Insurance: Mathematics and Economics 111 (2023) 279–287

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Pairwise counter-monotonicity

Jean-Gabriel Lauzier *, Liyuan Lin, Ruodu Wang

Department of Statistics and Actuarial Science, University of Waterloo, Canada

ARTICLE INFO

Article history:
Received February 2023
Received in revised form May 2023
Accepted 20 May 2023
Available online 29 May 2023

JEL classification:
C10
C71

Keywords:
Negative dependence
Mutual exclusivity
Risk sharing
Comotonicity
Joint mixability

ABSTRACT

We systematically study pairwise counter-monotonicity, an extremal notion of negative dependence. A stochastic representation and an invariance property are established for this dependence structure. We show that pairwise counter-monotonicity implies negative association, and it is equivalent to joint mix dependence if both are possible for the same marginal distributions. We find an intimate connection between pairwise counter-monotonicity and risk sharing problems for quantile agents. This result highlights the importance of this extremal negative dependence structure in optimal allocations for agents who are not risk averse in the classic sense.

1. Introduction

Dependence modeling is a crucial part of modern quantitative studies in economics, finance, and insurance (McNeil et al. (2015)). Comonotonicity and counter-monotonicity are known as the strongest forms of positive and negative dependence, respectively. In quantitative risk management, assuming knowledge of the marginal distributions, comonotonicity corresponds to the most dangerous dependence structure (Denneberg (1994) and Dhaene et al. (2002, 2006)) for the aggregate risk, whereas counter-monotonicity corresponds to the safest. In dimensions higher than 2, by counter-monotonicity we mean pairwise counter-monotonicity (Dall'Aglio (1972)), which has been studied under the name of mutual exclusivity in the actuarial literature (Dhaene and Denuit (1999) and Cheung and Lo (2014)).[1]

Despite the obvious similarity in their definitions, comonotonicity and counter-monotonicity are asymmetric in several major senses. For instance, comonotonicity admits a stochastic representation (see Lemma 1 below), but such a representation is not known for pairwise counter-monotonicity. Moreover, for any given tuple of marginal distributions, a comonotonic random vector with these marginal distributions always exists, but a pairwise counter-monotonic one may not exist unless quite restrictive conditions on the marginal distributions are satisfied, as first studied by Dall'Aglio (1972). In particular, a pairwise counter-monotonic random vector cannot have continuous marginal distributions. Comonotonicity has many important roles in economics, finance and actuarial science, and as such it has received great attention in the literature, as in axiomatization of preferences (Yaari (1987); Schmeidler (1989)), risk measures (Kusuoka (2001)) and premium principles (Wang et al. (1997)), risk sharing (Landsberger and Meilijson (1994); Jouini et al. (2008)), insurance design (Huberman et al. (1983); Carlier and Dana (2003)), risk aggregation (Embrechts et al. (2015)), and optimal transport (Rüschendorf (2013)).

In sharp contrast to the rich literature on comonotonicity, research on pairwise counter-monotonicity is quite limited. As a dependence concept, pairwise counter-monotonicity has been studied by Dall'Aglio (1972), Hu and Wu (1999), Dhaene and Denuit (1999) and Cheung and Lo (2014), but the list of relevant studies do not grow much longer. In contrast to the relatively limited studies on pairwise counter-monotonicity, this dependence structure appears naturally in many economic contexts, such as lottery tickets, Bitcoin mining, gambling, and mutual aid platforms, whenever payment events are mutually exclusive. In particular, the interest in studying pairwise counter-monotonicity has grown in the recent risk sharing literature. A pairwise counter-monotonic struc-

* Corresponding author.
E-mail addresses: jlauzier@uwaterloo.ca (J.G. Lauzier), liyuan.lin@uwaterloo.ca (L. Lin), wang@uwaterloo.ca (R. Wang).

[1] Mutual exclusivity is defined using joint exceedance probability (see Section 2). The two definitions are shown to be equivalent first by Dall'Aglio (1972, Lemma 2) and in a more precise form by Cheung and Lo (2014, Theorem 4.1).

https://doi.org/10.1016/j.insmatheco.2023.05.006
0167-6687/

ture is the essential building block of any optimal allocation for agents using Value-at-Risk (VaR, which are quantiles) and quantile-related risk measures; such problems are studied by Embrechts et al. (2018) and generalized by Weber (2018), Embrechts et al. (2020), Liu et al. (2022) and Xia et al. (2023). Moreover, counter-monotonicity, when possible, serves as the best-case dependence structure in risk aggregation for some common risk measures, and, in some contexts, it also serves as the worst-case dependence structure for VaR (see Example 1 in Section 2).

This paper is dedicated to a systematic study of pairwise counter-monotonicity. As comonotonicity and counter-monotonicity are classic and prominent concepts in mathematics and its applications with a long history, at least since the seminal work of Hardy et al. (1934), one may guess that there is not much more to discover about them. To our pleasant surprise, we offer, through the development of this paper, many new results on counter-monotonicity, some of which are motivated by recent developments in risk management.

We obtain a new stochastic representation for pairwise counter-monotonic random vectors using their component-wise sum in Theorem 1, which will be useful for many other results in the paper. The second result, Theorem 2, establishes that counter-monotonicity is preserved under increasing transforms on disjoint sets of components of a random vector, which is an invariance property proposed by Joag-Dev and Proschan (1983) satisfied by negative association (Alam and Saxena (1981)). Using this invariance property, we obtain in Theorem 3 that counter-monotonicity implies negative association. The notion of negative association is stronger than many other forms of negative dependence, such as negative orthant dependence (Block et al. (1982)) and negative supermodular dependence (Hu (2000)). In particular, Theorem 3 surpasses a result of Dhaene and Denuit (1999) showing that counter-monotonicity implies negative supermodular dependence.

Another negative dependence concept is joint mix dependence (Wang and Wang (2011, 2016)), which can be used to optimize many quantities in risk aggregation; see Wang et al. (2013) and Rüschendorf (2013). To connect counter-monotonicity and joint mix dependence, we fully characterize all Fréchet classes (Joe (1997)) which are compatible with both dependence concepts in Theorem 4; it turns out that the two notions, when both exist in the same Fréchet class, are equivalent. Finally, we show in Theorem 5 that in the context of risk sharing for quantile agents (Embrechts et al. (2018)), under some mild conditions on the total loss, there always exists a pairwise counter-monotonic Pareto-optimal allocation, and any pairwise counter-monotonic allocation is Pareto optimal for some agents. As a consequence, pairwise counter-monotonic random vectors are natural for agents that are not risk averse. This is in stark contrast to comonotonic allocations, which appear prominently for risk-averse agents (in the sense of Rothschild and Stiglitz (1970)) as a consequence of comonotonic improvements introduced by Landsberger and Meilijson (1994).

2. Preliminaries

We first define comonotonicity and counter-monotonicity for bivariate random variables. Fix a probability space $(\Omega, \mathcal{A}, \mathbb{P})$. The probability space does not need to be atomless in Sections 2–4. We treat almost surely (a.s.) equal random variables as identical; this means that all equalities and inequality for random variables hold in the a.s. sense, and we omit "a.s." in all our statements. Terms like "increasing" are in the non-strict sense. Let n be a positive integer and $[n] = \{1, \dots, n\}$. Throughout, we consider $n \geqslant 2$.

A bivariate random vector (X, Y) is *comonotonic* if there exist increasing functions f, g and a random variable Z such that $(X, Y) = (f(Z), g(Z))$. A bivariate random vector (X, Y) is *counter-monotonic* if $(X, -Y)$ is comonotonic. An equivalent formulation of comonotonicity is

$$(X(\omega) - X(\omega'))(Y(\omega) - Y(\omega')) \geqslant 0$$

for $(\mathbb{P} \times \mathbb{P})$-almost every $(\omega, \omega') \in \Omega^2$.

An equivalent formulation of counter-monotonicity is

$$(X(\omega) - X(\omega'))(Y(\omega) - Y(\omega')) \leqslant 0$$

for $(\mathbb{P} \times \mathbb{P})$-almost every $(\omega, \omega') \in \Omega^2$.

Next, we define these concepts in dimensions higher than 2. For $n \geqslant 3$, a random vector $\mathbf{X}$ taking values in $\mathbb{R}^n$ is *(pairwise) comonotonic* if each pair of its components is comonotonic, and it is *(pairwise) counter-monotonic* if each pair of its components is counter-monotonic.[2] We will say "pairwise counter-monotonicity" to emphasize the case $n \geqslant 3$ and simply say "counter-monotonicity" when we also include dimension 2. We always omit "pairwise" for comonotonicity, for which the distinction between dimensions $n = 2$ and $n \geqslant 3$ is unnecessary.

