{E}_u\big[e^{-v(\tau^+_{0,1}-\tau_0^-)}\mathbf{1}(\tau_0^-+\eta_1\ge\tau^+_{0,1})\\
&\quad\times\mathbf{E}_u(e^{-v\widetilde{T}+w\widetilde{X}_{\widetilde{T}}}\mathbf{1}(\widetilde{T}<\widetilde{\tau}_b^+)|\mathcal{F}_{\tau^+_{0,1}},\eta_1)\big|\mathcal{F}_{\tau_0^-}\big],
\end{aligned}$$

where we used the process $\widetilde{X}$ from (20) (and the relevant to it random times $\widetilde{T}, \widetilde{\tau}_b^+$) and the observation that the relation $T < \tau_b^+$ is equivalent to $\widetilde{T} < \widetilde{\tau}_b^+$ provided that $\tau_0^- < \tau_b^+$. Using the strong Markov property and the fact that $\widetilde{X}_0 = 0$, we conclude that

$$\begin{aligned}
E_2&=\mathbf{E}_u\big(e^{-v(\tau^+_{0,1}-\tau_0^-)}\mathbf{1}(\tau_0^-+\eta_1\ge\tau^+_{0,1})\big|\mathcal{F}_{\tau_0^-}\big)\\
&\quad\times\mathbf{E}_0(e^{-vT+wX_T};T<\tau_b^+)\\
&=\mathbf{E}_u\big(e^{-v\widehat{\tau}^+_{|\chi_1|}}\mathbf{1}(F^{\leftarrow}_{\chi_1}(U_1)\ge\widehat{\tau}^+_{|\chi_1|})\big|\chi_1\big)\mathbf{E}_0(e^{-vT+wX_T};T<\tau_b^+)\\
&=M_2(v,\chi_1)\mathbf{E}_0(e^{-vT+wX_T};T<\tau_b^+),
\end{aligned}$$

where $M_2(v, x)$ was introduced in (15).

Substituting the computed values for E_1 and E_2 into (24) yields

$$\begin{aligned}
&\mathbf{E}_u(e^{-vT+wX_T};T<\tau_b^+)\\
&\quad=\mathbf{E}_u\big[e^{-v\tau_0^-}\mathbf{1}(\tau_0^-<\tau_b^+)(M_1(v,w,\chi_1)\\
&\qquad+M_2(v,\chi_1)\mathbf{E}_0(e^{-vT+wX_T};T<\tau_b^+))\big]\\
&\quad=Q_{1,b}(u,v,w)+Q_{2,b}(u,v)\mathbf{E}_0(e^{-vT+wX_T};T<\tau_b^+).
\end{aligned}$$

Setting $u = 0$ we recover $\mathbf{E}_0(e^{-vT+wX_T}; T < \tau_b^+)$ as $Q_{1,b}(0, v, w)/(1 - Q_{2,b}(0, v))$. Substituting this back into the above formula completes the proof of (16). □

Proof of Corollary 1. As $b \to \infty$, by the monotone convergence theorem, the left-hand side of (16) converges to that of (17), whereas

$$\begin{aligned}
Q_{1,\infty}(u,v,w)&:=\lim_{b\to\infty}Q_{1,b}(u,v,w)\\
&=\mathbf{E}_u e^{-v\tau_0^-}\mathbf{1}(\tau_0^-<\infty)M_1(v,w,\chi_1).
\end{aligned}\tag{25}$$

From Corollary 10.2 in Kyprianou (2014), one gets, for $x < 0$,

$$\begin{aligned}
&\mathbf{E}_u\big(e^{-v\tau_0^-};\tau_0^-<\infty,\chi_1\in dx\big)\\
&\quad=\int_0^\infty\big(e^{-\Phi(v)y}W^{(v)}(u)-W^{(v)}(u-y)\big)\Pi(dx-y)dy.
\end{aligned}$$

Using this and then Fubini's theorem on the right-hand side of (25) yields

$$\int_0^\infty\bigg(\int_{-\infty}^0 M_1(v,w,x)\Pi(dx-y)\bigg)\big(e^{-\Phi(v)y}W^{(v)}(u)-W^{(v)}(u-y)\big)dy.$$

Changing variables by letting $\theta := x - y$ completes the derivation of the representation for $Q_{1,\infty}(u, v, w)$. The argument for deriving the stated expression for $Q_{2,\infty}(u, v)$ is basically the same. Corollary 1 is proved. □

