Insurance: Mathematics and Economics 113 (2023) 1–23

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Optimal risk sharing and dividend strategies under default contagion: A semi-analytical approach

Ming Qiu [a], Zhuo Jin [b,*], Shuanming Li [a]

[a] Centre for Actuarial Studies, Department of Economics, The University of Melbourne, VIC 3010, Australia
[b] Department of Actuarial Studies and Business Analytics, Macquarie University, NSW 2109, Australia

ARTICLE INFO

Article history:
Received July 2022
Received in revised form July 2023
Accepted 16 July 2023
Available online 1 August 2023

JEL classification:
C44
C61
G22

Keywords:
Optimal dividends
Risk sharing
Semi-analytical approach
Systemic risk
Default
Contagion

ABSTRACT

We investigate the risk control and dividend optimization problem of an insurance group in a general setting and propose an innovative semi-analytical approach to the problem. The group consists of multiple subsidiaries and is subject to exogenous default risk. The default intensity is subject to the contagious effect. The contagious effect refers to the increase in default intensities of surviving subsidiaries within the group when a default event occurs. The recursive system of Hamilton-Jacobi-Bellman variational inequalities (HJBVIs) is derived together with the verification theorem. We propose a semi-analytical approach that first finds the analytical solution in the continuation region and then the numerical solution in the risk exposure region. We further present a numerical example of a three-subsidiary insurance group to demonstrate the semi-analytical method and illustrate the recursive computation procedures that are extendible to cases with more subsidiaries.
© 2023 The Author(s). Published by Elsevier B.V. This is an open access article under the CC BY-NC-ND license (http://creativecommons.org/licenses/by-nc-nd/4.0/).

1. Introduction

The incorporation of systemic risk into the financial market modeling has demonstrated great importance since the 2007-2008 global financial crisis. Traditional measures of systemic risk included the conditional Value-at-Risk and conditional Expected Shortfall. Dhaene et al. (2022) introduced the classes of conditional distortion risk measures to quantify systemic risk. Brechmann et al. (2013) employed conditional copula simulations to illustrate the systemic risk among financial institutions and analyzed the contagion effects embedded. Mean-field model serves as another tool to study the interconnectedness among a large number of entities. The interactions among the entities are approximated by the average interactions; see Garnier et al. (2013), Carmona et al. (2015), and Hambly and Søjmark (2019) for details. Network-based model is another frequently used tool to describe a system with multiple dependent entities. For instance, the risk sharing structure among multiple insurance companies and reinsurance companies is investigated by establishing a reinsurance network in Lin et al. (2015) and Ettlin et al. (2020). Tang et al. (2022) proposed a static structural network model of three components, and the impact of a shock on the entities within the network was studied. Another way to describe the systemic risk is the intensity-based model, where the default intensities are dependent on the system's default state. In detail, the system is subject to the reduced-form default risk where the entities in the system are vulnerable to exogenous default events; see Jarrow and Turnbull (1995), Liang and Wang (2012), and Ballestra and Pacelli (2014). The contagion effect explains the increase of default intensities of the surviving entities when default events occur. In this paper, we aim to study the optimal decision-making problem of an insurance group subject to systemic risk by using the last approach.

The optimal decision-making problem on insurance companies has been extensively studied in the literature, mostly in a single insurance company. Since De Finetti (1957) formulated the optimal dividend problem on a random walk surplus process, the introduction

* Corresponding author.
E-mail addresses: ming.qiu@unimelb.edu.au (M. Qiu), zhuo.jin@mq.edu.au (Z. Jin), shli@unimelb.edu.au (S. Li).

https://doi.org/10.1016/j.insmatheco.2023.07.002
0167-6687/© 2023 The Author(s). Published by Elsevier B.V. This is an open access article under the CC BY-NC-ND license (http://creativecommons.org/licenses/by-nc-nd/4.0/).

of the diffusion approximation model enabled the application of stochastic control theory in the dividend optimization problem and the inclusion of other controls (Jeanblanc-Picqué and Shiryaev, 1995). Examples include controlling the risk exposure, the equity issuance, and the investment. Typically, optimization problems that control the risk exposure and dividend payout scheme simultaneously were investigated in the literature; see Irgens and Paulsen (2004), Taksar and Hunderup (2007), Jin et al. (2012), Jin et al. (2015), Feng et al. (2021), and references therein. Such problems are classified as mixed regular-singular stochastic control problems. By the dynamic programming principle, the variational inequalities are derived, and their solutions coincide with the solutions of the control problems (Fleming and Rishel, 1975).

As more sophisticated models are used to better describe the dynamics of surplus, closed-form solutions are sometimes not available for optimization problems. One viable alternative is referring to numerical solutions. For instance, Jin et al. (2021b) studied an optimal risk control, dividend, and investment problem associated with the jump-diffusion regime-switching model and developed a hybrid deep learning approach to the problem. However, obtaining an accurate numerical solution for a complex stochastic system is not easy, particularly high dimensional problems lead to "curse of dimensionality". Hence, when the solutions are partially available in closed-form, the semi-analytically approach can be utilized and shows its advantage in computation efficiency and accuracy. Lin et al. (2016) designed a semi-analytic algorithm to calculate the mean and variance of the move-based hedging cost. Besides, the semi-closed form of the option price could be written in the form of an integral that was dependent on a particular function, and it could only be solved numerically (Carr and Itkin, 2021). He et al. (2022) obtained a semi-analytical value function for an optimal asset allocation, consumption, and retirement time problem where the optimal retirement time was solved numerically. Due to the complexity of the problem we study in this paper, the explicit form solution is hardly available, and a semi-analytical approach is utilized alternatively.

In this paper, we establish a semi-analytical approach to the risk exposure-dividend optimization problem of an insurance group with multiple subsidiaries subject to contagious external default risk. We characterize the interconnectedness by modeling the group as a system subject to the reduced-form default with contagious intensities. Following a similar methodology in Bo and Capponi (2016) and Bo et al. (2019), we formulate a system of recursive HJBVIs and then show that the optimal strategies of the insurance group can be obtained by solving the system. For a particular HJBVI, we formulate a group of candidate functions and find the candidate function that coincides with the solution to the HJBVI. The solution obtained in the risk control region is expressed in terms of a differential equation. The solution obtained in the continuation region and dividend payments region are available analytically. We then propose a numerical algorithm that calculates the solution in the risk control region by transforming it into an initial value problem based on the closed-form solution in the continuation region. The system of HJBVIs is solved recursively from the base case, and each HJBVI is solved semi-analytically by the numerical algorithm.

The main contribution of our work is that we study a mixed regular-singular control problem in a very general setting and develop an innovative semi-analytical approach that solves the resulting system of HJBVIs of great complexity. To the best of our knowledge, the proposed work is the most generalized model to study the dividend and risk sharing for multiple subsidiaries within an insurance group. The general setting means that the assumption placed on the reinsurance strategy is alleviated compared to Qiu et al. (2022). The removal of the constraint on the regular control introduces significant non-linearity into each HJBVI within the system and considerably increases the complexity of the optimization problem. In addition, the non-trivial inhomogeneous terms introduced to the system of HJBVIs due to the reduced-form exogenous default risk result in a more intricate problem where a full explicit solution is generally unavailable. With the semi-analytical approach, we establish the numerical solution in the risk exposure region based on the explicit solution in the continuation and dividend payments regions, enhancing the computational efficiency and accuracy compared with a full numerical solution of the recursive HJBVI system. From the numerical demonstration of a three-subsidiary group, we observe that the optimal value function, the threshold for maximum risk level, and the optimal barrier all decrease when a default event occurs. In an economic view, the occurrence of a default event means the whole system becomes observably contagious. It is then optimal for the surviving subsidiaries to take the maximum level of risk sooner and make dividend payments sooner. In this way, the subsidiaries can bet on the increase of their reserves and make more dividend payouts before defaults.

The paper is organized in the following structure. In Section 2, we introduce the notation and formulate the problem. We later define a recursive system of variational inequalities and prove that the solutions of the variational inequalities coincide with the optimal value functions in Section 3. In Section 4, we first formulate the group of candidate functions for the solution of the variational inequality and derive the condition where the candidate function solves the variational inequality. Next, we propose a numerical algorithm that finds the semi-closed solution of the variational inequality and demonstrate how to apply the algorithm to a three-layer recursive system of variational inequalities. In Section 5, we provide the numerical results of the semi-closed solution of the three-layer recursive system discussed in the previous section. In Section 6, we summarize the results and offer more concluding remarks.

2. Model formulation

We consider an insurance group consisting of $N \geq 2$ subsidiaries. Let $(\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P})$ be a given complete filtered probability space. The filtration $\mathbb{G} := (\mathcal{G}_t)_{t\geq 0}$ satisfies the usual conditions. The filtration $\mathbb{F} := (\mathcal{F}_t)_{t\geq 0}$ is generated by an $\mathbb{R}^N$-valued process $\mathbf{W} = (W_1(t), \ldots, W_N(t))_{t\geq 0}$, where $W_i(t)$ is a standard Brownian motion for $1 \leq i \leq N$ and $N \geq 2$. The process $\mathbf{W}$ also satisfies that $\langle W_i, W_j \rangle_t = \rho_{ij} t$ for all $1 \leq i, j \leq N$ and for any $t \geq 0$ with the correlation coefficient $0 \leq |\rho_{ij}| \leq 1$. Let us define $\mathbf{Z} := (Z_1(t), \ldots, Z_N(t))_{t\geq 0}$ on the state space $\mathcal{S} := \{0, 1\}^N$ where $\mathbf{Z}$ is independent of $\mathbf{W}$ and generates the filtration $\mathbb{H} := (\mathcal{H}_t)_{t\geq 0}$ with $\mathcal{H}_t = \bigvee_{j=1}^N \sigma(Z_j(s), 0 \leq s \leq t)$. The global filtration $\mathbb{G}$ further satisfies $\mathbb{G} = \mathbb{F} \vee \mathbb{H}$.

2.1. Default indicator process

We model the occurrence of unexpected default events within the insurance group using the default indicator process $\mathbf{Z}$ that is defined before. In particular, these unexpected default events include sudden termination of the business and unanticipated financial distress due to changes in external factors, which are not limited to public policies, business cycles, and macroeconomic factors (Ballestra and Pacelli, 2014). For instance, we see empirical analysis of default within business groups (Beaver et al., 2019) capturing the dependency between macroeconomic fluctuations and the default events.

The default indicator process $\mathbf{Z} = (Z_1(t), \ldots, Z_N(t))_{t\geq 0}$ defined on $\mathcal{S}$ is assumed to be a continuous-time Markov chain, where $Z_i(t)$ denotes the default state of the ith subsidiary. Furthermore, $Z_i(t) = 1$ indicates that the ith subsidiary has defaulted by time t, and $Z_i(t) = 0$ otherwise. For $1 \leq i \leq N$, the default time of the ith subsidiary is then defined by

$$\nu_i := \inf\{t \geq 0 : Z_i(t) = 1\}.$$

Given that the ith subsidiary is alive by time t, the default indicator process $\mathbf{Z}$ is at state $\mathbf{Z}(t) = (Z_1(t), \ldots, Z_{i-1}(t), 0, Z_{i+1}(t), \ldots, Z_N(t))$ and could jump to the neighboring state $\mathbf{Z}^i(t) := (Z_1(t), \ldots, Z_{i-1}(t), 1, Z_{i+1}(t), \ldots, Z_N(t))$ at the stochastic rate of $\mathbb{1}_{\{Z_i(t)=0\}}\lambda_i(\mathbf{Z}(t))$ where $\lambda_i : \mathcal{S} \to \mathbb{R}_+$ for $1 \leq i \leq N$. In this paper, we assume that default events do not occur simultaneously, that is, $\Delta Z_i(t)\Delta Z_j(t) = 0$ for $1 \leq i, j \leq N$ where $\Delta Z_i(t) := Z_i(t) - Z_i(t-)$. Therefore, the default intensity of the ith subsidiary is dependent on the current default state and may change if any other subsidiary defaults (contagion effect). It can be shown that

$$M_i(t) := Z_i(t) - \int_0^{t\wedge\nu_i} \lambda_i(\mathbf{Z}(s))\, ds$$

is an $\mathbb{H}$-martingale for $1 \leq i \leq N$ and thus a $\mathbb{G}$-martingale (Bielecki and Rutkowski, 2004, section 5.1.4). Since $\mathbf{Z}$ is assumed to be independent of $\mathbf{W}$, it follows easily that $\mathbb{F}$ and $\mathbb{G}$ satisfy the condition (M.2a) specified in Bielecki and Rutkowski (2004, section 6.1.1), and thus $\mathbb{F}$ has the martingale invariance property with respect to $\mathbb{G}$. Therefore, we deduce that $\mathbf{W}$ is also a $\mathbb{G}$-Brownian motion. The construction of the global filtration $\mathbb{G}$ is similar to Bo et al. (2019). Due to the contagion effect, we assume that the default intensities of the surviving subsidiaries within the group increase if a default event occurs. The default indicator process subject to contagion effect, where the default intensities are dependent on the group's default state, is defined in a similar way and studied in the literature, see Frey and Backhaus (2008), Frey and Runggaldier (2010), Bo and Capponi (2017) and Birge et al. (2018).

2.2. Controlled surplus process

The reserve of each subsidiary is denoted by a diffusion process, which is given by $dR_i(t) = a_i dt - b_i dW_i(t)$ for $1 \leq i \leq N$ where a_i and b_i are positive constants. The diffusion approximation originates from the classical Cramér-Lundberg model, in which ruin-related problems have been studied extensively in the literature; see, for instance, Willmot and Lin (1998) and Lin and Willmot (2000). This approximation model has been widely used in actuarial mathematics; see Grandell (1991), Asmussen and Taksar (1997), Gerber and Shiu (2004), and references therein. To control the risk exposure and dividend distribution, we define the pair of $\mathbb{G}$-adapted processes $(\mathbf{P}, \mathbf{D})$ that describes the risk exposure and cumulative dividend payment of the insurance group. In particular, $\mathbf{P} = (p_1(t), \ldots, p_N(t))_{t\geq 0}$ and $\mathbf{D} = (D_1(t), \ldots, D_N(t))_{t\geq 0}$. For subsidiary i, the risk exposure at time t is $0 \leq p_i(t) \leq 1$ and the resulting surplus process satisfies $d\hat{X}_i(t) = p_i(t)a_i dt - p_i(t)b_i dW_i(t)$. The cumulative dividend payment $D_i(t)$ can be decomposed into the jump component $\Delta D_i(t) := D_i(t) - D_i(t-)$ and the continuous component $D_i^c(t) := D_i(t) - \sum_{0\leq s\leq t} \Delta D_i(s)$. It is natural to assume that no dividend is paid out if a default occurs. We first define $\tilde{X}_i(t) := (1 - Z_i(t))\hat{X}_i(t)$ for $1 \leq i \leq N$ as the surplus of subsidiary i is subject to external default risk. Then the surplus dynamic of subsidiary i with risk exposure and dividend controlled is described by

$$X_i(t) := (1 - Z_i(t))\left(\tilde{X}_i(t) - D_i(t)\right), \quad X_i(0) = x_i \geq 0, \quad 1 \leq i \leq N, \tag{2.1}$$

where x_i is the initial surplus. We denote the ruin time of subsidiary i by

$$\tau_i := \inf\{t \geq 0 : X_i(t) \leq 0\}, \quad 1 \leq i \leq N. \tag{2.2}$$

The class of admissible controls is defined as below.

Definition 2.1. The control pair $(\mathbf{P}, \mathbf{D})$ is admissible if it is a pair of $\mathbb{G}$-adapted processes taking values in $[0, 1]^N \times [0, \infty)^N$ with $\mathbf{D}$ being non-decreasing, càdlàg and satisfying $\mathbf{D}(0-) = \mathbf{0}$. Furthermore, the pair $(\mathbf{P}, \mathbf{D})$ is required to satisfy $\Delta D_i(t)\Delta Z_i(t) = 0$, $\Delta D_i(t) \leq X_i(t-)$ and $D_i(t) = D_i(t \wedge \tau_i)$ with $t \geq 0$ for $1 \leq i \leq N$. The set of all admissible controls is denoted by $\mathcal{U}$.

For the insurance group, we denote its surplus process by $\mathbf{X} = (X_1(t), \ldots, X_N(t))_{t\geq 0}$. Then the initial surplus is $\mathbf{X}(0) = \mathbf{x} = (x_1, \ldots, x_N) \in \mathcal{X} := [0, \infty)^N$. Also, the initial default state is given by $\mathbf{Z}(0) = \mathbf{z} = (z_1, \ldots, z_N) \in \mathcal{S}$. Given the surplus state $\mathbf{x}$ and default state $\mathbf{z}$, let $\mathbf{x}^{(l)}$ and $\mathbf{z}^l$ denote the surplus and default state of the system when subsidiary l defaults suddenly, i.e., $\mathbf{x}^{(l)} := (x_1, \cdots, x_{l-1}, 0, x_{l+1}, \cdots, x_N)$ and $\mathbf{z}^l := (z_1, \cdots, z_{l-1}, 1 - z_l, z_{l+1}, \cdots, z_N)$.

Remark 2.2. It is worth noting that we could also consider a stochastic factor describing the dynamics of macroeconomic environment in the current default contagion model. We see in literature of optimization problems that the stochastic factor corresponds to some continuous indicator of macroeconomics (Capponi and Frei, 2017; Bo and Capponi, 2018; Birge et al., 2018) or different states of macroeconomics (Bo and Capponi, 2016; Cheng et al., 2020). For both cases, the formulated HJBVIs and the optimal value functions are dependent on the dynamic of the stochastic factor, where the dimension of the HJBVIs is increased compared to the problems without the stochastic factor. In a similar vein to Bo and Capponi (2016); Cheng et al. (2020), we could use a continuous Markov chain to describe the dynamic of macroeconomics and construct a regime-switching model for the surplus of the insurance group. Therefore, the optimal risk sharing and dividend problem is also dependent on the state of the macroeconomics. The dimension of the existing recursive system of HJBVIs is increased due to the additional stochastic factor, and we see the HJBVIs are dependent on the optimal value functions associated with other initial states of the Markov chain and the transition probabilities between these states. However, the increased dimension of the recursive system results in greater complexity of solving the optimization problems, where the method of Markov chain approximation is demonstrated in Cheng et al. (2020) and could be implemented to our default contagion model accordingly.

2.3. Dividend optimization problem

We consider the optimization problem that the insurance group aims to maximize the sum of expected discounted dividend payments of all the subsidiaries before ruin. The aggregate expectation of discounted dividend payments prior to ruin is denoted by

$$J(\mathbf{x},\mathbf{z},\mathbf{P}(\cdot),\mathbf{D}(\cdot))=\mathbb{E}\left(\sum_{i=1}^{N}\int_0^{\tau_i}e^{-rt}dD_i(t)\right),\tag{2.3}$$

where r is the discount rate. By controlling the risk exposure and dividend payment strategy, we can maximize $J(\mathbf{x},\mathbf{z},\mathbf{P}(\cdot),\mathbf{D}(\cdot))$ and find the optimal $\mathbf{P}^*$ and $\mathbf{D}^*$. In other words, we aim to find the optimal value function that satisfies

$$f(\mathbf{x},\mathbf{z})=\sup_{(\mathbf{P},\mathbf{D})\in\mathcal{U}}J(\mathbf{x},\mathbf{z},\mathbf{P},\mathbf{D})=J\left(\mathbf{x},\mathbf{z},\mathbf{P}^*,\mathbf{D}^*\right).\tag{2.4}$$

3. Verification theorem

In this section, we first present the verification theorem following the similar idea as Bensoussana et al. (2014) but in a recursive form. Let us denote the number of alive subsidiaries by m. For any $1\le m\le N$, we assume that $\{j_1,\dots,j_m\}\subseteq\{1,\dots,N\}$ with $z_{j_l}=0$ for $1\le l\le m$ and $\{j_{m+1},\dots,j_N\}\subseteq\{1,\dots,N\}$ with $z_{j_l}=1$ for $m+1\le l\le N$. Without loss of generality, let us assume $\{j_1,\dots,j_m\}=\{1,\dots,m\}$ and $\{j_{m+1},\dots,j_N\}=\{m+1,\dots,N\}$.

Theorem 3.1. *For any $1\le m\le N$, let $\mathcal{W}_{\mathbf{z},h}:[0,\infty)\to[0,\infty)$ be of C^2 and concave for $1\le h\le N$. Further it satisfies*

$$\max\left\{\max_{0\le p_h\le 1}\left\{\mathcal{A}^{p_h,\mathbf{z},h}\mathcal{W}_{\mathbf{z},h}(x)+\sum_{l=1,l\neq h}^{m}\lambda_l(\mathbf{z})\mathcal{W}_{\mathbf{z}^l,h}(x)\right\},1-\mathcal{W}'_{\mathbf{z},h}(x)\right\}=0,\tag{3.1}$$

and $\mathcal{W}_{\mathbf{z},h}(0)=0$ where $\mathcal{A}^{p_h,\mathbf{z},h}\mathcal{W}_{\mathbf{z},h}(x):=-(r+\sum_{k=1}^m\lambda_k(\mathbf{z}))\mathcal{W}_{\mathbf{z},h}(x)+a_hp_h\mathcal{W}'_{\mathbf{z},h}(x)+\frac{1}{2}b_h^2p_h^2\mathcal{W}''_{\mathbf{z},h}(x)$ for $1\le h\le m$ and $\mathcal{W}_{\mathbf{z},h}(x)\equiv 0$ for $m+1\le h\le N$. For $1\le h\le m$, define $p_h(x,\mathbf{z}):=-\frac{a_h\mathcal{W}'_{\mathbf{z},h}(x)}{b_h^2\mathcal{W}''_{\mathbf{z},h}(x)}\wedge 1$ and $v_h(\mathbf{z}):=\inf\{x>0:1-\mathcal{W}'_{\mathbf{z},h}(x)=0\}$. Let $W(\mathbf{x},\mathbf{z}):=\sum_{h=1}^m\mathcal{W}_{\mathbf{z},h}(x_h)$ for $(\mathbf{x},\mathbf{z})\in\mathcal{X}\times\mathcal{S}$. Then W satisfies

$$\max_{1\le i\le m}\left\{\max_{0\le p_1,\cdots,p_m\le 1}\left\{\mathcal{L}^{\mathbf{P},\mathbf{z}}W(\mathbf{x},\mathbf{z})+\sum_{l=1}^{m}\lambda_l(\mathbf{z})W\left(\mathbf{x}^{(l)},\mathbf{z}^l\right)\right\},1-\frac{\partial W}{\partial x_i}(\mathbf{x},\mathbf{z})\right\}=0,\tag{3.2}$$

with $W(\mathbf{0},\mathbf{z})=0$ where

$$\begin{aligned}\mathcal{L}^{\mathbf{P},\mathbf{z}}W(\mathbf{x},\mathbf{z}):=&\sum_{l=1}^{m}\left(a_lp_l\frac{\partial W}{\partial x_l}(\mathbf{x},\mathbf{z})+\frac{1}{2}p_l^2b_l^2\frac{\partial^2W}{\partial x_l^2}(\mathbf{x},\mathbf{z})\right)-\left(r+\sum_{l=1}^{m}\lambda_l(\mathbf{z})\right)W(\mathbf{x},\mathbf{z})\\&+\sum_{l,k=1,l\neq k}^{m}b_lb_kp_lp_k\rho_{lk}\frac{\partial^2W}{\partial x_l\partial x_k}(\mathbf{x},\mathbf{z})\end{aligned}\tag{3.3}$$

and we have $W(\mathbf{x},\mathbf{z})\ge J(\mathbf{x},\mathbf{z},\mathbf{P},\mathbf{D})$ for any $(\mathbf{P},\mathbf{D})\in\mathcal{U}$. Moreover, it follows that $W(\mathbf{x},\mathbf{z})=\sup_{(\mathbf{P},\mathbf{D})\in\mathcal{U}}J(\mathbf{x},\mathbf{z},\mathbf{P},\mathbf{D})=J(\mathbf{x},\mathbf{z},\mathbf{P}^*,\mathbf{D}^*)=f(\mathbf{x},\mathbf{z})$. The optimal reinsurance strategy $\mathbf{P}^*=(p_1^*(t),\dots,p_N^*(t))_{t\ge0}$ satisfies $p_i^*(t):=p_i(X_i^*(t),\mathbf{Z}(t))$ and the optimal dividend strategy $\mathbf{D}^*=((D_1^*(t),\dots,D_N^*(t))_{t\ge0}$ is given by

$$\begin{cases}\int_0^{\infty}\mathbb{1}_{\{X_i^*(t)<v_i(\mathbf{Z}(t))\}}dD_i^*(t)=0,\\X_i^*(t)\le v_i(\mathbf{Z}(t)),\quad t\ge0,\end{cases}\tag{3.4}$$

where $\mathbf{X}^*=(X_1^*(t),\dots,X_N^*(t))_{t\ge0}$ is the surplus process defined by (2.1) under $(\mathbf{P}^*,\mathbf{D}^*)$.