There are many equivalent ways of formulating comonotonicity and counter-monotonicity; see Puccetti and Wang (2015, Section 3.2) for a review. For instance, they can be formulated using joint distributions. A comonotonic random vector and a counter-monotonic random vector have, respectively, the largest and the smallest joint distribution functions among all random vectors with the same marginals. With given marginals, the largest (resp. smallest) joint distribution function is known as the Fréchet-Hoeffding upper (resp. lower) bound.

A stochastic representation of comonotonicity, which follows from Denneberg (1994, Proposition 4.5), is presented in the next lemma.

Lemma 1 *(Denneberg (1994)). Let $(X_1, \dots, X_n)$ be a random vector and denote by $S = \sum_{i=1}^n X_i$. The following are equivalent.*

(i) $(X_1, \dots, X_n)$ is comonotonic.
(ii) There exist increasing functions $f_1, \dots, f_n$ and a random variable Z such that $X_i = f_i(Z)$ for all $i \in [n]$.
(iii) There exist continuously increasing functions $f_1, \dots, f_n$ such that $X_i = f_i(S)$ for all $i \in [n]$.

Lemma 1 implies that a comonotonic vector can be represented by increasing functions of the sum S. Such a representation result does not exist for pairwise counter-monotonicity, since the sum S cannot determine the components $(X_1, \dots, X_n)$ in the presence of negative dependence.

Although quite different from comonotonicity, pairwise counter-monotonicity also has a special structure, presented below in Lemma 2, which is a restatement of Lemma 2 and Theorem 3 of Dall'Aglio (1972). This result will be useful in a few places in the paper. The current form of this lemma can be found in Theorem 4.1 of Cheung and Lo (2014) and Proposition 3.2 of Puccetti and Wang (2015). Denote by ess-inf$X$ and ess-sup$X$ the essential infimum and essential supremum of a random variable X, respectively.

Lemma 2 *(Dall'Aglio (1972)). If at least three of $X_1, \dots, X_n$ are non-degenerate, pairwise counter-monotonicity of $(X_1, \dots, X_n)$ means that one of the following two cases holds true:*

[2] We also say that random variables $X_1, \dots, X_n$ are comonotonic (counter-monotonic), which means that the random vector $(X_1, \dots, X_n)$ is comonotonic (counter-monotonic).

$$\mathbb{P}(X_i > \text{ess-inf}X_i,\ X_j > \text{ess-inf}X_j) = 0 \text{ for all } i \neq j; \tag{1}$$

$$\mathbb{P}(X_i < \text{ess-sup}X_i,\ X_j < \text{ess-sup}X_j) = 0 \text{ for all } i \neq j. \tag{2}$$

A necessary condition for (1) is $\sum_{i=1}^n \mathbb{P}(X_i > \text{ess-inf}X_i) \leqslant 1$, and a necessary condition for (2) is $\sum_{i=1}^n \mathbb{P}(X_i < \text{ess-sup}X_i) \leqslant 1$.

In the actuarial literature, mutual exclusivity of $(X_1, \dots, X_n)$ is defined as either (1) or (2); see Cheung and Lo (2014).

Pairwise counter-monotonicity imposes strong constraints on the marginal distributions. For instance, the necessary condition in case of (1) is equivalent to $\sum_{i=1}^n \mathbb{P}(X_i = \text{ess-inf}X_i) \geqslant n-1$, and it implies, in particular, that $X_1, \dots, X_n$ are bounded from below. Moreover, given $n \geqslant 3$ non-degenerate marginal distributions, a pairwise counter-monotonic random vector exists if and only if one of the two necessary conditions on the marginal distributions holds (Theorem 3 of Dall'Aglio (1972)).

Example 1. We illustrate the special role of counter-monotonicity in risk aggregation with a simple model. Let $F_1, \dots, F_n$ be Bernoulli distributions with mean $\varepsilon \in (0, 1/n)$. These distributions may represent losses from credit default events in a pre-specified period, which usually occur with a small probability. In risk aggregation problems (e.g., Embrechts et al. (2013, 2015)), we are interested in the minimum (best-case) value or maximum (worst-case) value of

$$\rho\left(\sum_{i=1}^n X_i\right) \quad \text{with the marginal condition } X_i \sim F_i,\ i \in [n], \tag{3}$$

where ρ is a risk measure, and $\sum_{i=1}^n X_i$ represents the total loss from a portfolio of defaultable bonds, with the probability of default ε estimated from the credit rating of these bonds, assumed to be equal for simplicity. We consider two choices of ρ, which lead to opposite conclusions.

(a) Let ρ be a risk measure that is consistent with convex order. Such risk measures are characterized by Mao and Wang (2020), and they include all law-invariant coherent, as well as convex, risk measures, such as the Expected Shortfall (Föllmer and Schied (2016)). The *minimum* value of (3) is obtained by a counter-monotonic random vector $(X_1, \dots, X_n)$. This result holds for other marginal distributions as long as a counter-monotonic random vector with these marginal distributions exists; see e.g., Lemma 3.6 of Cheung and Lo (2014).

(b) Let $\rho : X \mapsto \inf\{x \in \mathbb{R} : \mathbb{P}(X \leqslant x) \geqslant 1-\alpha\}$, which is the risk measure VaR_α in Section 6. Further, assume that $\alpha/\varepsilon \in (n/2, n)$. The *maximum* value of (3) is obtained by a counter-monotonic random vector $(X_1, \dots, X_n)$, as explained below. First, since $\sum_{i=1}^n X_i$ only takes integer values, so does $\rho(\sum_{i=1}^n X_i)$. If $(X_1, \dots, X_n)$ is counter-monotonic, then $\sum_{i=1}^n X_i$ follows a Bernoulli distribution with mean $n\varepsilon > \alpha$, and hence $\rho(\sum_{i=1}^n X_i) = 1$. Moreover, for any $X_1, \dots, X_n$ with the specified marginal distributions, if $\rho(\sum_{i=1}^n X_i) \geqslant 2$ then $\mathbb{E}[\sum_{i=1}^n X_i] \geqslant 2\alpha > n\varepsilon$, a contradiction, thus showing $\rho(\sum_{i=1}^n X_i) \leqslant 1$.

The interpretation of the above two cases is that, for credit default losses, using a coherent risk measure and using VaR may lead to opposite conclusions on which dependence structure is safe or dangerous, and both cases highlight the important role of counter-monotonicity.

3. Stochastic representation of pairwise counter-monotonicity

We provide in this section a stochastic representation of pairwise counter-monotonicity. To explain the result, let Π_n be the set of all n-compositions of Ω, that is,

$$\Pi_n = \left\{(A_1, \dots, A_n) \in \mathcal{A}^n : \bigcup_{i \in [n]} A_i = \Omega \text{ and } A_1, \dots, A_n \text{ are disjoint}\right\}.$$

In other words, a composition of Ω is a partition of Ω with order. Denote by $\mathcal{X}_\pm$ the set of all nonnegative random variables and nonpositive random variables.

Theorem 1. *Let $(X_1, \dots, X_n)$ be a random vector and denote by $S = \sum_{i=1}^n X_i$. Suppose that at least three of $X_1, \dots, X_n$ are non-degenerate. The following are equivalent.*

(i) $(X_1, \dots, X_n)$ is pairwise counter-monotonic.

(ii) There exist $m_1, \dots, m_n \in \mathbb{R}$, $(A_1, \dots, A_n) \in \Pi_n$ and $Z \in \mathcal{X}_\pm$ such that

$$X_i = Z\mathbb{1}_{A_i} + m_i \quad \text{for all } i \in [n]. \tag{4}$$

(iii) There exists $(A_1, \dots, A_n) \in \Pi_n$ such that

$$X_i = (S-m)\mathbb{1}_{A_i} + m_i \quad \text{for all } i \in [n], \tag{5}$$

where either $m_i = \text{ess-inf}X_i$ for $i \in [n]$ or $m_i = \text{ess-sup}X_i$ for $i \in [n]$, and $m = \sum_{i=1}^n m_i$.

