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Optimal portfolio selection with VaR and portfolio insurance constraints under rank-dependent expected utility theory

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ABSTRACT

This paper investigates two optimal portfolio selection problems for a rank-dependent utility investor who needs to manage his risk exposure: one with a single Value-at-Risk (VaR) constraint and the other with joint VaR and portfolio insurance constraints. The two models generalize existing models under expected utility theory and behavioral theory. The martingale method, quantile formulation, and relaxation method are used to obtain explicit optimal solutions. We have specifically identified an equivalent condition under which the VaR constraint is effective. A numerical analysis is carried out to demonstrate theoretical results, and additional financial insights are presented. We find that, in bad market states, the risk of the optimal investment outcome is reduced when compared to existing models without or with one constraint.

1. Introduction

Portfolio selection/choice theory is a cornerstone of modern finance theory. It analyzes how to invest in the financial market when faced with many investment alternatives, frequently under uncertainty. It is not only of tremendous theoretical worth, but also of significant practical utility. It is useful not only for ordinary investors in asset allocation and risk management, but also for professional fund managers in their daily operations and insurance firms in insurance policy creation.

Risk preference is a major factor in developing a portfolio selection theory. Since its conception, the expected utility (EU, for short) theory, presented by von Neumann and Morgenstern, has been the dominant paradigm in risk preference theory. When comparing the outcomes of different portfolios, investors should focus on their expected utilities rather than their absolute values, according to this theory. When a preference relation $\succ$ satisfies the weak order axiom, the independence axiom and the Archimedean axiom, von Neumann and Morgenstern proved that the preference must satisfy

$$X \succ Y \quad \Longleftrightarrow \quad \mathbb{E}[u(X)] \geqslant \mathbb{E}[u(Y)]$$

for some affine unique function u. The function u is often called the utility function. If it is strictly concave (or convex), then the preference deduced by it is called risk averse (or risk seeking, respectively). The goal of an optimal portfolio selection problem under the EU theory, according to the above formulation, is to find a portfolio that maximizes the expected utility of its outcome across all potential portfolios.

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There are two frequently utilized tools to address EU maximizing problems when faced with continuous-time investment opportunities. The first one is stochastic control theory, see, e.g., Merton (1971); and the other is martingale theory, see, e.g. Pliska (1986), Karatzas et al. (1987), Cox and Huang (1989). We adopt the second method in this paper. The EU theory has remained alive and well to this day, for it can be deformed to model a variety of new emerging phenomena in risk management, optimal stopping, pricing, and other areas.

Despite its numerous applications in theory and practice, the EU theory has flaws. This theory, for instance, cannot provide satisfactory explanations for the well-known Allais paradox (Allais, 1953). Many alternative behavioral theories have been proposed by psychologists, sociologists, and economists in order to investigate the driving forces behind people's financial decisions; for example, Yarri's (1987) dual theory, Quggin's (1982) rank dependent expected utility (RDEU) theory, Kahneman and Tversky's (1992) cumulative prospect theory (CPT), and Lopes's (1986) security potential/aspiration (SP/A) theory. These theories take into account the impact of psychological biases on decision-making. The probability weighting function (also known as the distortion function) is a fundamental element of these behavioral theories that leads to improvements in the EU theory.

Suppose the possible outcomes of a portfolio are $x_1 < x_2 < \cdots < x_n$ with a distribution $\bar{p} = (p_1, p_2, \cdots, p_n)$ where $0 < p_i < 1$, $i = 1, \cdots, n$. Then according to the EU theory, the portfolio is evaluated as

$$\sum_{i=1}^{n} p_i u(x_i).$$

When applying this, two natural questions arise: first, whether investors have the ability to obtain the true distributions for all portfolios; second, whether they use the true probabilities to weight the utilities if they know them. Quggin is apprehensive about these questions. He believes that the weights used to evaluate a portfolio are related not only to the distribution of the outcomes, but also to their ranks. According to Quggin's RDEU theory, the portfolio should be evaluated as

$$\sum_{i=1}^{n} h_i(\bar{p}) u(x_i),$$

where $h_i(\bar{p}) = w\left(\sum_{j=i}^{n} p_j\right) - w\left(\sum_{j=i+1}^{n} p_j\right)$, and w is the so-called probability weighting function that distorts the true probabilities and reflects the subjective belief of the investor. Clearly, the weights h_1, h_2, $\cdots$, h_n depend not only on the distribution of the outcomes, but also on the ranks of them. If two potential outcomes x_i and x_j have the same chance to happen, that is, $p_i = p_j$, then under the EU theory, their utilities should be weighted by the same weight; by contrast, under the RDEU theory, they may not be weighted equally because $h_i(\bar{p})$ may not be equal to $h_j(\bar{p})$. For a general outcome X, it is evaluated as the distorted expected utility:

$$\int u(x) d(1 - w(1 - F_X(x))).$$

This is a Choquet expectation of $u(X)$, mathematically speaking. Because Choquet expectation is nonlinear, classical stochastic control theory, which is used to solve stochastic optimization problems with linear expectations, cannot be applied to stochastic optimization problems for a RDEU investor. To solve them, a new method must be developed.

Jin and Zhou (2008) develop a method for studying models with law-invariant behavioral preferences in the context of continuous-time portfolio selection theory. Their approach is to first transform the problem into its quantile formulation, i.e. to convert the decision variables from terminal wealths to their quantile functions, and then to solve the quantile optimization problem. In order to deal with the latter, they must assume a specific monotonic relationship between the pricing kernel[3] and the probability weighting function. Although their work has provided a solid foundation for future research, the flaw in their method is obvious: the monotonic assumption is overly strict. Even for some of the most commonly used probability weighting functions in the Black-Scholes market setting, the assumption may not hold. Economically, we should not expect such monotonicity to hold, because the pricing kernel is determined by the financial market, whereas the probability weighting function is determined by the subjective investor, which two appear to have no obvious connection.

He and Zhou (2010) explore and assess five models with various law-invariant preferences using the quantile technique. They successfully reduce Jin and Zhou's monotonicity hypothesis to a piecewise monotone assumption. Xia and Zhou (2014) make a significant breakthrough. They employ the calculus of variations method to solve a problem under the RDEU framework without making any monotonicity assumption. Their reasoning, however, is intricate and highly skilled, and there is also a paucity of discussion on the problem's feasibility, well-posedness, attainability and uniqueness. The second author (Xu, 2016) introduces a novel way, the so-called relaxation method, to solve their problem. His argument is more succinct and intuitive, making it easier to grasp. The feasibility, well-posedness, and other difficulties are resolved by relating the problem to one under the EU theory. Since then, a unified strategy for addressing portfolio selection problems under law-invariant preference has been developed. This paper will proceed in the same manner as Xu (2016).

Risk management, on the other hand, is becoming increasingly crucial in current financial practice. Many countries, for example, require financial organizations to keep the risks in their portfolios within particular limits. To measure risks, many risk metrics have been proposed. Since JP Morgan established it in the 1990s, Value-at-Risk (VaR) has been the most extensively used risk metric by financial firms and regulators. For instance, the VaR constraint must be satisfied for insurance companies operating under the Solvency II regulation which came into force since 2016. It is also applied to insurance companies operating under regulations created under Basel II as well. VaR controls the greatest possibility that the loss will exceed a predefined level. It has been condemned for neglecting to account for tail risks. Although it can minimize the likelihood of loss incidence to an extremely low level, when such a low probability event occurs, it can result in significant losses. The outbreak of the subprime mortgage meltdown in 2007 and the following worldwide financial crisis proved that the regulations with a solo VaR constraint were not adequate for curtailing the risks that some financial firms were taking, and the dangers they posed to the worldwide financial system. Therefore, it is necessary to introduce constraints beyond a solo VaR constraint. In

[3] It is also known as the state price density or discount factor in financial economics literature.

this paper, we will consider a portfolio selection problem under joint VaR and portfolio insurance (PI) constraints, aiming at achieving a better risk management strategy than that with a solo VaR constraint.

Basak and Shapiro (2001) first solve an EU maximization problem with a VaR restriction (we call it VaR-RM problem), and then suggest a limited expected losses technique to further control the size of the major loss, resulting in a better scheme than VaR-RM. Kraft and Steffensen (2013) propose a dynamic programming approach to the conventional EU maximization problem with VaR constraint. They reduce the problem to finding a one-dimensional payoff function of a European option (instead of finding a two-dimensional function). The optimal control and value function are expressed via the option value. Chen et al. (2018) investigate an EU maximization problem under joint VaR and PI constraints.

There has also been a lot of research done on the portfolio selection problem with risk management within the context of behavioral theory. For instance, Cahuich and Hernández-Hernández (2013) solve a quantile optimization problem under law-invariant comonotonic coherent risk measure[4]; He and Zhou (2016) add a VaR constraint to the SP/A model, and create the HF/A model to investigate how varying amounts of hope, fear, and aspiration influence the optimal ultimate wealth; Ding and Xu (2015) arrive at a general conclusion for the RDEU model under law-invariant comonotonic coherent risk measure. We point out that VaR is not a law-invariant comonotonic coherent risk measure, but they have a similar integral appearance.[5] In this paper, we will use this advantage to address our new issues.

Under the RDEU theory framework in a complete market setting, this study investigates two portfolio selection problems: one with a single VaR constraint and the other with joint VaR and PI requirements. We first transform the portfolio selection problem with a single VaR constraint into its quantile formulation, then use Basak and Shapiro's (2001) idea to reduce the latter to the study of some unconstrained Lagrangian quantile optimization problem, tackle it by Xu's (2016) relaxation method, and eventually obtain the optimal solution for the original problem. We show that the second model can be treated as a special instance of the first one, resulting in a solution instantly. We also do a numerical study to demonstrate our theoretical findings, compare them to the existing models, and provide additional financial insights for the specific numerical example.