Proof. First of all, we show that $W(\mathbf{x},\mathbf{z})=\sum_{h=1}^m\mathcal{W}_{\mathbf{z},h}(x_h)$ satisfies (3.2) for any $(\mathbf{x},\mathbf{z})\in\mathcal{X}\times\mathcal{S}$. When $m=1$, the initial state is given by $(\mathbf{x},\mathbf{z})=((x_1,0,\dots,0),(0,1,\dots,1))$ and $W(\mathbf{x},\mathbf{z})=\mathcal{W}_{\mathbf{z},1}(x_1)$ where $\mathcal{W}_{\mathbf{z},1}$ satisfies $\max\{\max_{0\le p_1\le1}\{\mathcal{A}^{p_1,\mathbf{z},1}\mathcal{W}_{\mathbf{z},1}(x)\},1-\mathcal{W}'_{\mathbf{z},1}(x)\}=0$ with $\mathcal{W}_{\mathbf{z},1}(0)=0$. Note that we have the above variational inequality since the term $\sum_{l=1,l\neq1}^m\lambda_l(\mathbf{z})\mathcal{W}_{\mathbf{z}^l,1}(x)$ vanishes when $m=1$. Also, the definition of W yields $W(\mathbf{x}^{(1)},\mathbf{z}^1)=W(\mathbf{0},\mathbf{1})=0$. Next, if we substitute $W(\mathbf{x},\mathbf{z})=\mathcal{W}_{\mathbf{z},1}(x_1)$ back into (3.2), we have $\max\{\max_{0\le p_1\le1}\{\mathcal{A}^{p_1,\mathbf{z},1}\mathcal{W}_{\mathbf{z},1}(x_1)\},1-\mathcal{W}'_{\mathbf{z},1}(x_1)\}=0$, which yields (3.2) when $m=1$.

For $m\ge2$, we have

$$\mathcal{L}^{\mathbf{P},\mathbf{z}}W(\mathbf{x},\mathbf{z})+\sum_{l=1}^{m}\lambda_l(\mathbf{z})W\left(\mathbf{x}^{(l)},\mathbf{z}^l\right)=\sum_{h=1}^{m}\mathcal{A}^{p_h,\mathbf{z},h}\mathcal{W}_{\mathbf{z},h}(x_h)+\sum_{l=1}^{m}\lambda_l(\mathbf{z})\sum_{k=1,k\neq l}^{m}\mathcal{W}_{\mathbf{z}^l,k}(x_k)$$

$$
\begin{aligned}
&= \sum_{h=1}^{m} \mathcal{A}^{p_h,\mathbf{z},h} \mathcal{W}_{\mathbf{z},h}(x_h) + \sum_{k=1}^{m} \sum_{l=1, l\neq k}^{m} \lambda_l(\mathbf{z}) \mathcal{W}_{\mathbf{z}^l,k}(x_k) \\
&= \sum_{h=1}^{m} \left[\mathcal{A}^{p_h,\mathbf{z},h} \mathcal{W}_{\mathbf{z},h}(x_h) + \sum_{l=1, l\neq h}^{m} \lambda_l(\mathbf{z}) \mathcal{W}_{\mathbf{z}^l,h}(x_h) \right].
\end{aligned}
$$

It holds that $\max_{0\le p_h\le 1}\{\mathcal{A}^{p_h,\mathbf{z},h}\mathcal{W}_{\mathbf{z},h}(x)+\sum_{l=1,l\neq h}^{m}\lambda_l(\mathbf{z})\mathcal{W}_{\mathbf{z}^l,h}(x)\}=0$ and $1-\mathcal{W}'_{\mathbf{z},h}(x)\le 0$ for $0<x\le v_h(\mathbf{z})$ because $\mathcal{W}_{\mathbf{z},h}$ is concave and satisfies (3.1). Similarly, we have $1-\mathcal{W}'_{\mathbf{z},h}(x)=0$ and $\max_{0\le p_h\le 1}\{\mathcal{A}^{p_h,\mathbf{z},h}\mathcal{W}_{\mathbf{z},h}(x)+\sum_{l=1,l\neq h}^{m}\lambda_l(\mathbf{z})\mathcal{W}_{\mathbf{z}^l,h}(x)\}\le 0$ for $x\ge v_h(\mathbf{z})$. Therefore, it follows naturally that

$$
\begin{aligned}
&\max_{1\le i\le m}\left\{ \max_{0\le p_1,\cdots,p_m\le 1}\left\{ \mathcal{L}^{\mathbf{P},\mathbf{z}} W(\mathbf{x},\mathbf{z}) + \sum_{l=1}^{m}\lambda_l(\mathbf{z}) W\left(\mathbf{x}^{(l)},\mathbf{z}^l\right)\right\}, 1-\frac{\partial W}{\partial x_i}(\mathbf{x},\mathbf{z})\right\} \\
&= \max_{1\le i\le m}\left\{ \sum_{h=1}^{m} \max_{0\le p_h\le 1}\left\{ \mathcal{A}^{p_h,\mathbf{z},h}\mathcal{W}_{\mathbf{z},h}(x_h) + \sum_{l=1,l\neq h}^{m}\lambda_l(\mathbf{z})\mathcal{W}_{\mathbf{z}^l,h}(x_h)\right\}, 1-\mathcal{W}'_{\mathbf{z},i}(x_i)\right\} = 0.
\end{aligned}
$$

Next we aim to show that $W(\mathbf{x},\mathbf{z})\ge J(\mathbf{x},\mathbf{z},\mathbf{P},\mathbf{D})$ for any $(\mathbf{P},\mathbf{D})\in\mathcal{U}$. Let τ and $(\mathbf{P},\mathbf{D})\in\mathcal{U}$ be arbitrary stopping time and strategies. Regarding $\mathbf{D}=(D_1(t),\ldots,D_N(t))_{t\ge 0}$, we could decompose the process $D_i(t)$ into the jump $\Delta D_i(t)$ and the continuous component $D_i^c(t):=D_i(t)-\sum_{0\le s\le t}\Delta D_i(s)$. As we have shown that $D_i(t)=0$ for $t\ge 0$ if $Z_i(0)=1$, it holds that $\sum_{i=1}^{N}\int_0^\tau e^{-rs}dD_i(s)=\sum_{i=1}^{m}\int_0^\tau e^{-rs}dD_i(s)$. Similarly, we define $\Delta\mathbf{D}^{(l)}(t):=(\Delta D_1(t),\ \ldots,\ \Delta D_{l-1}(t),\ 0,\ \Delta D_{l+1}(t),\ \ldots,\ \Delta D_N(t))$.

By Itô's lemma (Protter, 2005), we have

$$
\begin{aligned}
&e^{-r\tau}W(\mathbf{X}(\tau),\mathbf{Z}(\tau)) - W(\mathbf{x},\mathbf{z}) + \sum_{i=1}^{N}\int_0^\tau e^{-rs}dD_i(s) \\
&\quad = \int_0^\tau e^{-rs}\Bigg[\mathcal{L}^{\mathbf{P}(s),\mathbf{Z}(s)}W(\mathbf{X}(s),\mathbf{Z}(s)) + \sum_{\substack{l=1,\\ Z_l(s)=0}}^{m}\lambda_l(\mathbf{Z}(s))W\left(\mathbf{X}^{(l)}(s),\mathbf{Z}^l(s)\right)\Bigg]ds \\
&\quad + \sum_{i=1}^{m}\int_0^\tau e^{-rs}\left(1-\frac{\partial W}{\partial x_i}(\mathbf{X}(s),\mathbf{Z}(s))\right)dD_i^c(s) + M_\tau + \sum_{\substack{0<s\le\tau,\\ \Delta\mathbf{Z}(s)\neq 0}} e^{-rs}\sum_{j=1}^{m}\Delta Z_j(s) \\
&\quad \times\Bigg[\sum_{\substack{l=1,\\ l\neq j}}^{m}\Delta D_l(s) + W\left(\mathbf{X}^{(j)}(s),\mathbf{Z}^j(s-)\right) - W\left(\mathbf{X}^{(j)}(s)+\Delta\mathbf{D}^{(j)}(s),\mathbf{Z}^j(s-)\right)\Bigg] \\
&\quad + \sum_{\substack{0<s\le\tau,\\ \Delta\mathbf{Z}(s)=0}} e^{-rs}\Bigg[W(\mathbf{X}(s),\mathbf{Z}(s-)) - W(\mathbf{X}(s)+\Delta\mathbf{D}(s),\mathbf{Z}(s-)) + \sum_{l=1}^{m}\Delta D_l(s)\Bigg],
\end{aligned}
\tag{3.5}
$$

where M_τ is a local $\mathbb{G}$-martingale.

It has been shown that $\mathcal{W}'_{\mathbf{z},h}(x)\ge 1$ for $1\le h\le m$. Given $W(\mathbf{x},\mathbf{z})=\sum_{h=1}^{m}\mathcal{W}_{\mathbf{z},h}(x_h)$, there exists an $\epsilon_h\in(0,1)$ for $1\le h\le m$ that

$$
\begin{aligned}
&\sum_{l=1,l\neq j}^{m}\Delta D_l(s) + W\left(\mathbf{X}^{(j)}(s),\mathbf{Z}^j(s-)\right) - W\left(\mathbf{X}^{(j)}(s)+\Delta\mathbf{D}^{(j)}(s),\mathbf{Z}^j(s-)\right) \\
&= \sum_{l=1,l\neq j}^{m}\left[\Delta D_l(s) + \mathcal{W}_{\mathbf{Z}^j(s-),l}(X_l(s)) - \mathcal{W}_{\mathbf{Z}^j(s-),l}(X_l(s)+\Delta D_l(s))\right] \\
&= \sum_{l=1,l\neq j}^{m}\Delta D_l(s)\left[1-\mathcal{W}'_{\mathbf{Z}^j(s-),l}(X_l(s)+\epsilon_l\Delta D_l(s))\right]\le 0,
\end{aligned}
$$

where the last equality holds by mean value theorem. Likewise, we have

$$
W(\mathbf{X}(s),\mathbf{Z}(s-)) - W(\mathbf{X}(s)+\Delta\mathbf{D}(s),\mathbf{Z}(s-)) + \sum_{l=1}^{m}\Delta D_l(s)\le 0.
$$

As we have shown that W satisfies (3.2), it holds that

$$\mathcal{L}^{\mathbf{P}(s),\mathbf{Z}(s)}W(\mathbf{X}(s),\mathbf{Z}(s))+\sum_{l=1,Z_l(s)=0}^{m}\lambda_l(\mathbf{Z}(s))W(\mathbf{X}^{(l)}(s),\mathbf{Z}^l(s))\le 0.$$

We thus obtain $e^{-r\tau}W(\mathbf{X}(\tau),\mathbf{Z}(\tau))-W(\mathbf{x},\mathbf{z})+\sum_{i=1}^{N}\int_0^{\tau}e^{-rs}dD_i(s)\le M_\tau$. Since $M_{t\wedge\tau}$ is a local martingale, there exists a sequence of stopping time $\{T_n\}_{n=1}^{\infty}$ with $T_n\uparrow\infty$ such that

$$\begin{aligned}W(\mathbf{x},\mathbf{z})&\ge\lim_{n\to\infty}\mathbb{E}\left[e^{-r(\tau\wedge T_n)}W\left(\mathbf{X}(\tau\wedge T_n),\mathbf{Z}(\tau\wedge T_n)\right)+\sum_{i=1}^{N}\int_0^{\tau\wedge T_n}e^{-rs}dD_i(s)\right]-\lim_{n\to\infty}\mathbb{E}\left[M_{\tau\wedge T_n}\right]\\&\ge\mathbb{E}\left[\sum_{i=1}^{N}\int_0^{\tau}e^{-rs}dD_i(s)\right].\end{aligned}$$

By letting $\tau\to\infty$, we have $W(\mathbf{x},\mathbf{z})\ge\mathbb{E}[\sum_{i=1}^{N}\int_0^{\tau_i}e^{-rs}dD_i(s)]=J(\mathbf{x},\mathbf{z},\mathbf{P},\mathbf{D})$.

It remains to show that $W(\mathbf{x},\mathbf{z})=\sup_{(\mathbf{P},\mathbf{D})\in\mathcal{U}}J(\mathbf{x},\mathbf{z},\mathbf{P},\mathbf{D})=J(\mathbf{x},\mathbf{z},\mathbf{P}^*,\mathbf{D}^*)=f(\mathbf{x},\mathbf{z})$. Under $(\mathbf{P}^*,\mathbf{D}^*)$, the surplus process satisfies $X_i^*(t)=(1-Z_i(t))(\tilde{X}_i^*(t)-D_i^*(t))$ where $\tilde{X}_i^*(t)=(1-Z_i(t))\hat{X}_i^*(t)$ and $d\hat{X}_i^*(t)=p_i^*(t)a_idt-p_i^*(t)b_idW_i(t)$ for $1\le i\le m$. As we deduced before, it holds that $\mathcal{W}'_{\mathbf{Z}(s),i}(v_i(\mathbf{Z}(s)))=1$ if $Z_i(s)=0$. If $Z_i(s)=1$, no dividend will be paid out and $d(D_i^*)^c(s)=0$. By (3.4), $D_i^*(t)$ increases only if $X_i^*(t)$ is going to exceed the barrier $v_i(\mathbf{Z}(t))$, thus yielding $\sum_{i=1}^{m}\int_0^{\tau}e^{-rs}(1-\partial W(\mathbf{X}^*(s),\mathbf{Z}(s))/\partial x_i)d(D_i^*)^c(s)=\sum_{i=1}^{m}\int_0^{\tau}e^{-rs}(1-\mathcal{W}'_{\mathbf{Z}(s),i}(X_i^*(s)))d(D_i^*)^c(s)=0$. For $\Delta D_i^*(t)>0$, the amount of surplus exceeding $v_i(\mathbf{Z}(t))$ is paid out immediately, leaving $X_i^*(t)=v_i(\mathbf{Z}(t))$. Therefore, we have

$$\begin{aligned}&\sum_{\substack{l=1,\\l\ne j}}^{m}\Delta D_l^*(s)+W\left(\left(\mathbf{X}^*\right)^{(j)}(s),\mathbf{Z}^j(s-)\right)-W\left(\left(\mathbf{X}^*\right)^{(j)}(s)+\Delta\left(\mathbf{D}^*\right)^{(j)}(s),\mathbf{Z}^j(s-)\right)\\&=\sum_{l=1,l\ne j,Z_l(s-)=0}^{m}\mathbb{1}_{\{\Delta D_i^*(s)\ne0\}}\left[\mathcal{W}_{\mathbf{Z}^j(s-),l}\left(X_l^*(s)\right)-\mathcal{W}_{\mathbf{Z}^j(s-),l}\left(X_l^*(s)+\Delta D_i^*(s)\right)\right]+\sum_{l=1,l\ne j}^{m}\mathbb{1}_{\{\Delta D_l^*(s)\ne0\}}\Delta D_l^*(s)\\&=\sum_{\substack{l=1,l\ne j,\\\Delta D_l^*(s)\ne0}}^{m}\left[\Delta D_l^*(s)+\mathcal{W}_{\mathbf{Z}^j(s-),l}(v_i(\mathbf{Z}(s)))-\mathcal{W}_{\mathbf{Z}^j(s-),l}\left(v_i(\mathbf{Z}(s))+\Delta D_i^*(s)\right)\right]=0.\end{aligned}$$

The second last equality holds since $\Delta D_l^*(s)=0$ if $Z_l(s)=1$. Likewise, we have

$$\begin{aligned}&W\left(\left(\mathbf{X}^*\right)(s),\mathbf{Z}(s-)\right)-W\left(\left(\mathbf{X}^*\right)(s)+\Delta\mathbf{D}^*(s),\mathbf{Z}(s-)\right)+\sum_{l=1}^{m}\Delta D_l^*(s)\\&=\sum_{\substack{l=1,\\\Delta D_l^*(s)\ne0}}^{m}\left[\mathcal{W}_{\mathbf{Z}(s-),l}(v_i(\mathbf{Z}(s)))-\mathcal{W}_{\mathbf{Z}(s-),l}\left(v_i(\mathbf{Z}(s))+\Delta D_l^*(s)\right)+\Delta D_l^*(s)\right]=0.\end{aligned}$$

It follows naturally that $\arg\max_{0\le p_h\le1}\mathcal{A}^{p_h,\mathbf{z},h}\mathcal{W}_{\mathbf{z},h}(x)=-\frac{a_h\mathcal{W}'_{\mathbf{z},h}(x)}{b_h^2\mathcal{W}''_{\mathbf{z},h}(x)}\wedge1$, which coincides to the definition of $p_h(x,\mathbf{z})$. Given $p_i^*(t)=p_i(X_i^*(t),\mathbf{Z}(t))$, it holds that

$$\begin{aligned}&\max_{0\le p_i(s)\le1}\left\{\mathcal{A}^{p_i(s),\mathbf{Z}(s),i}\mathcal{W}_{\mathbf{Z}(s),i}\left(X_i^*(s)\right)+\sum_{\substack{l=1,l\ne i\\Z_l(s)=0}}^{m}\lambda_l(\mathbf{Z}(s))\mathcal{W}_{\mathbf{Z}^l(s),i}\left(X_i^*(s)\right)\right\}\\&=\mathcal{A}^{p_i^*(s),\mathbf{Z}(s),i}\mathcal{W}_{\mathbf{Z}(s),i}\left(X_i^*(s)\right)+\sum_{\substack{l=1,l\ne i\\Z_l(s)=0}}^{m}\lambda_l(\mathbf{Z}(s))\mathcal{W}_{\mathbf{Z}^l(s),i}\left(X_i^*(s)\right)=0\end{aligned}$$

for $X_i^*(s)\le v_i(\mathbf{Z}(s))$. Thus, if $X_i^*(t)\le v_i(\mathbf{Z}(t))$ for all $1\le i\le m$, we have

$$\begin{aligned}&\mathcal{L}^{\mathbf{P}^*(s),\mathbf{Z}(s)}W\left(\mathbf{X}^*(s),\mathbf{Z}(s)\right)+\sum_{\substack{l=1,\\Z_l(s)=0}}^{m}\lambda_l(\mathbf{Z}(s))W\left(\left(\mathbf{X}^*\right)^{(l)}(s),\mathbf{Z}^l(s)\right)\\&=\sum_{\substack{i=1,\\Z_i(s)=0}}^{m}\left[\mathcal{A}^{p_i^*(s),\mathbf{Z}(s),i}\mathcal{W}_{\mathbf{Z}(s),i}\left(X_i^*(s)\right)+\sum_{\substack{l=1,l\ne i\\Z_l(s)=0}}^{m}\lambda_l(\mathbf{Z}(s))\mathcal{W}_{\mathbf{Z}^l(s),i}\left(X_i^*(s)\right)\right]=0.\end{aligned}$$

Similarly, there exists a sequence of stopping time $\{T_n\}_{n=1}^{\infty}$ with $T_n \uparrow \infty$ and

$$\lim_{n\to\infty} \mathbb{E}\left[e^{-r(\tau\wedge T_n)} W\left(\mathbf{X}^*(\tau\wedge T_n), \mathbf{Z}(\tau\wedge T_n)\right) + \sum_{i=1}^{N} \int_0^{\tau\wedge T_n} e^{-rs} dD_i^*(s) - W(\mathbf{x},\mathbf{z}) \right]$$
$$= \mathbb{E}\left[e^{-r\tau} W\left(\mathbf{X}^*(\tau), \mathbf{Z}(\tau)\right) + \sum_{i=1}^{N} \int_0^{\tau} e^{-rs} dD_i^*(s) - W(\mathbf{x},\mathbf{z}) \right] = \lim_{n\to\infty} \mathbb{E}\left[M_{\tau\wedge T_n}\right] = 0.$$

Since $X_i^*(\tau)$ is non-negative and bounded by $v_i(\mathbf{Z}(\tau))$, we have that $W(\mathbf{X}^*(\tau), \mathbf{Z}(\tau))$ is also bounded. Thus by letting $\tau \to \infty$, we have

$$W(\mathbf{x},\mathbf{z}) = \mathbb{E}\left[\sum_{i=1}^{N} \int_0^{\tau_i} e^{-rs} dD_i^*(s) \right] = J\left(\mathbf{x},\mathbf{z},\mathbf{P}^*,\mathbf{D}^*\right), \qquad \text{a.s.}$$

Since $W(\mathbf{x},\mathbf{z}) \geq \mathbb{E}[\sum_{i=1}^{N} \int_0^{\tau_i} e^{-rs} dD_i(s)] = J(\mathbf{x},\mathbf{z},\mathbf{P},\mathbf{D})$ for any arbitrary $(\mathbf{P},\mathbf{D}) \in \mathcal{U}$, it follows naturally that $W(\mathbf{x},\mathbf{z}) = \sup_{(\mathbf{P},\mathbf{D})\in\mathcal{U}} J(\mathbf{x},\mathbf{z},\mathbf{P},\mathbf{D}) = f(\mathbf{x},\mathbf{z})$ and $(\mathbf{P}^*,\mathbf{D}^*)$ is the pair of optimal strategies.

Furthermore, if we apply Ito's lemma to $\mathcal{W}_{\mathbf{z},i}$ and follow the similar steps above, we can obtain

$$\mathbb{E}\left[\int_0^{\tau_i} e^{-rs} dD_i^*(s) \right] - \mathcal{W}_{\mathbf{z},i}(x_i) = 0, \quad 1 \leq i \leq m. \tag{3.6}$$

Such equality implies that the solution to (3.1) measures the cumulative expected discounted dividend for subsidiary i under the optimal strategies $(\mathbf{P}^*,\mathbf{D}^*)$ for the whole group and (p_i^*, D_i^*) is the pair of optimal strategies for subsidiary i in order to maximize group dividend.

We could also show the concavity of $\mathcal{W}_{\mathbf{z},h}$ from (3.6) in a similar vein to Højgaard and Taksar (1999). Assume $1 \leq h \leq m$, let us denote the expectation of discounted dividend payments prior to ruin of subsidiary h by $J_h(x_h,\mathbf{z},p_h,D_h)$ where $J_h(x_h,\mathbf{z},p_h,D_h) = \mathbb{E}[\int_0^{\tau_h} e^{-rs} dD_h(s)]$. Let $(\mathbf{P}_1,\mathbf{D}_1) = ((p_1(t), \ldots, p_{h-1}(t), \hat{p}_h(t), p_{h+1}(t), \ldots, p_N(t)), (D_1(t), \ldots, D_{h-1}(t), \hat{D}_h(t), D_{h+1}(t), \ldots, D_N(t)))$ be an admissible control pair for the initial surplus $\hat{\mathbf{x}} = (x_1, \ldots, x_{h-1}, y_1, x_{h+1}, \ldots, x_m, 0, \ldots, 0)$. Similarly, let $(\mathbf{P}_2,\mathbf{D}_2) = ((p_1(t),\ldots,p_{h-1}(t),\bar{p}_h(t),p_{h+1}(t),\ldots,p_N(t)), (D_1(t),\ldots,D_{h-1}(t), \bar{D}_h(t), D_{h+1}(t), \ldots, D_N(t)))$ be an admissible control pair for the initial surplus $\bar{\mathbf{x}} = (x_1, \ldots, x_{h-1}, y_2, x_{h+1}, \ldots, x_m, 0,\ldots,0)$. Here, we let the control strategies for all other subsidiaries be the same for $(\mathbf{P}_1,\mathbf{D}_1)$ and $(\mathbf{P}_2,\mathbf{D}_2)$ except for subsidiary h. For some $0 < \delta < 1$, let us define the next admissible control pair $(\mathbf{P}_3,\mathbf{D}_3) = ((p_1(t),\ldots,p_{h-1}(t),\delta\hat{p}_h(t)+(1-\delta)\bar{p}_h(t),p_{h+1}(t),\ldots,p_N(t)), (D_1(t),\ldots,D_{h-1}(t), \delta\hat{D}_h(t)+(1-\delta)\bar{D}_h(t), D_{h+1}(t), \ldots, D_N(t)))$. Subsequently, let us denote $X_{h,1}(t)$ by the surplus process of subsidiary h where its initial surplus is y_1, the pair of control is $(\hat{p}_h, \hat{D}_h)$. Similarly, we define $X_{h,2}(t)$ by the surplus process of subsidiary h where its initial surplus is y_2 and the pair of control is $(\bar{p}_h, \bar{D}_h)$. Since the dynamic of the surplus of subsidiary h is linear, it follows easily that $X_{h,3}(t) = \delta X_{h,1}(t) + (1-\delta)X_{h,2}(t)$ where $X_{h,3}(t)$ denotes the surplus process of subsidiary h with initial surplus $y_3 = \delta y_1 + (1-\delta)y_2$ governed by the pair of control $(\delta\hat{p}_h + (1-\delta)\bar{p}_h, \delta\hat{D}_h + (1-\delta)\bar{D}_h)$. We denote the ruin times associated with $X_{h,1}(t)$, $X_{h,2}(t)$, and $X_{h,3}(t)$ by $\tau_{h,1}$, $\tau_{h,2}$, and $\tau_{h,3}$, respectively. We further have $\tau_{h,3} = \tau_{h,1} \vee \tau_{h,2}$, and it follows easily that $(\mathbf{P}_3,\mathbf{D}_3)$ is also admissible.

Following the arguments above, it holds that

$$J_h\left(y_3,\mathbf{z},\delta\hat{p}_h + (1-\delta)\bar{p}_h, \delta\hat{D}_h + (1-\delta)\bar{D}_h\right) = \delta J_h\left(y_1,\mathbf{z},\hat{p}_h,\hat{D}_h\right) + (1-\delta) J_h\left(y_2,\mathbf{z},\bar{p}_h,\bar{D}_h\right).$$

For any $\epsilon > 0$, we can choose the pair of control (p_i, D_i) such that $J_i(x_i,\mathbf{z},p_i,D_i) \geq \mathcal{W}_{\mathbf{z},i}(x_i) - \epsilon/N$ for $1 \leq i \leq N$. Thus, it holds that

$$\sum_{i\neq h}^{N} J_i(x_i,\mathbf{z},p_i,D_i) + J_h\left(y_3,\mathbf{z},\delta\hat{p}_h + (1-\delta)\bar{p}_h, \delta\hat{D}_h + (1-\delta)\bar{D}_h\right) \geq \sum_{i\neq h}^{N} \mathcal{W}_{\mathbf{z},i}(x_i) + \delta\mathcal{W}_{\mathbf{z},h}(y_1) + (1-\delta)\mathcal{W}_{\mathbf{z},h}(y_2) - \epsilon.$$

Since $(\mathbf{P}_3,\mathbf{D}_3)$ is suboptimal, it follows that

$$\sum_{i\neq h}^{N} \mathcal{W}_{\mathbf{z},i}(x_i) + \mathcal{W}_{\mathbf{z},h}(y_3) \geq \sum_{i\neq h}^{N} J_i(x_i,\mathbf{z},p_i,D_i) + J_h\left(y_3,\mathbf{z},\delta\hat{p}_h + (1-\delta)\bar{p}_h, \delta\hat{D}_h + (1-\delta)\bar{D}_h\right).$$

Therefore, we arrive at the following inequality,

$$\mathcal{W}_{\mathbf{z},h}(y_3) = \mathcal{W}_{\mathbf{z},h}(\delta y_1 + (1-\delta)y_2) \geq \delta\mathcal{W}_{\mathbf{z},h}(y_1) + (1-\delta)\mathcal{W}_{\mathbf{z},h}(y_2) - \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we conclude that $\mathcal{W}_{\mathbf{z},h}$ is concave. $\square$

From Theorem 3.1, we can obtain the optimal value function $f(\mathbf{x},\mathbf{z})$ by solving (3.1) recursively given the initial default state $\mathbf{z}$ in a semi-analytical approach. The optimal controls are available once we solve (3.1).