Proof. The implication (iii)⇒(ii) is straightforward. To see (ii)⇒(i), take $i, j \in [n]$ with $i \neq j$, and we check a few cases of $\omega, \omega' \in \Omega$. If $\omega, \omega' \notin A_i$, then $X_i(\omega) = X_i(\omega') = m_i$, and hence

$$(X_i(\omega) - X_i(\omega'))(X_j(\omega) - X_j(\omega')) = 0. \tag{6}$$

Similarly, (6) holds if $\omega, \omega' \notin A_j$. If $(\omega, \omega') \in A_i \times A_j$ or $(\omega, \omega') \in A_j \times A_i$, then

$$(X_i(\omega) - X_i(\omega'))(X_j(\omega) - X_j(\omega')) = -Z(\omega)Z(\omega') \leqslant 0.$$

This shows that (X_i, X_j) is counter-monotonic, and hence, $(X_1, \dots, X_n)$ is pairwise counter-monotonic.

Next, we show the implication (i)⇒(iii). By Lemma 2, it suffices to consider (1) and (2). Suppose that (1) holds. Let $B_i = \{X_i > \text{ess-inf}X_i\}$ and $m_i = \text{ess-inf}X_i$ for $i \in [n]$. Clearly $B_1, \dots, B_n$ are (a.s.) disjoint events, and $S \geqslant \sum_{i=1}^n m_i = m$. Using (1), if event B_i occurs, then $X_j = m_j$ for $j \neq i$, and $S = X_i + \sum_{j=1}^n m_j - m_i$. Moreover, if B_i does not occur, then $X_i = m_i$. Therefore, we have

$$X_i = (S - m + m_i)\mathbb{1}_{B_i} + m_i\mathbb{1}_{B_i^c} = (S-m)\mathbb{1}_{B_i} + m_i, \quad \text{for } i \in [n]. \tag{7}$$

Let $B = \{S = m\}$ and it is clear that $(B, B_1, \dots, B_n)$ is a composition of Ω. Let $A_1 = B_1 \cup B$, and $A_2 = B_2, \dots, A_n = B_n$. Since $S - m = 0$ on B, (7) yields (5). If (2) holds instead of (1), then we can analogously show (5) with $m_i = \text{ess-sup}X_i$ for $i \in [n]$. □

Theorem 1 shows that pairwise counter-monotonicity can be represented by the sum S and a composition $(A_1, \dots, A_n)$. In contrast, comonotonicity can be represented by the sum S and increasing continuous functions $f_1, \dots, f_n$, as in Lemma 1. This representation result will be instrumental in proving the other results of this paper. Another direct consequence of Theorem 1 is that if at least three components of a pairwise counter-monotonic random vector are non-degenerate, then either the components are all bounded from below or they are all bounded from above; this can also be seen from Lemma 2.

Example 2. A simple pairwise counter-monotonic random vector in the form of (4) and (5), which will be referred to repeatedly in the following sections, is given by

$$X_i = \mathbb{1}_{A_i} \text{ for } i \in [n] \text{ where } (A_1, \dots, A_n) \in \Pi_n. \quad (8)$$

Such $(X_1, \dots, X_n)$ may represent the outcome of n lottery tickets, exactly one of which randomly wins a reward of 1, or the reward to Bitcoin miners computing the next block in the Bitcoin blockchain; see Leshno and Strack (2020).

Remark 1. In parts (ii) and (iii) of Theorem 1, we can replace $(A_1, \dots, A_n) \in \Pi_n$ by $A_1, \dots, A_n$ being disjoint events, and the equivalence relations in the theorem remain true.

In the case at most two components of $(X_1, \dots, X_n)$ are non-degenerate, the stochastic representation of counter-monotonicity is quite different from Theorem 1. When $n = 2$, (X_1, X_2) is counter-monotonic if and only if there exist increasing functions f_1, f_2 such that

$$X_1 = f_1(X_1 - X_2) \quad \text{and} \quad X_2 = f_2(X_2 - X_1);$$

this statement follows by applying Lemma 1 to the comonotonic random vector $(X_1, -X_2)$. Note that the difference $X_1 - X_2$ replaces the summation $S = X_1 + X_2$ in Lemma 1. The sum of two counter-monotonic random variables represents the loss from a hedged portfolio and it has been studied by Cheung et al. (2014) and Chaoubi et al. (2020).

4. Invariance property and negative association

Negative association appears in various natural probabilistic and statistical contexts, such as permutation distributions, sampling without replacement, negatively correlated Gaussian distributions and tournament scores; see Joag-Dev and Proschan (1983) and the more recent paper Chi et al. (2022) for many examples.

A random vector $\mathbf{X} = (X_1, \dots, X_n)$ is said to be *negatively associated* if for any disjoint subsets $I, J \subseteq [n]$, and any real-valued, coordinate-wise increasing functions f, g, we have

$$\mathrm{Cov}(f(\mathbf{X}_I), g(\mathbf{X}_J)) \leqslant 0, \quad (9)$$

where $\mathbf{X}_I = (X_k)_{k \in I}$ and $\mathbf{X}_J = (X_k)_{k \in J}$, provided that $f(\mathbf{X}_I)$ and $g(\mathbf{X}_J)$ have finite second moments. Negative association is stronger than many other notions of negative dependence, such as negative supermodular dependence (shown by Christofides and Vaggelatou (2004)) and negative orthant dependence (shown by Joag-Dev and Proschan (1983)).

Remark 2. Negative association is invariant under increasing marginal transforms. Therefore, if $f(\mathbf{X}_I)$ and $g(\mathbf{X}_J)$ are continuously distributed, then NA implies that (9) holds with the covariance operator replaced by Spearman's rank correlation coefficient or another similar concordance measure; see McNeil et al. (2015, Chapter 7).

We first present a self-consistency property of both comonotonicity and counter-monotonicity in the spirit of Property P6 of Joag-Dev and Proschan (1983) for negative association. To the best of our knowledge, this self-consistency property is not found in the literature even for the case of comonotonicity, although its proof is straightforward.

Theorem 2. *The following statements hold.*

(i) Increasing functions of subsets of a set of comonotonic random variables are comonotonic.
(ii) Increasing functions of disjoint subsets of a set of counter-monotonic random variables are counter-monotonic.

Proof. (i) Let $\mathbf{X} = (X_1, \dots, X_n)$ be a comonotonic random vector. By Lemma 1, there exist increasing functions $f_1, \dots, f_n$ and a random variable Z such that $X_i = f_i(Z)$ for all $i \in [n]$. For $I_1, \dots, I_m \subseteq [n]$ and increasing functions $g_j : \mathbb{R}^{|I_j|} \to \mathbb{R}$, $j \in [m]$, let $Y_j = g_j(\mathbf{X}_{I_j})$, $j \in [m]$, where $|\cdot|$ is the cardinality of a set. That is, $Y_j = g_j \circ f_{I_j}(Z)$ where $f_{I_j} = (f_i)_{i \in I_j}$. As the composition of increasing functions, $g_i \circ f_{I_j}$ is increasing on $\mathbb{R}$. Thus, $(Y_1, \dots, Y_m)$ is a comonotonic vector.

(ii) Let $\mathbf{X} = (X_1, \dots, X_n)$ be a pairwise counter-monotonic random vector. If at most two of $X_1, \dots, X_n$ are non-degenerate, the desired statement holds trivially. Next, we assume that at least three of $X_1, \dots, X_n$ are non-degenerate. For disjoint subsets $I_1, \dots, I_m$ of $[n]$ and increasing functions $g_j : \mathbb{R}^{|I_j|} \to \mathbb{R}$, $j \in [m]$, let $Y_j = g_j(\mathbf{X}_{I_j})$, $j \in [m]$. By Theorem 1, there exist $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{R}^n$, $(A_1, \dots, A_n) \in \Pi_n$ and $Z \in \mathcal{X}_\pm$ such that $X_i = Z\mathbb{1}_{A_i} + m_i$ for all $i \in [n]$. Without loss of generality, assume $Z \geqslant 0$. For $i \in [n]$ and $j \in [m]$, if A_i occurs, then $X_i = Z + m_i$ and $X_k = m_k$ for $k \neq i$, which means $Y_j = g_j(\mathbf{X}_{I_j}) \geqslant g_j(\mathbf{m}_{I_j})$. If A_i does not occur, then $Y_j = g_j(\mathbf{m}_{I_j})$. Let $Z_j = \sum_{i \in I_j} \left(g_j(\mathbf{X}_{I_j}) - g_j(\mathbf{m}_{I_j})\right) \mathbb{1}_{A_i} \geqslant 0$. It follows that

$$\begin{aligned} Y_j &= \sum_{i \in I_j} g_j(\mathbf{X}_{I_j}) \mathbb{1}_{A_i} + g_j(\mathbf{m}_{I_j}) \left(1 - \sum_{i \in I_j} \mathbb{1}_{A_i}\right) \\ &= Z_j \mathbb{1}_{\bigcup_{i \in I_j} A_i} + g_j(\mathbf{m}_{I_j}) = \left(\sum_{k=1}^m Z_k\right) \mathbb{1}_{\bigcup_{i \in I_j} A_i} + g_j(\mathbf{m}_{I_j}). \end{aligned}$$

By using Theorem 1 and the fact that $\sum_{k=1}^m Z_k \geqslant 0$, we conclude that $(Y_1, \dots, Y_m)$ is pairwise counter-monotonic. □

Remark 3. For Theorem 2 (i), an equivalent statement is that increasing functions of comonotonic random variables are comonotonic. This is because one can choose the subsets as $[n]$ and take functions on $\mathbb{R}^n$ which are constant in some dimensions. We use the current presentation of statement (i) to show a contrast to statement (ii).