Proof of Theorem 3. Arguing as in the proof of Theorem 1, we find that

$$\begin{aligned}&\mathbf{P}_u(N^\varepsilon=\infty)\\&\quad=\mathbf{P}_u(\tau_{-\varepsilon}^-=\infty)+\mathbf{E}_u\big[\mathbf{1}(\tau_{-\varepsilon}^-<\infty)\mathbf{E}_u\big(\mathbf{1}(N^\varepsilon=\infty)|\mathcal{F}_{\tau_{-\varepsilon}^-}\big)\big]\end{aligned}$$

and, on the event $\{\tau_{-\varepsilon}^-<\infty\}$,

$$\begin{aligned}&\mathbf{E}_u\big(\mathbf{1}(N^\varepsilon=\infty)|\mathcal{F}_{\tau_{-\varepsilon}^-}\big)\\&\quad=\mathbf{P}_0(N^\varepsilon=\infty)\mathbf{E}_u\big[\mathbf{E}_u\big(\mathbf{1}(\eta_1^\varepsilon\ge\tau_{0,1}^{+,\varepsilon}-\tau_{-\varepsilon}^-)|\mathcal{F}_{\tau_{0,1}^{+,\varepsilon}}\big)\big|\mathcal{F}_{\tau_{-\varepsilon}^-}\big],\quad(26)\end{aligned}$$

where $\mathbf{E}_u\big(\mathbf{1}(\eta_1^\varepsilon\ge\tau_{0,1}^{+,\varepsilon}-\tau_{-\varepsilon}^-)|\mathcal{F}_{\tau_{0,1}^{+,\varepsilon}}\big)=\overline{F}_{\chi_1^\varepsilon}(\tau_{0,1}^{+,\varepsilon}-\tau_{-\varepsilon}^-)$. Therefore, the second factor on the right-hand side of (26) is equal to

$$\begin{aligned}\mathbf{E}_u\big(\overline{F}_{\chi_1^\varepsilon}(\tau_{0,1}^{+,\varepsilon}-\tau_{-\varepsilon}^-)\big|\mathcal{F}_{\tau_{-\varepsilon}^-}\big)&=\mathbf{E}_u\big(\overline{F}_{\chi_1^\varepsilon}(\widehat{\tau}^+_{|\chi_1^\varepsilon|})\big|\mathcal{F}_{\tau_{-\varepsilon}^-}\big)\\&=\mathbf{E}_u\big(\overline{F}_{\chi_1^\varepsilon}(\widehat{\tau}^+_{|\chi_1^\varepsilon|})\big|\chi_1^\varepsilon\big)=K(\chi_1^\varepsilon).\end{aligned}$$

We conclude that

$$\begin{aligned}&\mathbf{P}_u(N^\varepsilon=\infty)\\&\quad=\mathbf{P}_u(\tau_{-\varepsilon}^-=\infty)+\mathbf{P}_0(N^\varepsilon=\infty)\mathbf{E}_u(K(\chi_1^\varepsilon);\,\tau_{-\varepsilon}^-<\infty)\\&\quad=\mathbf{P}_{u+\varepsilon}(\tau_0^-=\infty)+\mathbf{P}_0(N^\varepsilon=\infty)\mathbf{E}_{u+\varepsilon}(K(\chi_1-\varepsilon);\,\tau_0^-<\infty).\end{aligned}$$

Setting $u:=0$ to find $\mathbf{P}_0(N^\varepsilon=\infty)$ and using (23), we finally obtain (18). □

3. Examples

Example 1. Consider the classical CL model:

$$X_t=X_0+ct-\sum_{j=1}^{A_t}\xi_j,\quad t\ge0,\tag{27}$$

where $c>0$ is a constant premium payment rate and the Poisson claims arrival process $\{A_t\}_{t\ge0}$ with rate $\lambda>0$ is independent of the sequence of i.i.d. exponentially distributed claim sizes $\{\xi_n\}_{n\ge1}$ with rate $\alpha>0$.