4. Main results

In this section, we first find the semi-closed solution of (3.1) recursively and then present the numerical algorithm. Instead of tackling directly with the free boundary problem (3.1), we formulate a group of C^2 candidate functions where the function values and boundaries are available analytically or numerically. Assume the existence of a concave, increasing and C^2 solution to (3.1), we derive the condition where the candidate function solves the free boundary problem (3.1) if the condition is satisfied. The numerical algorithm based on the semi-closed solution is proposed later.

In addition, the semi-closed solution to (3.1) when $m=3$ is provided for illustration. We demonstrate the semi-analytical method by solving the recursive system of HJBVIs when $m=3$ because it provides sufficient details in obtaining the analytical solutions and applying the numerical algorithm recursively. For a recursive system of HJBVIs of more than three layers, we can extend the results by following the same steps without adding the complexity of the problem.

4.1. Semi-closed solution

We first examine the auxiliary variational inequality analytically.

Lemma 4.1. *Assume $m\geq 2$. Let $\Xi_l:[0,\infty)\to[0,\infty)$ be of class C^2, concave, monotonically increasing and satisfies $\Xi_l(0)=0$ for all $1\leq l\leq m+1$. Let $\xi>0$, $\kappa>0$, $r>0$ and $\Lambda_l>0$ for all $1\leq l\leq m+1$. Then the following differential equation*

$$\frac{\xi^2}{2}\zeta''(x)-\kappa\zeta'(x)-\left(r+\sum_{l=1}^{m+1}\Lambda_l\right)\zeta(x)+\sum_{l=1}^{m}\Lambda_l\Xi_l(v-x)=0,\quad 0\leq x\leq v, \tag{4.1}$$

for some $v>0$ with $\zeta'(0)=-1$, $\zeta''(0)=0$ admits a unique C^2 solution in the following form

$$\zeta(x)=\frac{\zeta(0)\left(\eta_2 e^{\eta_1 x}+\eta_1 e^{-\eta_2 x}\right)-e^{\eta_1 x}+e^{-\eta_2 x}}{\eta_1+\eta_2}+\frac{2}{\xi^2(\eta_1+\eta_2)}\int_0^x\sum_{l=1}^{m}\Lambda_l\Xi_l(v-s)\left[-e^{\eta_1(x-s)}+e^{-\eta_2(x-s)}\right]ds,\quad 0\leq x\leq v, \tag{4.2}$$

where $\eta_1>0$ and $-\eta_2<0$ are the two distinct roots of $\frac{1}{2}\xi^2 s^2-\kappa s-(r+\sum_{l=1}^{m+1}\Lambda_l)=0$ and $\zeta(0)=\frac{\sum_{l=1}^{m}\Lambda_l\Xi_l(v)+\kappa}{r+\sum_{l=1}^{m+1}\Lambda_l}$.

To maintain the flow of the paper, we include the proof of Lemma 4.1 in Appendix A, where we derived the unique C^2 solution to the differential equation in the form of (4.1) satisfying the specified conditions $\zeta'(0)=-1$ and $\zeta''(0)=0$. Next, we present two lemmas that are essential to prove some properties of the candidate functions later. Meanwhile, we can calculate the function values and the boundaries of the candidate functions analytically or numerically according to the following two lemmas. Then, we can formulate a group of C^2 candidate functions for $\mathcal{W}_{\mathbf{z},h}$ that satisfy the variational inequalities in similar forms as (3.1). Within the group of candidate functions, we next find the concave and increasing candidate function that satisfies a certain condition and thus solves the variational inequality (3.1).

Assume $m\geq 2$. Let $\eta_{\mathbf{z},h,1}>0$ and $-\eta_{\mathbf{z},h,2}<0$ be the two solutions of $\frac{1}{2}b_h^2 s^2-a_h s-(r+\sum_{k=1}^{m}\lambda_k(\mathbf{z}))=0$. Let $\mathcal{W}_{\mathbf{z}^l,h}(x)$ be the concave and increasing solution to (3.1) subject to $\mathcal{W}_{\mathbf{z}^l,h}(0)=0$ where the default state is $\mathbf{z}^l$ for all $1\leq l\leq m$. It holds that $\mathcal{W}'_{\mathbf{z}^l,h}(x)=1$ for $x\geq v_h(\mathbf{z}^l)$ from Theorem 3.1. For any $v>0$, let us define

$$H_{\mathbf{z},h,v}(x):=\frac{w_{\mathbf{z},h,v}\Phi_{\mathbf{z},h,2}(x)-\Phi'_{\mathbf{z},h,2}(x)\left(\eta_{\mathbf{z},h,1}\eta_{\mathbf{z},h,2}\right)^{-1}}{\eta_{\mathbf{z},h,1}+\eta_{\mathbf{z},h,2}}+\Phi_{\mathbf{z},h,v,1}(x),\quad 0\leq x\leq v, \tag{4.3}$$

where

$$\begin{aligned}
\Phi_{\mathbf{z},h,v,1}(x)&:=-\frac{2\int_0^x\sum_{l=1,l\neq h}^{m}\lambda_l(\mathbf{z})\mathcal{W}_{\mathbf{z}^l,h}(v-u)\Phi'_{\mathbf{z},h,2}(x-u)\,du}{b_h^2\left(\eta_{\mathbf{z},h,1}+\eta_{\mathbf{z},h,2}\right)\eta_{\mathbf{z},h,1}\eta_{\mathbf{z},h,2}},\\
\Phi_{\mathbf{z},h,2}(x)&:=\eta_{\mathbf{z},h,2}e^{\eta_{\mathbf{z},h,1}x}+\eta_{\mathbf{z},h,1}e^{-\eta_{\mathbf{z},h,2}x},\\
w_{\mathbf{z},h,v}&:=\frac{\sum_{l=1,l\neq h}^{m}\lambda_l(\mathbf{z})\mathcal{W}_{\mathbf{z}^l,h}(v)+a_h}{r+\sum_{k=1}^{m}\lambda_k(\mathbf{z})}.
\end{aligned}$$

By evaluating $H_{\mathbf{z},h,v}(x)$ at $x=0$, we obtain that $H_{\mathbf{z},h,v}(0)=w_{\mathbf{z},h,v}$. Since $\mathcal{W}_{\mathbf{z}^l,h}(x)$ is concave, increasing, and satisfies $\mathcal{W}_{\mathbf{z}^l,h}(0)=0$ for all $1\leq l\leq m$, it follows directly from Lemma 4.1 that $H_{\mathbf{z},h,v}(x)$ given by (4.3) is the unique C^2 solution to

$$\frac{b_h^2}{2}H''_{\mathbf{z},h,v}(x)-a_h H'_{\mathbf{z},h,v}(x)-\left(r+\sum_{k=1}^{m}\lambda_k(\mathbf{z})\right)H_{\mathbf{z},h,v}(x)+\sum_{l=1,l\neq h}^{m}\lambda_l(\mathbf{z})\mathcal{W}_{\mathbf{z}^l,h}(v-x)=0,\quad 0\leq x\leq v, \tag{4.4}$$

subject to $H'_{\mathbf{z},h,v}(0)=-1$ and $H''_{\mathbf{z},h,v}(0)=0$.

Lemma 4.2. *Assume $m \geq 2$. Fix $\mathbf{z} \in \mathcal{S}$ and $1 \leq h \leq m$. Then, there exists a set $\mathcal{O}_1$ that satisfies $[\mathcal{K}_{\mathbf{z},h,2}, \infty) \subseteq \mathcal{O}_1 \subseteq (\mathcal{K}_{\mathbf{z},h,1}, \infty)$ where*

$$\mathcal{K}_{\mathbf{z},h,1} := \inf\left\{x > 0 : \sum_{l=1,l\neq h}^{m} \lambda_l(\mathbf{z})\mathcal{W}'_{\mathbf{z}^l,h}(x) - \left(r + \sum_{k=1}^{m}\lambda_k(\mathbf{z})\right) \leq 0\right\}, \tag{4.5}$$

$$\mathcal{K}_{\mathbf{z},h,2} := \max_{1\leq l\leq m, l\neq h} v_h\left(\mathbf{z}^l\right) + \frac{a_h}{r + \lambda_h(\mathbf{z})}. \tag{4.6}$$

For any $v \in \mathcal{O}_1$, $H_{\mathbf{z},h,v}(x) \in C^2 : [0, v] \to \mathbb{R}$ given by (4.3) is concave, decreasing, and satisfies $a_h H'_{\mathbf{z},h,v}(x) \leq b_h^2 H''_{\mathbf{z},h,v}(x) \leq 0$ on $0 < x < v - n_{h,v}(\mathbf{z})$ where

$$n_{h,v}(\mathbf{z}) := v - \inf\left\{0 < x < v : \frac{a_h H'_{\mathbf{z},h,v}(x)}{b_h^2 H''_{\mathbf{z},h,v}(x)} = 1\right\}. \tag{4.7}$$

Proof. It follows naturally from (4.4) that $H_{\mathbf{z},h,v}(x)$ also satisfies

$$\frac{b_h^2}{2}H'''_{\mathbf{z},h,v}(x) - a_h H''_{\mathbf{z},h,v}(x) - \left(r + \sum_{k=1}^{m}\lambda_k(\mathbf{z})\right)H'_{\mathbf{z},h,v}(x) - \sum_{l=1,l\neq h}^{m}\lambda_l(\mathbf{z})\mathcal{W}'_{\mathbf{z}^l,h}(v - x) = 0,$$

for $0 \leq x \leq v$. Given $H'_{\mathbf{z},h,v}(0) = -1$ and $H''_{\mathbf{z},h,v}(0) = 0$, we immediately have

$$\frac{b_h^2}{2}H'''_{\mathbf{z},h,v}(0) = \sum_{l=1,l\neq h}^{m}\lambda_l(\mathbf{z})\mathcal{W}'_{\mathbf{z}^l,h}(v) - \left(r + \sum_{k=1}^{m}\lambda_k(\mathbf{z})\right).$$

Given $H''_{\mathbf{z},h,v}(0) = 0$, if $H_{\mathbf{z},h,v}(x)$ is concave, we have $H'''_{\mathbf{z},h,v}(0) \leq 0$. Also, $\mathcal{W}_{\mathbf{z}^l,h}(x)$ is concave on $x \geq 0$ for all $1 \leq l \leq m$ with $\mathcal{W}'_{\mathbf{z}^l,h}(x) \geq 1$. For $x > \max_{1\leq l\leq m,l\neq h} v_h(\mathbf{z}^l)$, we have $\mathcal{W}'_{\mathbf{z}^l,h}(x) = 1$ for all $1 \leq l \neq h \leq m$. If $v > \max_{1\leq l\leq m,l\neq h} v_h(\mathbf{z}^l)$, it follows immediately that $\sum_{l=1,l\neq h}^{m}\lambda_l(\mathbf{z})\mathcal{W}'_{\mathbf{z}^l,h}(v) - (r + \sum_{k=1}^{m}\lambda_k(\mathbf{z})) = -r - \lambda_h(\mathbf{z}) < 0$. Thus, $\mathcal{K}_{\mathbf{z},h,1}$ defined by (4.5) exists and satisfies $0 \leq \mathcal{K}_{\mathbf{z},h,1} \leq \max_{1\leq l\leq m,l\neq h} v_h(\mathbf{z}^l) < \mathcal{K}_{\mathbf{z},h,2}$. Consequently, it holds that $H'''_{\mathbf{z},h,v}(0) < 0$ if $v > \mathcal{K}_{\mathbf{z},h,1}$, which is necessary for $H_{\mathbf{z},h,v}(x)$ to be concave and decreasing.

Given $v > \mathcal{K}_{\mathbf{z},h,1}$, we have $H'''_{\mathbf{z},h,v}(0) < 0$ and $\lim_{x\downarrow 0}\frac{a_h H'_{\mathbf{z},h,v}(x)}{b_h^2 H''_{\mathbf{z},h,v}(x)} = \infty$. Let us assume $v > \mathcal{K}_{\mathbf{z},h,2}$. For $0 < x < \frac{a_h}{r+\lambda_h(\mathbf{z})}$, we have

$$\begin{aligned}\frac{b_h^2}{2}H'''_{\mathbf{z},h,v}(x) &= a_h H''_{\mathbf{z},h,v}(x) + \left(r + \sum_{k=1}^{m}\lambda_k(\mathbf{z})\right)H'_{\mathbf{z},h,v}(x) + \sum_{l=1,l\neq h}^{m}\lambda_l(\mathbf{z})\mathcal{W}'_{\mathbf{z}^l,h}(v - x)\\ &= a_h H''_{\mathbf{z},h,v}(x) + \left(r + \sum_{k=1}^{m}\lambda_k(\mathbf{z})\right)H'_{\mathbf{z},h,v}(x) + \sum_{l=1,l\neq h}^{m}\lambda_l(\mathbf{z})\end{aligned}$$

with $b_h^2 H'''_{\mathbf{z},h,v}(0)/2 = -r - \lambda_h(\mathbf{z}) < 0$. Given $H'_{\mathbf{z},h,v}(0) = -1$ and $H''_{\mathbf{z},h,v}(0) = 0$, we deduce that $H'_{\mathbf{z},h,v}(x)$, $H''_{\mathbf{z},h,v}(x)$, and $H'''_{\mathbf{z},h,v}(x)$ are negative and decreasing for $0 < x < \frac{a_h}{r+\lambda_h(\mathbf{z})}$.

Next, we rearrange (4.4) and have

$$\frac{b_h^2}{2}H''_{\mathbf{z},h,v}(x) - a_h H'_{\mathbf{z},h,v}(x) = \left(r + \sum_{k=1}^{m}\lambda_k(\mathbf{z})\right)H_{\mathbf{z},h,v}(x) - \sum_{l=1,l\neq h}^{m}\lambda_l(\mathbf{z})\mathcal{W}_{\mathbf{z}^l,h}(v - x). \tag{4.8}$$

Immediately, we have $(r + \sum_{k=1}^{m}\lambda_k(\mathbf{z}))H_{\mathbf{z},h,v}(0) - \sum_{l=1,l\neq h}^{m}\lambda_l(\mathbf{z})\mathcal{W}_{\mathbf{z}^l,h}(v) = a_h$. Since $H'_{\mathbf{z},h,v}(x)$ is negative and decreasing for $0 < x < \frac{a_h}{r+\lambda_h(\mathbf{z})}$, it holds that

$$\begin{aligned}&\left(r + \sum_{k=1}^{m}\lambda_k(\mathbf{z})\right)H'_{\mathbf{z},h,v}(x) + \sum_{l=1,l\neq h}^{m}\lambda_l(\mathbf{z})\mathcal{W}'_{\mathbf{z}^l,h}(v - x)\\ &\quad= \left(r + \sum_{k=1}^{m}\lambda_k(\mathbf{z})\right)H'_{\mathbf{z},h,v}(x) + \sum_{l=1,l\neq h}^{m}\lambda_l(\mathbf{z}) < -r - \lambda_h(\mathbf{z}) < 0, \quad 0 < x < \frac{a_h}{r + \lambda_h(\mathbf{z})}.\end{aligned}$$

Furthermore, $(r + \sum_{k=1}^{m}\lambda_k(\mathbf{z}))H'_{\mathbf{z},h,v}(x) + \sum_{l=1,l\neq h}^{m}\lambda_l(\mathbf{z})\mathcal{W}'_{\mathbf{z}^l,h}(v - x)$ is decreasing for $0 < x < \frac{a_h}{r+\lambda_h(\mathbf{z})}$. Therefore, the right-hand-side of (4.8) is decreasing for $0 < x < \frac{a_h}{r+\lambda_h(\mathbf{z})}$. It follows naturally that the left-hand-side of (4.8), $b_h^2 H''_{\mathbf{z},h,v}(x)/2 - a_h H'_{\mathbf{z},h,v}(x)$, is monotonically decreasing where $b_h^2 H'''_{\mathbf{z},h,v}(x)/2 - a_h H''_{\mathbf{z},h,v}(x) < -r - \lambda_h(\mathbf{z})$ for $0 < x < \frac{a_h}{r+\lambda_h(\mathbf{z})}$. With $(r + \sum_{k=1}^{m}\lambda_k(\mathbf{z}))H_{\mathbf{z},h,v}(0) - \sum_{l=1,l\neq h}^{m}\lambda_l(\mathbf{z})\mathcal{W}_{\mathbf{z}^l,h}(v) = a_h$ and $(r + \sum_{k=1}^{m}\lambda_k(\mathbf{z}))H'_{\mathbf{z},h,v}(x) + \sum_{l=1,l\neq h}^{m}\lambda_l(\mathbf{z})\mathcal{W}'_{\mathbf{z}^l,h}(v - x) < -r - \lambda_h(\mathbf{z})$, we can find an M_1 that satisfies

$$\left(r + \sum_{k=1}^{m}\lambda_k(\mathbf{z})\right)H_{\mathbf{z},h,v}(M_1) - \sum_{l=1,l\neq h}^{m}\lambda_l(\mathbf{z})\mathcal{W}_{\mathbf{z}^l,h}(v - M_1) = 0, \quad 0 < M_1 < \frac{a_h}{r + \lambda_h(\mathbf{z})}.$$

Then, we have $\frac{b_h^2}{2}H''_{\mathbf{z},h,v}(M_1) - a_h H'_{\mathbf{z},h,v}(M_1) = 0$ and $\frac{a_h H'_{\mathbf{z},h,v}(M_1)}{b_h^2 H''_{\mathbf{z},h,v}(M_1)} = \frac{1}{2}$. Since $H'_{\mathbf{z},h,v}(x)$ and $H''_{\mathbf{z},h,v}(x)$ are negative and decreasing for $0 < x < \frac{a_h}{r+\lambda_h(\mathbf{z})}$ with $\lim_{x\downarrow 0} \frac{a_h H'_{\mathbf{z},h,v}(x)}{b_h^2 H''_{\mathbf{z},h,v}(x)} = \infty$, we can find at least one M_2 that satisfies $0 < M_2 < M_1 < \frac{a_h}{r+\lambda_h(\mathbf{z})}$ and $\frac{a_h H'_{\mathbf{z},h,v}(M_2)}{b_h^2 H''_{\mathbf{z},h,v}(M_2)} = 1$. Thus, there exists at least one $0 < M_2 < v$ that $\frac{a_h H'_{\mathbf{z},h,v}(M_2)}{b_h^2 H''_{\mathbf{z},h,v}(M_2)} = 1$ for any $v \geq \mathcal{K}_{\mathbf{z},h,2}$. We denote the smallest M_2 by $v - n_{h,v}(\mathbf{z})$, which is equivalent to the definition given by (4.7). Thus, it is sufficient for $H_{\mathbf{z},h,v}(x)$ to be concave, decreasing, and satisfy $a_h H'_{\mathbf{z},h,v}(x) \leq b_h^2 H''_{\mathbf{z},h,v}(x) \leq 0$ on $0 < x < v - n_{h,v}(\mathbf{z})$ if $v \geq \mathcal{K}_{\mathbf{z},h,2}$. Consequently, we can find a set $\mathcal{O}_1$ that satisfies $[\mathcal{K}_{\mathbf{z},h,2}, \infty) \subseteq \mathcal{O}_1 \subseteq (\mathcal{K}_{\mathbf{z},h,1}, \infty)$, and for any $v \in \mathcal{O}_1$, the associated $H_{\mathbf{z},h,v}(x)$ is concave, decreasing, and satisfies $a_h H'_{\mathbf{z},h,v}(x) \leq b_h^2 H''_{\mathbf{z},h,v}(x) \leq 0$ on $0 < x < v - n_{h,v}(\mathbf{z})$ and $a_h H'_{\mathbf{z},h,v}(v - n_{h,v}(\mathbf{z})) = b_h^2 H''_{\mathbf{z},h,v}(v - n_{h,v}(\mathbf{z}))$. $\square$

In the following part, we consider $v \in \mathcal{O}_1$. Let us define $G_{\mathbf{z},h,v}(x)$ on $v - n_{h,v}(\mathbf{z}) \leq x \leq v$ that satisfies

$$-\frac{a_h^2 \left(G'_{\mathbf{z},h,v}(x)\right)^2}{2b_h^2 G''_{\mathbf{z},h,v}(x)} - \left(r + \sum_{k=1}^{m} \lambda_k(\mathbf{z})\right) G_{\mathbf{z},h,v}(x) + \sum_{l=1,l\neq h}^{m} \lambda_l(\mathbf{z}) \mathcal{W}_{\mathbf{z}^l,h}(v-x) = 0, \tag{4.9}$$

where

$$G_{\mathbf{z},h,v}\left(v - n_{h,v}(\mathbf{z})\right) = H_{\mathbf{z},h,v}\left(v - n_{h,v}(\mathbf{z})\right),$$
$$G'_{\mathbf{z},h,v}\left(v - n_{h,v}(\mathbf{z})\right) = H'_{\mathbf{z},h,v}\left(v - n_{h,v}(\mathbf{z})\right).$$

Lemma 4.3. *Assume $m \geq 2$. Fix $\mathbf{z} \in \mathcal{S}$ and $1 \leq h \leq m$. For any $v \in \mathcal{O}_1$, let $G_{\mathbf{z},h,v}(x)$ satisfy (4.9), $G_{\mathbf{z},h,v}(v - n_{h,v}(\mathbf{z})) = H_{\mathbf{z},h,v}(v - n_{h,v}(\mathbf{z}))$ and $G'_{\mathbf{z},h,v}(v - n_{h,v}(\mathbf{z})) = H'_{\mathbf{z},h,v}(v - n_{h,v}(\mathbf{z}))$. Then, there exists a set $\mathcal{O}_2$ where $[\mathcal{K}_{\mathbf{z},h,2}, \infty) \subseteq \mathcal{O}_2 \subseteq \mathcal{O}_1 \subseteq (\mathcal{K}_{\mathbf{z},h,1}, \infty)$. For any $v \in \mathcal{O}_2$, $G_{\mathbf{z},h,v}(x)$ is concave and decreasing with $b_h^2 G''_{\mathbf{z},h,v}(x) \leq a_h G'_{\mathbf{z},h,v}(x) < 0$ on $v - n_{h,v}(\mathbf{z}) < x < \hat{v}_{h,v}(\mathbf{z})$ where*

$$\hat{v}_{h,v}(\mathbf{z}) := \min\left\{\inf\left\{v - n_{h,v}(\mathbf{z}) < x \leq v : \left(r + \sum_{k=1}^{m}\lambda_k(\mathbf{z})\right) G_{\mathbf{z},h,v}(x) = \sum_{l=1,l\neq h}^{m} \lambda_l(\mathbf{z}) \mathcal{W}_{\mathbf{z}^l,h}(v-x)\right\}, v\right\}. \tag{4.10}$$

Proof. Let us rearrange (4.9) and have

$$-\frac{a_h^2 \left(G'_{\mathbf{z},h,v}(x)\right)^2}{2b_h^2 G''_{\mathbf{z},h,v}(x)} = \left(r + \sum_{k=1}^{m} \lambda_k(\mathbf{z})\right) G_{\mathbf{z},h,v}(x) - \sum_{l=1,l\neq h}^{m} \lambda_l(\mathbf{z}) \mathcal{W}_{\mathbf{z}^l,h}(v-x). \tag{4.11}$$

Assume $v > \mathcal{K}_{\mathbf{z},h,2}$. Following the argument in the proof of Lemma 4.2, we can find at least one M_2 that satisfies $0 < M_2 < \frac{a_h}{r+\lambda_h(\mathbf{z})} < \mathcal{K}_{\mathbf{z},h,2} < v$ and $\frac{a_h H'_{\mathbf{z},h,v}(M_2)}{b_h^2 H''_{\mathbf{z},h,v}(M_2)} = 1$, where we define $v - n_{h,v}(\mathbf{z})$ to be the smallest M_2. Thus, it holds that $G'_{\mathbf{z},h,v}(v - n_{h,v}(\mathbf{z})) < -1$ and

$$\left(r + \sum_{k=1}^{m} \lambda_k(\mathbf{z})\right) G'_{\mathbf{z},h,v}\left(v - n_{h,v}(\mathbf{z})\right) + \sum_{l=1,l\neq h}^{m} \lambda_l(\mathbf{z}) \mathcal{W}'_{\mathbf{z}^l,h}\left(n_{h,v}(\mathbf{z})\right) < -r - \lambda_h(\mathbf{z}) < 0.$$

We next show that $G''_{\mathbf{z},h,v}(x)$ and $G'_{\mathbf{z},h,v}(x)$ are negative and decreasing for $x > v - n_{h,v}(\mathbf{z})$ until the right-hand-side of (4.11) is zero.