What we will use from Theorem 2 is the second statement, which leads to the next result in this section; that is, counter-monotonicity implies negative association. Since negative association is stronger than negative supermodular dependence, this result surpasses Theorem 12 of Dhaene and Denuit (1999), which states that counter-monotonicity is stronger than negative supermodular dependence.

Theorem 3. *Counter-monotonicity implies negative association.*

Proof. Let $\mathbf{X}$ be an n-dimensional counter-monotonic random vector. Take disjoint subsets $I, J \subseteq [n]$ and coordinate-wise increasing functions $f : \mathbb{R}^{|I|} \to \mathbb{R}$ and $g : \mathbb{R}^{|J|} \to \mathbb{R}$, where $|\cdot|$ is the cardinality of a set. By Theorem 2 (ii), $f(\mathbf{X}_I)$ and $g(\mathbf{X}_J)$ are counter-monotonic. The Fréchet-Hoeffding inequality (see e.g., Corollary 3.28 of Rüschendorf (2013)) yields $\mathbb{E}[f(\mathbf{X}_I)g(\mathbf{X}_J)] \leqslant \mathbb{E}[f(\mathbf{X}_I)]\mathbb{E}[g(\mathbf{X}_J)]$ provided that the expectations exist. Hence, $\mathbf{X}$ is negatively associated. □

Joag-Dev and Proschan (1983, Theorem 2.11) already noted that the lottery-type random vector (8) in Example 2 is negatively associated.

The result in Theorem 3 has a straightforward interpretation, as counter-monotonicity is the extreme form of negative dependence, which intuitively should imply other notions of negative dependence, among which negative association is considered a strong notion; see Amini et al. (2013) for a comparison of several notions of negative dependence.

Counter-monotonicity is also stronger than several other notions of negative dependence which are not implied by negative association. These notions include conditional decreasing in sequence and negative dependence in sequence (see Joag-Dev and Proschan (1983, Remark 2.16)) and negative dependence through stochastic ordering (see Block et al. (1985)). These implications can be checked directly with Theorem 2, thus highlighting its usefulness.

Remark 4. A random vector $\mathbf{X}$ is positively associated if $\mathrm{Cov}(f(\mathbf{X}), g(\mathbf{X})) \geqslant 0$ for all real-valued, coordinate-wise increasing functions f, g (Esary et al. (1967)). Comonotonicity implies positive association because $(f(\mathbf{X}), g(\mathbf{X}))$ is comonotonic by Theorem 2, and the covariance of a comonotonic pair of random variables is non-negative due to the Fréchet-Hoeffding inequality.

5. Joint mix dependence and Fréchet classes

Another type of extremal negative dependence structure is the notion of joint mixes. In this section, we study the connection between counter-monotonicity and joint mix dependence.

From now on, assume that the probability space $(\Omega, \mathcal{A}, \mathbb{P})$ is atomless. A random vector $(X_1, \dots, X_n)$ is a *joint mix* if $\sum_{i=1}^n X_i$ is a constant c, and in this case we say that *joint mix dependence* holds for $(X_1, \dots, X_n)$. The constant c is called the center of $(X_1, \dots, X_n)$, and it is obvious that $c = \sum_{i=1}^n \mathbb{E}[X_i]$ if the expectations of $X_1, \dots, X_n$ are finite. Joint mix dependence is regarded as a concept of extremal negative dependence due to its opposite role to comonotonicity in risk aggregation problems; see Puccetti and Wang (2015) and Wang and Wang (2016).

The lottery-type random vector in Example 2 satisfies both counter-monotonicity and joint mix dependence. In the case $n = 2$, joint mix dependence is strictly stronger than counter-monotonicity. This result cannot be extended to $n \geqslant 3$. For example, $(X, X, -2X)$ is a joint mix that is not counter-monotonic. A weaker notion than joint mix dependence is proposed by Lee and Ahn (2014), which does not imply, and is not implied by, counter-monotonicity in dimension $n \geqslant 3$.

Joint mix dependence and counter-monotonicity share some similarities. First, for a random vector $(X_1, \dots, X_n)$ with its sum $S = X_1 + \cdots + X_n$, if either pairwise counter-monotonicity or joint mix dependence holds, then X_i and $S - X_i$ are counter-monotonic for each $i \in [n]$. The case of pairwise counter-monotonicity is verified by Theorem 2, and the case of joint mix dependence is verified by definition. Second, both dependence notions impose strong conditions on the marginal distributions. The condition for pairwise counter-monotonicity is given in Lemma 2, and that for joint mix dependence is much more sophisticated; see Wang and Wang (2016) for some sufficient conditions as well as necessary ones. This is in contrast to concepts such as comonotonicity, independence, and negative association, for which the existence of the corresponding random vectors is always guaranteed for any given marginal distribution. Both joint mix dependence and counter-monotonicity are used in the tail region to obtain lower bounds for risk aggregation with given marginal distributions, as studied by Bernard et al. (2014) and Cheung et al. (2017), respectively.

The next result characterizes marginal distributions that are compatible with both counter-monotonicity and joint mix dependence. For this, we need some notation and terminology. In what follows, we will use distribution functions to represent distributions. For an n-tuple $(F_1, \dots, F_n)$ of distributions on $\mathbb{R}$, a *Fréchet class* (see Joe (1997, Chapter 3)) is defined as

$$\mathcal{F}_n(F_1, \dots, F_n) = \{\text{distribution of } (X_1, \dots, X_n) : X_i \sim F_i,\ i \in [n]\}.$$

We say that a Fréchet class $\mathcal{F}_n(F_1, \dots, F_n)$ supports counter-monotonicity (resp. joint mix dependence) if there exists a counter-monotonic random vector (resp. a joint mix) whose distribution is in this class. Let δ_x be the distribution function of a point-mass at $x \in \mathbb{R}$, and denote by Θ_n the standard n-simplex, that is, $\Theta_n = \{(p_1, \dots, p_n) \in [0,1]^n : \sum_{i=1}^n p_i = 1\}$. Two distributions F and G are symmetric if $F(x) = 1 - G(c - x)$, $x \in \mathbb{R}$ for some $c \in \mathbb{R}$. In other words, if X has distribution F, then $c - X$ has distribution G.