The following are the primary contributions of this study. To begin, this study, motivated by Chen et al. (2018), considers a behavioral portfolio selection problem with joint VaR and PI constraints in order to have better risk management than that with VaR constraint alone. Existing behavioral models in the literature, by contrast, only take into account one/none risk constraint. Second, He and Zhou (2016) study a model identical to our first, but their method can only handle reversed S-shaped probability weighting functions. We investigate the topic using a novel approach and provide a comprehensive solution to general probability weighting functions. Lastly, we have specifically identified an equivalent condition under which the VaR constraint is effective.

The remainder of this paper is structured as follows. Section 2 describes the financial market and the portfolio selection problem with a single VaR constraint under the RDEU theory. In Section 3, we first translate the problem into its quantile formulation, then use Xu's approach to find the optimal solution. Section 4 studies several special cases and presents the numerical analysis. Section 5 brings the paper to a close. The appendices contain several proofs.

Notation

In the paper, we may suppress "almost surely" (a.s.) and "almost everywhere" (a.e.) for notation simplicity when no confusion occurs. A function is said to be increasing (or decreasing) if it is non-decreasing (respectively, non-increasing).

2. Problem formulation

In this section we will describe the financial market and the portfolio selection problem with a single VaR constraint under the RDEU theory.

2.1. Financial market

Throughout this study, we set a specific investment time frame $[0, T]$. The market is represented by a filtered complete probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{0\leqslant t\leqslant T})$ on which is defined a standard Brownian motion $W(\cdot)$. It is assumed that $\mathcal{F}_t = \sigma\{W(s) : s \leqslant t\}$, augmented by all the P-null sets and $\mathcal{F}_T = \mathcal{F}$.

In the market, one risk-free asset (called bond) and one risky asset (called stock) are traded continuously. The bond price process follows an ordinary differential equation:

$$dB_0(t) = r(t)B_0(t)dt, \quad t \in [0, T]; \quad B_0(0) = B_0 > 0,$$

where $r(\cdot)$ is the interest rate process, and the stock price process follows a stochastic differential equation (SDE)

$$dS(t) = S(t)[b(t)dt + \sigma(t)dW(t)], \quad t \in [0, T]; \quad S(0) = S_0 > 0,$$

where $b(\cdot)$ and $\sigma(\cdot)$ represent the appreciation rate and volatility rate processes of the stock, respectively. Throughout the study, we assume the following assumption.

Assumption 2.1.

- *The processes $r(\cdot)$, $b(\cdot)$ and $\sigma(\cdot)$ are $\{\mathcal{F}_t\}$-progressively measurable and uniformly bounded.*

[4] Artzner et al. (1999) contend that a proper risk measure should address actual demands. They introduce the concept of coherent risk measure, which satisfies the axioms of monotonicity, cash invariance, subadditivity, and positive homogeneity.

[5] Denneberg (1994) proves that a law-invariant comonotonic coherent risk measure must be of the form $R_\psi(X) = \int_{-\infty}^{0}(\psi(P(-X > t)) - 1)dt + \int_0^{+\infty}\psi(P(-X > t))dt$, where ψ is a concave function. VaR can also be expressed in this form by setting $\psi(t) = 1_{(1-\alpha,1]}(t)$ at the cost of losing concavity.

- *There exists a unique, uniformly bounded, $\{\mathcal{F}_t\}$-progressively measurable process $\theta(\cdot)$ such that $\sigma(t)\theta(t) = b(t) - r(t)$ a.s. a.e. $t \in [0, T]$.*

The process $\theta(\cdot)$ is called the market price of risk (or the Sharpe ratio) process. The market would provide arbitrage opportunities if no such $\theta(\cdot)$ does exist. We define the state price density process as

$$\rho(t) = e^{-\int_0^t \left[r(s)+\frac{1}{2}\theta^2(s)\right]ds - \int_0^t \theta(s)dW(s)}, \quad t \in [0, T]. \tag{2.1}$$

It satisfies the following SDE

$$d\rho(t) = -\rho(t)[r(t)dt + \theta(t)dW(t)].$$

Consider an agent (“He”), endowed with an initial capital $x_0 > 0$. His total wealth at time t is denoted by $X(t)$. Let $\pi(t)$ denote the dollar amount invested in the risky asset at time t. Assume that trading takes place continuously in a self-financing fashion and there are no transaction costs or taxes. Then the wealth process $X(\cdot)$ satisfies the following SDE (see, e.g., Karatzas and Shreve (1999))

$$dX(t) = r(t)X(t)dt + \pi(t)\sigma(t)[\theta(t)dt + dW(t)], \quad t \in [0, T]. \tag{2.2}$$

A portfolio $\pi(\cdot)$ is said to be admissible if it is an $\{\mathcal{F}_t\}$-progressively measurable process with

$$\int_0^T \left[|\pi(t)b(t)| + |\pi(t)\sigma(t)|^2 \right] dt < \infty,$$

and the corresponding wealth process $X(\cdot)$ given by (2.2) satisfies $X(t) \geqslant cB_0(t)$ a.e. $t \in [0, T]$ for some constant c (which is allowed to be negative and varies for different portfolios). In this case the process $X(\cdot)$ is said to be admissible and $\pi(\cdot)$ to be tame. We only consider admissible controls from now on.

Definition 2.2. Let $\mathscr{A}$ denote the collection of $\mathcal{F}_T$-measurable random variables X such that $\mathbb{E}[\rho(T)X] \leqslant x_0$ and $X \geqslant cB_0(T)$ for some constant c.

Under Assumption 2.1, one can easily show by Itô’s lemma that $\mathbb{E}[\rho(T)X(T)] \leqslant x_0$ for any admissible process $X(\cdot)$. By the definition of admissible process, we see $X(T) \in \mathscr{A}$. The market is complete in the following sense.

Lemma 2.3. *Suppose Assumption 2.1 holds. Then for any $X \in \mathscr{A}$, there exists an admissible process $X(\cdot)$ such that $X(T) = X$.*

We call $\rho(T)$ the pricing kernel of the market. For notation simplicity, we write ρ instead of $\rho(T)$ from now on when no confusion occurs. Under Assumption 2.1, $\mathbb{E}[\rho] < \infty$.

Assumption 2.4. *The pricing kernel ρ admits no atom, i.e., its cumulative distribution function $F_\rho(\cdot)$ is continuous.*[6] *The function $F_\rho(\cdot)$ is strictly increasing and* essinf $\rho = 0$.

This assumption is satisfied if all the market parameters are constants and $\theta \neq 0$. Throughout the study, Assumptions 2.1 and 2.4 are put in force explicitly or implicitly.

2.2. Portfolio selection problem for a RDEU investor with VaR constraint

We now formulate the portfolio selection problem for a RDEU investor with VaR restriction. The RDEU agent evaluates a random payoff X as

$$\int_0^\infty u(x)d(1 - w(1 - F_X(x))),$$

where $u(\cdot)$ and $w(\cdot)$ denote, respectively, his utility and probability weighting functions. Given an initial wealth x_0, the agent attempts to maximize his evaluation of the terminal wealth $X(T)$ at time T. Simultaneously, in order to control his risk, a VaR restriction must be met. In terms of mathematics, the RDEU agent must solve the following portfolio selection problem with VaR constraint:

$$\begin{aligned} &\sup_{\pi} \int_0^\infty u(x)d(1 - w(1 - F_{X(T)}(x))) \\ &\text{subject to } \; dX(t) = r(t)X(t)dt + \pi(t)\sigma(t)[\theta(t)dt + dW(t)], \; t \in [0, T], \\ &\qquad X(0) = x_0, \; X(t) \geqslant 0, \; t \in [0, T], \\ &\qquad \mathbb{P}(X(T) \geqslant A) \geqslant \alpha. \end{aligned} \tag{2.3}$$

[6] This assumption is indeed redundant as we can deal with atomic pricing kernel using the result of Xu (2014). We put this assumption in this paper for the simplicity of presentation.

The VaR level is represented by $A > 0$, and the confidence level is represented by $\alpha \in (0, 1]$. If there is no probability weighting in the target, this problem reduces to the standard portfolio selection problem for an EU agent with VaR constraint investigated in Basak and Shapiro (2001) and Kraft and Steffensen (2013).

There are two key challenges in solving problem (2.3). First, the target is a Choquet expectation (see Choquet (1953)) which is not linear unless the weighting function is trivial, i.e. $w(x) \equiv x$. As a result, employing classical stochastic control theory, which primarily deals with linear expectation, fails. Despite the fact that we have control theory for nonlinear expectation such as Peng's g-expectation theory (see Peng (1997, 1999)), unfortunately, Choquet expectation and g-expectation are different except for a trivial case (i.e., the linear mathematical expectation), so we cannot immediately apply Peng's theory to our problem. When there is no weighting function, Chen et al. (2005) use the Lagrangian method to study optimal investment under VaR-regulation and minimum insurance constraints. Under the same setting, Kraft and Steffensen (2013) propose a dynamic programming approach to solve the conventional EU maximization problem with VaR constraint. Although their method is elegant, it seems hard or impossible to apply it to behavioral problems with weighting function even if there is no VaR constraint. Second, the VaR constraint in the formulation makes the problem not just more realistic economically, but also less tractable mathematically. Because this restriction is not convex, the problem is non-concave even if the target reduces to a linear expectation.