We begin by proving $G''_{\mathbf{z},h,v}(v - n_{h,v}(\mathbf{z})) = H''_{\mathbf{z},h,v}(v - n_{h,v}(\mathbf{z}))$. From (4.11), (4.8), and $G_{\mathbf{z},h,v}(v - n_{h,v}(\mathbf{z})) = H_{\mathbf{z},h,v}(v - n_{h,v}(\mathbf{z}))$, we have

$$-\frac{a_h^2 \left(G'_{\mathbf{z},h,v}\left(v - n_{h,v}(\mathbf{z})\right)\right)^2}{2b_h^2 G''_{\mathbf{z},h,v}\left(v - n_{h,v}(\mathbf{z})\right)} = \frac{b_h^2}{2} H''_{\mathbf{z},h,v}\left(v - n_{h,v}(\mathbf{z})\right) - a_h H'_{\mathbf{z},h,v}\left(v - n_{h,v}(\mathbf{z})\right).$$

According to the definition of $n_{h,v}(\mathbf{z})$ given by (4.7), it follows that $b_h^2 H''_{\mathbf{z},h,v}(v - n_{h,v}(\mathbf{z})) = a_h H'_{\mathbf{z},h,v}(v - n_{h,v}(\mathbf{z}))$, which further yields $-\frac{a_h^2 (G'_{\mathbf{z},h,v}(v-n_{h,v}(\mathbf{z})))^2}{2b_h^2 G''_{\mathbf{z},h,v}(v-n_{h,v}(\mathbf{z}))} = -\frac{a_h}{2} H'_{\mathbf{z},h,v}(v - n_{h,v}(\mathbf{z}))$ and consequently $\frac{a_h (G'_{\mathbf{z},h,v}(v-n_{h,v}(\mathbf{z})))^2}{b_h^2 G''_{\mathbf{z},h,v}(v-n_{h,v}(\mathbf{z}))} = H'_{\mathbf{z},h,v}(v - n_{h,v}(\mathbf{z}))$. Recall that $G'_{\mathbf{z},h,v}(v - n_{h,v}(\mathbf{z})) = H'_{\mathbf{z},h,v}(v - n_{h,v}(\mathbf{z}))$, it follows immediately that $b_h^2 G''_{\mathbf{z},h,v}(v - n_{h,v}(\mathbf{z})) = a_h G'_{\mathbf{z},h,v}(v - n_{h,v}(\mathbf{z})) = a_h H'_{\mathbf{z},h,v}(v - n_{h,v}(\mathbf{z})) = b_h^2 H''_{\mathbf{z},h,v}(v - n_{h,v}(\mathbf{z}))$ as $H'_{\mathbf{z},h,v}(v - n_{h,v}(\mathbf{z}))$ is non-zero and negative from Lemma 4.2. Therefore, we obtain that $G''_{\mathbf{z},h,v}(v - n_{h,v}(\mathbf{z})) = H''_{\mathbf{z},h,v}(v - n_{h,v}(\mathbf{z}))$.

From Lemma 4.2, we immediately have $H''_{\mathbf{z},h,v}(v - n_{h,v}(\mathbf{z})) = G''_{\mathbf{z},h,v}(v - n_{h,v}(\mathbf{z})) < 0$ and $H'_{\mathbf{z},h,v}(v - n_{h,v}(\mathbf{z})) = G'_{\mathbf{z},h,v}(v - n_{h,v}(\mathbf{z})) < -1$ for any $v \in \mathcal{O}_1$, which also holds for $v > \mathcal{K}_{\mathbf{z},h,2}$. It also has been shown that the right-hand-side of (4.8) is decreasing. Given $a_h H'_{\mathbf{z},h,v}(v - n_{h,v}(\mathbf{z})) = b_h^2 H''_{\mathbf{z},h,v}(v - n_{h,v}(\mathbf{z}))$, $H'_{\mathbf{z},h,v}(v - n_{h,v}(\mathbf{z})) < -1$ and $H_{\mathbf{z},h,v}(v - n_{h,v}(\mathbf{z})) = G_{\mathbf{z},h,v}(v - n_{h,v}(\mathbf{z}))$, we obtain from (4.8) that

$$\frac{a_h}{2} < -\frac{a_h H'_{\mathbf{z},h,v}\left(v - n_{h,v}(\mathbf{z})\right)}{2} = \left(r + \sum_{k=1}^{m} \lambda_k(\mathbf{z})\right) G_{\mathbf{z},h,v}\left(v - n_{h,v}(\mathbf{z})\right) - \sum_{l=1,l\neq h}^{m} \lambda_l(\mathbf{z}) \mathcal{W}_{\mathbf{z}^l,h}\left(n_{h,v}(\mathbf{z})\right) < a_h.$$

Therefore, we have $(r + \sum_{k=1}^{m} \lambda_k(\mathbf{z})) G_{\mathbf{z},h,v}(x) - \sum_{l=1,l\neq h}^{m} \lambda_l(\mathbf{z}) \mathcal{W}_{\mathbf{z}^l,h}(v-x) > 0$, $G''_{\mathbf{z},h,v}(x) < 0$, and $G'_{\mathbf{z},h,v}(x) < -1$ on $v - n_{h,v}(\mathbf{z}) < x < v - n_{h,v}(\mathbf{z}) + \epsilon < \frac{a_h}{r+\lambda_h(\mathbf{z})}$ for some $\epsilon > 0$. Thus, it follows that $(r + \sum_{k=1}^{m} \lambda_k(\mathbf{z})) G'_{\mathbf{z},h,v}(x) + \sum_{l=1,l\neq h}^{m} \lambda_l(\mathbf{z}) \mathcal{W}'_{\mathbf{z}^l,h}(v-x) < -r - \lambda_h(\mathbf{z})$ for

$v-n_{h,v}(\mathbf{z})<x<v-n_{h,v}(\mathbf{z})+\epsilon$. We conclude that the right-hand-side of (4.11) is positive and decreasing $v-n_{h,v}(\mathbf{z})<x<v-n_{h,v}(\mathbf{z})+\epsilon$, which also holds for the left-hand-side of (4.11). Note that $G'_{\mathbf{z},h,v}(x)<-1$ is decreasing due to $G''_{\mathbf{z},h,v}(x)<0$ for $v-n_{h,v}(\mathbf{z})<x<v-n_{h,v}(\mathbf{z})+\epsilon$, $G''_{\mathbf{z},h,v}(x)<0$ is also decreasing for $v-n_{h,v}(\mathbf{z})<x<v-n_{h,v}(\mathbf{z})+\epsilon$ in order to guarantee that the left-hand-side of (4.11) is decreasing. Thus, we conclude that $G''_{\mathbf{z},h,v}(x)<0$ and $G'_{\mathbf{z},h,v}(x)<-1$ are both decreasing as x increases from $v-n_{h,v}(\mathbf{z})$ given that the right-hand-side of (4.11) is positive and $x<\frac{a_h}{r+\lambda_h(\mathbf{z})}$.

We next show that the right-hand-side of (4.11) decreases to zero before x hits $\frac{a_h}{r+\lambda_h(\mathbf{z})}$. Recall that the right-hand-side of (4.8) decreases from a_h to $-\frac{a_h H'_{\mathbf{z},h,v}(v-n_{h,v}(\mathbf{z}))}{2}>0$ for $0\le x\le v-n_{h,v}(\mathbf{z})$. Since $(r+\sum_{k=1}^m \lambda_k(\mathbf{z}))H'_{\mathbf{z},h,v}(x)+\sum_{l=1,l\neq h}^m \lambda_l(\mathbf{z})\mathcal{W}'_{\mathbf{z}^l,h}(v-x)<-r-\lambda_h(\mathbf{z})$ for $0<x<v-n_{h,v}(\mathbf{z})$, it holds that

$$0<v-n_{h,v}(\mathbf{z})<\frac{a_h+\frac{a_h H'_{\mathbf{z},h,v}(v-n_{h,v}(\mathbf{z}))}{2}}{r+\lambda_h(\mathbf{z})}.$$

Rearranging the above inequality leads to

$$-\frac{a_h H'_{\mathbf{z},h,v}(v-n_{h,v}(\mathbf{z}))}{2}<(r+\lambda_h(\mathbf{z}))\left[\frac{a_h}{r+\lambda_h(\mathbf{z})}-\left(v-n_{h,v}(\mathbf{z})\right)\right]. \tag{4.12}$$

Also, it has been shown that $(r+\sum_{k=1}^m \lambda_k(\mathbf{z}))G'_{\mathbf{z},h,v}(x)+\sum_{l=1,l\neq h}^m \lambda_l(\mathbf{z})\mathcal{W}'_{\mathbf{z}^l,h}(v-x)<-r-\lambda_h(\mathbf{z})$ as x increases from $v-n_{h,v}(\mathbf{z})$ given that $x<\frac{a_h}{r+\lambda_h(\mathbf{z})}$ and $(r+\sum_{k=1}^m \lambda_k(\mathbf{z}))G_{\mathbf{z},h,v}(x)-\sum_{l=1,l\neq h}^m \lambda_l(\mathbf{z})\mathcal{W}_{\mathbf{z}^l,h}(v-x)$ decreases from $-\frac{a_h H'_{\mathbf{z},h,v}(v-n_{h,v}(\mathbf{z}))}{2}>0$ but remains positive. We show that the right-hand-side of (4.11) first attains zero at some $v-n_{h,v}(\mathbf{z})<M_3<\frac{a_h}{r+\lambda_h(\mathbf{z})}$ by contradiction. Assume the right-hand-side of (4.11) remains positive for $v-n_{h,v}(\mathbf{z})<x<\frac{a_h}{r+\lambda_h(\mathbf{z})}$. Then, we would have $G'_{\mathbf{z},h,v}(x)<-1$, $\mathcal{W}'_{\mathbf{z}^l,h}(v-x)=1$, and $(r+\sum_{k=1}^m \lambda_k(\mathbf{z}))G'_{\mathbf{z},h,v}(x)+\sum_{l=1,l\neq h}^m \lambda_l(\mathbf{z})\mathcal{W}'_{\mathbf{z}^l,h}(v-x)<-r-\lambda_h(\mathbf{z})$ for $v-n_{h,v}(\mathbf{z})<x<\frac{a_h}{r+\lambda_h(\mathbf{z})}$. Then, we would obtain

$$-\frac{a_h H'_{\mathbf{z},h,v}(v-n_{h,v}(\mathbf{z}))}{2}>(r+\lambda_h(\mathbf{z}))\left[\frac{a_h}{r+\lambda_h(\mathbf{z})}-\left(v-n_{h,v}(\mathbf{z})\right)\right],$$

which contradicts to (4.12). Thus, we deduce that the right-hand-side of (4.11) first hits zero at some $v-n_{h,v}(\mathbf{z})<M_3<\frac{a_h}{r+\lambda_h(\mathbf{z})}$, where M_3 is equivalent to $\hat{v}_{h,v}(\mathbf{z})$ given by (4.10).

We proceed to show that $b_h^2 G''_{\mathbf{z},h,v}(x)\le a_h G'_{\mathbf{z},h,v}(x)<0$ on $v-n_{h,v}(\mathbf{z})\le x<\hat{v}_{h,v}(\mathbf{z})$. We have deduced that $G''_{\mathbf{z},h,v}(x)<0$ and $G'_{\mathbf{z},h,v}(x)<-1$ are both decreasing as x increases from $v-n_{h,v}(\mathbf{z})$ given that the right-hand-side of (4.11) is positive and $x<\frac{a_h}{r+\lambda_h(\mathbf{z})}$. We also showed that the right-hand-side of (4.11) decreases to zero at $\hat{v}_{h,v}(\mathbf{z})$ where $\hat{v}_{h,v}(\mathbf{z})<\frac{a_h}{r+\lambda_h(\mathbf{z})}$. Therefore, we conclude that for $v-n_{h,v}(\mathbf{z})<x<\hat{v}_{h,v}(\mathbf{z})$, $G''_{\mathbf{z},h,v}(x)<0$, $G'_{\mathbf{z},h,v}(x)<-1$, and $-\frac{a_h^2(G'_{\mathbf{z},h,v}(x))^2}{2b_h^2 G''_{\mathbf{z},h,v}(x)}>0$ are decreasing. Thus, $-G''_{\mathbf{z},h,v}(x)>0$ is increasing faster than $(G'_{\mathbf{z},h,v}(x))^2>1$, where $(G'_{\mathbf{z},h,v}(x))^2>1$ is increasing faster than $-G'_{\mathbf{z},h,v}(x)>1$ for $v-n_{h,v}(\mathbf{z})<x<\hat{v}_{h,v}(\mathbf{z})$. Hence, $\frac{a_h G'_{\mathbf{z},h,v}(x)}{b_h^2 G''_{\mathbf{z},h,v}(x)}>0$ is decreasing for $v-n_{h,v}(\mathbf{z})<x<\hat{v}_{h,v}(\mathbf{z})$. Recall that $a_h H'_{\mathbf{z},h,v}(v-n_{h,v}(\mathbf{z}))=b_h^2 H''_{\mathbf{z},h,v}(v-n_{h,v}(\mathbf{z}))$, $H''_{\mathbf{z},h,v}(v-n_{h,v}(\mathbf{z}))=G''_{\mathbf{z},h,v}(v-n_{h,v}(\mathbf{z}))<0$ and $H'_{\mathbf{z},h,v}(v-n_{h,v}(\mathbf{z}))=G'_{\mathbf{z},h,v}(v-n_{h,v}(\mathbf{z}))<-1$, it holds that $0<\frac{a_h G'_{\mathbf{z},h,v}(x)}{b_h^2 G''_{\mathbf{z},h,v}(x)}\le 1$ for $v-n_{h,v}(\mathbf{z})\le x<\hat{v}_{h,v}(\mathbf{z})$, which leads to $b_h^2 G''_{\mathbf{z},h,v}(x)\le a_h G'_{\mathbf{z},h,v}(x)<0$ on $v-n_{h,v}(\mathbf{z})\le x<\hat{v}_{h,v}(\mathbf{z})$.

The above results are deduced based on $v\in\mathcal{O}_1$ and $v>\mathcal{K}_{\mathbf{z},h,2}$. Thus, we can find a set $\mathcal{O}_2$ that satisfies $[\mathcal{K}_{\mathbf{z},h,2},\infty)\subseteq\mathcal{O}_2\subseteq\mathcal{O}_1\subseteq(\mathcal{K}_{\mathbf{z},h,1},\infty)$, and for any $v\in\mathcal{O}_2$, the associated $G_{\mathbf{z},h,v}(x)$ is concave, decreasing, and satisfies $b_h^2 G''_{\mathbf{z},h,v}(x)\le a_h G'_{\mathbf{z},h,v}(x)<0$ on $v-n_{h,v}(\mathbf{z})<x<\hat{v}_{h,v}(\mathbf{z})$. $\square$

Note that $(r+\sum_{k=1}^m \lambda_k(\mathbf{z}))G_{\mathbf{z},h,v}(\hat{v}_{h,v}(\mathbf{z}))-\sum_{l=1,l\neq h}^m \lambda_l(\mathbf{z})\mathcal{W}_{\mathbf{z}^l,h}(v-\hat{v}_{h,v}(\mathbf{z}))=0$, it follows that $\lim_{x\uparrow\hat{v}_{h,v}(\mathbf{z})}-\frac{a_h^2(G'_{\mathbf{z},h,v}(x))^2}{2b_h^2 G''_{\mathbf{z},h,v}(x)}=0$. Thus, $\lim_{x\uparrow\hat{v}_{h,v}(\mathbf{z})}G''_{\mathbf{z},h,v}(x)=-\infty$ given $\lim_{x\uparrow\hat{v}_{h,v}(\mathbf{z})}G'_{\mathbf{z},h,v}(x)<-1$, which further leads to $\lim_{x\uparrow\hat{v}_{h,v}(\mathbf{z})}\frac{a_h G'_{\mathbf{z},h,v}(x)}{b_h^2 G''_{\mathbf{z},h,v}(x)}=0$. Let us rewrite (4.9) in the following form,

$$G''_{\mathbf{z},h,v}(x)=-\frac{a_h^2\left(G'_{\mathbf{z},h,v}(x)\right)^2}{2b_h^2\left[\left(r+\sum_{k=1}^m \lambda_k(\mathbf{z})\right)G_{\mathbf{z},h,v}(x)-\sum_{l=1,l\neq h}^m \lambda_l(\mathbf{z})\mathcal{W}_{\mathbf{z}^l,h}(v-x)\right]}. \tag{4.13}$$

For $v-n_{h,v}(\mathbf{z})\le x<\hat{v}_{h,v}(\mathbf{z})$, the right-hand-side of (4.13) is continuous and the associated initial value problem admits solution of C^2 class. In addition, we assume that the initial value problem (4.9) with $G_{\mathbf{z},h,v}(v-n_{h,v}(\mathbf{z}))=H_{\mathbf{z},h,v}(v-n_{h,v}(\mathbf{z}))$ and $G'_{\mathbf{z},h,v}(v-n_{h,v}(\mathbf{z}))=H'_{\mathbf{z},h,v}(v-n_{h,v}(\mathbf{z}))$ is well-posed on $v-n_{h,v}(\mathbf{z})\le x\le\hat{v}_{h,v}(\mathbf{z})$.

Remark 4.4. Due to the complexity of the non-linear ODE (4.9), we assumed the well-posedness of the associated initial value problem. In the following, we will further discuss local well-posedness of this problem to show that the proposed assumption is reasonable. As proven in Lemma 4.3, $G_{\mathbf{z},h,v}(x)$ is continuous and satisfies $G_{\mathbf{z},h,v}(\hat{v}_{h,v}(\mathbf{z}))\le G_{\mathbf{z},h,v}(x)\le H_{\mathbf{z},h,v}(v-n_{h,v}(\mathbf{z}))$ for $v-n_{h,v}(\mathbf{z})\le x\le\hat{v}_{h,v}(\mathbf{z})$ where $G_{\mathbf{z},h,v}(\hat{v}_{h,v}(\mathbf{z}))\ge 0$. In addition, it has been shown that $G'_{\mathbf{z},h,v}(x)$ and $G''_{\mathbf{z},h,v}(x)$ are continuous and bounded for $v-n_{h,v}(\mathbf{z})\le x\le\hat{v}_{h,v}(\mathbf{z})-\epsilon$

where $0<\epsilon<\hat{v}_{h,v}(\mathbf{z})-v+n_{h,v}(\mathbf{z})$. Thus, the right-hand-side of (4.13) is continuous in x, $G_{\mathbf{z},h,v}(x)$ and $G'_{\mathbf{z},h,v}(x)$ for $v-n_{h,v}(\mathbf{z})\le x<\hat{v}_{h,v}(\mathbf{z})$ and Lipschitz continuous in $G_{\mathbf{z},h,v}(x)$ and $G'_{\mathbf{z},h,v}(x)$ for $v-n_{h,v}(\mathbf{z})\le x\le\hat{v}_{h,v}(\mathbf{z})-\epsilon$ where $0<\epsilon<\hat{v}_{h,v}(\mathbf{z})-v+n_{h,v}(\mathbf{z})$. Therefore, the initial value problem (4.9) is well-posed (see e.g., Burden and Faires (2010, section 5.1)) and admits a unique solution on $v-n_{h,v}(\mathbf{z})\le x\le\hat{v}_{h,v}(\mathbf{z})-\epsilon$ for any $0<\epsilon<\hat{v}_{h,v}(\mathbf{z})-v+n_{h,v}(\mathbf{z})$. Also, there exists continuous solution on $v-n_{h,v}(\mathbf{z})\le x\le\hat{v}_{h,v}(\mathbf{z})$ where $G_{\mathbf{z},h,v}(\hat{v}_{h,v}(\mathbf{z}))=\frac{\sum_{l=1,l\neq h}^{m}\lambda_l(\mathbf{z})\mathcal{W}_{\mathbf{z}^l,h}(v-\hat{v}_{h,v}(\mathbf{z}))}{r+\sum_{k=1}^{m}\lambda_k(\mathbf{z})}\ge 0$. Given the local well-posedness and the existence of continuous solution for $v-n_{h,v}(\mathbf{z})\le x\le\hat{v}_{h,v}(\mathbf{z})$ with $G_{\mathbf{z},h,v}(\hat{v}_{h,v}(\mathbf{z}))=\frac{\sum_{l=1,l\neq h}^{m}\lambda_l(\mathbf{z})\mathcal{W}_{\mathbf{z}^l,h}(v-\hat{v}_{h,v}(\mathbf{z}))}{r+\sum_{k=1}^{m}\lambda_k(\mathbf{z})}$, we assume the well-posedness for $v-n_{h,v}(\mathbf{z})\le x\le\hat{v}_{h,v}(\mathbf{z})$.

Similar to Yao et al. (2011), we assume in Theorem 3.1 that there exists an increasing, concave and C^2 solution to the free boundary problem (3.1) and show that the solution divides the interval $[0,\infty)$ into three regions: the risk control region $\mathcal{R}:=\{x\ge 0: 0\le-\frac{a_h\mathcal{W}'_{\mathbf{z},h}(x)}{b_h^2\mathcal{W}''_{\mathbf{z},h}(x)}<1, 1-\mathcal{W}'_{\mathbf{z},h}(x)<0\}$, continuation region $\mathcal{C}:=\{x\ge 0:-\frac{a_h\mathcal{W}'_{\mathbf{z},h}(x)}{b_h^2\mathcal{W}''_{\mathbf{z},h}(x)}\ge 1, 1-\mathcal{W}'_{\mathbf{z},h}(x)<0\}$, and dividend payments region $\mathcal{D}:=\{x\ge 0:1-\mathcal{W}'_{\mathbf{z},h}(x)=0\}$. As deduced in Theorem 3.1, $\mathcal{W}_{\mathbf{z},h}$ measures the cumulative expected dividend under optimal strategies of the group where the optimal reinsurance strategy of subsidiary h is given by $p_h^*(t)=p_h(X_h^*(t),\mathbf{Z}(t))$. Choulli et al. (2003) has shown that it is optimal for a subsidiary to minimize the risk exposure when the reserve is low, gradually increase the exposure while the reserve level increases, and maintain the maximum risk level when the reserve level exceeds a certain threshold. Given the insurance group's initial default state $\mathbf{z}$ and subsidiary h's initial surplus x, it is optimal for subsidiary h to adjust the risk exposure level to $p_h(x,\mathbf{z})=-\frac{a_h\mathcal{W}'_{\mathbf{z},h}(x)}{b_h^2\mathcal{W}''_{\mathbf{z},h}(x)}\wedge 1$, i.e., adjust to $-\frac{a_h\mathcal{W}'_{\mathbf{z},h}(x)}{b_h^2\mathcal{W}''_{\mathbf{z},h}(x)}$ when the surplus level x does not hit the threshold $n_h(\mathbf{z}):=\inf\{x>0:-\frac{a_h\mathcal{W}'_{\mathbf{z},h}(x)}{b_h^2\mathcal{W}''_{\mathbf{z},h}(x)}=1\}$ given the existence of $n_h(\mathbf{z})$, or to the maximum risk level 1 if $x\ge n_h(\mathbf{z})$. In our problem setting, the dividend payout rate is unbounded, and the dividend can be distributed immediately when the reserve is high. Thus, it is optimal for the subsidiary to maintain the maximum risk level and not to distribute dividends until the reserve hits the optimal dividend barrier, as described by (3.4).

It is common in the existing works for the optimal control problems to assume the existence of an increasing and concave solution to the HJB equation of certain differentiability class, formulate the candidate functions for the solution, and show that the candidate function indeed solves the optimal control problems; see Yao et al. (2011) and Feng et al. (2021). In a similar vein, we assume the existence of an increasing, concave and C^2 solution to (3.1) and its boundaries $0\le n_h(\mathbf{z})\le v_h(\mathbf{z})<\infty$, and conclude that $\mathcal{R}=[0,n_h(\mathbf{z}))$, $\mathcal{C}=[n_h(\mathbf{z}),v_h(\mathbf{z}))$, and $\mathcal{D}=[v_h(\mathbf{z}),\infty)$. In the following proposition, we formulate the group of functions $\hat{\mathcal{W}}_{\mathbf{z},h,v}(x)$ for all $v\in\mathcal{O}_2$ that satisfy the HJBVIs similar to (3.1), where the group of functions satisfy $v=\inf\{x>0:1-\hat{\mathcal{W}}'_{\mathbf{z},h,v}(x)=0\}$ and $n_{h,v}(\mathbf{z})=\inf\{x>0:-\frac{a_h\hat{\mathcal{W}}'_{\mathbf{z},h,v}(x)}{b_h^2\hat{\mathcal{W}}''_{\mathbf{z},h,v}(x)}=1\}$. Given the existence of $0\le n_h(\mathbf{z})\le v_h(\mathbf{z})<\infty$, we show later that if v satisfies certain condition, the associated candidate function $\hat{\mathcal{W}}_{\mathbf{z},h,v}(x)$ solves the (3.1) subject to $\hat{\mathcal{W}}_{\mathbf{z},h,v}(0)=0$ where $v=v_h(\mathbf{z})$ and $n_{h,v}(\mathbf{z})=n_h(\mathbf{z})$.