It turns out that all Fréchet classes $\mathcal{F}_n(F_1, \dots, F_n)$ which support both counter-monotonicity and joint mix dependence can be characterized explicitly. If at least three of $F_1, \dots, F_n$ are non-degenerate, then $F_1, \dots, F_n$ are two-point distributions given by

$$F_i = p_i \delta_{a+m_i} + (1 - p_i)\delta_{m_i} \text{ for } i \in [n],$$
$$\text{where } a, m_1, \dots, m_n \in \mathbb{R} \text{ and } (p_1, \dots, p_n) \in \Theta_n. \tag{10}$$

If at most two of $F_1, \dots, F_n$ are non-degenerate, then

$$F_i \text{ and } F_j \text{ are symmetric for some } i, j \in [n],$$
$$\text{and } F_k \text{ is degenerate for all } k \in [n] \setminus \{i, j\}. \tag{11}$$

Theorem 4. *A Fréchet class supports both counter-monotonicity and joint mix dependence if and only if one of (10) and (11) holds. In case both are supported, counter-monotonicity and joint mix dependence are equivalent for this Fréchet class.*

Proof. We first prove the equivalence statement in the last part of the theorem. Suppose that the Fréchet class $\mathcal{F}_n(F_1, \dots, F_n)$ supports both counter-monotonicity and joint mix dependence. Puccetti and Wang (2015, Theorem 3.8) shows that if a Fréchet class supports a counter-monotonic random vector, then a random vector is counter-monotonic if and only if it is Σ-counter-monotonic, and moreover, a joint mix is always Σ-counter-monotonic. Using these two facts, a joint mix is counter-monotonic for this Fréchet class. For the converse statement, note that in $\mathcal{F}_n(F_1, \dots, F_n)$ there exists a unique distribution function

$$F(x_1, \dots, x_n) = \left(\sum_{i=1}^n F_i(x_i) - d + 1\right)_+, \quad (x_1, \dots, x_n) \in \mathbb{R}^n$$

of a counter-monotonic random vector (Theorem 3.3 of Puccetti and Wang (2015)). Since a joint mix with marginal distributions $F_1, \dots, F_n$ is counter-monotonic, its distribution must coincide with F. This shows that F is the distribution of a joint mix.

Next, we prove the first part of the theorem. For the "if" statement, assume that a Fréchet class $\mathcal{F}_n(F_1, \dots, F_n)$ supports both counter-monotonicity and joint mix dependence. By the above argument, $\mathcal{F}_n(F_1, \dots, F_n)$ supports a pairwise counter-monotonic joint mix $(X_1, \dots, X_n)$. First, consider the case that at least three of $F_1, \dots, F_n$ are non-degenerate. Using (5),

$$X_i = (c - m)\mathbb{1}_{A_i} + m_i, \quad \text{for} \quad i \in [n],$$

where $(A_1, \dots, A_n) \in \Pi_n$, c is the center of $(X_1, \dots, X_n)$, either $m_i = \text{ess-inf}(X_i)$ for all $i \in [n]$ or $m_i = \text{ess-sup}(X_i)$ for all $i \in [n]$, and $m = \sum_{i=1}^n m_i$. It is clear that F_i has the form (10) by setting $a = c - m$. If at most two of $F_1, \dots, F_n$ are degenerate, say F_i and F_j, then a joint mix $(X_1, \dots, X_n)$ with marginal distributions $F_1, \dots, F_n$ satisfies $X_i = c - X_j$ for some $c \in \mathbb{R}$, and X_k is a constant for each $k \in [n] \setminus \{i, j\}$. This implies (11).

Finally, we verify the converse statement. If $(F_1, \dots, F_n)$ has the form (10), then take $X_i = a\mathbb{1}_{A_i} + m_i$ with $(A_1, \dots, A_n) \in \Pi_n$ satisfying $\mathbb{P}(A_i) = p_i$ for $i \in [n]$, and we have $(X_1, \dots, X_n)$ is counter-monotonic by Theorem 1 and $\sum_{i=1}^n X_i = a + \sum_{i=1}^n m_i$. If $(F_1, \dots, F_n)$ has the form (11), then by taking X_i with distribution F_i, $X_j = c - X_i$ with distribution F_j and $c \in \mathbb{R}$, and X_k with distribution F_k for each $k \in [n] \setminus \{i, j\}$, we can directly verify that $(X_1, \dots, X_n)$ is a counter-monotonic joint mix. □

From the proof of Theorem 4 (ii), if at least three components of a pairwise counter-monotonic joint mix $\mathbf{X} = (X_1, \dots, X_n)$ are non-degenerate, then it has the form

$$X_i = a\mathbb{1}_{A_i} + m_i, \quad \text{for} \quad i \in [n]$$

where $(A_1, \dots, A_n) \in \Pi_n$, $a \in \mathbb{R}$ and $\mathbf{m} = (m_1, \dots, m_n) \in \mathbb{R}^n$. If $a \neq 0$, then the random vector $(\mathbf{X} - \mathbf{m})/a$ has a categorical distribution with n categories and probability vector $(\mathbb{P}(A_1), \dots, \mathbb{P}(A_n))$.

Remark 5. Theorem 4 characterizes a Fréchet class that supports both counter-monotonicity and joint mix dependence. Fréchet classes that support (non-degenerate) pairwise counter-monotonicity are fully described by the conditions in Lemma 2. Whether a given Fréchet class supports joint mix dependence is a very challenging problem, with existing result summarized in Puccetti and Wang (2015) and Wang and Wang (2016). In risk aggregation problems, the notion of joint mix dependence is more relevant, because a joint mix usually "approximately exists" for large dimensions, which leads to the main idea behind the Rearrangement Algorithm; see Embrechts et al. (2013, 2014), Bernard and Vanduffel (2015) and Bernard et al. (2017). In contrast, counter-monotonicity is more relevant for risk sharing problems, which we discuss in the next section.

6. Optimal allocations in risk sharing for quantile agents

We now formally establish the link between counter-monotonicity and Pareto-optimal allocations in risk sharing problems for quantile agents.

We first describe the basic setting. A quantile agent assesses risk by its quantile, also known as the risk measure Value-at-Risk (VaR) in risk management. Following the convention of Embrechts et al. (2018), the VaR at level $\alpha \in (0, 1)$ is defined as

$$\mathrm{VaR}_\alpha(X) = \inf\{x \in \mathbb{R} : \mathbb{P}(X \leqslant x) \geqslant 1 - \alpha\}, \quad X \in \mathcal{X},$$

where $\mathcal{X}$ is the set of all random variables in the probability space. Moreover, write $\mathrm{VaR}_\alpha = -\infty$ on $\mathcal{X}$ for $\alpha \geqslant 1$, although our agents use VaR_α for $\alpha \in (0, 1)$. It is important to highlight that quantile agents with level $\alpha \in (0, 1)$ are not risk averse (Rothschild and Stiglitz (1970)).

We consider the risk sharing problem for $n \geqslant 3$ quantile agents with levels $\alpha_1, \dots, \alpha \in (0, 1)$. For a given $S \in \mathcal{X}$, the set of *allocations* of S is

$$\mathbb{A}_n(S) = \left\{(X_1, \dots, X_n) \in \mathcal{X}^n : \sum_{i=1}^n X_i = S\right\}.$$

An allocation $(X_1, \dots, X_n) \in \mathbb{A}_n(S)$ is *Pareto optimal* if for any $(Y_1, \dots, Y_n) \in \mathbb{A}_n(S)$ satisfying $\mathrm{VaR}_{\alpha_i}(Y_i) \leqslant \mathrm{VaR}_{\alpha_i}(X_i)$ for all $i \in [n]$, we have $\mathrm{VaR}_{\alpha_i}(Y_i) = \mathrm{VaR}_{\alpha_i}(X_i)$ for all $i \in [n]$. Pareto optimality of $(X_1, \dots, X_n) \in \mathbb{A}_n(S)$ is equivalent to

$$\sum_{i=1}^n \mathrm{VaR}_{\alpha_i}(X_i) = \inf\left\{\sum_{i=1}^n \mathrm{VaR}_{\alpha_i}(Y_i) : (Y_1, \dots, Y_n) \in \mathbb{A}_n(S)\right\} = \mathrm{VaR}_{\sum_{i=1}^n \alpha_i}(S), \tag{12}$$

where the first equality is Embrechts et al. (2018, Proposition 1) and the second equality is Embrechts et al. (2018, Corollary 2). Using (12), we obtain that the existence of a Pareto-optimal allocation is equivalent to $\sum_{i=1}^n \alpha_i < 1$; this is also given by Theorem 3.6 of Wang and Wei (2020). For this reason, we say that the n quantile agents are *compatible* if $\sum_{i=1}^n \alpha_i < 1$ holds, meaning that a Pareto-optimal allocation exists for some S, and equivalently, for every S.

The following theorem shows that, under some conditions of the total risk S to share, the risk sharing problem for any quantile agents admits a pairwise counter-monotonic Pareto-optimal allocation, and every pairwise counter-monotonic allocation is Pareto optimal for some agents. Moreover, comonotonic allocations are never Pareto optimal. Recall that by Lemma 2, a pairwise counter-monotonic random vector $(X_1, \dots, X_n)$ satisfies either (1) or (2).