We can adopt the martingale theory to solve problem (2.3). By virtue of Lemma 2.3, it is sufficient to examine the following (reduced) static problem to solve the dynamic problem (2.3):

$$\begin{aligned} \sup_{X} \quad & \int_0^{\infty} u(x) d(1 - w(1 - F_X(x))) \\ \text{subject to} \quad & \mathbb{E}[\rho X] \leqslant x_0, \; X \geqslant 0, \\ & \mathbb{P}(X \geqslant A) \geqslant \alpha, \end{aligned} \tag{2.4}$$

where X is an $\mathcal{F}_T$-measurable random variable, representing the random payoff of some investment strategy $\pi(\cdot)$. Here we used the fact that the dynamic infinity dimensional constraint $X(t) \geqslant 0$ for all $t \in [0, T]$ is equivalent to the static constraint $X(T) \geqslant 0$, which can be proved by virtue of Itô's lemma. Also the constraints $\mathbb{E}[\rho X] \leqslant x_0$ and $X \geqslant 0$ imply $X \in \mathscr{A}$.

Neither the objective function nor the VaR constraint in problem (2.4) is concave or convex with regard to the underlying decision variable X. As a result, if we want to solve the problem directly, as in the model for the classical anticipated utility agent, we must make additional assumptions. Actually, one needs to assume the probability function $w(\cdot)$ and the pricing kernel ρ to satisfy some monotonicity condition, see, e.g., Jin and Zhou (2008) and Xu (2016). However, since ρ reflects the market environment, whereas $w(\cdot)$ is subjective and may differ for various investors, expecting them to have a monotonicity relationship is unrealistic. To overcome the difficulty, new approaches must be devised.

3. Quantile formulation and solution

Fortunately, the objective of problem (2.4) is law-invariant, that is, it just depends on the distribution of X, therefore quantile formulation may be used. The quantile function of a random variable is defined as the right-continuous inverse function of its cumulative distribution function.[7] The original problem (2.4), with decision variable X, can be transformed into an optimization problem over its quantile function using existing theory. The two problems are equivalent in the sense that their optimal solutions, i.e., an optimal random variable and an optimal quantile function respectively, have one definite relationship; see He and Zhou (2010) and Xu (2014) for further discussion. This enables quantile formulation to be used to address the portfolio choice problem for RDEU agents.

Xu (2016) propose a change of variable and relaxation method to solve a class of quantile optimization problems. This method eliminates the requirement for the previously described monotonicity assumptions while revealing the essence of the portfolio choice problem. In this paper, we use his procedure to solve problem (2.4).

Following the literature, we put the following conventional assumptions on the utility and probability weighting functions.

Assumption 3.1. *The utility function $u(\cdot) : \mathbb{R}^+ \mapsto \mathbb{R}^+$ satisfies the following conditions: (i) it is strictly increasing and second-order differentiable with $u''(\cdot) < 0$; (ii) it satisfies the usual Inada conditions, i.e., $\lim_{x \downarrow 0} u'(x) = +\infty$ and $\lim_{x \uparrow \infty} u'(x) = 0$.*

Assumption 3.2. *The probability weighting function $w(\cdot) : [0, 1] \mapsto [0, 1]$ is differentiable with $0 < w'(\cdot) < \infty$ on $[0, 1]$, and satisfies $w(0) = 0$ and $w(1) = 1$.*

We first introduce the quantile formulation of problem (2.4). By Proposition C.1, Jin and Zhou (2008) or Theorem 9, Xu (2014), the quantile formulation problem for (2.4) is given as

[7] Some literatures prefer to use the left-continuous inverse. The two definitions are only different at a set of zero Lebesgue measure. This will not affect the results as they give the value of integrals.

$$\begin{aligned}\sup_{G\in\mathscr{Q}} \quad & \int_0^1 u(G(z))d(1-w(1-z)) \\ \text{subject to} \quad & \int_0^1 F_\rho^{-1}(1-z)G(z)dz \leqslant x_0, \\ & G(1-\alpha) \geqslant A,\end{aligned} \tag{3.1}$$

where $\mathscr{Q}$ denotes the set of quantile functions of all the nonnegative random variables:

$$\mathscr{Q} = \{G(\cdot) : (0,1) \mapsto [0,\infty),\ \text{increasing and right-continuous with left limits (RCLL)}\}.$$

The relation between problems (2.4) and (3.1) is revealed by Proposition C.1, Jin and Zhou (2008) or Theorem 9, Xu (2014), that is, if G^* is an optimal solution to problem (3.1), then

$$X^* = G^*(1 - F_\rho(\rho)) \tag{3.2}$$

is an optimal solution to problem (2.4), vice versa. We remark that problem (3.1) is a concave optimization problem while problem (2.4) is not.

Before solving problem (3.1), we need to study its feasibility issue, that is, whether the admissible set is empty. For any $G \in \mathscr{Q}$ that satisfies the constraint of problem (3.1), as G is nonnegative and increasing, we have

$$\begin{aligned} x_0 &\geqslant \int_0^1 F_\rho^{-1}(1-z)G(z)dz \\ &\geqslant \int_{1-\alpha}^1 F_\rho^{-1}(1-z)G(z)dz \\ &\geqslant \int_{1-\alpha}^1 F_\rho^{-1}(1-z)G(1-\alpha)dz \\ &\geqslant A\int_{1-\alpha}^1 F_\rho^{-1}(1-z)dz \\ &= A\int_0^\alpha F_\rho^{-1}(z)dz. \end{aligned}$$

Hence

$$A\int_0^\alpha F_\rho^{-1}(z)dz \leqslant x_0. \tag{3.3}$$

If (3.3) is not satisfied, then problem (3.1) is ill-posed as there is no admissible $G \in \mathscr{Q}$ to satisfy its constraints. In fact in this case the constraints $\mathbb{E}[\rho X] \leqslant x_0$, $X \geqslant 0$ and $\mathbb{P}(X \geqslant A) \geqslant \alpha$ in problem (2.4) can not be satisfied simultaneously.

If the equation holds in (3.3), then the only admissible (thus optimal) $G \in \mathscr{Q}$ is given by

$$G(z) = \begin{cases} 0, & z < 1-\alpha; \\ A, & 1-\alpha \leqslant z < 1. \end{cases}$$

Consequently, the optimal solution to problem (2.4) is given by $X^* = A\,\mathbf{1}_{\rho \leqslant F_\rho^{-1}(\alpha)}$.

It is only left to study the non trivial case

$$A\int_0^\alpha F_\rho^{-1}(z)dz < x_0. \tag{3.4}$$

Clearly, in this case there are infinitely many feasible solutions to problem (3.1).

3.1. On functions $\varphi(\cdot)$ and $\delta(\cdot)$

Before we tackle problems (2.4) and (3.1), we go over two crucial functions: $\varphi(\cdot)$ and $\delta(\cdot)$ which are introduced by Xu (2016). They will be heavily used in our following analyses. Here we present their properties.

We define

$$\varphi(x) := -\int_0^{w^{-1}(1-x)} F_\rho^{-1}(z)dz,\ x \in [0,1].$$

This is a differentiable and strictly increasing function with $\varphi(0) = -\mathbb{E}[\rho]$ and $\varphi(1) = 0$.

Let $\delta(\cdot)$ to be the concave envelope function of $\varphi(\cdot)$, that is, the smallest concave function dominating $\varphi(\cdot)$ on $[0,1]$ with $\delta(0) = \varphi(0)$ and $\delta(1) = \varphi(1)$. We extend $\delta(\cdot)$ to the whole real line by setting $\delta(z) = \delta'(0+)z + \delta(0)$ for $z < 0$ and $\delta(z) = \delta'(1-)(z-1) + \delta(1)$ for $z > 1$. Define

$$(\delta')^{-1}(x) = \inf\{z \in \mathbb{R} : \delta'(z) < x\}, \quad x > 0. \tag{3.5}$$

We may use the following properties in our subsequent analysis without claim.

Lemma 3.3. *We have $\delta \in C^1(-\infty,\infty)$ and $(\delta')^{-1}(\cdot)$ is decreasing and left continuous. For any $z \in \mathbb{R}$ and $x > 0$,*

$$\delta'\big((\delta')^{-1}(z)\big) = z. \tag{3.6}$$

$$\varphi'\big((\delta')^{-1}(x)\big) = x. \tag{3.7}$$

$$\delta'(z) < x \quad \text{if and only if} \quad (\delta')^{-1}(x) < z. \tag{3.8}$$

$$(\delta')^{-1}(\delta'(z)) \geqslant z. \tag{3.9}$$

$$\delta'(z) > x \quad \text{implies} \quad (\delta')^{-1}(x) > z. \tag{3.10}$$

Proof. Since $\varphi(\cdot)$ is differentiable and strictly increasing on $(0,1)$, we see that $\delta'(\cdot)$ is positive, continuous and decreasing on $(0,1)$. So $\delta \in C^1(-\infty,\infty)$ and $(\delta')^{-1}(\cdot)$ is decreasing. If $(\delta')^{-1}(\cdot)$ was not left continuous at x, then $(\delta')^{-1}(x-\varepsilon) > z > (\delta')^{-1}(x)$ for some fixed z and any small $\varepsilon > 0$. From (3.5), it would yield that $x - \varepsilon \leqslant \delta'(z) < x$, leading to a contradiction by sending $\varepsilon \to 0$. So we proved that $(\delta')^{-1}(\cdot)$ is left continuous.

By (3.5) we have $\delta'\big((\delta')^{-1}(z)+\varepsilon\big) < z$ and $\delta'\big((\delta')^{-1}(z)-\varepsilon\big) \geqslant z$ for any $\varepsilon > 0$. So (3.6) follows from the continuity of $\delta'(\cdot)$. The claim (3.7) is a well-known property of concave envelope.