Proposition 4.5. *Assume $m\ge 2$ and $v\in\mathcal{O}_2$. Fix $\mathbf{z}\in\mathcal{S}$ and $1\le h\le m$. For $x\ge v-\hat{v}_{h,v}(\mathbf{z})$, let us define $\hat{\mathcal{W}}_{\mathbf{z},h,v}(x)$ by*

$$\hat{\mathcal{W}}_{\mathbf{z},h,v}(x)=\begin{cases}G_{\mathbf{z},h,v}(v-x), & v-\hat{v}_{h,v}(\mathbf{z})\le x<n_{h,v}(\mathbf{z}),\\ H_{\mathbf{z},h,v}(v-x), & n_{h,v}(\mathbf{z})\le x<v,\\ w_{\mathbf{z},h,v}+x-v, & x\ge v,\end{cases}\tag{4.14}$$

where $w_{\mathbf{z},h,v}=H_{\mathbf{z},h,v}(0)=\frac{\sum_{l=1,l\neq h}^{m}\lambda_l(\mathbf{z})\mathcal{W}_{\mathbf{z}^l,h}(v)+a_h}{r+\sum_{k=1}^{m}\lambda_k(\mathbf{z})}$. Then, $\hat{\mathcal{W}}_{\mathbf{z},h,v}(x)$ is of C^2, concave and increasing that satisfies

$$\max\left\{\max_{0\le p_h\le 1}\left\{\mathcal{A}^{p_h,\mathbf{z},h}\hat{\mathcal{W}}_{\mathbf{z},h,v}(x)+\sum_{l=1,l\neq h}^{m}\lambda_l(\mathbf{z})\mathcal{W}_{\mathbf{z}^l,h}(x)\right\},1-\hat{\mathcal{W}}'_{\mathbf{z},h,v}(x)\right\}=0,\quad x>v-\hat{v}_{h,v}(\mathbf{z}),\tag{4.15}$$

with $\hat{\mathcal{W}}_{\mathbf{z},h,v}(v-\hat{v}_{h,v}(\mathbf{z}))\ge 0$. Furthermore, we assume the existence of an increasing, concave and C^2 solution to (3.1) and its boundaries $0\le n_h(\mathbf{z})\le v_h(\mathbf{z})<\infty$. Then $\hat{\mathcal{W}}_{\mathbf{z},h,v}(x)$ satisfies the variational inequality (3.1) with initial condition $\hat{\mathcal{W}}_{\mathbf{z},h,v}(0)=0$ if

$$\inf\left\{v-n_{h,v}(\mathbf{z})<x\le v:\left(r+\sum_{k=1}^{m}\lambda_k(\mathbf{z})\right)G_{\mathbf{z},h,v}(x)=\sum_{l=1,l\neq h}^{m}\lambda_l(\mathbf{z})\mathcal{W}_{\mathbf{z}^l,h}(v-x)\right\}=v.\tag{4.16}$$

Proof. We have deduced before that $H_{\mathbf{z},h,v}(x)$ and $G_{\mathbf{z},h,v}(x)$ are concave and decreasing on $0\le x\le v-n_{h,v}(\mathbf{z})$ and $v-n_{h,v}(\mathbf{z})\le x\le\hat{v}_{h,v}(\mathbf{z})$ respectively for $v\in\mathcal{O}_2$. Therefore, $G_{\mathbf{z},h,v}(v-x)$ and $H_{\mathbf{z},h,v}(v-x)$ are concave and increasing on $v-\hat{v}_{h,v}(\mathbf{z})\le x\le n_{h,v}(\mathbf{z})$ and $n_{h,v}(\mathbf{z})\le x\le v$ respectively for $v\in\mathcal{O}_2$. Also, it can be easily shown that $\hat{\mathcal{W}}_{\mathbf{z},h,v}(x)$ is of C^2 since $G''_{\mathbf{z},h,v}(v-n_{h,v}(\mathbf{z}))=H''_{\mathbf{z},h,v}(v-n_{h,v}(\mathbf{z}))$. Given that $G_{\mathbf{z},h,v}(x)$ satisfies (4.9) and $H_{\mathbf{z},h,v}(x)$ satisfies (4.4), it follows immediately that $\hat{\mathcal{W}}_{\mathbf{z},h,v}(x)$ given by (4.14) satisfies

$$\begin{cases}-\dfrac{a_h^2\left(\hat{\mathcal{W}}'_{\mathbf{z},h,v}(x)\right)^2}{2b_h^2\hat{\mathcal{W}}''_{\mathbf{z},h,v}(x)}-\left(r+\sum_{k=1}^{m}\lambda_k(\mathbf{z})\right)\hat{\mathcal{W}}_{\mathbf{z},h,v}(x)+\sum_{l=1,l\neq h}^{m}\lambda_l(\mathbf{z})\mathcal{W}_{\mathbf{z}^l,h}(x)=0, & v-\hat{v}_{h,v}(\mathbf{z})<x<n_{h,v}(\mathbf{z}),\\ \dfrac{b_h^2}{2}\hat{\mathcal{W}}''_{\mathbf{z},h,v}(x)+a_h\hat{\mathcal{W}}'_{\mathbf{z},h,v}(x)-\left(r+\sum_{k=1}^{m}\lambda_k(\mathbf{z})\right)\hat{\mathcal{W}}_{\mathbf{z},h,v}(x)+\sum_{l=1,l\neq h}^{m}\lambda_l(\mathbf{z})\mathcal{W}_{\mathbf{z}^l,h}(x)=0, & n_{h,v}(\mathbf{z})\le x<v,\\ 1-\hat{\mathcal{W}}'_{\mathbf{z},h,v}(x)=0, & x\ge v.\end{cases}\tag{4.17}$$

Since $\hat{\mathcal{W}}_{\mathbf{z},h,v}(x)$ is concave with $\hat{\mathcal{W}}'_{\mathbf{z},h,v}(v)=1$, it holds that $\hat{\mathcal{W}}'_{\mathbf{z},h,v}(x)\geq 1$ for $v-\hat{v}_{h,v}(\mathbf{z})<x<v$. Further, $\hat{\mathcal{W}}_{\mathbf{z},h,v}(x)$ satisfies $\max_{0\leq p_h\leq 1}\{\mathcal{A}^{p_h,\mathbf{z},h}\hat{\mathcal{W}}_{\mathbf{z},h,v}(x)+\sum_{l=1,l\neq h}^{m}\lambda_l(\mathbf{z})\mathcal{W}_{\mathbf{z}^l,h}(x)\}=0$ where $p_h=-\frac{a_h\hat{\mathcal{W}}'_{\mathbf{z},h,v}(x)}{b_h^2\hat{\mathcal{W}}''_{\mathbf{z},h,v}(x)}$ for $v-\hat{v}_{h,v}(\mathbf{z})<x<n_{h,v}(\mathbf{z})$ since we have $0<G'_{\mathbf{z},h,v}(x)/G''_{\mathbf{z},h,v}(x)<b_h^2/a_h$ for $v-n_{h,v}(\mathbf{z})<x<\hat{v}_{h,v}(\mathbf{z})$. Also, $\hat{\mathcal{W}}_{\mathbf{z},h,v}(x)$ satisfies $\max_{0\leq p_h\leq 1}\{\mathcal{A}^{p_h,\mathbf{z},h}\hat{\mathcal{W}}_{\mathbf{z},h,v}(x)+\sum_{l=1,l\neq h}^{m}\lambda_l(\mathbf{z})\mathcal{W}_{\mathbf{z}^l,h}(x)\}=0$ where $p_h=1$ for $n_{h,v}(\mathbf{z})\leq x<v$ as we have $\frac{a_hH'_{\mathbf{z},h,v}(v-x)}{b_h^2H''_{\mathbf{z},h,v}(v-x)}\geq 1$ and therefore $-\frac{a_h\hat{\mathcal{W}}'_{\mathbf{z},h,v}(x)}{b_h^2\hat{\mathcal{W}}''_{\mathbf{z},h,v}(x)}\geq 1$ for $n_{h,v}(\mathbf{z})\leq x<v$.

It remains to show $\hat{\mathcal{W}}_{\mathbf{z},h,v}(x)$ satisfies $\max_{0\leq p_h\leq 1}\{\mathcal{A}^{p_h,\mathbf{z},h}\hat{\mathcal{W}}_{\mathbf{z},h,v}(x)+\sum_{l=1,l\neq h}^{m}\lambda_l(\mathbf{z})\mathcal{W}_{\mathbf{z}^l,h}(x)\}\leq 0$ for $x\geq v$. It holds that $p_h=1$ for $x\geq v$ since $\mathcal{A}^{p_h,\mathbf{z},h}\hat{\mathcal{W}}_{\mathbf{z},h,v}(x)=a_hp_h-(r+\sum_{k=1}^{m}\lambda_k(\mathbf{z}))\hat{\mathcal{W}}_{\mathbf{z},h,v}(x)$. We deduce $\lim_{x\uparrow v}\hat{\mathcal{W}}'''_{\mathbf{z},h,v}(x)\geq 0$ since $\hat{\mathcal{W}}_{\mathbf{z},h,v}(x)$ is concave for $v-\hat{v}_{h,v}(\mathbf{z})<x<v$ and $\hat{\mathcal{W}}''_{\mathbf{z},h,v}(v)=0$. It follows naturally that for $n_{h,v}(\mathbf{z})<x<v$, $\hat{\mathcal{W}}_{\mathbf{z},h,v}(x)$ also satisfies

$$\frac{b_h^2}{2}\hat{\mathcal{W}}'''_{\mathbf{z},h,v}(x)+a_h\hat{\mathcal{W}}''_{\mathbf{z},h,v}(x)-\left(r+\sum_{k=1}^{m}\lambda_k(\mathbf{z})\right)\hat{\mathcal{W}}'_{\mathbf{z},h,v}(x)+\sum_{l=1,l\neq h}^{m}\lambda_l(\mathbf{z})\mathcal{W}'_{\mathbf{z}^l,h}(x)=0. \tag{4.18}$$

Subsequently, we deduce $-(r+\sum_{k=1}^{m}\lambda_k(\mathbf{z}))+\sum_{l=1,l\neq h}^{m}\lambda_l(\mathbf{z})\mathcal{W}'_{\mathbf{z}^l,h}(v)=-\lim_{x\uparrow v}b_h^2\hat{\mathcal{W}}'''_{\mathbf{z},h,v}(x)/2\leq 0$. Since $\mathcal{W}_{\mathbf{z}^l,h}(x)$ is concave, we have $\sum_{l=1,l\neq h}^{m}\lambda_l(\mathbf{z})\mathcal{W}'_{\mathbf{z}^l,h}(x)\leq\sum_{l=1,l\neq h}^{m}\lambda_l(\mathbf{z})\mathcal{W}'_{\mathbf{z}^l,h}(v)$ for $x\geq v$ and thus $-(r+\sum_{k=1}^{m}\lambda_k(\mathbf{z}))+\sum_{l=1,l\neq h}^{m}\lambda_l(\mathbf{z})\mathcal{W}'_{\mathbf{z}^l,h}(x)\leq 0$ for $x\geq v$. Since $p_h=1$ for $x>n_{h,v}(\mathbf{z})$ and we have $\frac{b_h^2}{2}\hat{\mathcal{W}}''_{\mathbf{z},h,v}(v)+a_h\hat{\mathcal{W}}'_{\mathbf{z},h,v}(v)-(r+\sum_{k=1}^{m}\lambda_k(\mathbf{z}))\hat{\mathcal{W}}_{\mathbf{z},h,v}(v)+\sum_{l=1,l\neq h}^{m}\lambda_l(\mathbf{z})\mathcal{W}_{\mathbf{z}^l,h}(v)=0$, we conclude $\frac{b_h^2}{2}\hat{\mathcal{W}}''_{\mathbf{z},h,v}(x)+a_h\hat{\mathcal{W}}'_{\mathbf{z},h,v}(x)-(r+\sum_{k=1}^{m}\lambda_k(\mathbf{z}))\hat{\mathcal{W}}_{\mathbf{z},h,v}(x)+\sum_{l=1,l\neq h}^{m}\lambda_l(\mathbf{z})\mathcal{W}_{\mathbf{z}^l,h}(x)\leq 0$ for $x\geq v$. Thus, $\hat{\mathcal{W}}_{\mathbf{z},h,v}(x)$ satisfies $\max_{0\leq p_h\leq 1}\{\mathcal{A}^{p_h,\mathbf{z},h}\hat{\mathcal{W}}_{\mathbf{z},h,v}(x)+\sum_{l=1,l\neq h}^{m}\lambda_l(\mathbf{z})\mathcal{W}_{\mathbf{z}^l,h}(x)\}\leq 0$ for $x\geq v$. It follows easily that $\hat{\mathcal{W}}_{\mathbf{z},h,v}(x)$ satisfies (4.15).

We next show that $\hat{\mathcal{W}}_{\mathbf{z},h,v}(v-\hat{v}_{h,v}(\mathbf{z}))\geq 0$ and deduce the criterion that $\hat{\mathcal{W}}_{\mathbf{z},h,v}(x)$ satisfies (3.1). If $\hat{v}_{h,v}(\mathbf{z})<v$, it follows directly from (4.10) that we can find the $\hat{v}_{h,v}(\mathbf{z})$ that satisfies $v-n_{h,v}(\mathbf{z})<\hat{v}_{h,v}(\mathbf{z})<v$ and

$$\left(r+\sum_{k=1}^{m}\lambda_k(\mathbf{z})\right)G_{\mathbf{z},h,v}\left(\hat{v}_{h,v}(\mathbf{z})\right)=\sum_{l=1,l\neq h}^{m}\lambda_l(\mathbf{z})\mathcal{W}_{\mathbf{z}^l,h}\left(v-\hat{v}_{h,v}(\mathbf{z})\right)>0.$$

Note that we have $\sum_{l=1,l\neq h}^{m}\lambda_l(\mathbf{z})\mathcal{W}_{\mathbf{z}^l,h}(v-\hat{v}_{h,v}(\mathbf{z}))>0$ since $\mathcal{W}_{\mathbf{z}^l,h}(x)$ is increasing subject to $\mathcal{W}_{\mathbf{z}^l,h}(0)=0$ for all $1\leq 1\neq h\leq m$. Therefore, we have $\hat{\mathcal{W}}_{\mathbf{z},h,v}(v-\hat{v}_{h,v}(\mathbf{z}))=G_{\mathbf{z},h,v}(\hat{v}_{h,v}(\mathbf{z}))>0$.

Next, if $\hat{v}_{h,v}(\mathbf{z})=v$, it follows that

$$\left(r+\sum_{k=1}^{m}\lambda_k(\mathbf{z})\right)G_{\mathbf{z},h,v}(x)-\sum_{l=1,l\neq h}^{m}\lambda_l(\mathbf{z})\mathcal{W}_{\mathbf{z}^l,h}(v-x)\geq 0,\quad v-n_{h,v}(\mathbf{z})\leq x\leq v.$$

We first consider the case where the above inequality is strictly positive for $v-n_{h,v}(\mathbf{z})\leq x\leq v$. Since $\sum_{l=1,l\neq h}^{m}\lambda_l(\mathbf{z})\mathcal{W}_{\mathbf{z}^l,h}(v-x)\geq 0$ for $v-n_{h,v}(\mathbf{z})\leq x\leq v$, it holds that $G_{\mathbf{z},h,v}(x)>0$ for all $v-n_{h,v}(\mathbf{z})\leq x\leq v$, which follows $\hat{\mathcal{W}}_{\mathbf{z},h,v}(v-\hat{v}_{h,v}(\mathbf{z}))>0$. If $\hat{v}_{h,v}(\mathbf{z})=v$ and the equality holds at $x=v$, we have

$$\left(r+\sum_{k=1}^{m}\lambda_k(\mathbf{z})\right)G_{\mathbf{z},h,v}(v)-\sum_{l=1,l\neq h}^{m}\lambda_l(\mathbf{z})\mathcal{W}_{\mathbf{z}^l,h}(0)=0.$$

We obtain $G_{\mathbf{z},h,v}(v)=G_{\mathbf{z},h,v}(\hat{v}_{h,v}(\mathbf{z}))=0$ since $\mathcal{W}_{\mathbf{z}^l,h}(0)=0$ for all $1\leq h\leq m$. Consequently, we have $G_{\mathbf{z},h,v}(\hat{v}_{h,v}(\mathbf{z}))=\hat{\mathcal{W}}_{\mathbf{z},h,v}(v-\hat{v}_{h,v}(\mathbf{z}))=\hat{\mathcal{W}}_{\mathbf{z},h,v}(0)=0$. Thus, we conclude that $\hat{\mathcal{W}}_{\mathbf{z},h,v}(x)$ satisfies

$$\max\left\{\max_{0\leq p_h\leq 1}\left\{\mathcal{A}^{p_h,\mathbf{z},h}\hat{\mathcal{W}}_{\mathbf{z},h,v}(x)+\sum_{l=1,l\neq h}^{m}\lambda_l(\mathbf{z})\mathcal{W}_{\mathbf{z}^l,h}(x)\right\},1-\hat{\mathcal{W}}'_{\mathbf{z},h,v}(x)\right\}=0,\quad x>0,$$

with initial condition $\hat{\mathcal{W}}_{\mathbf{z},h,v}(0)=0$ when

$$\inf\left\{v-n_{h,v}(\mathbf{z})<x\leq v:\left(r+\sum_{k=1}^{m}\lambda_k(\mathbf{z})\right)G_{\mathbf{z},h,v}(x)=\sum_{l=1,l\neq h}^{m}\lambda_l(\mathbf{z})\mathcal{W}_{\mathbf{z}^l,h}(v-x)\right\}=v.$$

It follows naturally that $n_{h,v}(\mathbf{z})=n_h(\mathbf{z})$ and $v=v_h(\mathbf{z})$. $\square$

Remark 4.6. Indeed, if $v\in[\mathcal{K}_{\mathbf{z},h,2},\infty)$, it is shown in Lemma 4.3 that $v-n_{h,v}(\mathbf{z})<M_3=\hat{v}_{h,v}(\mathbf{z})<v$. Meanwhile, we showed in Proposition 4.5 that if $\hat{v}_{h,v}(\mathbf{z})<v$, the associated $\hat{\mathcal{W}}_{\mathbf{z},h,v}(x)$ satisfies (4.15) on $v-\hat{v}_{h,v}(\mathbf{z})\leq x\leq v$ with $\hat{\mathcal{W}}_{\mathbf{z},h,v}(v-\hat{v}_{h,v}(\mathbf{z}))>0$, which is not the original free boundary problem (3.1). Therefore, we can further infer that for some $v\in\mathcal{O}_2\setminus[\mathcal{K}_{\mathbf{z},h,2},\infty)\subseteq(\mathcal{K}_{\mathbf{z},h,1},\mathcal{K}_{\mathbf{z},h,2})$, the formulated $\hat{\mathcal{W}}_{\mathbf{z},h,v}(x)$ given by (4.14) solves (3.1) with $\hat{\mathcal{W}}_{\mathbf{z},h,v}(0)=0$, which suggests the choice of the initial search window for v in the following section of numerical algorithm.

Next, we find the solution to (4.9) and the value of $v_h(\mathbf{z})$ by a numerical algorithm.

4.2. Numerical algorithm

From Proposition 4.5, we have shown that $\hat{\mathcal{W}}_{\mathbf{z},h,v}(x) = G_{\mathbf{z},h,v}(v-x)$ on $v - \hat{v}_{h,v}(\mathbf{z}) \le x < n_{h,v}(\mathbf{z})$ where $G_{\mathbf{z},h,v}(x)$ satisfies (4.9) and corresponding conditions. For such a non-linear second-order ODE, Runge-Kutta method (Burden and Faires, 2010) is appropriate.

Motivated by Proposition 4.5 and Remark 4.6, we propose a numerical method starting from the search window $(\mathcal{K}_{\mathbf{z},h,1}, \mathcal{K}_{\mathbf{z},h,2})$ where we choose a fixed $v \in (\mathcal{K}_{\mathbf{z},h,1}, \mathcal{K}_{\mathbf{z},h,2})$ and formulate the candidate function $\hat{\mathcal{W}}_{\mathbf{z},h,v}(x)$ given by (4.14). Since $(\mathcal{K}_{\mathbf{z},h,1}, \mathcal{K}_{\mathbf{z},h,2}) \supseteq \mathcal{O}_2 \setminus [\mathcal{K}_{\mathbf{z},h,2}, \infty)$, it is possible that the existence of $n_{h,v}(\mathbf{z})$ is not guaranteed and $-a_h\hat{\mathcal{W}}'_{\mathbf{z},h,v}(x)/(b_h^2\hat{\mathcal{W}}''_{\mathbf{z},h,v}(x)) > 1$ for all $x \in (0, v)$. It implies that current v is too small and we should try a greater guess. In this case, we adjust the lower bound of the search window to the current v. If $0 < n_{h,v}(\mathbf{z}) < v$ exists, we can solve (4.9) numerically with initial conditions $G_{\mathbf{z},h,v}(v - n_{h,v}(\mathbf{z})) = \hat{\mathcal{W}}_{\mathbf{z},h,v}(n_{h,v}(\mathbf{z}))$ and $G'_{\mathbf{z},h,v}(v - n_{h,v}(\mathbf{z})) = -\hat{\mathcal{W}}'_{\mathbf{z},h,v}(n_{h,v}(\mathbf{z}))$, and found $\hat{v}_{h,v}(\mathbf{z})$ defined by (4.10). As discussed in Proposition 4.5, if $\hat{v}_{h,v}(\mathbf{z}) = v$ and

$$\left(r + \sum_{k=1}^{m} \lambda_k(\mathbf{z})\right) G_{\mathbf{z},h,v}(v) - \sum_{l=1, l\neq h}^{m} \lambda_l(\mathbf{z}) \mathcal{W}_{\mathbf{z}^l,h}(0) = 0,$$

we conclude that $\hat{v}_{h,v}(\mathbf{z}) = v = v_h(\mathbf{z})$ and the formulated $\hat{\mathcal{W}}_{\mathbf{z},h,v}(x)$ solves (3.1) with $\hat{\mathcal{W}}_{\mathbf{z},h,v}(0) = 0$ and $\lim_{x\downarrow 0}\hat{\mathcal{W}}''_{\mathbf{z},h,v}(x) = -\infty$. Otherwise, there are two possible scenarios. First, if $\hat{v}_{h,v}(\mathbf{z}) < v$, i.e.,

$$\inf\left\{ v - n_{h,v}(\mathbf{z}) < x \le v : \left(r + \sum_{k=1}^{m} \lambda_k(\mathbf{z})\right) G_{\mathbf{z},h,v}(x) = \sum_{l=1, l\neq h}^{m} \lambda_l(\mathbf{z}) \mathcal{W}_{\mathbf{z}^l,h}(v-x) \right\} < v,$$

$\hat{\mathcal{W}}_{\mathbf{z},h,v}(x)$ satisfies (4.15) on $x > v - \hat{v}_{h,v}(\mathbf{z}) > 0$ with $\lim_{x\downarrow v-\hat{v}_{h,v}(\mathbf{z})}\hat{\mathcal{W}}''_{\mathbf{z},h,v}(x) = \lim_{x\uparrow \hat{v}_{h,v}(\mathbf{z})} G''_{\mathbf{z},h,v}(x) = -\infty$ and $\hat{\mathcal{W}}_{\mathbf{z},h,v}(v - \hat{v}_{h,v}(\mathbf{z})) > 0$. It can be inferred that v is too large compared with $v_h(\mathbf{z})$ and $G''_{\mathbf{z},h,v}(x)$ decreases to negative infinity before x hits v. Therefore, we adjust the upper bound of the search window to the current v. Second, if $\hat{v}_{h,v}(\mathbf{z}) = v$ and

$$\left(r + \sum_{k=1}^{m} \lambda_k(\mathbf{z})\right) G_{\mathbf{z},h,v}(x) - \sum_{l=1, l\neq h}^{m} \lambda_l(\mathbf{z}) \mathcal{W}_{\mathbf{z}^l,h}(v-x) > 0, \quad v - n_{h,v}(\mathbf{z}) \le x \le v,$$

$\hat{\mathcal{W}}_{\mathbf{z},h,v}(x)$ satisfies (4.15) on $x > v - \hat{v}_{h,v}(\mathbf{z}) = 0$ with $\lim_{x\downarrow 0}\hat{\mathcal{W}}''_{\mathbf{z},h,v}(x) = \lim_{x\uparrow v} G''_{\mathbf{z},h,v}(x) < -\infty$ and $\hat{\mathcal{W}}_{\mathbf{z},h,v}(0) > 0$. It can be inferred that v is too small compared with $v_h(\mathbf{z})$ and $G''_{\mathbf{z},h,v}(x)$ has not decreased to negative infinity when x hits v. In this case, the lower bound of the search window is updated to the current value of v.

We further discuss in detail how to adjust the next guess according to the numerical results. In practice, we start with the search window $(v^L, v^U) = (\mathcal{K}_{\mathbf{z},h,1}, \mathcal{K}_{\mathbf{z},h,2})$. At each iteration, we let v be the midpoint of the search window and denote the midpoint at ith iteration by $v^{(i)}$. During iteration i, if $\hat{v}_{h,v^{(i)}}(\mathbf{z}) < v^{(i)}$, the upper bound v^U is decreased to $v^{(i)}$. On the contrary, if $n_{h,v^{(i)}}(\mathbf{z})$ does not exist or if $\hat{v}_{h,v^{(i)}}(\mathbf{z}) = v^{(i)}$ and

$$\left(r + \sum_{k=1}^{m} \lambda_k(\mathbf{z})\right) G_{\mathbf{z},h,v^{(i)}}(x) - \sum_{l=1, l\neq h}^{m} \lambda_l(\mathbf{z}) \mathcal{W}_{\mathbf{z}^l,h}\left(v^{(i)} - x\right) > 0, \quad v^{(i)} - n_{h,v^{(i)}}(\mathbf{z}) \le x \le v^{(i)},$$

we increase the lower bound v^L to $v^{(i)}$. At iteration $i+1$, we let $v^{(i+1)} = (v^L + v^U)/2$ where v^L or v^U is updated in the last iteration. The above process is repeated until $\hat{v}_{h,v^{(i)}}(\mathbf{z}) = v^{(i)}$ and $(r + \sum_{k=1}^m \lambda_k(\mathbf{z}))G_{\mathbf{z},h,v^{(i)}}(v^{(i)}) = \sum_{l=1,l\neq h}^m \lambda_l(\mathbf{z})\mathcal{W}_{\mathbf{z}^l,h}(0) = 0$ or the width of the search window is within the tolerance.