Theorem 5. *For $S \in \mathcal{X}$, the following hold.*

(i) If S is bounded from below, then for any compatible quantile agents there exists a pairwise counter-monotonic allocation of S which is Pareto optimal.
(ii) If $\mathbb{P}(S = \text{ess-inf}S) > 0$, then every type-(1) pairwise counter-monotonic allocation of S is Pareto optimal for some quantile agents.
(iii) If S is continuously distributed, then a comonotonic allocation of S is never Pareto optimal for any quantile agents.

Proof. (i) Let $\alpha_1, \dots, \alpha_n \in (0, 1)$ be the VaR levels of the quantile agents. Compatibility of the agents means $\sum_{i=1}^n \alpha_i < 1$. In this case, a Pareto-optimal allocation $(X_1, \dots, X_n)$ of S is given by Theorem 2 of Embrechts et al. (2018), with the form

$$X_i = (X - m)\mathbb{1}_{A_i}, \ i \in [n-1] \ \text{ and } \ X_n = (X - m)\mathbb{1}_{A_n} + m$$

for some $(A_1, \dots, A_n) \in \Pi_n$. By setting $m = \text{ess-inf}S$, $(X_1, \dots, X_n)$ is pairwise counter-monotonic by Theorem 1.

(ii) Note that shifting $X_1, \dots, X_n$ by arbitrary constants, and adjusting S correspondingly, does not affect its Pareto optimality due to (12). Moreover, (1) guarantees that at most one of $X_1, \dots, X_n$ is not bounded from below, and further $\mathbb{P}(S = \text{ess-inf}S) > 0$ guarantees that this can only happen if all $X_1, \dots, X_n$ are bounded from below. Therefore, we can, without loss of generality, assume $\text{ess-inf}X_i = 0$ for each $i \in [n]$.

Let $B = \{S = \text{ess-inf}S\}$ and $A = \bigcup_{i=1}^n \{X_i > 0\}$. First, if $\mathbb{P}(B \cap A) = 0$, then we let $\alpha_i = \mathbb{P}(X_i > 0) + \mathbb{P}(B)/(2n) > 0$ for $i \in [n]$. Note that

$$\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \mathbb{P}(X_i > 0) + \frac{1}{2}\mathbb{P}(B) = \mathbb{P}(A) + \frac{1}{2}\mathbb{P}(B) < \mathbb{P}(A) + \mathbb{P}(B) = \mathbb{P}(A \cup B) \leqslant 1.$$

It is clear that $\mathrm{VaR}_{\alpha_i}(X_i) = 0$ for each $i \in [n]$, leading to $\sum_{i=1}^n \mathrm{VaR}_{\alpha_i}(X_i) = 0 \leqslant \text{ess-inf}S \leqslant \mathrm{VaR}_{\sum_{i=1}^n \alpha_i}(S)$. Note that

$$\sum_{i=1}^n \mathrm{VaR}_{\alpha_i}(X_i) \leqslant \mathrm{VaR}_{\sum_{i=1}^n \alpha_i}(S) \implies (X_1, \dots, X_n) \text{ is Pareto optimal.} \tag{13}$$

This is because Corollary 1 of Embrechts et al. (2018) gives $\sum_{i=1}^n \mathrm{VaR}_{\alpha_i}(X_i) \geqslant \mathrm{VaR}_{\sum_{i=1}^n \alpha_i}(S)$, and this leads to $\sum_{i=1}^n \mathrm{VaR}_{\alpha_i}(X_i) = \mathrm{VaR}_{\sum_{i=1}^n \alpha_i}(S)$ in (12), which gives Pareto optimality of $(X_1, \dots, X_n)$ as we see in part (i).

Next, assume $\mathbb{P}(B \cap A) > 0$. Then there exists $j \in [n]$ such that $\mathbb{P}(B \cap \{X_j > 0\}) > 0$. Let $\varepsilon = \mathbb{P}(B \cap \{X_j > 0\})/(2n)$. Take $\alpha_i = \mathbb{P}(X_i > 0) + \varepsilon > 0$ for $i \in [n] \setminus \{j\}$ and $\alpha_j = \mathbb{P}(\{X_j > 0\} \setminus B) + \varepsilon$. By Lemma 2,

$$1 \geqslant \sum_{i=1}^n \mathbb{P}(X_i>0) = \sum_{i=1}^n (\alpha_i - \varepsilon) + \mathbb{P}(B \cap \{X_j>0\}) = \sum_{i=1}^n \alpha_i + n\varepsilon,$$

and hence $\sum_{i=1}^n \alpha_i < 1$. By definition of $\alpha_1, \dots, \alpha_n$, we have $\mathrm{VaR}_{\alpha_i}(X_i) = 0$ for $i \in [n] \setminus \{j\}$. Moreover, note that $X_j = S$ on $\{X_j > 0\}$ and

$$\mathbb{P}(\{X_j = \text{ess-inf}S\} \cap \{X_j>0\}) = \mathbb{P}(B \cap \{X_j>0\}) = 2n\varepsilon,$$

which implies $\mathbb{P}(X_j > \text{ess-inf}S) = \mathbb{P}(X_j>0) - 2n\varepsilon < \alpha_j$. Therefore, $\mathrm{VaR}_{\alpha_j}(X_j) \leqslant \text{ess-inf}S$, leading to $\sum_{i=1}^n \mathrm{VaR}_{\alpha_i}(X_i) \leqslant \text{ess-inf}S \leqslant \mathrm{VaR}_{\sum_{i=1}^n \alpha_i}(S)$. Hence, we obtain Pareto optimality of $(X_1, \dots, X_n)$ via (13).

(iii) For a comonotonic allocation $(X_1, \dots, X_n)$ of S, using decreasing monotonicity of $\alpha \mapsto \mathrm{VaR}_\alpha$ and comonotonic additivity of VaR_α, we have

$$\sum_{i=1}^n \mathrm{VaR}_{\alpha_i}(X_i) \geqslant \sum_{i=1}^n \mathrm{VaR}_{\beta}(X_i) = \mathrm{VaR}_\beta(S),$$

where we write $\beta = \max\{\alpha_1, \dots, \alpha_n\}$. As S is continuously distributed, $\mathrm{VaR}_\alpha(S)$ is strictly decreasing in α. Noting that $\beta < \sum_{i=1}^n \alpha_i$, we have $\mathrm{VaR}_\beta(S) > \mathrm{VaR}_{\sum_{i=1}^n \alpha_i}(S)$. Therefore, the comonotonic allocation $(X_1, \dots, X_n)$ is not Pareto optimal by (12). □

Theorem 5 states that allocations with a pairwise counter-monotonic structure solve the problem of sharing risk among quantile agents. For instance, the lottery-type allocation in Example 2 is Pareto optimal for some quantile agents. Further, Theorem 5 (iii) states that comonotonic allocations can never be Pareto optimal for quantile agents if the total risk is continuously distributed. As mentioned, this is in stark contrast with the risk sharing problem with risk-averse agents, for which comonotonic allocations are always optimal. The latter result, due to the comonotonic improvements of Landsberger and Meilijson (1994), is well-known; see also Jouini et al. (2008) and Rüschendorf (2013). Moreover, when all agents are strictly risk averse, only comonotonic allocations are Pareto optimal (see Lauzier et al. (2023, Proposition 4) for the case when preferences are modeled by strictly concave distortion functions).

As a symmetric statement to Theorem 5, if a random vector $(X_1, \dots, X_n)$ is pairwise counter-monotonic of type (2), then it is the maximizer of a risk sharing problem for some quantile agents.

Theorem 5 (i) assumes that S is bounded from below. This is needed because any type-(1) pairwise counter-monotonic allocation is bounded from below. Theorem 5 (ii) assumes $\mathbb{P}(S = \text{ess-inf}S) > 0$. In case $\mathbb{P}(S > \text{ess-inf}S) = 0$, a pairwise counter-monotonic allocation of type (1) may not be Pareto optimal for any quantile agents with levels in $(0, 1)$. A counter-example is provided in Example 3 below. Theorem 5 (iii) assumes that S is continuously distributed. This condition is also needed for the result to hold. For instance, if $S = 1$, then the allocation $(1/n, \dots, 1/n)$ is Pareto optimal for any compatible quantile agents, violating the impossibility statement on Pareto optimality.