If $\delta'(z) < x$, then by the continuity of δ', we have $\delta'(z-\varepsilon) < x$ for some small $\varepsilon > 0$. It follows from (3.5) that $(\delta')^{-1}(x) \leqslant z - \varepsilon < z$. On the other hand, if $(\delta')^{-1}(x) < z$, then by (3.5), we have $\delta'(z) < x$. Therefore, we get the equivalency (3.8), which implies (3.9) immediately.

If $\delta'(z) > x$, then it follows from (3.5) that $(\delta')^{-1}(x) \geqslant z$. But if $(\delta')^{-1}(x) = z$, then $\delta'(z) = x$ by (3.6), contradicting to $\delta'(z) > x$. Hence, (3.10) follows. □

Lemma 3.4. *Let*

$$\lambda_{\min} := \frac{u'(A)}{\delta'(1-w(\alpha))}. \tag{3.11}$$

Then

$$\lambda_{\min} = \max\left\{\lambda > 0 : \int_0^1 1_{(u')^{-1}(\lambda\delta'(1-w(z)))\geqslant A} dz \geqslant \alpha\right\}. \tag{3.12}$$

Proof. By the monotonicity of $u'(\cdot)$, $\delta'(\cdot)$ and $w(\cdot)$,

$$\int_0^1 1_{(u')^{-1}(\lambda_{\min}\delta'(1-w(z)))\geqslant A} dz \geqslant \int_0^\alpha 1_{(u')^{-1}(\lambda_{\min}\delta'(1-w(z)))\geqslant A} dz = \alpha.$$

If $\lambda > \lambda_{\min}$, then by the continuity of $\delta'(1-w(\cdot))$, there exists $\varepsilon > 0$ such that $\lambda\delta'(1-w(\alpha-\varepsilon)) > u'(A)$, that is, $(u')^{-1}(\lambda\delta'(1-w(\alpha-\varepsilon))) < A$. So, by monotonicity,

$$\begin{aligned}\int_0^1 1_{(u')^{-1}(\lambda\delta'(1-w(z)))\geqslant A} dz &= \int_0^{\alpha-\varepsilon} 1_{(u')^{-1}(\lambda\delta'(1-w(z)))\geqslant A} dz + \int_{\alpha-\varepsilon}^1 1_{(u')^{-1}(\lambda\delta'(1-w(z)))\geqslant A} dz \\ &= \int_0^{\alpha-\varepsilon} 1_{(u')^{-1}(\lambda\delta'(1-w(z)))\geqslant A} dz \\ &< \alpha.\end{aligned}$$

This completes the proof. □

3.2. Solution in the trivial case

When the VaR constraint $\mathbb{P}(X \geqslant A) \geqslant \alpha$ is not present in problem (2.4), the problem is solved by Xia and Zhou (2014) and Xu (2016) using different approaches. Their result is summarized in the following theorem.

Theorem 3.5 *(Theorem 3.3 in Xia and Zhou (2014); Theorem 4.1 in Xu (2016)). Suppose Assumptions 2.4, 3.1 and 3.2 hold, then the optimal solution to the following problem*

$$\begin{aligned} &\sup_X \int_0^1 u(x) d(1 - w(1 - F_X(x))) \\ &\text{subject to} \quad \mathbb{E}[\rho X] \leqslant x_0, \quad X \geqslant 0, \end{aligned} \tag{3.13}$$

is given by

$$\overline{X} = (u')^{-1}\big(\bar{\lambda}\delta'\big(1 - w(F_\rho(\rho))\big)\big),$$

where $\bar{\lambda} > 0$ is determined by $\mathbb{E}[\rho \overline{X}] = x_0$.

If $\mathbb{P}(\overline{X} \geqslant A) \geqslant \alpha$, then $\overline{X}$ satisfies the VaR constraint in problem (2.4) so that it is optimal. The following result fully characterizes this trivial case.

Corollary 3.6. *Under the setting of Theorem 3.5, $\overline{X}$ is an optimal solution to problem (2.4) if and only if*

$$\int_0^1 F_\rho^{-1}(z)(u')^{-1}(\lambda_{\min}\delta'(1 - w(z)))dz \leqslant x_0. \tag{3.14}$$

Proof. Assumption 2.4 implies $F_\rho(\rho)$ is uniformly distributed on $(0, 1)$, so

$$\begin{aligned} x_0 &= \mathbb{E}[\rho \overline{X}] \\ &= \mathbb{E}\Big[F_\rho^{-1}\big(F_\rho(\rho)\big)(u')^{-1}\big(\bar{\lambda}\delta'\big(1 - w(F_\rho(\rho))\big)\big)\Big] \\ &= \int_0^1 F_\rho^{-1}(z)(u')^{-1}(\bar{\lambda}\delta'(1 - w(z)))dz. \end{aligned} \tag{3.15}$$

Because $(u')^{-1}$ is strictly decreasing, comparing this to (3.14), we conclude that (3.14) holds if and only if $\bar{\lambda} \leqslant \lambda_{\min}$.

- If $\overline{X}$ is an optimal solution to problem (2.4), then it satisfies the last constraint in problem (2.4), that is,

$$\begin{aligned} \alpha \leqslant \mathbb{P}(\overline{X} \geqslant A) &= \mathbb{P}\Big((u')^{-1}\big(\bar{\lambda}\delta'\big(1 - w(F_\rho(\rho))\big)\big) \geqslant A\Big) \\ &= \int_0^1 1_{(u')^{-1}(\bar{\lambda}\delta'(1-w(z)))\geqslant A} dz. \end{aligned}$$

  By (3.12), we see $\bar{\lambda} \leqslant \lambda_{\min}$. As a consequence, (3.14) holds.
- If (3.14) holds, then $\bar{\lambda} \leqslant \lambda_{\min}$. By the monotonicity of $(u')^{-1}(\cdot)$ and (3.12),

$$\begin{aligned} \mathbb{P}(\overline{X} \geqslant A) &= \int_0^1 1_{(u')^{-1}(\bar{\lambda}\delta'(1-w(z)))\geqslant A} dz \\ &\geqslant \int_0^1 1_{(u')^{-1}(\lambda_{\min}\delta'(1-w(z)))\geqslant A} dz \\ &\geqslant \alpha. \end{aligned}$$

  This means $\overline{X}$ satisfies the constraints of problem (2.4), so it is optimal.

The proof is complete. □

Since now, we only need to concentrate on the unresolved case

$$\int_0^1 F_\rho^{-1}(z)(u')^{-1}(\lambda_{\min}\delta'(1-w(z)))dz > x_0. \tag{3.16}$$

3.3. Solution in the nontrivial case

In this section, we will attempt to solve problem (3.1) for the last unsolved case: both (3.4) and (3.16) hold, namely

$$\int_0^1 F_\rho^{-1}(z)(u')^{-1}(\lambda_{\min}\delta'(1-w(z)))dz > x_0 > A\int_0^\alpha F_\rho^{-1}(z)dz. \tag{3.17}$$

Following the idea of Xu (2016), we allow the objective of problem (3.1) does not explicitly contain the probability weighting function by a change-of-variable approach. Let ν be defined as follows:

$$\nu(x) = 1 - w^{-1}(1-x),\ x \in [0,1].$$

It is often called the dual probability weighting function for $w(\cdot)$. By the definition of $w(\cdot)$, it is clear to see that $\nu(\cdot)$ is differentiable with $0 < \nu'(\cdot) < \infty$ on $[0,1]$. We also have $\nu(0) = 0$ and $\nu(1) = 1$.

We rewrite the objective in (3.1) as

$$\begin{aligned}\int_0^1 u(G(z))d(1-w(1-z)) &= \int_0^1 u(G(z))d\nu^{-1}(z)\\ &= \int_0^1 u(G(\nu(z)))dz\\ &= \int_0^1 u(Q(z))dz,\end{aligned}$$

where

$$Q(z) := G(\nu(z)),\ z \in (0,1).$$

Because $G \in \mathcal{Q}$ is a quantile function and ν is differentiable and strictly increasing on $[0,1]$, Q is also increasing and RCLL on $(0,1)$. So $Q \in \mathcal{Q}$, which means it can be regarded as the quantile function for some nonnegative random variable.

Also, we can rewrite the constraint in problem (3.1) as

$$\begin{aligned}x_0 &\geqslant \int_0^1 F_\rho^{-1}(1-z)G(z)dz\\ &= \int_0^1 F_\rho^{-1}(1-\nu(z))G(\nu(z))d\nu(z)\\ &= \int_0^1 F_\rho^{-1}(1-\nu(z))\nu'(z)Q(z)dz\\ &= \int_0^1 \varphi'(z)Q(z)dz,\end{aligned}$$

where the last equation follows from the fact that

$$\varphi(x) = -\int_0^{w^{-1}(1-x)} F_\rho^{-1}(z)dz = -\int_0^{1-\nu(x)} F_\rho^{-1}(z)dz.$$

As $G(1-\alpha) = Q(\nu^{-1}(1-\alpha)) = Q(1-w(\alpha))$, after the above change-of-variables, problem (3.1) turns into:

$$\begin{aligned} &\sup_{Q\in\mathscr{Q}} \int_0^1 u(Q(z))dz \\ &\text{subject to} \int_0^1 \varphi'(z)Q(z)dz \leqslant x_0, \\ &\qquad Q(1-w(\alpha)) \geqslant A. \end{aligned} \tag{3.18}$$

Because u is strictly concave with respect to the decision variable Q and the constraint set is convex, problem (3.18) admits at most one optimal solution. Clearly, if Q^* is an optimal solution to (3.18), then

$$G^*(z) = Q^*(1-w(1-z)),\ z\in(0,1),$$

is an optimal solution to (3.1), and vice versa. The concavity of the objective functional is recovered after the foregoing change-of-variables, which transform the decision variable from the random outcome to its quantile function, so we may apply the Lagrangian approach to solve quantile optimization problem (3.18).