There are several advantages of solving the problem in the semi-analytical approach. First of all, compared with a purely numerical solution, we can calculate $\mathcal{W}_{\mathbf{z},h}(x)$ explicitly on $(n_h(\mathbf{z}), \infty)$ and then numerically on $(0, n_h(\mathbf{z}))$. The semi-analytical approach not only enhances the computation efficiency but also provides accurate initial conditions for the numerical part. Note that we formulate the group of candidate functions starting from a fixed v and then find the one that coincides with $\mathcal{W}_{\mathbf{z},h}(x)$. If we tackle with the original free boundary problem (3.1), the explicit forms of the boundaries are not straightforward. As for the numerical methods, we start with only the boundary condition $\mathcal{W}_{\mathbf{z},h}(0) = 0$ without any further information regarding the derivatives at $x = 0$. The numerical methods applicable to such boundary problems are limited and the accuracy of the numerical approximation is a concern.

4.3. Semi-closed solution for three subsidiaries

In this section we present the semi-closed expression of $\mathcal{W}_{\mathbf{z},h}(x)$ when $m = 3$. By Theorem 3.1, the solution to $\mathcal{W}_{\mathbf{z},h}(x)$ is solved by a recursive system of HJBVIs with three layers. We start from the base layer, where an explicit form of the solution has been solved in the literature. We proceed to the second layer and find the semi-closed solution to the system of HJBVIs according to Algorithm 4.1. Note that the semi-analytical solution $(w_{1,0}, \dots, w_{1,q})$ obtained from Algorithm 4.1 are in backward direction. Therefore, we flip the semi-analytical solution and obtain $(w_{1,q}, \dots, w_{1,0})$, which is the numerical solution to $(\mathcal{W}_{\mathbf{z},h}(x_0), \dots, \mathcal{W}_{\mathbf{z},h}(x_q))$ The same steps are repeated for the third layer. The details on constructing the recursive solution are discussed below.

Without loss of generality, we consider a group with three subsidiaries, i.e., $N = 3$. For the rest of the section, let $\mathbf{z} = (0,0,0)$ and $\mathbf{x} = (x_1, x_2, x_3)$. We aim to find the semi-closed solution to $\mathcal{W}_{\mathbf{z},h}(x)$ for some $1 \le h \le 3$ and begin with decomposing the dependent structure from the top layer. By Proposition 4.5, $\mathcal{W}_{\mathbf{z},h}(x)$ is dependent on $\mathcal{W}_{\mathbf{z}^l,h}(x)$ where $1 \le l \le 3$ and $l \neq h$. For instance, $\mathcal{W}_{\mathbf{z},1}(x)$ is dependent on $\mathcal{W}_{\mathbf{z}^2,1}(x)$ and $\mathcal{W}_{\mathbf{z}^3,1}(x)$, where $\mathbf{z}^2 = (0,1,0)$ and $\mathbf{z}^3 = (0,0,1)$. As for $\mathcal{W}_{\mathbf{z}^2,1}(x)$ and $\mathcal{W}_{\mathbf{z}^3,1}(x)$, both depend on $\mathcal{W}_{\mathbf{z}_1,1}(x)$, which is the solution to the base layer of the recursive system. Note that we let $\mathbf{z}_1 = (0,1,1)$, the default state that subsidiary 1 is the

Algorithm 4.1 Algorithm for Solving the Auxiliary Problem Semi-analytically.

Input: : $v^{(0)}$, v^L, v^U, k, M, ϵ
$N \leftarrow v^{(U)}/k$
for $j = 0, 1, 2, \dots, N$ **do**
  $x_j \leftarrow jk$
end for
for $i = 0, 1, 2, \dots, M$ **do**
  $N \leftarrow v^{(i)}/k$
  for $j = 0, 1, 2, \dots, N$ **do**
    $w_{1,j} \leftarrow H_{\mathbf{z},h,v^{(i)}}(x_j)$
    $w_{2,j} \leftarrow H'_{\mathbf{z},h,v^{(i)}}(x_j)$
    if $H'_{\mathbf{z},h,v^{(i)}}(x_j)/H''_{\mathbf{z},h,v^{(i)}}(x_j) \le b_h^2/a_h$ **then**
      $q \leftarrow j$
      break
    end if
  end for
  if $j == N$ **AND** $H'_{\mathbf{z},h,v^{(i)}}(x_j)/H''_{\mathbf{z},h,v^{(i)}}(x_j) > b_h^2/a_h$ **then**
    $v^L \leftarrow v^{(i)}$
    $v^{(i+1)} \leftarrow (v^L + v^U)/2$
    continue
  end if
  while $(r + \sum_{k=1}^m \lambda_k(\mathbf{z}))w_{1,q} - \sum_{l=1,l\neq h}^m \lambda_l(\mathbf{z})\mathcal{W}_{\mathbf{z}^l,h}(v^{(i)} - x_q) > 0$ **OR** $q < N$ **do**
    obtain $w_{1,q+1}$ and $w_{2,q+1}$ by Runge-Kutta method
    $q \leftarrow q + 1$
  end while
  if $q \ge N$ **then**
    $v^L \leftarrow v^{(i)}$
  else if $|x_q - v^{(i)}| < \epsilon$ **OR** $|v^U - v^L| < \epsilon$ **then**
    break
  else
    $v^U \leftarrow v^{(i)}$
  end if
  $v^{(i+1)} \leftarrow (v^L + v^U)/2$
  $i \leftarrow i + 1$
end for
return $v^{(i)}$, $w_{1,0}, \dots, w_{1,q}$

only alive subsidiary within the group. Therefore, we start from the explicit form of $\mathcal{W}_{\mathbf{z}_1,1}(x)$, calculate $\mathcal{W}_{\mathbf{z}^2,1}(x)$ and $\mathcal{W}_{\mathbf{z}^3,1}(x)$ in a semi-analytical approach, and then derive the semi-closed solution to $\mathcal{W}_{\mathbf{z},1}(x)$. We can find the semi-closed solutions of $\mathcal{W}_{\mathbf{z},2}(x)$ and $\mathcal{W}_{\mathbf{z},3}(x)$ in the same way and the optimal value function satisfies $W(\mathbf{x},\mathbf{z}) = \sum_{h=1}^3 \mathcal{W}_{\mathbf{z},h}(x_h)$.

The HJBVI in the base layer has been solved explicitly in the literature (Højgaard and Taksar, 1999). Let $\theta_1 > 0$ and $-\theta_2 < 0$ be the two distinct roots of the equation $b_1^2 s^2/2 + a_1 s - (r + \lambda_1(\mathbf{z}_1)) = 0$. Further we define $\gamma := (a_1^2(2b_1^2(r+\lambda_1(\mathbf{z}_1)))^{-1} + 1)^{-1}$, $n_1(\mathbf{z}_1) := a_1^{-1} b_1^2(1-\gamma)$, $v_1(\mathbf{z}_1) := n_1(\mathbf{z}_1) + (\theta_1+\theta_2)^{-1}(\ln\theta_2 - \ln\theta_1)$ and

$$K := \left[\left(\frac{\theta_2}{\theta_1}\right)^{\frac{\theta_1}{\theta_1+\theta_2}} + \left(\frac{\theta_1}{\theta_2}\right)^{\frac{\theta_2}{\theta_1+\theta_2}}\right]^{-1}.$$

Then, the closed-form expression of $\mathcal{W}_{\mathbf{z}_1,1}(x)$ is

$$\mathcal{W}_{\mathbf{z}_1,1}(x_1) = \begin{cases} c_1 x_1^{\gamma}, & 0 \le x_1 \le n_1(\mathbf{z}_1), \\ c_2 e^{\theta_1 x_1} + c_3 e^{-\theta_2 x_1}, & n_1(\mathbf{z}_1) < x_1 \le v_1(\mathbf{z}_1), \\ c_4 + x_1, & x_1 > v_1(\mathbf{z}_1), \end{cases} \tag{4.19}$$

where $c_1 = a_1 K[(r+\lambda_1(\mathbf{z}_1))n_1(\mathbf{z}_1)^{\gamma}]^{-1}$, $c_2 = K\theta_1^{-1}e^{-\theta_1 n_1(\mathbf{z}_1)}$, $c_3 = -K\theta_2^{-1}e^{\theta_2 n_1(\mathbf{z}_1)}$ and $c_4 = a_1(r+\lambda_1(\mathbf{z}_1))^{-1} - v_1(\mathbf{z}_1)$.

We proceed to the second layer and formulate the candidate functions for $\mathcal{W}_{\mathbf{z}^l,1}(x)$ for $l = 2$ and 3 respectively. We first determine $n_{1,v}(\mathbf{z}^l)$ according to (4.7) where $v \in (\mathcal{K}_{\mathbf{z}^l,1,1}, \mathcal{K}_{\mathbf{z}^l,1,2})$. From (4.3), we can rewrite the definition of $n_{h,v}(\mathbf{z})$ as

$$n_{h,v}(\mathbf{z}) = v - \inf\left\{0 < x < v : \frac{-\Phi_{\mathbf{z},h,2}(x) - \left(\eta_{\mathbf{z},h,1} + \eta_{\mathbf{z},h,2}\right)\hat{\Phi}_{\mathbf{z},h,v,1}(x)}{-\Phi'_{\mathbf{z},h,2}(x) - \left(\eta_{\mathbf{z},h,1} + \eta_{\mathbf{z},h,2}\right)\hat{\Phi}'_{\mathbf{z},h,v,1}(x)} = \frac{b_h^2}{a_h}\right\}, \tag{4.20}$$

where

$$\hat{\Phi}_{\mathbf{z},h,v,1}(x) := -\frac{2\int_0^x \sum_{l=1,l\neq h}^m \lambda_l(\mathbf{z})\mathcal{W}'_{\mathbf{z}^l,h}(v-u)\,\Phi'_{\mathbf{z},h,2}(x-u)\,du}{b_h^2\left(\eta_{\mathbf{z},h,1} + \eta_{\mathbf{z},h,2}\right)\eta_{\mathbf{z},h,1}\eta_{\mathbf{z},h,2}}.$$

Closed-form expressions of $\hat{\Phi}_{\mathbf{z}^l,1,v,1}(x)$ and $\hat{\Phi}'_{\mathbf{z}^l,1,v,1}(x)$ for $l = 2$ and 3 are given by

$$\hat{\Phi}_{\mathbf{z}^l,1,v,1}(x) = -\frac{2\sum_{k=2,k\neq l}^3 \lambda_k\left(\mathbf{z}^l\right)\left[e^{\eta_{\mathbf{z}^l,1,1}x}\tilde{\Psi}_{\mathbf{z}^l,1,l,v,1}(x) - e^{-\eta_{\mathbf{z}^l,1,2}x}\tilde{\Psi}_{\mathbf{z}^l,1,l,v,2}(x)\right]}{b_1^2\left(\eta_{\mathbf{z}^l,1,1} + \eta_{\mathbf{z}^l,1,2}\right)},$$

$$\hat{\Phi}'_{\mathbf{z}^l,1,v,1}(x) = -\frac{2\sum_{k=2,k\neq l}^{3}\lambda_k\left(\mathbf{z}^l\right)\left[\eta_{\mathbf{z}^l,1,1}e^{\eta_{\mathbf{z}^l,1,1}x}\tilde{\Psi}_{\mathbf{z}^l,1,l,v,1}(x) + \eta_{\mathbf{z}^l,1,2}e^{-\eta_{\mathbf{z}^l,1,2}x}\tilde{\Psi}_{\mathbf{z}^l,1,l,v,2}(x)\right]}{b_1^2\left(\eta_{\mathbf{z}^l,1,1}+\eta_{\mathbf{z}^l,1,2}\right)},$$

where

$$\tilde{\Psi}_{\mathbf{z}^l,1,l,v,2}(x) = \int_0^x \mathcal{W}'_{\mathbf{z}_1,1}(v-s)\,e^{\eta_{\mathbf{z}^l,1,2}s}ds$$
$$= \begin{cases} \dfrac{e^{\eta_{\mathbf{z}^l,1,2}x}-1}{\eta_{\mathbf{z}^l,1,2}x}, & 0\le x\le v-v_1(\mathbf{z}_1), \\ \tilde{\Psi}_{\mathbf{z}^l,1,l,v,2}(v-v_1(\mathbf{z}_1)) + c_2\theta_1 e^{\theta_1 v}\left[\dfrac{e^{\left(\eta_{\mathbf{z}^l,1,2}-\theta_1\right)x}-e^{\left(\eta_{\mathbf{z}^l,1,2}-\theta_1\right)(v-v_1(\mathbf{z}_1))}}{\eta_{\mathbf{z}^l,1,2}-\theta_1}\right] & \\ -c_3\theta_2 e^{-\theta_2 v}\left[\dfrac{e^{\left(\theta_2+\eta_{\mathbf{z}^l,1,2}\right)x}-e^{\left(\theta_2+\eta_{\mathbf{z}^l,1,2}\right)(v-v_1(\mathbf{z}_1))}}{\theta_2+\eta_{\mathbf{z}^l,1,2}}\right], & v-v_1(\mathbf{z}_1)<x\le v-n_1(\mathbf{z}_1). \end{cases}$$

By replacing the $\eta_{\mathbf{z}^l,1,2}$ and $\tilde{\Psi}_{\mathbf{z}^l,1,l,v,2}(v-v_1(\mathbf{z}_1))$ in $\tilde{\Psi}_{\mathbf{z}^l,1,l,v,2}(x)$ by $-\eta_{\mathbf{z}^l,1,1}$ and $\tilde{\Psi}_{\mathbf{z}^l,1,l,v,1}(v-v_1(\mathbf{z}_1))$ respectively, we have the explicit form of $\tilde{\Psi}_{\mathbf{z}^l,1,l,v,1}(x)$. Then we can determine $n_{1,v}(\mathbf{z}^l)$ according to (4.20).

We proceed to find the analytical expression of $\hat{\mathcal{W}}_{\mathbf{z}^l,1,v}(x)$ on $(n_{1,v}(\mathbf{z}^l),v)$, which is dependent on the analytical expression of $\Phi_{\mathbf{z}^l,1,v,1}(x)$. The analytical expression of $\Phi_{\mathbf{z}^l,1,v,1}(x)$ is

$$\Phi_{\mathbf{z}^l,1,v,1}(x) = -\frac{2\sum_{k=2,k\neq l}^{3}\lambda_k\left(\mathbf{z}^l\right)\left[e^{\eta_{\mathbf{z}^l,1,1}x}\Psi_{\mathbf{z}^l,1,l,v,1}(x) - e^{-\eta_{\mathbf{z}^l,1,2}x}\Psi_{\mathbf{z}^l,1,l,v,2}(x)\right]}{b_1^2\left(\eta_{\mathbf{z}^l,1,1}+\eta_{\mathbf{z}^l,1,2}\right)},$$

where

$$\Psi_{\mathbf{z}^l,1,l,v,2}(x) = \int_0^x \mathcal{W}_{\mathbf{z}_1,1}(v-s)\,e^{\eta_{\mathbf{z}^l,1,2}s}ds$$
$$= \begin{cases} (c_4+v)\left(\dfrac{e^{\eta_{\mathbf{z}^l,1,2}x}-1}{\eta_{\mathbf{z}^l,1,2}}\right) - \dfrac{xe^{\eta_{\mathbf{z}^l,1,2}x}}{\eta_{\mathbf{z}^l,1,2}} + \left(\dfrac{e^{\eta_{\mathbf{z}^l,1,2}x}-1}{\eta_{\mathbf{z}^l,1,2}^2}\right), & 0\le x\le v-v_1(\mathbf{z}_1), \\ \Psi_{\mathbf{z}^l,1,l,v,2}(v-v_1(\mathbf{z}_1)) + c_2 e^{\theta_1 v}\left[\dfrac{e^{\left(\eta_{\mathbf{z}^l,1,2}-\theta_1\right)x}-e^{\left(\eta_{\mathbf{z}^l,1,2}-\theta_1\right)(v-v_1(\mathbf{z}_1))}}{\eta_{\mathbf{z}^l,1,2}-\theta_1}\right] & \\ +c_3 e^{-\theta_2 v}\left[\dfrac{e^{\left(\theta_2+\eta_{\mathbf{z}^l,1,2}\right)x}-e^{\left(\theta_2+\eta_{\mathbf{z}^l,1,2}\right)(v-v_1(\mathbf{z}_1))}}{\theta_2+\eta_{\mathbf{z}^l,1,2}}\right], & v-v_1(\mathbf{z}_1)<x\le v-n_1(\mathbf{z}_1). \end{cases}$$

Similarly, $\Psi_{\mathbf{z}^l,1,l,v,1}(x)$ can be obtained by replacing the $\eta_{\mathbf{z}^l,1,2}$ and $\Psi_{\mathbf{z}^l,1,l,v,2}(v-v_1(\mathbf{z}_1))$ in $\Psi_{\mathbf{z}^l,1,l,v,2}(x)$ by $-\eta_{\mathbf{z}^l,1,1}$ and $\Psi_{\mathbf{z}^l,1,l,v,1}(v-v_1(\mathbf{z}_1))$ respectively. Then, the analytical expressions of $\hat{\mathcal{W}}_{\mathbf{z}^l,1,v}(x)$ and $\hat{\mathcal{W}}'_{\mathbf{z}^l,1,v}(x)$ on $(n_{1,v}(\mathbf{z}^l),\infty)$ are available. Then we can solve $G_{\mathbf{z}^l,1,v}(x)$ numerically given $G_{\mathbf{z}^l,1,v}(v-n_{1,v}(\mathbf{z}^l)) = \hat{\mathcal{W}}_{\mathbf{z}^l,1,v}(n_{1,v}(\mathbf{z}^l))$ and $G'_{\mathbf{z}^l,1,v}(v-n_{1,v}(\mathbf{z}^l)) = -\hat{\mathcal{W}}'_{\mathbf{z}^l,1,v}(n_{1,v}(\mathbf{z}^l))$. In this way, the group of candidate functions are available semi-analytically, where we can find the candidate function that coincides with the value function $\mathcal{W}_{\mathbf{z}^l,1}(x)$ and the boundaries $n_1(\mathbf{z}^l)$ and $v_1(\mathbf{z}^l)$ follow naturally.

As we have clarified before, the analytical expression of $\mathcal{W}_{\mathbf{z},1}(x)$ is dependent on $\mathcal{W}_{\mathbf{z}^2,1}(x)$ and $\mathcal{W}_{\mathbf{z}^3,1}(x)$. Once we have the semi-closed solutions of $\mathcal{W}_{\mathbf{z}^2,1}(x)$ and $\mathcal{W}_{\mathbf{z}^3,1}(x)$, we can follow Algorithm 4.1 again to compute the semi-closed solution of $\mathcal{W}_{\mathbf{z},1}(x)$. Similarly, we first formulate the candidate functions for $\mathcal{W}_{\mathbf{z},1}(x)$. Closed-form expressions of $\hat{\Phi}_{\mathbf{z},1,v,1}(x)$ and $\hat{\Phi}'_{\mathbf{z},1,v,1}(x)$ are given by

$$\hat{\Phi}_{\mathbf{z},1,v,1}(x) = -\frac{2\left[e^{\eta_{\mathbf{z},1,1}x}\sum_{l=2}^{3}\lambda_l(\mathbf{z})\,\tilde{\Psi}_{\mathbf{z},1,l,v,1}(x) - e^{-\eta_{\mathbf{z},1,2}x}\sum_{l=2}^{3}\lambda_l(\mathbf{z})\,\tilde{\Psi}_{\mathbf{z},1,l,v,2}(x)\right]}{b_1^2\left(\eta_{\mathbf{z},1,1}+\eta_{\mathbf{z},1,2}\right)},$$

where

$$\tilde{\Psi}_{\mathbf{z},1,l,v,2}(x) = \begin{cases} \dfrac{e^{\eta_{\mathbf{z},1,2}x}-1}{\eta_{\mathbf{z},1,2}}, & 0 \le x \le v - v_1\left(\mathbf{z}^l\right), \\ \tilde{\Psi}_{\mathbf{z},1,l,v,2}\left(v - v_1\left(\mathbf{z}^l\right)\right) \\ +\dfrac{\eta_{\mathbf{z}^l,1,2}e^{\eta_{\mathbf{z}^l,1,1}\left(v_1(\mathbf{z}^l)-v\right)}}{\eta_{\mathbf{z}^l,1,1}+\eta_{\mathbf{z}^l,1,2}} \times \dfrac{e^{\left(\eta_{\mathbf{z}^l,1,1}+\eta_{\mathbf{z},1,2}\right)x} - e^{\left(\eta_{\mathbf{z}^l,1,1}+\eta_{\mathbf{z},1,2}\right)\left(v-v_1\left(\mathbf{z}^l\right)\right)}}{\eta_{\mathbf{z}^l,1,1}+\eta_{\mathbf{z},1,2}} \\ +\dfrac{\eta_{\mathbf{z}^l,1,1}e^{-\eta_{\mathbf{z}^l,1,2}\left(v_1(\mathbf{z}^l)-v\right)}}{\eta_{\mathbf{z}^l,1,1}+\eta_{\mathbf{z}^l,1,2}} \times \dfrac{e^{\left(-\eta_{\mathbf{z}^l,1,2}+\eta_{\mathbf{z},1,2}\right)x} - e^{\left(-\eta_{\mathbf{z}^l,1,2}+\eta_{\mathbf{z},1,2}\right)\left(v-v_1\left(\mathbf{z}^l\right)\right)}}{-\eta_{\mathbf{z}^l,1,2}+\eta_{\mathbf{z},1,2}} \\ +\dfrac{2}{b_1^2\left(\eta_{\mathbf{z}^l,1,1}+\eta_{\mathbf{z}^l,1,2}\right)}\Big[-e^{\eta_{\mathbf{z}^l,1,1}\left(v_1\left(\mathbf{z}^l\right)-v\right)}\sum_{k=2,k\neq l}^{3}\lambda_k\left(\mathbf{z}^l\right)\tilde{\chi}_{\mathbf{z},1,l,k,v,2,1}(x) \\ +e^{-\eta_{\mathbf{z}^l,1,2}\left(v_1\left(\mathbf{z}^l\right)-v\right)}\sum_{k=2,k\neq l}^{3}\lambda_k\left(\mathbf{z}^l\right)\tilde{\chi}_{\mathbf{z},1,l,k,v,2,2}(x)\Big], & v - v_1\left(\mathbf{z}^l\right) < x \le v - n_1\left(\mathbf{z}^l\right). \end{cases}$$