Clearly, in this case one has $\psi(\theta)=c\theta+\lambda(\frac{\alpha}{\alpha+\theta}-1)$, $\theta>-\alpha$, so that condition (2) turns into

$$\mathbf{E}_0X_1=c-\lambda/\alpha>0.$$

Elementary computation yields

$$\Phi(q)=\frac{1}{2c}\Big(\sqrt{(\alpha c-\lambda-q)^2+4q\alpha c}-(\alpha c-\lambda-q)\Big),\quad q\ge0,$$

and

$$W(x)=\frac{\alpha}{\alpha c-\lambda}\Big(1-\frac{\lambda}{\alpha c}e^{-(\alpha-\lambda/c)x}\Big)\mathbf{1}(x\ge0),\quad\text{with }W(0)=\frac{1}{c}$$

(see p. 251 in Kyprianou, 2014). To find the q-scale function, we note that $\psi'(\theta)=c-\lambda\alpha/(\alpha+\theta)^2$ and that, for $q>0$, the equation $\psi(\theta)=q$ has two solutions: $\Phi(q)>0$ and $\zeta(q):=(\lambda+q)/c-\alpha-\Phi(q)\in(-\alpha,0)$. Hence, after some elementary algebra, we obtain

$$\begin{aligned}\psi'(\Phi(q))&=c-\frac{\lambda\alpha}{(\alpha+\Phi(q))^2}=\frac{c(\Phi(q)-\zeta(q))}{\alpha+\Phi(q)},\\\psi'(\zeta(q))&=-\frac{c(\Phi(q)-\zeta(q))}{\alpha+\zeta(q)}.\end{aligned}$$

Therefore, by Lemma 9.1 in Kyprianou (2013),

$$W^{(q)}(x)=\frac{(\alpha+\Phi(q))e^{\Phi(q)x}-(\alpha+\zeta(q))e^{\zeta(q)x}}{c(\Phi(q)-\zeta(q))}\mathbf{1}(x\ge0)$$

(see also Example 5.3 in Behme and Oechsler (2020) for an alternative representation).

It is well-known that, for this model, one has

$$\mathbf{P}_u(\tau_0^-<\infty)=\frac{\lambda}{\alpha c}e^{-(\alpha-\lambda/c)u},\quad u\ge0\tag{28}$$

(see, e.g., p. 78 in Asmussen and Albrecher, 2010). Due to the memoryless property of the exponential distribution, the conditional distribution of $-\chi_1$ (given that X ever turns negative) will coincide with the distribution of ξ_1, so that

$$\begin{aligned}\mathbf{P}_u(\chi_1\le x,\,\tau_0^-<\infty)&=\mathbf{P}_u(\chi_1\le x\,|\,\tau_0^-<\infty)\mathbf{P}_u(\tau_0^-<\infty)\\&=\frac{\lambda}{\alpha c}e^{\alpha x-(\alpha-\lambda/c)u},\quad x\le0.\end{aligned}$$

Therefore

$$h_u(x)=\frac{\lambda}{c}e^{\alpha x-(\alpha-\lambda/c)u},\quad x<0,$$

and hence

$$H(v)=\int_{-\infty}^0K(x)h_v(x)\,dx=H(0)e^{-(\alpha-\lambda/c)v},$$

$$H(0)=\frac{\lambda}{c}\int_{-\infty}^0e^{\alpha x}K(x)\,dx.\tag{29}$$

Substituting the obtained expressions into (13) yields

$$\mathbf{P}_u(N<\infty)=\frac{\lambda}{\alpha c}\Big[1-\frac{(\alpha c-\lambda)H(0)}{\lambda(1-H(0))}\Big]e^{-(\alpha-\lambda/c)u},\quad u\ge0.\tag{30}$$

Comparing this with (28), we see that, for the CL risk reserve process model, the Parisian ruin probability differs from the "usual" one (28) by the "square bracket factor" that does not exceed one and does not depend on the initial reserve u.

To compute the value of $H(0)$ in (30), we need to specify the distribution of the delay window length. We will consider two special cases.