Example 3. Suppose that S is uniformly distributed on $[0, 1]$, and $X_i = S\mathbb{1}_{A_i}$ for $(A_1, \dots, A_n) \in \Pi_n$ independent of S with $\mathbb{P}(A_i) > 0$ for each $i \in [n]$. We will see that the pairwise counter-monotonic allocation $(X_1, \dots, X_n)$ is not Pareto optimal for any quantile agents with levels $\alpha_1, \dots, \alpha_n \in (0, 1)$. If $\sum_{i=1}^n \alpha_i \geqslant 1$, there does not exist any Pareto-optimal allocation. If $\sum_{i=1}^n \alpha_i < 1$, then

$$\sum_{i=1}^n \mathrm{VaR}_{\alpha_i}(X_i) = \sum_{i=1}^n \left(1 - \frac{\alpha_i}{\mathbb{P}(A_i)}\right)_+ = \sum_{i=1}^n \left(\frac{\mathbb{P}(A_i) - \alpha_i}{\mathbb{P}(A_i)}\right)_+$$

and

$$\mathrm{VaR}_{\sum_{i=1}^n \alpha_i}(S) = 1 - \sum_{i=1}^n \alpha_i = \sum_{i=1}^n (\mathbb{P}(A_i) - \alpha_i)$$
$$\leqslant \sum_{i=1}^n \left(\frac{\mathbb{P}(A_i) - \alpha_i}{\mathbb{P}(A_i)}\right)_+ = \sum_{i=1}^n \mathrm{VaR}_{\alpha_i}(X_i).$$

Using the condition (12), if $(X_1, \dots, X_n)$ is Pareto optimal, then the inequality above is an equality; this implies $\alpha_i = \mathbb{P}(A_i)$ for each $i \in [n]$. However, this further implies $\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \mathbb{P}(A_i) = 1$ conflicting $\sum_{i=1}^n \alpha_i < 1$.

The next example illustrates that for the same S in Example 3 and compatible quantile agents, a pairwise counter-monotonic Pareto-optimal allocation exists as implied by Theorem 5 (i).

Example 4. Let S be uniformly distributed on $[0, 1]$ and $\alpha_1, \dots, \alpha_n \in (0, 1)$ with $\sum_{i=1}^n \alpha_i < 1$. Take $(A_1, \dots, A_n) \in \Pi_n$ such that $\bigcup_{i=1}^{n-1} A_i = \{S \geqslant 1 - \sum_{i=1}^{n-1} \alpha_i\}$ and $\mathbb{P}(A_i) = \alpha_i$ for $i \in [n-1]$. Let $X_i = S\mathbb{1}_{A_i}$ for $i \in [n]$. We can verify that $\mathrm{VaR}_{\alpha_i}(X_i) = 0$ for $i \in [n-1]$ and

$$\mathrm{VaR}_{\alpha_n}(X_n) = \mathrm{VaR}_{\alpha_n}\left(S\mathbb{1}_{\{S < 1 - \sum_{i=1}^{n-1} \alpha_i\}}\right)$$
$$= 1 - \sum_{i=1}^n \alpha_i = \mathrm{VaR}_{\sum_{i=1}^n \alpha_i}(S).$$

This shows that $(X_1, \dots, X_n)$ is Pareto optimal. It is also pairwise counter-monotonic by Theorem 1. Note that although the allocation $(X_1, \dots, X_n)$ here has the same form $(S\mathbb{1}_{A_1}, \dots, S\mathbb{1}_{A_n})$ as the one in Example 3, the specification of $(A_1, \dots, A_n)$ is different in the two examples, leading to opposite conclusions on optimality.

Remark 6. One may notice that the condition on S in Theorem 5 part (ii) and that in part (iii), although both quite weak, are actually conflicting. This is not a coincidence, because comonotonicity and counter-monotonicity have a non-empty intersection: A random vector is both comonotonic and counter-monotonic if and only if it has at most one non-degenerate component. Therefore, we cannot have both conclusions in parts (ii) and (iii) for the same S.

Remark 7. As shown by Embrechts et al. (2018), the same pairwise counter-monotonic allocation which is Pareto optimal for quantile agents is also optimal for the more general Range Value-at-Risk (RVaR) agents. Therefore, the conclusion in Theorem 5 also applies to the RVaR agents. Another appearance of pairwise counter-monotonicity in optimal allocations is obtained by Lauzier et al. (2023), where it is shown that for agents using inter-quantile differences, a Pareto-optimal allocation is the sum of two pairwise counter-monotonic random vectors. All discussions above assume homogeneous beliefs; that is, all agents use the same probability measure $\mathbb{P}$. In the setting of heterogeneous beliefs, Embrechts et al. (2020) showed that for Expected Shortfall agents, a Pareto-optimal allocation above certain constant level also has a pairwise counter-monotonic structure; see their Proposition 3. Generally, agents using the dual utility model of Yaari (1987), including the quantile-based models above, have quite different features in risk sharing and other optimization problems compared to those with expected utility agents. For the optimal payoff of Yaari agents in portfolio choice, see Boudt et al. (2022).

Example 5. We illustrate that counter-monotonicity may also be the structure of an optimal allocation outside the dual utility of

Yaari (1987). Let $(\Omega, \mathcal{A}, \mathbb{P})$ be an atomless probability space, $S = 1$ and $\alpha > 0$. Consider the problem

$$\text{to maximize} \quad \sum_{i=1}^{n} \mathbb{E}\left[\alpha \mathbb{1}_{\{X_i \geqslant 1\}}\right]$$

subject to $(X_1, \dots, X_n) \in \mathbb{A}_n(S)$ and $X_i \geqslant 0$, for $i \in [n]$.

It is straightforward to verify that the set of maximizers is

$$\mathbb{A}^* = \left\{(\mathbb{1}_{A_1}, \dots, \mathbb{1}_{A_n}) \in \mathbb{A}_n(S) : (A_1, \dots, A_n) \in \Pi_n\right\},$$

which contains only counter-monotonic allocations. This problem can be interpreted as the problem of sharing $S = 1$ among n expected utility maximizers with common utility function $u(x) = \alpha \mathbb{1}_{\{x \geqslant 1\}}$ for $\alpha > 0$. The optimization problem is thus a social planner's problem, and the set $\mathbb{A}^*$ contains all Pareto-optimal allocations for this problem. The allocations satisfying $\mathbb{P}(A_i) = \mathbb{P}(A_j)$ for every $i \neq j$ are of particular interest, as they are common in auction theory as the *random tie-breaking rule*. The variable S can be understood as an indivisible good that was auctioned, and the parameter α as the net utility of a series of n agents with quasi-linear utilities $v(X, t) = \theta X - t$ having bid the same amount $0 \leqslant t < \theta$. It is straightforward to see that these allocations are the only *fair* allocations, in the sense that all agents have the same expected utility. In other words, a fair lottery (which is counter-monotonic) is the only fair way to distribute the indivisible good among people who value it equally.

7. Conclusion

We provide a series of technical results on the representation (Theorem 1) and invariance property (Theorem 2) of pairwise counter-monotonicity, as well as their connection to negative association (Theorem 3), joint mix dependence (Theorem 4), and optimal allocations for quantile agents (Theorem 5). Our paper is motivated by the recently increasing attention in counter-monotonicity and negative dependence, and it fills the gap between the relatively scarce studies on pairwise counter-monotonicity in the literature and the wide appearance of this dependence structure in modern applications, in particular, in risk sharing problems with agents that are not using expected utilities.

In general, studies of negative dependence and positive dependence are highly asymmetric in nature, with negative dependence being more challenging to study in various applications of risk management and statistics. In addition to the negative dependence concepts we considered in this paper, some other notions have been studied in the recent literature, and the interested reader is referred to Amini et al. (2013), Lee and Ahn (2014), Lee et al. (2017) and Chi et al. (2022), as well as the monographs of Joe (1997, 2014).

Declaration of competing interest

None.

Data availability

No data was used for the research described in the article.

Acknowledgements

We thank the handling editor and two anonymous referees for helpful comments on a previous version of the paper. Ruodu Wang acknowledges support from the Canada Research Chairs (CRC-2022-00141) and the Natural Sciences and Engineering Research Council of Canada (NSERC Grant No. RGPIN-2018-03823).