The last constraint in problem (3.18) is, however, not in an integral form and cannot be removed immediately by the Lagrangian approach. We now rewrite it in an integral form. Notice if $Q(x) \geqslant A$, then by monotonicity

$$\int_0^1 1_{\{Q(z)<A\}}dz \leqslant \int_0^1 1_{\{Q(z)<Q(x)\}}dz = \int_0^x 1_{\{Q(z)<Q(x)\}}dz \leqslant x.$$

On the other hand, if $Q(x) < A$, then $Q(x+\epsilon) < A$ for some $\epsilon > 0$ as Q is RCLL, so

$$\int_0^1 1_{\{Q(z)<A\}}dz \geqslant \int_0^{x+\epsilon} 1_{\{Q(z)<A\}}dz = x+\epsilon > x.$$

Therefore, we conclude $Q(x) \geqslant A$ if and only if

$$\int_0^1 1_{\{Q(z)<A\}}dz \leqslant x.$$

By virtue of this result, we can rewrite problem (3.18) as

$$\begin{aligned} &\sup_{Q\in\mathscr{Q}} \int_0^1 u(Q(z))dz \\ &\text{subject to} \int_0^1 \varphi'(z)Q(z)dz \leqslant x_0, \\ &\qquad \int_0^1 1_{\{Q(z)<A\}}dz \leqslant 1-w(\alpha). \end{aligned} \tag{3.19}$$

This problem admits at most one solution as so does problem (3.18).

We are now ready to apply the Lagrangian approach to problem (3.19). We begin with the following unconstrained Lagrangian problem, inspired by Basak and Shapiro (2001):

$$\sup_{Q\in\mathscr{Q}} \int_0^1 \Big[u(Q(z)) - \lambda\delta'(z)Q(z) - \mu(\lambda)1_{\{Q(z)<A\}}\Big]dz, \tag{3.20}$$

where $\lambda > 0$ is a Lagrangian multiplier to be chosen and

$$\mu(\lambda) := u\Big((u')^{-1}\big(\lambda\delta'(1-w(\alpha))\big)\Big) - u(A) - \lambda\delta'(1-w(\alpha))\Big[(u')^{-1}\big(\lambda\delta'(1-w(\alpha))\big) - A\Big]. \tag{3.21}$$

By the concavity of u, we have $u(x) - u(A) \geqslant u'(x)(x-A)$. Taking $x = (u')^{-1}\big(\lambda\delta'(1-w(\alpha))\big)$ into it, we obtain $\mu(\lambda) \geqslant 0$. The following proposition characterizes the optimal solution to problem (3.20).

Proposition 3.7. *If $\lambda > \lambda_{\min}$, then the optimal solution to (3.20) is given by*

$$Q_\lambda^*(z) = \begin{cases} (u')^{-1}(\lambda\delta'(z)) & \text{if } 0 < z < z_{\min}; \\ A & \text{if } z_{\min} \leqslant z < z_{\max}(\lambda); \\ (u')^{-1}(\lambda\delta'(z)) & \text{if } z_{\max}(\lambda) \leqslant z < 1, \end{cases} \tag{3.22}$$

where

$$z_{\min} := 1 - w(\alpha), \quad z_{\max}(\lambda) := (\delta')^{-1}\left(\frac{u'(A)}{\lambda}\right).$$

Proof. Suppose $\lambda > \lambda_{\min}$. Thanks to (3.11), $\delta'(1 - w(\alpha)) > \frac{u'(A)}{\lambda}$. This implies $z_{\min} < z_{\max}(\lambda)$ by virtue of (3.10). Moreover, $\delta'(z) \geqslant \delta'(1 - w(\alpha)) > \frac{u'(A)}{\lambda}$ for $z < z_{\min}$, so

$$(u')^{-1}(\lambda\delta'(z)) < A, \quad 0 < z < z_{\min}. \tag{3.23}$$

By virtue of (3.6), $\delta'(z_{\max}(\lambda)) = \frac{u'(A)}{\lambda}$, so $(u')^{-1}(\lambda\delta'(z_{\max}(\lambda))) = A$. By monotonicity,

$$(u')^{-1}(\lambda\delta'(z)) \leqslant A, \quad 0 < z < z_{\max}(\lambda); \tag{3.24}$$

$$(u')^{-1}(\lambda\delta'(z)) \geqslant A, \quad z_{\max}(\lambda) \leqslant z < 1. \tag{3.25}$$

By these inequalities and monotonicity, we conclude Q_λ^* defined in (3.22) is increasing and RCLL, so $Q_\lambda^* \in \mathcal{Q}$.

To show $Q_\lambda^*(\cdot)$ is an optimal solution to problem (3.20), it suffices to show, for every fixed $z \in (0, 1)$, $Q_\lambda^*(z)$ is an optimal solution to the following problem:

$$\sup_{x>0}\Big[u(x) - \lambda\delta'(z)x - \mu(\lambda)1_{\{x<A\}}\Big]. \tag{3.26}$$

The above problem (3.26) can be rewritten as

$$\max\Big\{S_{1,\max},\ S_{2,\max}\Big\}$$

where

$$S_{1,\max} = \sup_{x\geqslant A} S_1(x), \qquad S_1(x) = u(x) - \lambda\delta'(z)x,$$

$$S_{2,\max} = \sup_{0<x<A} S_2(x), \qquad S_2(x) = u(x) - \lambda\delta'(z)x - \mu(\lambda).$$

Since $\mu(\lambda) \geqslant 0$, we have $S_1(x) \geqslant S_2(x)$ for all $x > 0$. We will use this fact in the subsequent analysis without claim. By the concavity of u, it is easy to see

$$\begin{cases} S_{1,\max} = S_1\big(\max\{(u')^{-1}(\lambda\delta'(z)),\ A\}\big), \\ S_{2,\max} = S_2\big(\min\{(u')^{-1}(\lambda\delta'(z)),\ A\}\big). \end{cases} \tag{3.27}$$

First consider the case $z_{\max}(\lambda) \leqslant z < 1$. By (3.25), $(u')^{-1}(\lambda\delta'(z)) \geqslant A$, so by (3.27),

$$S_1(Q_\lambda^*(z)) = S_1\big((u')^{-1}(\lambda\delta'(z))\big) = S_{1,\max} = \sup_{x\geqslant A} S_1(x) \geqslant S_1\big(A\big) \geqslant S_2\big(A\big) = S_{2,\max}.$$

So $Q_\lambda^*(z)$ is an optimal solution to problem (3.26).

Now consider the case $0 < z < z_{\max}(\lambda)$. In this case, by (3.24), $(u')^{-1}\big(\lambda\delta'(z)\big) \leqslant A$. So by (3.27),

$$\begin{aligned} S_{1,\max} - S_{2,\max} &= S_1\big(A\big) - S_2\big((u')^{-1}(\lambda\delta'(z))\big) \\ &= S_1\big(A\big) - S_1\big((u')^{-1}(\lambda\delta'(z))\big) + \mu(\lambda) \\ &= u\Big((u')^{-1}\big(\lambda\delta'(1 - w(\alpha))\big)\Big) - \lambda\delta'(1 - w(\alpha))\Big[(u')^{-1}\big(\lambda\delta'(1 - w(\alpha))\big)\Big] \\ &\quad - u\Big((u')^{-1}\big(\lambda\delta'(z)\big)\Big) + \lambda\delta'(z)\Big[(u')^{-1}\big(\lambda\delta'(z)\big)\Big] \\ &\quad + A\Big[\lambda\delta'(1 - w(\alpha)) - \lambda\delta'(z)\Big] \\ &= f\big((u')^{-1}\big(\lambda\delta'(z_{\min})\big)\big) - f\big((u')^{-1}\big(\lambda\delta'(z)\big)\big), \end{aligned}$$

where

$$f(x) = u(x) - u'(x)x + Au'(x).$$

Clearly $f'(x) = u''(x)(A - x)$, so f is decreasing when $x \leqslant A$. There are two cases:

- If $0 < z < z_{\min}$, then by (3.24),

  $$(u')^{-1}\big(\lambda\delta'(z)\big) \leqslant (u')^{-1}\big(\lambda\delta'(z_{\min})\big) \leqslant A.$$

  So

  $$S_{1,\max} - S_{2,\max} = f\Big((u')^{-1}\big(\lambda\delta'(z_{\min})\big)\Big) - f\Big((u')^{-1}\big(\lambda\delta'(z)\big)\Big) \leqslant 0.$$

  Notice $S_2(Q_\lambda^*(z)) = S_2\big((u')^{-1}(\lambda\delta'(z))\big) = S_{2,\max}$, so $Q_\lambda^*(z)$ is an optimal solution to problem (3.26).
- If $z_{\min} \leqslant z < z_{\max}(\lambda)$, then by (3.24),

  $$(u')^{-1}\big(\lambda\delta'(z_{\min})\big) \leqslant (u')^{-1}\big(\lambda\delta'(z)\big) \leqslant A.$$

  So

  $$S_{1,\max} - S_{2,\max} = f\Big((u')^{-1}\big(\lambda\delta'(z_{\min})\big)\Big) - f\Big((u')^{-1}\big(\lambda\delta'(z)\big)\Big) \geqslant 0.$$

  Notice $S_1(Q_\lambda^*(z)) = S_1\big(A\big) = S_{1,\max}$, so $Q_\lambda^*(z)$ is an optimal solution to problem (3.26).