Case 1. Assume that the conditional distribution of the window length is exponential with parameter depending on the deficit: there is a Borel function $r:(-\infty,0)\to(0,\infty]$ such that $\overline{F}_x(t)=e^{-r(x)t}$, $t>0$ (where the value $r(x)=\infty$ means immediate ruin when χ_1 is equal to that x). Then, by Theorem 3.12 in Kyprianou (2014),

$$K(x)=\int_0^\infty\overline{F}_x(t)dG_{|x|}(t)=\mathbf{E}_0e^{-r(x)\tau^+_{|x|}}=e^{\Phi(r(x))x},\quad x<0,\tag{31}$$

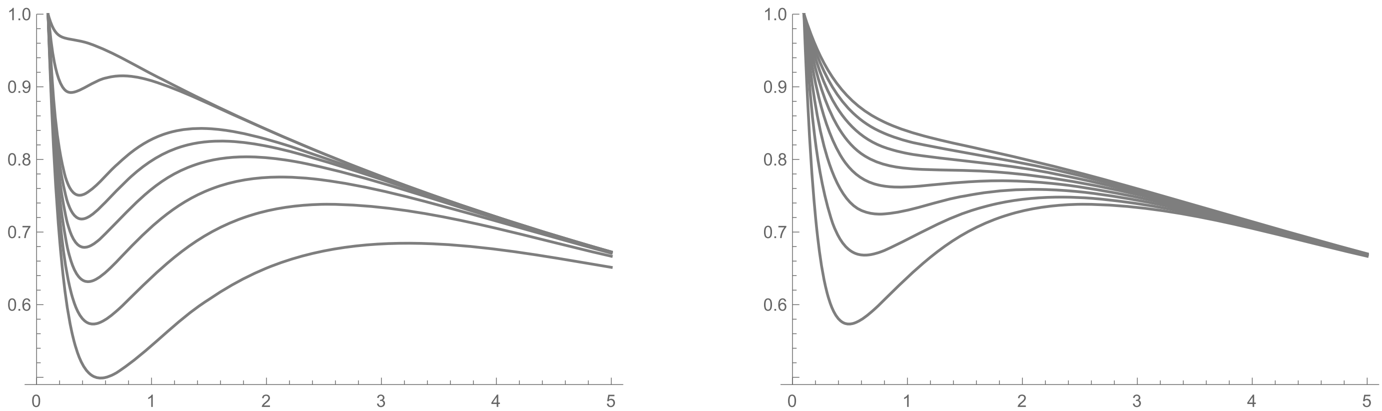
and hence

$$H(0)=\frac{\lambda}{c}\int_{-\infty}^0e^{[\alpha+\Phi(r(x))]x}dx.\tag{32}$$

This quantity can be explicitly evaluated, for instance, in the special case when $r(x)$ is piece-wise constant:

$$r(x):=\sum_{k=1}^nr_k\mathbf{1}(x\in(a_{k-1},a_k])\tag{33}$$

for some $n\ge1$, $r_k\in(0,\infty]$, $k=1,\dots,n$, and $-\infty=:a_0<a_1<\dots<a_{n-1}<a_n:=0$. Then (32) turns into



(a) $a_1 = -0.75$ (the bottom curve); -1; -1.25; -1.5; -1.75; -2; -4; -8 (the top curve); $r_1 = 0.1$ in all eight cases.

(b) $r_1 = 0.1$ (the bottom curve); 0.2; 0.3; 0.4; 0.5; 0.6; 0.7; 0.8 (the top curve); $a_1 = -1$ in all eight cases.

Fig. 1. The square brackets factor in the Parisian ruin probability (30) as a function of the claim size rate α for the model from Example 1, Case 1 with $c = 1$, $\lambda = 0.1$, $n = 2$, $r_2 = 10$.

$$H(0) = \frac{\lambda}{c}\sum_{k=1}^{n}\int_{a_{k-1}}^{a_k} e^{(\alpha+\Phi(r_k))x}dx = \frac{\lambda}{c}\sum_{k=1}^{n}\frac{e^{(\alpha+\Phi(r_k))a_k} - e^{(\alpha+\Phi(r_k))a_{k-1}}}{\alpha+\Phi(r_k)},$$

the terms in the sum with $r_k = \infty$ being equal to 0.

To illustrate how the introduction of the "Parisian ruin mechanism" can affect the ruin probabilities, we will consider the case where r is a step function of the form (33) with $n = 2$ and show how the square bracket factor in (30) can "modify" the "usual ruin probability" (28) depending on the choice of the delay windows' parameters. We set $c = 1$, $\lambda = 0.1$, $r_2 = 10$. The curves in Fig. 1a show the behavior of that factor as a function of the claim size rate α when $r_1 = 0.1$ is fixed, while the threshold a_1 (at which the rate of the random exponential delay switches from r_1 to r_2) changes from -0.75 (the bottom curve) to -8 (the top curve). As the Parisian ruin is always less likely than the "usual" one, the value of the factor is never greater than one. The interesting phenomenon we observe here is that as the severity of claims increases (i.e., α decreases), in the case of relatively small threshold a_1 values, that factor can actually decrease (before eventually increasing to one for very large claims).