References

Alam, K., Saxena, K.M.L., 1981. Positive dependence in multivariate distributions. Communications in Statistics - Theory and Methods 10 (12), 1183–1196.
Amini, M., Nili Sani, H.R., Bozorgnia, A., 2013. Aspects of negative dependence structures. Communications in Statistics - Theory and Methods 42 (5), 907–917.
Bernard, C., Jiang, X., Wang, R., 2014. Risk aggregation with dependence uncertainty. Insurance: Mathematics and Economics 54, 93–108.
Bernard, C., Rüschendorf, L., Vanduffel, S., 2017. VaR bounds with variance constraint. The Journal of Risk and Insurance 84 (3), 923–959.
Bernard, C., Vanduffel, S., 2015. A new approach to assessing model risk in high dimensions. Journal of Banking & Finance 58, 166–178.
Block, H.W., Savits, T.H., Shaked, M., 1982. Some concepts of negative dependence. Annals of Probability 10 (3), 765–772.
Block, H.W., Savits, T.H., Shaked, M., 1985. A concept of negative dependence using stochastic ordering. Statistics & Probability Letters 3 (2), 81–86.
Boudt, K., Dragun, K., Vanduffel, S., 2022. The optimal payoff for a Yaari investor. Quantitative Finance 22 (10), 1839–1852.
Carlier, G., Dana, R.A., 2003. Pareto efficient insurance contracts when the insurer's cost function is discontinuous. Economic Theory 21 (4), 871–893.
Chaoubi, I., Cossette, H., Gadoury, S.-P., Marceau, E., 2020. On sums of two counter-monotonic risks. Insurance: Mathematics and Economics 92, 47–60.
Cheung, K.C., Denuit, M., Dhaene, J., 2017. Tail mutual exclusivity and Tail-VaR lower bounds. Scandinavian Actuarial Journal 2017 (1), 88–104.
Cheung, K.C., Dhaene, J., Lo, A., Tang, Q., 2014. Reducing risk by merging counter-monotonic risks. Insurance: Mathematics and Economics 54, 58–65.
Cheung, K.C., Lo, A., 2014. Characterizing mutual exclusivity as the strongest negative multivariate dependence structure. Insurance: Mathematics and Economics 55, 180–190.
Chi, Z., Ramdas, A., Wang, R., 2022. Multiple testing under negative dependence. arXiv:2212.09706.
Christofides, T.C., Vaggelatou, E., 2004. A connection between supermodular ordering and positive/negative association. Journal of Multivariate Analysis 88 (1), 138–151.
Dall'Aglio, G., 1972. Fréchet classes and compatibility of distribution functions. Symposia Mathematica 9, 131–150.
Denneberg, D., 1994. Non-additive Measure and Integral. Springer Science & Business Media.
Dhaene, J., Denuit, M., 1999. The safest dependence structure among risks. Insurance: Mathematics and Economics 25 (1), 11–21.
Dhaene, J., Denuit, M., Goovaerts, M.J., Kaas, R., Vynche, D., 2002. The concept of comonotonicity in actuarial science and finance: theory. Insurance: Mathematics and Economics 31 (1), 3–33.
Dhaene, J., Vanduffel, S., Goovaerts, M.J., Kaas, R., Tang, Q., Vynche, D., 2006. Risk measures and comonotonicity: a review. Stochastic Models 22, 573–606.
Embrechts, P., Liu, H., Wang, R., 2018. Quantile-based risk sharing. Operations Research 66 (4), 936–949.
Embrechts, P., Liu, H., Mao, T., Wang, R., 2020. Quantile-based risk sharing with heterogeneous beliefs. Mathematical Programming Series B 181 (2), 319–347.
Embrechts, P., Puccetti, G., Rüschendorf, L., 2013. Model uncertainty and VaR aggregation. Journal of Banking & Finance 37 (8), 2750–2764.
Embrechts, P., Puccetti, G., Rüschendorf, L., Wang, R., Beleraj, A., 2014. An academic response to Basel 3.5. Risks 2 (1), 25–48.
Embrechts, P., Wang, B., Wang, R., 2015. Aggregation-robustness and model uncertainty of regulatory risk measures. Finance and Stochastics 19 (4), 763–790.
Esary, J.D., Proschan, F., Walkup, D.W., 1967. Association of random variables, with applications. The Annals of Mathematical Statistics 38 (5), 1466–1474.
Föllmer, H., Schied, A., 2016. Stochastic Finance. An Introduction in Discrete Time, fourth edition. Walter de Gruyter, Berlin.
Hardy, G.H., Littlewood, J.E., Pólya, G., 1934. Inequalities. Cambridge University Press.
Hu, T., Wu, Z., 1999. On dependence of risks and stop-loss premiums. Insurance: Mathematics and Economics 24 (3), 323–332.
Hu, T., 2000. Negatively superadditive dependence of random variables with applications. Chinese Journal of Applied Probability and Statistics 16 (2), 133–144.
Huberman, G., Mayers, D., Smith Jr, C.W., 1983. Optimal insurance policy indemnity schedules. Bell Journal of Economics 14 (2), 415–426.
Joag-Dev, K., Proschan, F., 1983. Negative association of random variables with applications. The Annals of Statistics 11, 286–295.
Joe, H., 1997. Multivariate Models and Dependence Concepts. Chapman & Hall, London.
Joe, H., 2014. Dependence Modeling with Copulas. Chapman & Hall, London.
Jouini, E., Schachermayer, W., Touzi, N., 2008. Optimal risk sharing for law invariant monetary utility functions. Mathematical Finance 18 (2), 269–292.
Kusuoka, S., 2001. On law invariant coherent risk measures. Advances in Mathematical Economics 3, 83–95.
Landsberger, M., Meilijson, I., 1994. Co-monotone allocations, Bickel-Lehmann dispersion and the Arrow-Pratt measure of risk aversion. Annals of Operations Research 52 (2), 97–106.
Lauzier, J.G., Lin, L., Wang, R., 2023. Risk sharing, measuring variability, and distortion riskmetrics. arXiv:2302.04034.

Lee, W., Ahn, J.Y., 2014. On the multidimensional extension of countermonotonicity and its applications. Insurance: Mathematics and Economics 56, 68–79.
Lee, W., Cheung, K.C., Ahn, J.Y., 2017. Multivariate countermonotonicity and the minimal copulas. Journal of Computational and Applied Mathematics 317, 589–602.
Leshno, J.D., Strack, P., 2020. Bitcoin: an axiomatic approach and an impossibility theorem. American Economic Review: Insights 2 (3), 269–286.
Liu, F., Mao, T., Wang, R., Wei, L., 2022. Inf-convolution, optimal allocations, and model uncertainty for tail risk measures. Mathematics of Operations Research 47 (3), 2494–2519.
Mao, T., Wang, R., 2020. Risk aversion in regulatory capital calculation. SIAM Journal on Financial Mathematics 11 (1), 169–200.
McNeil, A.J., Frey, R., Embrechts, P., 2015. Quantitative Risk Management: Concepts, Techniques and Tools, revised edition. Princeton University Press, Princeton, NJ.
Puccetti, G., Wang, R., 2015. Extremal dependence concepts. Statistical Science 30 (4), 485–517.
Rothschild, M., Stiglitz, J.E., 1970. Increasing risk: I. A definition. Journal of Economic Theory 2 (3), 225–243.
Rüschendorf, L., 2013. Mathematical Risk Analysis. Dependence, Risk Bounds, Optimal Allocations and Portfolios. Springer, Heidelberg.
Schmeidler, D., 1989. Subjective probability and expected utility without additivity. Econometrica 57 (3), 571–587.
Wang, B., Wang, R., 2011. The complete mixability and convex minimization problems with monotone marginal densities. Journal of Multivariate Analysis 102 (10), 1344–1360.
Wang, B., Wang, R., 2016. Joint mixability. Mathematics of Operations Research 41 (3), 808–826.
Wang, R., Peng, L., Yang, J., 2013. Bounds for the sum of dependent risks and worst value-at-risk with monotone marginal densities. Finance and Stochastics 17 (2), 395–417.
Wang, R., Wei, Y., 2020. Characterizing optimal allocations in quantile-based risk sharing. Insurance: Mathematics and Economics 93, 288–300.
Wang, S., Young, V.R., Panjer, H.H., 1997. Axiomatic characterization of insurance prices. Insurance: Mathematics and Economics 21 (2), 173–183.
Weber, S., 2018. Solvency II, or how to sweep the downside risk under the carpet. Insurance: Mathematics and Economics 82, 191–200.
Xia, Z., Zou, Z., Hu, T., 2023. Inf-convolution and optimal allocations for mixed-VaRs. Insurance: Mathematics and Economics 108, 156–164.
Yaari, M.E., 1987. The dual theory of choice under risk. Econometrica 55 (1), 95–115.