The proof is complete. □

Corollary 3.8. *If $\lambda > \lambda_{\min}$, then the function $Q_\lambda^*(\cdot)$ defined by (3.22) is an optimal solution to the following problem*

$$\sup_{Q\in\mathscr{Q}} \int_0^1 \Big[u(Q(z)) - \lambda\varphi'(z)Q(z) - \mu(\lambda)1_{\{Q(z)<A\}}\Big]dz. \tag{3.28}$$

Proof. Because δ is the concave envelope of φ on [0,1],

$$\delta(0) = \varphi(0) = -\mathbb{E}[\rho], \quad \delta(1) = \varphi(1) = 0, \quad \delta(z) \geqslant \varphi(z),\ z \in [0,1].$$

By Fubini's Theorem, for any $Q \in \mathscr{Q}$, we have

$$\int_0^1 (\varphi'(z) - \delta'(z))Q(z)dz = \int_{(0,1)} (\delta(z) - \varphi(z))dQ(z) \geqslant 0,$$

hence,

$$\begin{aligned}
&\int_0^1 \Big[u(Q(z)) - \lambda\varphi'(z)Q(z) - \mu(\lambda)1_{\{Q(z)<A\}}\Big]dz \\
\leqslant &\int_0^1 \Big[u(Q(z)) - \lambda\delta'(z)Q(z) - \mu(\lambda)1_{\{Q(z)<A\}}\Big]dz \\
\leqslant &\int_0^1 \Big[u(Q_\lambda^*(z)) - \lambda\delta'(z)Q_\lambda^*(z) - \mu(\lambda)1_{\{Q_\lambda^*(z)<A\}}\Big]dz,
\end{aligned} \tag{3.29}$$

where the last inequality follows from Proposition 3.7. Because δ is the concave envelope of φ, it is affine whenever $\delta \neq \varphi$, applying Fubini's Theorem, we have

$$\begin{aligned}
\int_0^1 (\varphi'(z) - \delta'(z))Q_\lambda^*(z)dz &= \int_{(0,1)} (\delta(z) - \varphi(z))dQ_\lambda^*(z) \\
&= \int_{(0,z_{\min}]\cup[z_{\max}(\lambda),1)} (\delta(z) - \varphi(z))d\big((u')^{-1}(\lambda\delta'(z))\big) + \int_{(z_{\min},z_{\max}(\lambda))} (\delta(z) - \varphi(z))dA \\
&= 0,
\end{aligned}$$

so

$$\int_0^1 \varphi'(z)Q_\lambda^*(z)dz = \int_0^1 \delta'(z)Q_\lambda^*(z)dz.$$

Hence

$$\int_0^1 [u(Q_\lambda^*(z)) - \lambda\delta'(z)Q_\lambda^*(z) - \mu(\lambda)1_{\{Q_\lambda^*(z)<A\}}]dz = \int_0^1 [u(Q_\lambda^*(z)) - \lambda\varphi'(z)Q_\lambda^*(z) - \mu(\lambda)1_{\{Q_\lambda^*(z)<A\}}]dz.$$

Coming this and (3.29), we see Q_λ^* is an optimal solution to problem (3.28). □

The following result provides the optimal solution to problem (3.19) in the nontrivial case.

Proposition 3.9. *Suppose*

$$\int_0^1 F_\rho^{-1}(z)(u')^{-1}\left(\lambda_{\min}\delta'(1-w(z))\right)dz > x_0 > A\int_0^\alpha F_\rho^{-1}(z)dz,$$

and let $Q_\lambda^*(\cdot)$ be defined by (3.22). Then there exists a unique constant $\lambda^* > \lambda_{\min}$ such that

$$\int_0^1 \varphi'(z)Q_{\lambda^*}^*(z)dz = x_0.$$

Furthermore, $Q_{\lambda^*}^*$ is the unique optimal solution to problem (3.19).

Proof. Let

$$L(\lambda) = \int_0^1 \varphi'(z)Q_\lambda^*(z)dz.$$

Then by virtue of (3.22), (3.24) and (3.25),

$$L(\lambda) = \int_0^{z_{\min}} \varphi'(z)(u')^{-1}(\lambda\delta'(z))dz + \int_{z_{\min}}^1 \varphi'(z)\max\left\{(u')^{-1}(\lambda\delta'(z)),\ A\right\}dz. \tag{3.30}$$

The right hand side shows $L(\cdot)$ is a strictly decreasing function on $(\lambda_{\min}, \infty)$. By the monotone convergence theorem it is continuous as well. Moreover,

$$\begin{aligned}\lim_{\lambda\downarrow\lambda_{\min}} L(\lambda) &= \int_0^{z_{\min}} \varphi'(z)(u')^{-1}(\lambda_{\min}\delta'(z))dz + \int_{z_{\min}}^1 \varphi'(z)\max\left\{(u')^{-1}(\lambda_{\min}\delta'(z)),\ A\right\}dz\\ &\geqslant \int_0^1 \varphi'(z)(u')^{-1}(\lambda_{\min}\delta'(z))dz\\ &= \int_0^1 F_\rho^{-1}(1-\nu(z))\nu'(z)(u')^{-1}(\lambda_{\min}\delta'(z))dz\\ &= \int_0^1 F_\rho^{-1}(z)(u')^{-1}(\lambda_{\min}\delta'(1-w(z)))dz\\ &> x_0,\end{aligned}$$

and

$$\begin{aligned}\lim_{\lambda\uparrow\infty} L(\lambda) &= \int_0^{z_{\min}} \varphi'(z)(u')^{-1}(\infty)dz + \int_{z_{\min}}^1 \varphi'(z)\max\left\{(u')^{-1}(\infty),\ A\right\}dz\\ &= \int_{z_{\min}}^1 \varphi'(z)Adz = A\int_0^\alpha F_\rho^{-1}(z)dz < x_0.\end{aligned}$$

Therefore, there exists a unique constant $\lambda^* > \lambda_{\min}$ such that $L(\lambda^*) = x_0$. It means $Q_{\lambda^*}^*$ satisfies the first constraint of problem (3.19) with equation.

We next show $Q_{\lambda^*}^*$ satisfies the second constraint of problem (3.19) with equation too. In fact, for any $\lambda > \lambda_{\min}$, by virtue of (3.23), we have

$$\int_0^1 1_{\{Q_\lambda^*(z)<A\}}dz = \int_0^{z_{\min}} 1_{\{Q_\lambda^*(z)<A\}}dz = z_{\min} = 1-w(\alpha).$$

In particular, it is true for $Q_{\lambda^*}^*$.

The preceding argument shows that $Q_{\lambda^*}^*$ is a feasible solution to problem (3.19). We next show it is indeed optimal. Let Q be any feasible solution to problem (3.19). Then

$$\int_0^1 \varphi'(z)Q(z)dz \leqslant x_0 = \int_0^1 \varphi'(z)Q_{\lambda^*}^*(z)dz$$

and

$$\int_0^1 1_{\{Q(z)<A\}}dz \leqslant 1-w(\alpha) = \int_0^1 1_{\{Q_{\lambda^*}^*(z)<A\}}dz.$$

Therefore,

$$\begin{aligned}
&\int_0^1 u(Q_{\lambda^*}^*(z))dz - \int_0^1 u(Q(z))dz \\
&\geqslant \int_0^1 u(Q_{\lambda^*}^*(z))dz - \int_0^1 u(Q(z))dz + \lambda^*\left(\int_0^1 \varphi'(z)Q(z)dz - \int_0^1 \varphi'(z)Q_{\lambda^*}^*(z)dz\right) \\
&\quad + \mu(\lambda^*)\left(\int_0^1 1_{\{Q(z)<A\}}dz - \int_0^1 1_{\{Q_{\lambda^*}^*(z)<A\}}dz\right) \\
&= \int_0^1 \left[u(Q_{\lambda^*}^*(z)) - \lambda^*\varphi'(z)Q_{\lambda^*}^*(z) - \mu(\lambda^*)1_{\{Q_{\lambda^*}^*(z)<A\}}\right]dz \\
&\quad - \int_0^1 \left[u(Q(z)) - \lambda^*\varphi'(z)Q(z) - \mu(\lambda^*)1_{\{Q(z)<A\}}\right]dz \\
&\geqslant 0,
\end{aligned}$$

where the last inequality is due to Corollary 3.8. Hence we proved that $Q_{\lambda^*}^*$ is an optimal solution to problem (3.19).

As earlier mentioned, problem (3.19) admits at most one solution, so $Q_{\lambda^*}^*$ is indeed the unique optimal solution to it. □

Theorem 3.10. *Suppose*

$$\int_0^1 F_\rho^{-1}(z)(u')^{-1}\left(\lambda_{\min}\delta'(1-w(z))\right)dz > x_0 > A\int_0^\alpha F_\rho^{-1}(z)dz.$$

Then

$$X^* = \begin{cases} (u')^{-1}(\lambda^*\delta'(1-w(F_\rho(\rho)))) & \text{if } \rho \leqslant \frac{u'(A)}{\lambda^*}w'\left(w^{-1}\left(1-(\delta')^{-1}\left(\frac{u'(A)}{\lambda^*}\right)\right)\right) \\ A & \text{if } \frac{u'(A)}{\lambda^*}w'\left(w^{-1}\left(1-(\delta')^{-1}\left(\frac{u'(A)}{\lambda^*}\right)\right)\right) < \rho \leqslant F_\rho^{-1}(\alpha); \\ (u')^{-1}(\lambda^*\delta'(1-w(F_\rho(\rho)))) & \text{if } \rho > F_\rho^{-1}(\alpha), \end{cases} \tag{3.31}$$

is an optimal solution to problem (2.4), where $\lambda^* > \lambda_{\min}$ is the unique constant such that $\mathbb{E}[\rho X^*] = x_0$.