In addition, $\tilde{\chi}_{\mathbf{z},1,l,k,v,2,1}(x)$ and $\tilde{\chi}_{\mathbf{z},1,l,k,v,2,2}(x)$ are defined by

$$\tilde{\chi}_{\mathbf{z},1,l,k,v,2,1}(x) := \int_{v-v_1(\mathbf{z}^l)}^{x} e^{\eta_{\mathbf{z}^l,1,1}s}e^{\eta_{\mathbf{z},1,2}s}\tilde{\Psi}_{\mathbf{z}^l,1,k,v_1(\mathbf{z}^l),1}\left(v_1\left(\mathbf{z}^l\right)-v+s\right)ds,$$

$$\tilde{\chi}_{\mathbf{z},1,l,k,v,2,2}(x) := \int_{v-v_1(\mathbf{z}^l)}^{x} e^{-\eta_{\mathbf{z}^l,1,2}s}e^{\eta_{\mathbf{z},1,2}s}\tilde{\Psi}_{\mathbf{z}^l,1,k,v_1(\mathbf{z}^l),2}\left(v_1\left(\mathbf{z}^l\right)-v+s\right)ds,$$

for $v - v_1(\mathbf{z}^l) \le x \le v - n_1(\mathbf{z})$ where

$$\tilde{\chi}_{\mathbf{z},1,l,k,v,2,2}(x) = \begin{cases} \dfrac{1}{-\eta_{\mathbf{z}^l,1,2}}\left[\dfrac{e^{\left(-\eta_{\mathbf{z}^l,1,2}+\eta_{\mathbf{z},1,2}\right)x} - e^{\left(-\eta_{\mathbf{z}^l,1,2}+\eta_{\mathbf{z},1,2}\right)\left(v-v_1\left(\left(\mathbf{z}^l\right)\right)\right)}}{-\eta_{\mathbf{z}^l,1,2}+\eta_{\mathbf{z},1,2}}\right] \\ -\dfrac{e^{-\eta_{\mathbf{z}^l,1,2}\left(v-v_1\left(\mathbf{z}^l\right)\right)}}{-\eta_{\mathbf{z}^l,1,2}}\left[\dfrac{e^{\eta_{\mathbf{z},1,2}x} - e^{\eta_{\mathbf{z},1,2}\left(v-v_1\left(\left(\mathbf{z}^l\right)\right)\right)}}{\eta_{\mathbf{z},1,2}}\right], & v - v_1\left(\mathbf{z}^l\right) \le x \le v - v_1(\mathbf{z}_1), \\ \tilde{\chi}_{\mathbf{z},1,l,k,v,2,2}\left(v - v_1(\mathbf{z}_1)\right) \\ +\tilde{\Psi}_{\mathbf{z}^l,1,k,v_1(\mathbf{z}^l),2}\left(v_1\left(\mathbf{z}^l\right)-v_1(\mathbf{z}_1)\right)\left[\dfrac{e^{-\left(\eta_{\mathbf{z}^l,1,2}-\eta_{\mathbf{z},1,2}\right)x} - e^{-\left(\eta_{\mathbf{z}^l,1,2}-\eta_{\mathbf{z},1,2}\right)(v-v_1(\mathbf{z}_1))}}{-\left(\eta_{\mathbf{z}^l,1,2}-\eta_{\mathbf{z},1,2}\right)}\right] \\ +\dfrac{c_2\theta_1 e^{\theta_1 v_1\left(\mathbf{z}^l\right)}}{\eta_{\mathbf{z}^l,1,2}-\theta_1}\Bigg[-e^{\left(\eta_{\mathbf{z}^l,1,2}-\theta_1\right)\left(v_1\left(\mathbf{z}^l\right)-v_1(\mathbf{z}_1)\right)}\dfrac{e^{-\left(\eta_{\mathbf{z}^l,1,2}-\eta_{\mathbf{z},1,2}\right)x} - e^{-\left(\eta_{\mathbf{z}^l,1,2}-\eta_{\mathbf{z},1,2}\right)(v-v_1(\mathbf{z}_1))}}{-\left(\eta_{\mathbf{z}^l,1,2}-\eta_{\mathbf{z},1,2}\right)} \\ +e^{\left(\eta_{\mathbf{z}^l,1,2}-\theta_1\right)\left(v_1\left(\mathbf{z}^l\right)-v\right)}\dfrac{e^{-(\theta_1-\eta_{\mathbf{z},1,2})x} - e^{-(\theta_1-\eta_{\mathbf{z},1,2})(v-v_1(\mathbf{z}_1))}}{-(\theta_1-\eta_{\mathbf{z},1,2})}\Bigg] \\ -\dfrac{c_3\theta_2 e^{-\theta_2 v_1\left(\mathbf{z}^l\right)}}{\theta_2+\eta_{\mathbf{z}^l,1,2}}\Bigg[-e^{\left(\theta_2+\eta_{\mathbf{z}^l,1,2}\right)\left(v_1\left(\mathbf{z}^l\right)-v_1(\mathbf{z}_1)\right)}\dfrac{e^{-\left(\eta_{\mathbf{z}^l,1,2}-\eta_{\mathbf{z},1,2}\right)x} - e^{-\left(\eta_{\mathbf{z}^l,1,2}-\eta_{\mathbf{z},1,2}\right)(v-v_1(\mathbf{z}_1))}}{-\left(\eta_{\mathbf{z}^l,1,2}-\eta_{\mathbf{z},1,2}\right)} \\ +e^{\left(\theta_2+\eta_{\mathbf{z}^l,1,2}\right)\left(v_1\left(\mathbf{z}^l\right)-v\right)}\dfrac{e^{(\theta_2+\eta_{\mathbf{z},1,2})x} - e^{(\theta_2+\eta_{\mathbf{z},1,2})(v-v_1(\mathbf{z}_1))}}{\theta_2+\eta_{\mathbf{z},1,2}}\Bigg], \\ & v - v_1(\mathbf{z}_1) < x \le v - n_1(\mathbf{z}_1), \end{cases}$$

and $\tilde{\chi}_{\mathbf{z},1,l,k,v,2,1}(x)$ can be obtained by replacing $\tilde{\chi}_{\mathbf{z},1,l,k,v,2,2}(v - v_1(\mathbf{z}_1))$, $\tilde{\Psi}_{\mathbf{z}^l,1,k,v_1(\mathbf{z}^l),2}(v_1(\mathbf{z}^l) - v_1(\mathbf{z}_1))$ and $\eta_{\mathbf{z}^l,1,2}$ by $\tilde{\chi}_{\mathbf{z},1,l,k,v,2,1}(v - v_1(\mathbf{z}_1))$, $\tilde{\Psi}_{\mathbf{z}^l,1,k,v_1(\mathbf{z}^l),1}(v_1(\mathbf{z}^l)-v_1(\mathbf{z}_1))$ and $-\eta_{\mathbf{z}^l,1,1}$ respectively. Similarly, $\tilde{\Psi}_{\mathbf{z},1,l,v,1}(x)$ can be obtained by replacing the $\eta_{\mathbf{z},1,2}$, $\tilde{\chi}_{\mathbf{z},1,l,k,v,2,1}(x)$, $\tilde{\chi}_{\mathbf{z},1,l,k,v,2,2}(x)$ and $\tilde{\Psi}_{\mathbf{z},1,l,v,2}(v - v_1(\mathbf{z}^l))$ in $\tilde{\Psi}_{\mathbf{z},1,l,v,2}(x)$ by $-\eta_{\mathbf{z},1,1}$, $\tilde{\chi}_{\mathbf{z},1,l,k,v,1,1}(x)$, $\tilde{\chi}_{\mathbf{z},1,l,k,v,1,2}(x)$ and $\tilde{\Psi}_{\mathbf{z},1,l,v,1}(v - v_1(\mathbf{z}^l))$ respectively. Also, $\tilde{\chi}_{\mathbf{z},1,l,k,v,1,2}(x)$ can be obtained by replacing $\eta_{\mathbf{z},1,2}$ by $-\eta_{\mathbf{z},1,1}$ in $\tilde{\chi}_{\mathbf{z},1,l,k,v,2,2}(x)$. Additionally, $\tilde{\chi}_{\mathbf{z},1,l,k,v,1,1}(x)$ can be obtained by replacing $\eta_{\mathbf{z},1,2}$ by $-\eta_{\mathbf{z},1,1}$ in $\tilde{\chi}_{\mathbf{z},1,l,k,v,2,1}(x)$.

With the above expressions, we can calculate $\hat{\Phi}_{\mathbf{z},1,v,1}(x)$ and determine $n_{1,v}(\mathbf{z})$ by (4.7). The next step is to find the analytical expression of $\hat{\mathcal{W}}_{\mathbf{z},1,v}(x)$ for $n_{1,v}(\mathbf{z}) \le x \le v$. In a similar vein, we derive the analytical expression of $\Phi_{\mathbf{z},1,v,1}(x)$, which is given by

$$\Phi_{\mathbf{z},1,v,1}(x) = -\frac{2\left[e^{\eta_{\mathbf{z},1,1}x}\sum_{l=2}^{3}\lambda_l(\mathbf{z})\Psi_{\mathbf{z},1,l,v,1}(x) - e^{-\eta_{\mathbf{z},1,2}x}\sum_{l=2}^{3}\lambda_l(\mathbf{z})\Psi_{\mathbf{z},1,l,v,2}(x)\right]}{b_1^2\left(\eta_{\mathbf{z},1,1}+\eta_{\mathbf{z},1,2}\right)},$$

where

$$\Psi_{\mathbf{z},1,l,v,2}(x) = \begin{cases} \frac{\left(w_{\mathbf{z}^l,1,v_1(\mathbf{z}^l)} + v - v_1(\mathbf{z}^l)\right)\left(e^{\eta_{\mathbf{z},1,2}x}-1\right)}{\eta_{\mathbf{z},1,2}} - \frac{xe^{\eta_{\mathbf{z},1,2}x}}{\eta_{\mathbf{z},1,2}} + \frac{e^{\eta_{\mathbf{z},1,2}x}-1}{\eta_{\mathbf{z},1,2}^2}, & 0 \le x \le v - v_1(\mathbf{z}^l), \\ \Psi_{\mathbf{z},1,l,v,2}\left(v - v_1(\mathbf{z}^l)\right) \\ +\frac{\left(\eta_{\mathbf{z}^l,1,2}w_{\mathbf{z}^l,1,v_1(\mathbf{z}^l)} - 1\right)e^{\eta_{\mathbf{z}^l,1,1}\left(v_1(\mathbf{z}^l)-v\right)}}{\eta_{\mathbf{z}^l,1,1}+\eta_{\mathbf{z}^l,1,2}}\frac{e^{\left(\eta_{\mathbf{z}^l,1,1}+\eta_{\mathbf{z},1,2}\right)x} - e^{\left(\eta_{\mathbf{z}^l,1,1}+\eta_{\mathbf{z},1,2}\right)\left(v-v_1(\mathbf{z}^l)\right)}}{\left(\eta_{\mathbf{z}^l,1,1}+\eta_{\mathbf{z},1,2}\right)} \\ +\frac{\left(\eta_{\mathbf{z}^l,1,1}w_{\mathbf{z}^l,1,v_1(\mathbf{z}^l)} + 1\right)e^{-\eta_{\mathbf{z}^l,1,2}\left(v_1(\mathbf{z}^l)-v\right)}}{\eta_{\mathbf{z}^l,1,1}+\eta_{\mathbf{z}^l,1,2}}\frac{e^{\left(-\eta_{\mathbf{z}^l,1,2}+\eta_{\mathbf{z},1,2}\right)x} - e^{\left(-\eta_{\mathbf{z}^l,1,2}+\eta_{\mathbf{z},1,2}\right)\left(v-v_1(\mathbf{z}^l)\right)}}{\left(-\eta_{\mathbf{z}^l,1,2}+\eta_{\mathbf{z},1,2}\right)} \\ +\frac{2\sum_{k=1,k\neq 1,l}^{3}\lambda_k(\mathbf{z}^l)}{b_1^2\left(\eta_{\mathbf{z}^l,1,1}+\eta_{\mathbf{z}^l,1,2}\right)}\left[-e^{\eta_{\mathbf{z}^l,1,1}\left(v_1(\mathbf{z}^l)-v\right)}\chi_{\mathbf{z},1,l,k,v,2,1}(x)\right. \\ \left.+e^{-\eta_{\mathbf{z}^l,1,2}\left(v_1(\mathbf{z}^l)-v\right)}\chi_{\mathbf{z},1,l,k,v,2,2}(x)\right], & v - v_1(\mathbf{z}^l) < x \le v - n_{1,v}(\mathbf{z}). \end{cases}$$

We define $\chi_{\mathbf{z},1,l,k,v,2,1}(x)$ and $\chi_{\mathbf{z},1,l,k,v,2,2}(x)$ as

$$\chi_{\mathbf{z},1,l,k,v,2,1}(x) := \int_{v-v_1(\mathbf{z}^l)}^{x} e^{\eta_{\mathbf{z}^l,1,1}s}e^{\eta_{\mathbf{z},1,2}s}\Psi_{\mathbf{z}^l,1,k,v_1(\mathbf{z}^l),1}\left(v_1\left(\mathbf{z}^l\right) - v + s\right)ds,$$

$$\chi_{\mathbf{z},1,l,k,v,2,2}(x) := \int_{v-v_1(\mathbf{z}^l)}^{x} e^{-\eta_{\mathbf{z}^l,1,2}s}e^{\eta_{\mathbf{z},1,2}s}\Psi_{\mathbf{z}^l,1,k,v_1(\mathbf{z}^l),2}\left(v_1\left(\mathbf{z}^l\right) - v + s\right)ds$$

for $v - v_1(\mathbf{z}^l) \le x \le v - n_{1,v}(\mathbf{z})$ where $\chi_{\mathbf{z},1,l,k,v,2,2}(x)$ is given by

$$\chi_{\mathbf{z},1,l,k,v,2,2}(x) = \begin{cases} \left(-\frac{c_4 + v_1(\mathbf{z}^l)}{\eta_{\mathbf{z}^l,1,2}} - \frac{1}{\eta_{\mathbf{z}^l,1,2}^2}\right)\left(\frac{e^{\left(-\eta_{\mathbf{z}^l,1,2}+\eta_{\mathbf{z},1,2}\right)x} - e^{\left(-\eta_{\mathbf{z}^l,1,2}+\eta_{\mathbf{z},1,2}\right)\left(v-v_1(\mathbf{z}^l)\right)}}{-\eta_{\mathbf{z}^l,1,2}+\eta_{\mathbf{z},1,2}}\right) \\ +e^{\eta_{\mathbf{z}^l,1,2}\left(v_1(\mathbf{z}^l)-v\right)}\left(\frac{c_4 + v_1(\mathbf{z}^l)}{\eta_{\mathbf{z}^l,1,2}} - \frac{v_1(\mathbf{z}^l) - v}{\eta_{\mathbf{z}^l,1,2}} + \frac{1}{\eta_{\mathbf{z}^l,1,2}^2}\right) \\ \times\frac{e^{\eta_{\mathbf{z},1,2}x} - e^{\eta_{\mathbf{z},1,2}\left(v-v_1(\mathbf{z}^l)\right)}}{\eta_{\mathbf{z},1,2}} + \left(-\frac{e^{\eta_{\mathbf{z}^l,1,2}\left(v_1(\mathbf{z}^l)-v\right)}}{\eta_{\mathbf{z}^l,1,2}}\right) \\ \times\left\{\frac{xe^{\eta_{\mathbf{z},1,2}x} - \left(v - v_1(\mathbf{z}^l)\right)e^{\eta_{\mathbf{z},1,2}\left(v-v_1(\mathbf{z}^l)\right)}}{\eta_{\mathbf{z},1,2}} - \frac{e^{\eta_{\mathbf{z},1,2}x} - e^{\eta_{\mathbf{z},1,2}\left(v-v_1(\mathbf{z}^l)\right)}}{\eta_{\mathbf{z},1,2}^2}\right\}, & v - v_1(\mathbf{z}^l) \le x \le v - v_1(\mathbf{z}_1), \\ \chi_{\mathbf{z},1,l,k,v,2,2}(v - v_1(\mathbf{z}_1)) + \left[\frac{e^{\left(-\eta_{\mathbf{z}^l,1,2}+\eta_{\mathbf{z},1,2}\right)s} - e^{\left(-\eta_{\mathbf{z}^l,1,2}+\eta_{\mathbf{z},1,2}\right)(v-v_1(\mathbf{z}_1))}}{-\eta_{\mathbf{z}^l,1,2}+\eta_{\mathbf{z},1,2}}\right] \\ \times\left[\Psi_{\mathbf{z}^l,1,k,v_1(\mathbf{z}^l),2}\left(v_1(\mathbf{z}^l) - v_1(\mathbf{z}_1)\right) - c_2e^{\theta_1 v_1(\mathbf{z}^l)}\frac{e^{\left(\eta_{\mathbf{z}^l,1,2}-\theta_1\right)\left(v_1(\mathbf{z}^l)-v_1(\mathbf{z}_1)\right)}}{\eta_{\mathbf{z}^l,1,2}-\theta_1}\right. \\ \left.-c_3e^{-\theta_2 v_1(\mathbf{z}^l)}\frac{e^{\left(\theta_2+\eta_{\mathbf{z}^l,1,2}\right)\left(v_1(\mathbf{z}^l)-v_1(\mathbf{z}_1)\right)}}{\theta_2+\eta_{\mathbf{z}^l,1,2}}\right] + c_2e^{\theta_1 v_1(\mathbf{z}^l)}\frac{e^{\left(\eta_{\mathbf{z}^l,1,2}-\theta_1\right)\left(v_1(\mathbf{z}^l)-v\right)}}{\eta_{\mathbf{z}^l,1,2}-\theta_1} \\ \times\left[\frac{e^{(\eta_{\mathbf{z},1,2}-\theta_1)x} - e^{(\eta_{\mathbf{z},1,2}-\theta_1)(v-v_1(\mathbf{z}_1))}}{\eta_{\mathbf{z},1,2}-\theta_1}\right] + c_3e^{-\theta_2 v_1(\mathbf{z}^l)}\frac{e^{\left(\theta_2+\eta_{\mathbf{z}^l,1,2}\right)\left(v_1(\mathbf{z}^l)-v\right)}}{\theta_2+\eta_{\mathbf{z}^l,1,2}} \\ \times\left[\frac{e^{(\eta_{\mathbf{z},1,2}+\theta_2)x} - e^{(\eta_{\mathbf{z},1,2}+\theta_2)(v-v_1(\mathbf{z}_1))}}{\eta_{\mathbf{z},1,2}+\theta_2}\right], & v - v_1(\mathbf{z}_1) < x \le v - n_1(\mathbf{z}_1), \end{cases}$$

and $\chi_{\mathbf{z},1,l,k,v,2,1}(x)$ can be obtained by replacing $\chi_{\mathbf{z},1,l,k,v,2,2}(v - v_1(\mathbf{z}_1))$, $\Psi_{\mathbf{z}^l,1,k,v_1(\mathbf{z}^l),2}(v_1(\mathbf{z}^l) - v_1(\mathbf{z}_1))$ and $\eta_{\mathbf{z}^l,1,2}$ by $\chi_{\mathbf{z},1,l,k,v,2,1}(v - v_1(\mathbf{z}_1))$, $\Psi_{\mathbf{z}^l,1,k,v_1(\mathbf{z}^l),1}(v_1(\mathbf{z}^l) - v_1(\mathbf{z}_1))$ and $-\eta_{\mathbf{z}^l,1,1}$ respectively. Similarly, $\Psi_{\mathbf{z},1,l,v,1}(x)$ can be obtained by replacing the $\eta_{\mathbf{z},1,2}$, $\chi_{\mathbf{z},1,l,k,v,2,1}(x)$,

$\chi_{\mathbf{z},1,l,k,v,2,2}(x)$ and $\Psi_{\mathbf{z},1,l,v,2}(v - v_1(\mathbf{z}^l))$ in $\Psi_{\mathbf{z},1,l,v,2}(x)$ by $-\eta_{\mathbf{z},1,1}$, $\chi_{\mathbf{z},1,l,k,v,1,1}(x)$, $\chi_{\mathbf{z},1,l,k,v,1,2}(x)$ and $\Psi_{\mathbf{z},1,l,v,1}(v - v_1(\mathbf{z}^l))$ in $\Psi_{\mathbf{z},1,l,v,2}(x)$ respectively. Also, $\chi_{\mathbf{z},1,l,k,v,1,2}(x)$ can be obtained by replacing $\eta_{\mathbf{z},1,2}$ by $-\eta_{\mathbf{z},1,1}$ in $\chi_{\mathbf{z},1,l,k,v,2,2}(x)$. Additionally, $\chi_{\mathbf{z},1,l,k,v,1,1}(x)$ can be obtained by replacing $\eta_{\mathbf{z},1,2}$ by $-\eta_{\mathbf{z},1,1}$ in $\chi_{\mathbf{z},1,l,k,v,2,1}(x)$.

Thus, we can calculate $\hat{\mathcal{W}}_{\mathbf{z},1,v}(x)$ on $[n_{1,v}(\mathbf{z}), v]$ and then obtain $G_{\mathbf{z},1,v}(v - n_1(\mathbf{z})) = \hat{\mathcal{W}}_{\mathbf{z},1,v}(n_{1,v}(\mathbf{z}))$ and $G'_{\mathbf{z},1,v}(v - n_1(\mathbf{z})) = -\hat{\mathcal{W}}'_{\mathbf{z},1,v}(n_{1,v}(\mathbf{z}))$. Thus, we can obtain $G_{\mathbf{z},1,v}(x)$ numerically and thus the candidate function on its domain. Then, we can determine the candidate function that coincides with the value function $\mathcal{W}_{\mathbf{z},1}(x)$.

Remark 4.7. The semi-analytical approach can be extended to a system with more than three entities without changing the complexity of the problem. When we consider more entities within the system, the number of layers within the recursive system of HJBVIs increases. As more layers are involved, the round-off error accumulates as Runge-Kutta method is utilized to find the numerical solution in the risk exposure region. Thus, the convergence criterion for the base case should be chosen carefully to guarantee the accuracy of the solution.

5. Numerical examples

In this section, we apply the semi-analytical approach to an insurance group consisting of three subsidiaries and present the numerical results of the optimal value functions and the optimal controls. We focus on the behaviors of subsidiary 1's optimal strategies subject to different default states because we can calculate the numerical results for other subsidiaries in the same way. The premium rate and the volatility of subsidiary 1's reserve process are assigned as $a_1 = 1$ and $b_1 = 2$, respectively. The discount rate is assigned as $r = 0.05$. We follow the work of Feng et al. (2021) for the selection of the above parameters. For the default indicator process $\mathbf{Z}$, the parameters are assigned to reflect the default clustering effects embedded in the unexpected distress events.

To begin with, we consider the case where two out of three subsidiaries are alive. Chronologically, all three subsidiaries are alive initially, and then one of them defaults. In this section, we discuss the two default states in reverse order, which corresponds to the order of solving the system of HJBVIs from the base layer to the third layer. We recall that the default indicator process jumps from zero to one if a default event occurs. Given a default state $\mathbf{z}$, $\mathbf{z}^l$ denotes the default state of the group when subsidiary l defaults suddenly. In this section, we denote the state that all three subsidiaries are alive by $\mathbf{z} = (0, 0, 0)$. Thereby, it follows naturally that both $\mathbf{z}^2 = (0, 1, 0)$ and $\mathbf{z}^3 = (0, 0, 1)$ represent the state of two alive subsidiaries, but the one that goes default is subsidiary 2 for $\mathbf{z}^2$ and subsidiary 3 for $\mathbf{z}^3$, respectively. For the state of one alive subsidiary, we use $\mathbf{z}_i$ where subsidiary i is the surviving one. Thus, $\mathbf{z}_1 = (0, 1, 1)$ denotes the base case that only subsidiary 1 survives.

We use the parameters in Table 5.1 to compute subsidiary 1's semi-closed optimal value functions and optimal strategies subject to default states $\mathbf{z}^3$ and $\mathbf{z}^2$ respectively. The purpose is to explore the impact of other subsidiaries' default intensities on subsidiary 1's optimal strategies. Due to the contagion effect, subsidiary 1's default intensity is highest when the default state is $\mathbf{z}_1$. To provide a more convincing comparison result, we assign the same value to the default intensities of subsidiary 1 subject to default states $\mathbf{z}^2$ and $\mathbf{z}^3$, i.e., $\lambda_1(\mathbf{z}^2) = \lambda_1(\mathbf{z}^3) = 0.03$. The default intensity of subsidiary 2 is considerably greater than that of subsidiary 3, i.e., $\lambda_2(\mathbf{z}^3) = 0.20 > \lambda_3(\mathbf{z}^2) = 0.05$. We intentionally choose a large number for $\lambda_2(\mathbf{z}^3)$ so that the gap between $\lambda_2(\mathbf{z}^3)$ and $\lambda_3(\mathbf{z}^2)$ is relatively large. In this way, when comparing subsidiary 1's optimal strategies subject to $\mathbf{z}^3$ and $\mathbf{z}^2$, the differences are more observable. It turns out that different default risks confronted by other subsidiaries indeed influence the behaviors of the optimal strategies of subsidiary 1, which can be identified in Fig. 5.1. In Fig. 5.1a, the optimal value function associated with the default state $\mathbf{z}^2 = (0, 1, 0)$ (the blue curve) is higher than that associated with $\mathbf{z}^3 = (0, 0, 1)$ (the red curve), whereas both curves are higher than the optimal value function associated with the default state $\mathbf{z}_1 = (0, 1, 1)$ (the green curve). We also observe differences among the optimal dividend barriers, where $v_1(\mathbf{z}^2) = 5.5027$ (the blue dashed line) is greater than $v_1(\mathbf{z}^3) = 5.1697$ (the red dash-dotted line), and both are greater than $v_1(\mathbf{z}_1) = 4.0253$ (the green dotted line). As for the optimal reinsurance strategies in Fig. 5.1b, all of them increase the risk exposure from zero to one, but the threshold of maintaining the maximum risk level is highest for default state $\mathbf{z}^2$ and lowest for $\mathbf{z}_1$, i.e., $n_1(\mathbf{z}^2) = 2.2944 > n_1(\mathbf{z}^3) = 2.0918 > n_1(\mathbf{z}_1) = 1.8182$.

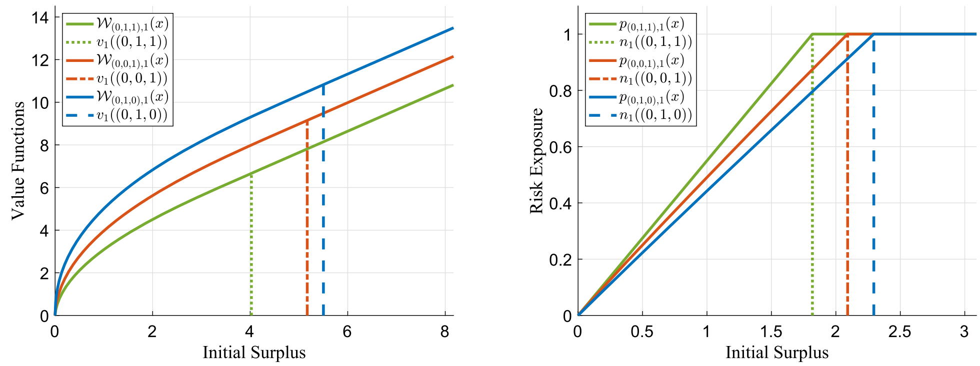
From Fig. 5.1, we first notice that subsidiary 1 takes riskier strategies and accumulates fewer expected discounted dividends if the default state jumps from $\mathbf{z}^2$ or $\mathbf{z}^3$ to $\mathbf{z}_1$. We also observe that greater default intensity of the other subsidiary results in a smaller optimal dividend barrier and fewer expected dividends accumulated before ruin. The threshold of maintaining maximum risk exposure also decreases if the default intensity of the other subsidiary increases. Although subsidiary 1's default intensity is the same in both cases, we observe the above differences because a greater possibility of the other subsidiary defaulting increases the chance of $\mathbf{Z}$ jumping to $\mathbf{z}_1 = (0, 1, 1)$, where $\lambda_1(\mathbf{z}_1)$ is greater than $\lambda_1(\mathbf{z}^2)$ and $\lambda_1(\mathbf{z}^3)$ due to the contagious effect. Since the chance of jumping to $\mathbf{z}_1$ from $\mathbf{z}^3$ is greater, and the expectation of discounted dividend is lowest in state $\mathbf{z}_1$, subsidiary 1 is expected to accumulate fewer dividends in state $\mathbf{z}^3$ than in state $\mathbf{z}^2$. It explains the result that the red curve is below the blue curve. Moreover, although $\lambda_1(\mathbf{z}^2)$ and $\lambda_1(\mathbf{z}^3)$ are the same, subsidiary 1 has greater chance of jumping to $\mathbf{z}_1$ where it faces the greatest default risk. Thus, subsidiary 1 implicitly bears a greater default risk by staying in state $\mathbf{z}^3$ rather than $\mathbf{z}^2$. Implicitly threatened by the greater default risk, subsidiary 1 is compelled to take riskier strategies when the reserve is low in order to gamble on the advance towards the continuation region and the dividend payout region, which is consistent with the results by Choulli et al. (2003). Likewise, subsidiary 1 is inclined to make dividend payments earlier and therefore set a lower barrier. The observations regarding the optimal barrier strategies are consistent with Jin et al. (2021a), where they explained that the default contagion effect forces the surviving subsidiary to pay the dividends sooner.