This can be explained as follows: as the deficit at "ordinary ruin" has the same exponential distribution as the claim size, when α decreases the deficit is more likely to hit the lower region, where the random delay window rate is $r_1 = 0.1$ and hence the delay window is typically long giving more time for the company to recover. For very large claims, the presence of the Parisian ruin mechanism does not make much difference and hence the factor tends to 1. In addition, the lower the threshold a_1 is located, the smaller the above-described effect.

Fig. 1b depicts the behavior of the square bracket factor in (30) as a function of α when the threshold location $a_1 = -1$ is fixed, whereas the rate r_1 of the exponential delay time given the deficit is below a_1 varies from 0.1 (the bottom curve) to 0.8 (the top curve). We observe here a similar phenomenon: in the presence of longer delay windows in the lower region, the increasing severity of claims can somewhat counter-intuitively make smaller the value of the "correction factor" for the Parisian ruin probability.

In all the cases, the factor vanishes as $\alpha \to \infty$, which is due to the fact when the claim sizes are very small, even short delay windows are likely to be enough for the company to recover.

Case 2. Assume now that the conditional distribution of the window length is a finite mixture of Erlang distributions with parameters depending on the deficit: for an $m \geq 1$, there are Borel functions $p_j : (-\infty, 0) \to [0, 1]$, $\sum_{j=1}^m p_j(x) \equiv 1$, $r_j : (-\infty, 0) \to (0, \infty]$, and $\nu_j(x) : (-\infty, 0) \to \mathbb{N}$, $j = 1, \dots, m$, such that, for $x < 0$,

$$\overline{F}_x(t) = \sum_{j=1}^{m} p_j(x) \sum_{\ell=0}^{\nu_j(x)-1} \frac{(r_j(x)t)^\ell}{\ell!}e^{-r_j(x)t}, \quad t > 0,$$

is the right distribution tail of a mixture of (up to) m components that are Erlang distributions with the respective shape and rate parameters $\nu_j(x)$, $r_j(x)$, $j = 1, \dots, m$. Such mixtures form a rather large class: it is well-known to be everywhere dense in the weak convergence topology in the class of all probability distributions on $(0, \infty)$ (see, e.g., p. 153 in Tijms, 1994).

In this case,

$$K(x) = \sum_{j=1}^{m} p_j(x) \sum_{\ell=0}^{\nu_j(x)-1} \frac{r_j^\ell(x)}{\ell!}\int_0^\infty t^\ell e^{-r_j(x)t}dG_{|x|}(t) = \sum_{j=1}^{m} p_j(x) \sum_{\ell=0}^{\nu_j(x)-1} r_j^\ell(x)\phi_\ell(r_j(x), x), \tag{34}$$

where we used the fact that, by (31) and the well-known property of Laplace transforms,

$$\int_0^\infty t^\ell e^{-rt}dG_{|x|}(t) = \ell!\phi_\ell(r, x), \quad \phi_\ell(r, x) := \frac{(-1)^\ell}{\ell!}\frac{\partial^\ell}{\partial r^\ell}e^{\Phi(r)x}. \tag{35}$$

As in Case 1, we now assume that the functions participating in the definition of $\overline{F}_x$ are piece-wise constant. Namely, there exist $-\infty =: a_0 < a_1 < \cdots < a_{n-1} < a_n := 0$ such that whenever the deficit at the beginning of a negative excursion of the risk reserve process hits the interval $(a_{k-1}, a_k]$, the applicable delay window length will have one and the same distribution given by a finite mixture of Erlang distributions. More formally, for some $p_{j,k} \in [0, 1]$ ($\sum_{j=1}^m p_{j,k} = 1$), $r_{j,k} \in (0, \infty]$, and $\nu_{j,k} \in \mathbb{N}$, one has $r_j(x) := \sum_{k=1}^n r_{j,k}\mathbf{1}(x \in (a_{k-1}, a_k])$, $p_j(x) := \sum_{k=1}^n p_{j,k}\mathbf{1}(x \in (a_{k-1}, a_k])$ and $\nu_j(x) := \sum_{k=1}^n \nu_{j,k}\mathbf{1}(x \in (a_{k-1}, a_k])$. Then from (29) and (35) we get the following expression that can be evaluated for any set of the model parameters:

$$H(0)=\frac{\lambda}{c}\sum_{k=1}^{n}\sum_{j=1}^{m}p_{j,k}\sum_{\ell=0}^{\nu_{j,k}-1}r_{j,k}^{\ell}\int_{a_{k-1}}^{a_k}e^{\alpha x}\phi_\ell(r_{j,k},x)dx.$$