Proof. Notice

$$\varphi'(x) = \frac{F_\rho^{-1}(w^{-1}(1-x))}{w'(w^{-1}(1-x))}, \tag{3.32}$$

so

$$F_\rho^{-1}\left(w^{-1}\left(1-(\delta')^{-1}(x)\right)\right) = \varphi'\left((\delta')^{-1}(x)\right)w'\left(w^{-1}\left(1-(\delta')^{-1}(x)\right)\right) = xw'\left(w^{-1}\left(1-(\delta')^{-1}(x)\right)\right),$$

where we used (3.7) to get the last equation. By virtue of this, the previous proposition and (3.2), the claim follows. □

Corollary 3.11 *(Concave φ). If φ is a concave function, then the optimal terminal wealth to problem (2.4) is*

$$X^*=\begin{cases}(u')^{-1}\left(\frac{\lambda^*\rho}{w'(F_\rho(\rho))}\right) & \text{if } \rho\leqslant\frac{u'(A)}{\lambda^*}w'\left(w^{-1}\left(1-(\varphi')^{-1}\left(\frac{u'(A)}{\lambda^*}\right)\right)\right);\\ A & \text{if } \frac{u'(A)}{\lambda^*}w'\left(w^{-1}\left(1-(\varphi')^{-1}\left(\frac{u'(A)}{\lambda^*}\right)\right)\right)<\rho\leqslant F_\rho^{-1}(\alpha);\\ (u')^{-1}\left(\frac{\lambda^*\rho}{w'(F_\rho(\rho))}\right) & \text{if } \rho>F_\rho^{-1}(\alpha).\end{cases}\tag{3.33}$$

Proof. When φ is a concave function, $\delta=\varphi$. Thanks to (3.32), the claim follows. □

Remark 3.12. When w is concave, φ is also concave. Indeed, if w is concave, then since F_ρ^{-1} is increasing and w' is decreasing, their ratio F_ρ^{-1}/w' is increasing. It follows from (3.32) that φ' is decreasing, so φ is concave.

In particular, φ is concave when w is the identity function, so the optimal terminal wealth by Corollary 3.11 reduces to

$$X^*=\begin{cases}(u')^{-1}(\lambda^*\rho) & \text{if } \rho\leqslant\frac{u'(A)}{\lambda^*};\\ A & \text{if } \frac{u'(A)}{\lambda^*}<\rho\leqslant F_\rho^{-1}(\alpha);\\ (u')^{-1}(\lambda^*\rho) & \text{if } \rho>F_\rho^{-1}(\alpha).\end{cases}\tag{3.34}$$

This recovers (Basak and Shapiro, 2001, Proposition 1). As a result, Theorem 3.10 extends their result by allowing for nonlinear probability weighting function. Comparing (3.34) to (3.31), we can see that the probability weighting function scales the ranges for $X^*=A$ and $X^*>A$ as well as the value of X^* when $X^*\neq A$.

Corollary 3.13 *(S-shaped φ). Suppose φ is an S-shaped function. Let $c=\inf\{z>0:\varphi(z)=\delta(z)\}$.*

- *If $c<1-w(\alpha)$, then the optimal terminal wealth is*

$$X^*=\begin{cases}(u')^{-1}(\frac{\lambda^*\rho}{w'(F_\rho(\rho))}) & \text{if } \rho\leqslant\frac{u'(A)}{\lambda^*}w'(w^{-1}(1-(\varphi')^{-1}(\frac{u'(A)}{\lambda^*})));\\ A & \text{if } \frac{u'(A)}{\lambda^*}w'(w^{-1}(1-(\varphi')^{-1}(\frac{u'(A)}{\lambda^*})))<\rho\leqslant F_\rho^{-1}(\alpha);\\ (u')^{-1}(\frac{\lambda^*\rho}{w'(F_\rho(\rho))}) & \text{if } F_\rho^{-1}(\alpha)<\rho\leqslant F_\rho^{-1}(w^{-1}(1-c));\\ (u')^{-1}(\frac{\lambda^*F_\rho^{-1}(w^{-1}(1-c))}{w'(w^{-1}(1-c))}) & \text{if } \rho>F_\rho^{-1}(w^{-1}(1-c)).\end{cases}$$

- *If $1-w(\alpha)\leqslant c<(\varphi')^{-1}(\frac{u'(A)}{\lambda^*})$, then the optimal terminal wealth is*

$$X^*=\begin{cases}(u')^{-1}(\frac{\lambda^*\rho}{w'(F_\rho(\rho))}) & \text{if } \rho\leqslant\frac{u'(A)}{\lambda^*}w'(w^{-1}(1-(\varphi')^{-1}(\frac{u'(A)}{\lambda^*})));\\ A & \text{if } \frac{u'(A)}{\lambda^*}w'(w^{-1}(1-(\varphi')^{-1}(\frac{u'(A)}{\lambda^*})))<\rho\leqslant F_\rho^{-1}(\alpha);\\ (u')^{-1}(\frac{\lambda^*F_\rho^{-1}(w^{-1}(1-c))}{w'(w^{-1}(1-c))}) & \text{if } \rho>F_\rho^{-1}(\alpha).\end{cases}$$

A proof of this corollary is provided in Appendix A.

4. Special cases and numerical analysis

In this section, we study several special cases and perform numerical analysis to demonstrate our theoretical findings and provide some financial insights. We study a concave distortion function case in Section 4.1 an inverse S-shaped distortion function case in Section 4.2, and a case with both VaR and PI constraints in Section 4.3, respectively.

4.1. Concave distortion function

Throughout this subsection, we assume that the utility function is

$$u(x)=\frac{x^{1-\eta}-1}{1-\eta},\quad x>0,\tag{4.1}$$

where $0<\eta\neq1$ is the relative risk aversion coefficient, and the probability distortion function is

$$w(x)=\Phi\big(\Phi^{-1}(x)+\beta\big),\tag{4.2}$$

where $\beta>0$ is a constant and $\Phi(\cdot)$ denotes the standard normal distribution function. The above probability distortion function w is first introduced by Wang (2000), so it is often called the Wang transform in the literature. As $w\big(\Phi(x)\big)=\Phi\big(x+\beta\big)$, we see

$$w'\big(\Phi(x)\big)=\frac{\Phi'\big(x+\beta\big)}{\Phi'(x)}=e^{-\beta(x+\beta/2)}$$

is a decreasing function, so w is concave. By Remark 3.12 and Corollary 3.11, the optimal terminal wealth X^* is given by (3.33).

For the sake of simplicity, we assume that the risk-free asset's return rate, as well as the risky asset's appreciation and volatility rates, are all constants. Consequently, the pricing kernel ρ follows a lognormal distribution:

$$\ln\rho \sim N\left(-\left(r+\tfrac{\theta^2}{2}\right)T, \quad \theta^2 T\right). \tag{4.3}$$

Remark 4.1. For further discussion, we here use a power utility function; however, it is simple to confirm that all of the conclusions we derive in this part largely remain unchanged if we use the exponential or logarithmic utilities.

We first give the optimal terminal wealth and portfolio.

Theorem 4.2. *Let the utility $u(\cdot)$ and the probability distortion function $w(\cdot)$ be, respectively, given by (4.1) and (4.2). Assume the market coefficients $r(\cdot)$, $b(\cdot)$, $\sigma(\cdot)$ are all constants with $b > r$ and $\sigma > 0$. Then, the optimal dollar amount invested in the risky asset for problem (2.3), as a feedback function of time t and the stock price $S(t)$, is given by*

$$\pi^*(t, S(t)) = f_x(t, S(t))S(t)$$

and the corresponding optimal wealth process at time t is

$$X^*(t) = f(t, S(t)),$$

where

$$\begin{aligned}
f(t,x) &= e^{-r(T-t)}\,\mathbb{E}\left[g\left(xe^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(W(T)-W(t))}\right)\right],\\
g(x) &= C_1 x^{\kappa_1}1_{x<C_2} + A1_{C_2\leqslant x<C_3} + C_1x^{\kappa_1}1_{x\geqslant C_3},\\
\varphi(x) &= -\int_0^{\Phi(\Phi^{-1}(1-x)-\beta)} F_\rho^{-1}(z)dz,\\
C_1 &= \left(\lambda^* e^{\frac{\beta^2}{2}+\frac{\beta(r+\frac{1}{2}\theta^2)\sqrt{T}}{\theta}}\right)^{-\frac{1}{\eta}} \varpi^{\kappa_1},\\
C_2 &= \left[F_\rho^{-1}(\alpha)\right]^{\sigma/\theta}\varpi^{-1},\\
C_3 &= \left[F_\rho^{-1}\left(\Phi\left(\Phi^{-1}\left(1-(\varphi')^{-1}\left((\lambda^*)^{-1}A^{-\eta}\right)\right)-\beta\right)\right)\right]^{\sigma/\theta}\varpi^{-1},\\
\varpi &= (S(0))^{-1}\exp\left(\left(-(b-\frac{1}{2}\sigma^2)+\frac{\sigma}{\theta}(r+\frac{1}{2}\theta^2)\right)T\right),\\
\kappa_1 &= \frac{\theta}{\sigma\eta}\left(1+\frac{\beta}{\theta\sqrt{T}}\right),\\
\theta &= \sigma^{-1}(b-r),
\end{aligned}$$

and the constant λ^* is uniquely determined by

$$\mathbb{E}\left[e^{-(r+\frac{1}{2}\theta^2)T-\theta W(T)}g\left(S(0)e^{(b-\frac{1}{2}\sigma^2)T+\sigma W(T)}\right)\right] = x_0.$$

A proof of this theorem is provided in Appendix B.

Assuming there is no probability weighting function, Kraft and Steffensen (2013) propose an elegant dynamic programming approach to solve the classical EU maximization problem with VaR constraint. The optimal control is expressed by a feedback function of time and the value of an option. In Theorem 4.2, the optimal control is similarly expressed by a feedback function of time and the stock price. From (4.4) below, we see that the optimal control can also be interpreted as a feedback function of time and option.