For the sake of comparison, we also present the numerical results of two alive subsidiaries generated according to the parameters in Table 5.2, where we compare directly the effect of a greater default intensity. By letting $\lambda_2(\mathbf{z}^3) = \lambda_3(\mathbf{z}^2)$, their impacts on subsidiary 1's optimal strategies are the same. Compared to the previous example, we use Table 5.2 to investigate the impact of subsidiary 1's default intensity on its optimal strategies. In particular, we let $\lambda_1(\mathbf{z}^3) = 0.08 > \lambda_1(\mathbf{z}^2) = 0.03$ and examine the effect of a greater default intensity on its optimal strategies.

As shown in Fig. 5.2a, the optimal value function in state $\mathbf{z}^2$ (the blue curve) is much higher than that in state $\mathbf{z}^3$ (the red curve), while the optimal value function in state $\mathbf{z}_1$ (the green curve) is still the lowest. The optimal barrier of dividend payment subject to the default state $\mathbf{z}^3$ (the red dash-dotted line) is much lower compared to the default state $\mathbf{z}^2$ (the blue dashed line), i.e., $v_1(\mathbf{z}^3) = 4.3508 < v_1(\mathbf{z}^2) = 5.5027$. Likewise, the threshold of maintaining maximum risk exposure is also much smaller, i.e., $n_1(\mathbf{z}^3) = 1.9338 < n_1(\mathbf{z}^2) = 2.2944$. Still,

Table 5.1
Default intensities associated with two alive subsidiaries where $\lambda_1(\mathbf{z}^2) = \lambda_1(\mathbf{z}^3)$.

default state	$\mathbf{z}^2 = (0, 1, 0)$	$\mathbf{z}^3 = (0, 0, 1)$	$\mathbf{z}_1 = (0, 1, 1)$
subsidiary 1	0.03	0.03	0.10
subsidiary 2	NA	0.20	NA
subsidiary 3	0.05	NA	NA

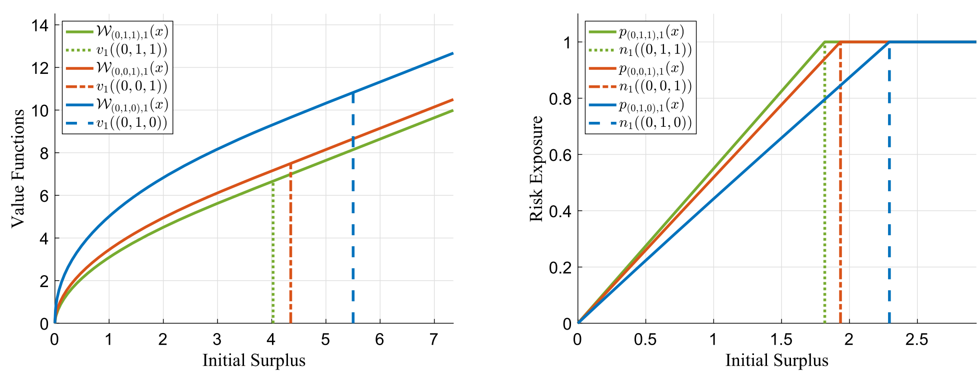


(a) value functions (b) optimal reinsurance strategies

Fig. 5.1. The value functions and optimal reinsurance strategies for subsidiary 1 with default intensities given by Table 5.1. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

Table 5.2
Default intensities associated with two alive subsidiaries where $\lambda_2(\mathbf{z}^3) = \lambda_3(\mathbf{z}^2)$.

default state	$\mathbf{z}^2 = (0, 1, 0)$	$\mathbf{z}^3 = (0, 0, 1)$	$\mathbf{z}_1 = (0, 1, 1)$
subsidiary 1	0.03	0.08	0.10
subsidiary 2	NA	0.05	NA
subsidiary 3	0.05	NA	NA



(a) value functions (b) optimal reinsurance strategies

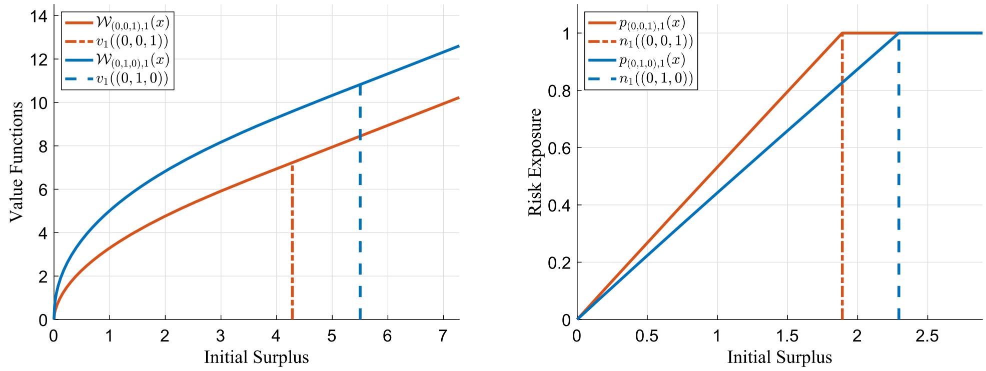
Fig. 5.2. The value functions and optimal reinsurance strategies for subsidiary 1 with default intensities given by Table 5.2.

Table 5.3
Default intensities associated with three alive subsidiaries.

default state	$\mathbf{z} = (0, 0, 0)$	$\mathbf{z}^2 = (0, 1, 0)$	$\mathbf{z}^3 = (0, 0, 1)$	$\mathbf{z}_1 = (0, 1, 1)$
subsidiary 1	0.01	0.03	0.08	0.10
subsidiary 2	0.08	NA	0.20	NA
subsidiary 3	0.03	0.05	NA	NA

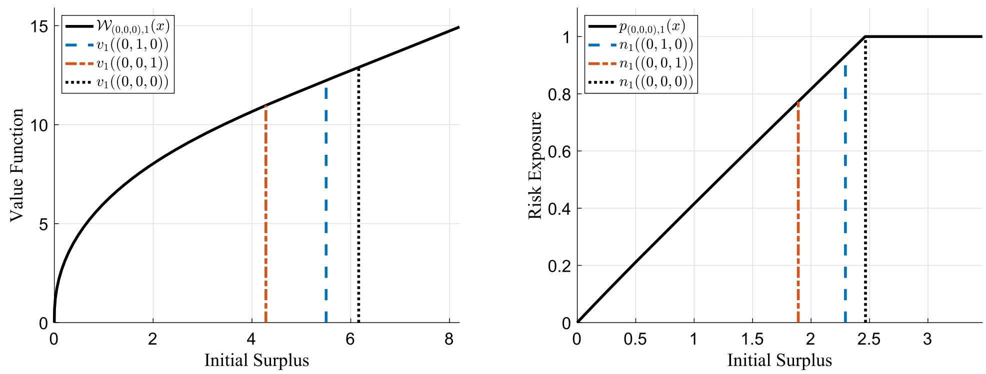
both $v_1(\mathbf{z}_1)$ and $n_1(\mathbf{z}_1)$ are the lowest. The reasons behind the observations are the same as before but more directly, and the optimal strategies exhibit more noticeable differences. It is because the threat of greater default risk originates directly from the greater default intensity of subsidiary 1, instead of an implicit result from the other subsidiary. The explanation is further supported by the graphical demonstrations in Fig. 5.3, where we increase $\lambda_2(\mathbf{z}^3)$ to 0.20. In Fig. 5.3, the gap between $v_1(\mathbf{z}^3)$ and $v_1(\mathbf{z}_2)$ and the gap between $n_1(\mathbf{z}^3)$ and $n_1(\mathbf{z}_2)$ are both greater than those in Fig. 5.1 and Fig. 5.2, i.e., $v_1(\mathbf{z}^3) = 4.2832 < v_1(\mathbf{z}^2) = 5.5027$ and $n_1(\mathbf{z}^3) = 1.8908 < n_1(\mathbf{z}^2) = 2.2944$. They are the joint results of the implicit greater default risk demonstrated in Fig. 5.1 and the direct greater default risk demonstrated in Fig. 5.2.

Subsequently, we consider the case where all three subsidiaries within the insurance group are alive initially. It corresponds to the setting in Section 4 that $m = 3$. We list the parameters of the default intensities in Table 5.3. The default intensities, when there are no



(a) value functions (b) optimal reinsurance strategies

Fig. 5.3. The value functions and optimal reinsurance strategies for subsidiary 1 with default intensities given by Table 5.3.



(a) value function (b) optimal reinsurance strategy

Fig. 5.4. The value function and optimal reinsurance strategy for subsidiary 1 with default intensities given by Table 5.3.

defaulted subsidiaries within the group, are smaller compared to other default states. As default events occur, the intensities of other alive subsidiaries increase correspondingly. As we explained in Section 4, the recursive system of HJBVIs has three layers for an insurance group with three alive subsidiaries. The solution of the base layer is explicit, and we solve $\mathcal{W}_{\mathbf{z}^2,1}(x)$ and $\mathcal{W}_{\mathbf{z}^3,1}(x)$ in the second layer semi-analytically where the numerical results are given in Fig. 5.3. Then we can compute the semi-closed solution of $\mathcal{W}_{\mathbf{z},1}(x)$ in the third layer and present the numerical results in Fig. 5.4. In Fig. 5.4a, we directly identify that the optimal barrier of dividend payment subject to default state $\mathbf{z} = (0,0,0)$ is greater than any of the other two barriers, i.e., $v_1((0,1,0))$ and $v_1((0,0,1))$. Moreover, the threshold of maintaining maximum risk exposure is also the greatest, as shown in Fig. 5.4b. Likewise, we attribute these observations to the lower default risk confronted by subsidiary 1 in default state $\mathbf{z} = (0,0,0)$. The default contagion effect embedded in the insurance group implies that subsidiary 1 is concerned least about the default risk when all three subsidiaries are alive. Being less sensitive to distress risk, it is not necessary for subsidiary 1 to take as risky strategies as in previous examples. Instead, the subsidiary acts more conservatively and distributes the dividends later to ensure its ability to continue as a going concern.

6. Concluding remarks

We propose a semi-analytical approach to the risk control and dividend optimization problem for a multi-subsidiary insurance group subject to external default contagion. Due to the complexity of the system of HJBVIs derived, the risk exposure region is solved numerically based on the analytical solutions in the continuation region. We demonstrate the semi-analytical approach on a three-subsidiary insurance group with the analytical solution in the continuation region derived. We also compare the numerical results of the optimal value functions and optimal strategies in different default states, which aligns with the economic intuition that it is optimal for the subsidiaries to take riskier actions if they are subject to greater default risk.

The innovative practice of the semi-analytical approach in the mixed regular-singular control problem under a contagious system demonstrates its great potential in solving optimization problems that lack analytical solutions. First of all, it provides more flexibility in the selection of controls, such as capital injection and equity issuance. Although the resulting system of HJBVIs are of great complexity, we can solve them semi-analytically. Second, we can apply this framework to a system subject to competitive default risk, where the default intensity decreases if a default event occurs. There are insurance groups whose subsidiaries are selling similar products. The competence among the subsidiaries suggests that a competitive system should be proper to describe the group. Besides, we can further improve the numerical method used for solving the initial value problem. Currently, Runge-Kutta method is utilized, and the round-off error increases as we move forward from the base case. Alternatively, we can consider the hybrid deep learning Markov chain approximation method (Cheng et al., 2020) to avoid the accumulation of round-off error.

Declaration of competing interest

The authors declare that they have no conflict of interest.

Data availability

No data was used for the research described in the article.

Acknowledgements

This research was supported in part by the Research Grants Council of the Hong Kong Special Administrative Region (project No. 17304921) and an Australian Government Research Training Program (RTP) Scholarship.

Appendix A. Proof of Lemma 4.1

Proof. Since $\zeta(x)$ satisfies the differential equation (4.1) and the initial conditions $\zeta'(0)=-1$ and $\zeta''(0)=0$, it follows immediately from (4.1) by letting $x=0$ that

$$\zeta(0)=\frac{\kappa+\sum_{l=1}^{m}\Lambda_l\Xi(v)}{r+\sum_{l=1}^{m+1}\Lambda_l}.$$

Thus the initial value problem defined in Lemma 4.1 is equivalent to the initial value problem satisfying (4.1) with initial conditions $\zeta(0)=\frac{\kappa+\sum_{l=1}^{m}\Lambda_l\Xi(v)}{r+\sum_{l=1}^{m+1}\Lambda_l}$ and $\zeta'(0)=-1$. We have $\sum_{l=1}^{m}\Lambda_l\Xi_l(v-x)$ is continuous on $0\le x\le v$ since it is assumed that $\Xi_l(x)$ is of C^2 for all $1\le l\le m+1$ on $[0,\infty)$. Also, the differential equation (4.1) is a linear nonhomogeneous differential equation with constant coefficients. Therefore, the solution to (4.1) with $\zeta(0)=\frac{\kappa+\sum_{l=1}^{m}\Lambda_l\Xi(v)}{r+\sum_{l=1}^{m+1}\Lambda_l}$ and $\zeta'(0)=-1$ has a unique solution on $0\le x\le v$ (Adkins and Davidson, 2012, section 4.1). Equivalently, the initial value problem defined in Lemma 4.1 admits a unique solution.

Next we find the unique solution. Let $\boldsymbol{\zeta}(x):=(\zeta(x),\zeta'(x))^T$. Then we can rewrite (4.1) in matrix form as

$$\frac{d}{dx}\boldsymbol{\zeta}(x)=\begin{pmatrix}0 & 1\\ \frac{2}{\xi^2}\left(r+\sum_{l=1}^{m+1}\Lambda_l\right) & \frac{2}{\xi^2}\kappa\end{pmatrix}\boldsymbol{\zeta}(x)+\begin{pmatrix}0\\ -\frac{2}{\xi^2}\sum_{l=1}^{m}\Lambda_l\Xi(v-x)\end{pmatrix}=\mathbf{A}\boldsymbol{\zeta}(x)+\mathbf{F}(x),$$

where $\boldsymbol{\zeta}'(0)=(-1,0)^T$. Also, the definitions of η_1 and $-\eta_2$ show that they are the eigenvalues of $\mathbf{A}$. Therefore, the homogeneous solution to (4.1) is given by

$$\boldsymbol{\zeta}_H(x)=\begin{pmatrix}e^{\eta_1x} & e^{-\eta_2x}\\ \eta_1e^{\eta_1x} & -\eta_2e^{-\eta_2x}\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix}=\Psi(x)\mathbf{c},$$

where c_1 and c_2 are constants and $\Psi(x)$ is the fundamental matrix. Thus, we can obtain the normalized fundamental matrix as

$$\Phi_0(x)=\Psi(x)\Psi^{-1}(0)=-\frac{1}{\eta_1+\eta_2}\begin{pmatrix}-\eta_2e^{\eta_1x}-\eta_1e^{-\eta_2x} & -e^{\eta_1x}+e^{-\eta_2x}\\ -\eta_1\eta_2e^{\eta_1x}+\eta_1\eta_2e^{-\eta_2x} & -\eta_1e^{\eta_1x}-\eta_2e^{-\eta_2x}\end{pmatrix}.$$

Then, the closed-form expression of $\boldsymbol{\zeta}(x)$ can be written as

$$\boldsymbol{\zeta}(x)=\Phi_0(x)\left(\int_0^x\Phi_0^{-1}(s)\mathbf{F}(s)\,ds+\boldsymbol{\zeta}(0)\right).$$

We have obtained that the initial condition $\boldsymbol{\zeta}(0)$ is given by

$$\boldsymbol{\zeta}(0)=\begin{pmatrix}\frac{\kappa+\sum_{l=1}^{m}\Lambda_l\Xi(v)}{r+\sum_{l=1}^{m+1}\Lambda_l}\\ -1\end{pmatrix}.$$

Therefore, we have

$$\Phi_0(x)\boldsymbol{\zeta}(0)=-\frac{1}{\eta_1+\eta_2}\begin{pmatrix}\zeta(0)\left(-\eta_2e^{\eta_1x}-\eta_1e^{-\eta_2x}\right)+e^{\eta_1x}-e^{-\eta_2x}\\ \cdots\end{pmatrix}.$$

Here we omit the second element of the vector because only the first element is required to find the expression of $\zeta(x)$.

Next we evaluate the following integral as

$$\int_0^x\Phi_0^{-1}(s)\mathbf{F}(s)\,ds=-\frac{2}{\xi^2(\eta_1+\eta_2)}\int_0^x\begin{pmatrix}\left(-e^{\eta_2s}+e^{-\eta_1s}\right)\sum_{l=1}^{m}\Lambda_l\Xi(v-s)\\ \left(\eta_2e^{\eta_2s}+\eta_1e^{-\eta_1s}\right)\sum_{l=1}^{m}\Lambda_l\Xi(v-s)\end{pmatrix}ds.$$

Then we can obtain the particular solution as

$$\Phi_0(x)\int_0^x \Phi_0^{-1}(s)\,\mathbf{F}(s)\,ds=\begin{pmatrix}\frac{2}{\xi^2(\eta_1+\eta_2)}\int_0^x\sum_{l=1}^m\Lambda_l\Xi_l(v-s)\left(-e^{\eta_1(x-s)}+e^{-\eta_2(x-s)}\right)ds\\ \cdots\end{pmatrix}.$$

Similarly, we omit the result of the second element of the vector. Therefore, we obtain the analytical expression of $\zeta(x)$ as

$$\zeta(x)=\frac{\zeta(0)\left(\eta_2e^{\eta_1x}+\eta_1e^{-\eta_2x}\right)-e^{\eta_1x}+e^{-\eta_2x}}{\eta_1+\eta_2}+\frac{2}{\xi^2(\eta_1+\eta_2)}\int_0^x\sum_{l=1}^m\Lambda_l\Xi_l(v-s)\left[-e^{\eta_1(x-s)}+e^{-\eta_2(x-s)}\right]ds,$$

where $\zeta(0)=\frac{\sum_{l=1}^m\Lambda_l\Xi_l(v)+\kappa}{r+\sum_{l=1}^{m+1}\Lambda_l}$. $\square$

References

Adkins, W.A., Davidson, M.G., 2012. Ordinary Differential Equations. Springer, New York.
Asmussen, S., Taksar, M., 1997. Controlled diffusion models for optimal dividend pay-out. Insurance: Mathematics and Economics 20 (1), 1–15.
Ballestra, L.V., Pacelli, G., 2014. Valuing risky debt: a new model combining structural information with the reduced-form approach. Insurance: Mathematics and Economics 55, 261–271.
Beaver, W.H., Cascino, S., Correia, M., McNichols, M.F., 2019. Group affiliation and default prediction. Management Science 65 (8), 3559–3584.
Bensoussana, A., Siu, C.C., Yam, S.C.P., Yang, H., 2014. A class of non-zero-sum stochastic differential investment and reinsurance games. Automatica 50 (8), 2025–2037.
Bielecki, T.R., Rutkowski, M., 2004. Credit Risk: Modeling, Valuation and Hedging. Springer, New York.
Birge, J.R., Bo, L., Capponi, A., 2018. Risk-sensitive asset management and cascading defaults. Mathematics of Operations Research 43 (1), 1–28.
Bo, L., Capponi, A., 2016. Optimal investment in credit derivatives portfolio under contagion risk. Mathematical Finance 26 (4), 785–834.
Bo, L., Capponi, A., 2017. Optimal investment under information driven contagious distress. SIAM Journal on Control and Optimization 55 (2), 1020–1068.
Bo, L., Capponi, A., 2018. Portfolio choice with market–credit-risk dependencies. SIAM Journal on Control and Optimization 56 (4), 3050–3091.
Bo, L., Liao, H., Yu, X., 2019. Risk sensitive portfolio optimization with default contagion and regime-switching. SIAM Journal on Control and Optimization 57 (1), 366–401.
Brechmann, E.C., Hendrich, K., Czado, C., 2013. Conditional copula simulation for systemic risk stress testing. Insurance: Mathematics and Economics 53 (3), 722–732.
Burden, R.L., Faires, J.D., 2010. Numerical Analysis, 9th edition. Cengage Learning, Boston.
Capponi, A., Frei, C., 2017. Systemic influences on optimal equity-credit investment. Management Science 63 (8), 2756–2771.
Carmona, R., Fouque, J.-P., Sun, L.-H., 2015. Mean field games and systemic risk. Communications in Mathematical Sciences 13 (4), 911–933.
Carr, P., Itkin, A., 2021. Semi-analytical solutions for barrier and American options written on a time-dependent Ornstein-Uhlenbeck process. The Journal of Derivatives 29 (1), 9–26.
Cheng, X., Jin, Z., Yang, H., 2020. Optimal insurance strategies: a hybrid deep learning Markov chain approximation approach. ASTIN Bulletin 50 (2), 449–477.
Choulli, T., Taksar, M., Zhou, X.Y., 2003. A diffusion model for optimal dividend distribution for a company with constraints on risk control. SIAM Journal on Control and Optimization 41 (6), 1946–1979.
De Finetti, B., 1957. Su un'impostazione alternativa della teoria collettiva del rischio. Transactions of the XVth International Congress of Actuaries 2 (1), 433–443.
Dhaene, J., Laeven, R.J., Zhang, Y., 2022. Systemic risk: conditional distortion risk measures. Insurance: Mathematics and Economics 102, 126–145.
Ettlin, N., Farkas, W., Kull, A., Smirnow, A., 2020. Optimal risk-sharing across a network of insurance companies. Insurance: Mathematics and Economics 95, 39–47.
Feng, Y., Zhu, J., Siu, T.K., 2021. Optimal risk exposure and dividend payout policies under model uncertainty. Insurance: Mathematics and Economics 100, 1–29.
Fleming, W.H., Rishel, R.W., 1975. Deterministic and Stochastic Optimal Control. Springer, New York.
Frey, R., Backhaus, J., 2008. Pricing and hedging of portfolio credit derivatives with interacting default intensities. International Journal of Theoretical and Applied Finance 11 (06), 611–634.
Frey, R., Runggaldier, W., 2010. Pricing credit derivatives under incomplete information: a nonlinear-filtering approach. Finance and Stochastics 14 (4), 496–526.
Garnier, J., Papanicolaou, G., Yang, T.-W., 2013. Large deviations for a mean field model of systemic risk. SIAM Journal on Financial Mathematics 4 (1), 151–184.
Gerber, H.U., Shiu, E.S.W., 2004. Optimal dividends. North American Actuarial Journal 8 (1), 1–20.
Grandell, J., 1991. Aspects of Risk Theory. Springer, New York.
Hambly, B., Søjmark, A., 2019. An spde model for systemic risk with endogenous contagion. Finance and Stochastics 23 (3), 535–594.
He, L., Liang, Z., Song, Y., Ye, Q., 2022. Optimal asset allocation, consumption and retirement time with the variation in habitual persistence. Insurance: Mathematics and Economics 102, 188–202.
Højgaard, B., Taksar, M., 1999. Controlling risk exposure and dividends payout schemes: insurance company example. Mathematical Finance 9 (2), 153–182.
Irgens, C., Paulsen, J., 2004. Optimal control of risk exposure, reinsurance and investments for insurance portfolios. Insurance: Mathematics and Economics 35 (1), 21–51.
Jarrow, R.A., Turnbull, S.M., 1995. Pricing derivatives on financial securities subject to credit risk. The Journal of Finance 50 (1), 53–85.
Jeanblanc-Picqué, M., Shiryaev, A.N., 1995. Optimization of the flow of dividends. Uspehi Matematičeskih Nauk 50 (2), 25–46.
Jin, Z., Liao, H., Yang, Y., Yu, X., 2021a. Optimal dividend strategy for an insurance group with contagious default risk. Scandinavian Actuarial Journal 2021 (4), 335–361.
Jin, Z., Yang, H., Yin, G., 2015. Optimal debt ratio and dividend payment strategies with reinsurance. Insurance: Mathematics and Economics 64, 351–363.
Jin, Z., Yang, H., Yin, G., 2021b. A hybrid deep learning method for optimal insurance strategies: algorithms and convergence analysis. Insurance: Mathematics and Economics 96, 262–275.
Jin, Z., Yin, G., Zhu, C., 2012. Numerical solutions of optimal risk control and dividend optimization policies under a generalized singular control formulation. Automatica 48 (8), 1489–1501.
Liang, X., Wang, G., 2012. On a reduced form credit risk model with common shock and regime switching. Insurance: Mathematics and Economics 51 (3), 567–575.
Lin, X.S., Willmot, G.E., 2000. The moments of the time of ruin, the surplus before ruin, and the deficit at ruin. Insurance: Mathematics and Economics 27 (1), 19–44.
Lin, X.S., Wu, P., Wang, X., 2016. Move-based hedging of variable annuities: a semi-analytic approach. Insurance: Mathematics and Economics 71, 40–49.
Lin, Y., Yu, J., Peterson, M.O., 2015. Reinsurance networks and their impact on reinsurance decisions: theory and empirical evidence. The Journal of Risk and Insurance 82 (3), 531–569.
Protter, P.E., 2005. Stochastic Integration and Differential Equations, 2nd edition. Springer, New York.
Qiu, M., Jin, Z., Li, S., 2022. Optimal dividend strategies with reinsurance under contagious systemic risk. SIAM Journal on Control and Optimization 60 (3), 1269–1293.
Taksar, M., Hunderup, C.L., 2007. The influence of bankruptcy value on optimal risk control for diffusion models with proportional reinsurance. Insurance: Mathematics and Economics 40 (2), 311–321.
Tang, Q., Tong, Z., Xun, L., 2022. Insurance risk analysis of financial networks vulnerable to a shock. European Journal of Operational Research 301 (2), 756–771.
Willmot, G.E., Lin, X.S., 1998. Exact and approximate properties of the distribution of surplus before and after ruin. Insurance: Mathematics and Economics 23 (1), 91–110.
Yao, D., Yang, H., Wang, R., 2011. Optimal dividend and capital injection problem in the dual model with proportional and fixed transaction costs. European Journal of Operational Research 211 (3), 568–576.