Example 2. Here we consider the classical CL model (27), where the claim distribution is now hyperexponential: for some $\pi_i>0$, $i=1,\dots,n$, such that $\sum_{i=1}^n\pi_i=1$, and $\alpha_n>\alpha_{n-1}>\dots>\alpha_1>0$, one has

$$\mathbf{P}(\xi_1>x)=\sum_{i=1}^{n}\pi_i e^{-\alpha_i x},\quad x>0,$$

so that the jump measure of the process X is of the form

$$\Pi(dx)=\lambda\sum_{i=1}^{n}\pi_i\alpha_i e^{\alpha_i x}\mathbf{1}(x<0)dx.$$

In this case, the Laplace exponent of X is clearly equal to $\psi(\theta)=\theta\big(c-\lambda\sum_{i=1}^n\frac{\pi_i}{\alpha_i+\theta}\big)$, $\theta>-\alpha_1$. Denote by ψ_0 the analytic extension of ψ to $\mathbb{C}\setminus\{\alpha_1,\dots,\alpha_n\}$. It is known (see p. 80 in Kyprianou (2013)) that, for any $q>0$, the equation $\psi_0(\theta)=q$ has exactly $n+1$ distinct solutions in the domain of ψ_0, all of them being real. One of these roots is positive (our $\Phi(q)$) and the other n are negative; denote the latter by $\zeta_i(q)$, $i=1,\dots,n$, in the descending order. One can easily see that

$$-\alpha_n<\zeta_n(q)<\dots<-\alpha_2<\zeta_2(q)<-\alpha_1<\zeta_1(q)<0<\zeta_0(q):=\Phi(q).$$

By Lemma 9.1 from Kyprianou (2013), for any $q>0$, the q-scale function is given by

$$W^{(q)}(x)=\sum_{j=0}^{n}\frac{e^{\zeta_j(q)x}}{\psi_0'(\zeta_j(q))}\mathbf{1}(x\ge 0).$$

For $q=0$, one has $W(x)=\frac{1}{\mathbf{E}_0X_1}+\sum_{j=1}^n\frac{e^{\zeta_j x}}{\psi_0'(\zeta_j)}\mathbf{1}(x\ge 0)$ where $\zeta_j:=\zeta_j(0+)$, $j=1,\dots,n$. Therefore we obtain from (8) that

$$\begin{aligned}h_u(x)&=\lambda\sum_{i=1}^{n}\pi_i\int_0^u e^{\alpha_i(x+z-u)}W'(z)dz+\lambda W(0)\sum_{i=1}^{n}\pi_i e^{\alpha_i(x-u)}\\&=\lambda\sum_{i=1}^{n}\pi_i e^{\alpha_i x}\left(\sum_{j=1}^{n}\frac{\zeta_j(e^{\zeta_j u}-e^{-\alpha_i u})}{\psi_0'(\zeta_j)(\alpha_i+\zeta_j)}+W(0)e^{-\alpha_i u}\right).\end{aligned}$$

Now turning to (12), we get $H(u)=\lambda\sum_{i=1}^n g_i(u)\int_{-\infty}^0 e^{\alpha_i x}K(x)dx$, where

$$g_i(u):=\pi_i\left(\sum_{j=1}^{n}\frac{\zeta_j(e^{\zeta_j u}-e^{-\alpha_i u})}{\psi_0'(\zeta_j)(\alpha_i+\zeta_j)}+W(0)e^{-\alpha_i u}\right).$$

In particular, $H(0)=\lambda W(0)\sum_{i=1}^n\pi_i\int_{-\infty}^0 e^{\alpha_i x}K(x)dx$. The above calculations provide, together with (10) and (11), all the components for computing the Parisian ruin probability (13) in both cases of the delay window distribution structure considered in Example 1 (cf. (31), (34)).

Declaration of competing interest

There is no competing interest to declare.

Data availability

No data was used for the research described in the article.

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