Remark 4.3. Here the strategy is pre-committed, that is, optimal at time 0. Because the strategy depends on the initial stock price $S(0)$, it is time inconsistent, that is, the strategy will change if one revisits the problem after the initial time. This is a common feature for optimal investment problems under behavioral finance theories.

Remark 4.4. We have the explicit expression

$$\begin{aligned}
f(t,x) &= C_1x^{\kappa_1}e^{(r+\frac{\kappa_1\sigma^2}{2})(\kappa_1-1)(T-t)}\Phi\left(\frac{\log C_2-\log x-(r-\frac{1}{2}\sigma^2+\kappa_1\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)\\
&\quad + Ae^{-r(T-t)}\Phi\left(\frac{\log C_3-\log x-(r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)
\end{aligned}$$

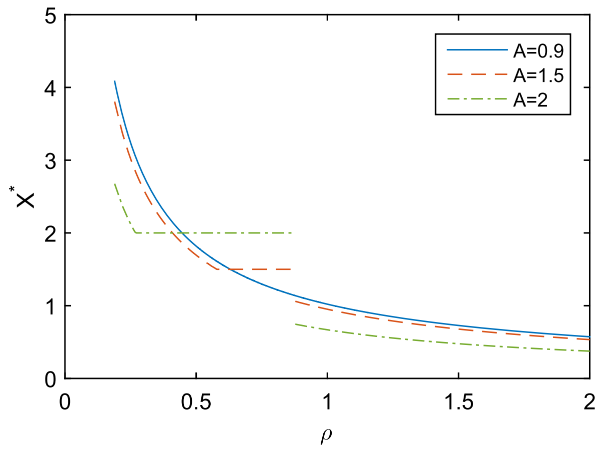


Fig. 1. The effect of benchmark A on the optimal terminal wealth X^*.

$$- Ae^{-r(T-t)}\Phi\left(\frac{\log C_2 - \log x - (r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right)$$
$$+ C_1 x^{\kappa_1} e^{(r+\frac{\kappa_1\sigma^2}{2})(\kappa_1-1)(T-t)}\Phi\left(-\frac{\log C_3 - \log x - (r-\frac{1}{2}\sigma^2+\kappa_1\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right). \tag{4.4}$$

This equation is proved in Appendix C.

In the following, we perform numerical analysis.

4.1.1. Effect of benchmark A

Because the optimal terminal wealth X^* in (3.33), as a function of pricing kernel ρ, is divided into three regions, we call them, respectively, the good market states (the first interval with small ρ), the intermediate market states (the second interval with medium ρ) and the bad market states (the last interval with big ρ). This section will look at how the benchmark affects the optimal terminal wealth.

Observe (3.33). If A rises, the intermediate states will account for a bigger fraction of the total, and the terminal wealth rises. To meet the restriction $L(\lambda^*) = x_0$, the Lagrangian multiplier λ^* should be increased by (3.30). Given two benchmarks $A_1 < A_2$ that fulfill the nontrivial condition (3.17), the corresponding Lagrangian multipliers λ_1 and λ_2 must satisfy $\lambda_1 < \lambda_2$. It thus follows

$$\frac{(u')^{-1}\left(\frac{\lambda_2\rho}{w'(F_\rho(\rho))}\right)}{(u')^{-1}\left(\frac{\lambda_1\rho}{w'(F_\rho(\rho))}\right)} = \frac{\left(\frac{\lambda_2\rho}{w'(F_\rho(\rho))}\right)^{-\frac{1}{\eta}}}{\left(\frac{\lambda_1\rho}{w'(F_\rho(\rho))}\right)^{-\frac{1}{\eta}}} = \left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{\eta}} < 1.$$

Economically speaking, as A arises, the terminal wealth will fall in both the good and bad market states. This phenomenon is depicted in Fig. 1, where the parameters are set as follows:

$$r = 0.05, \quad \theta = 0.4, \quad x_0 = 1, \quad \eta = 1.5, \quad \beta = 0.1, \quad \alpha = 0.5.$$

Fig. 1 shows the optimal terminal wealth as a function of ρ when the benchmark A is set to 0.9, 1.5, or 2.

When $A = 0.9$, the benchmark is too low such that the VaR constraint holds automatically, hence the optimal wealth X^* is equivalent to the one without the constraint as defined in Theorem 3.5. When A rises, there is a discontinuity in X^* as a function of ρ. To meet the VaR restriction, the function hops. Furthermore, the intermediate states of agent 1 with $A = 2$ account for a greater share of the total than the intermediate states of agent 2 with $A = 1.5$. Aside from the intermediate states, agent 1 has a lower terminal wealth than agent 2 in every market state, good or bad. This implies that, in order to obtain additional security, the agent must give up some of his returns in both good and bad market circumstances.

4.1.2. Effect of confidence level α

To meet the constraint $L(\lambda^*) = x_0$, a rise in α (that is a drop in $z_{\min}$) causes an increase in the unique Lagrangian multiplier λ^* by (3.30). Given two confidence levels $0 < \alpha_1 < \alpha_2 < 1$, we get

$$\frac{(u')^{-1}\left(\frac{\lambda_2\rho}{w'(F_\rho(\rho))}\right)}{(u')^{-1}\left(\frac{\lambda_1\rho}{w'(F_\rho(\rho))}\right)} = \frac{\left(\frac{\lambda_2\rho}{w'(F_\rho(\rho))}\right)^{-\frac{1}{\eta}}}{\left(\frac{\lambda_1\rho}{w'(F_\rho(\rho))}\right)^{-\frac{1}{\eta}}} = \left(\frac{\lambda_1}{\lambda_2}\right)^{\frac{1}{\eta}} < 1.$$

In addition, when the confidence level rises, the intermediate states will account for a larger part of the total. This behavior is depicted in Fig. 2. The parameters r, θ, x_0, η, and β are set as described previously, and the benchmark A is set to 2. Fig. 2 shows the related optimal terminal wealth as a function of ρ when α is 0.05, 0.5, or 1.

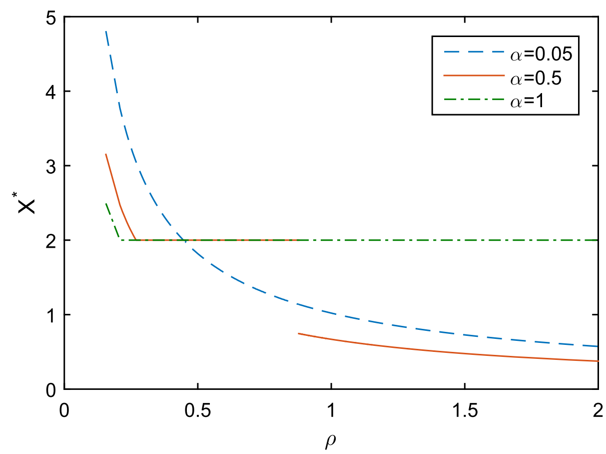


Fig. 2. The effect of confidence level α on the optimal terminal wealth X^*.

When $\alpha = 0.05$, the confidence level is too low such that the VaR constraint holds automatically, indicating that the terminal wealth function of ρ is continuous. As the α increases, implying that the agent intends for $X \geqslant A$ to have a larger probability, he must accept a lower terminal wealth in both good and bad market states. In the worst-case scenario, $\alpha = 1$, the agent wants his terminal wealth to be no less than 2 for sure. This is often referred to as portfolio insurance. The terminal wealth in good market states is lower than in bad market situations, but the agent can ensure that his terminal wealth will never be less than 2 in any scenario. Economically, he preserves his future earnings in bad market situations by foregoing some of his advantages in good market states.

4.2. Inverse S-shaped distortion function

In this subsection, we assume that the utility function is given by (4.1), and consider the Prelec weighting function:

$$w(x) = e^{-\beta_1(-\ln x)^{\alpha_1}}, \quad \alpha_1 > 0, \quad \beta_1 > 0.$$

When $0 < \alpha_1 < 1$, the function $w(\cdot)$ is inverse S-shaped; see Prelec (1998).

We have

$$w'(x) = \frac{\alpha_1\beta_1}{x}(-\ln x)^{\alpha_1-1}e^{-\beta_1(-\ln x)^{\alpha_1}},$$

and

$$w^{-1}(1-x) = e^{-(-\frac{\ln(1-x)}{\beta_1})^{\frac{1}{\alpha_1}}}.$$

Under the assumption of (4.3), we have

$$\varphi(x) = -e^{\mu_\rho + \frac{\sigma_\rho^2}{2}}\Phi\Big(\Phi^{-1}\big(e^{-(-\frac{\ln(1-x)}{\beta_1})^{\frac{1}{\alpha_1}}}\big) - \sigma_\rho\Big),$$

and

$$\varphi'(x) = \frac{1}{\alpha_1\beta_1(1-x)}\Big(-\frac{\ln(1-x)}{\beta_1}\Big)^{-\frac{\alpha_1-1}{\alpha_1}}e^{-(-\frac{\ln(1-x)}{\beta_1})^{\frac{1}{\alpha_1}}}e^{\mu_\rho+\sigma_\rho\Phi^{-1}(e^{-(-\frac{\ln(1-x)}{\beta_1})^{\frac{1}{\alpha_1}}})}.$$

According to He and Zhou (2016) and Xu (2016), $\varphi'(\cdot)$ is first strictly increasing and then strictly decreasing, i.e., $\varphi(\cdot)$ is a strictly S-shaped. By Corollary 3.13, we can get the optimal terminal wealth X^*.

It is interesting to explore the effects of different distortion parameters α_1 and β_1 on the optimal terminal wealth. The parameters are as follows:

$$r = 0.05, \ \theta = 0.5, \ x_0 = 1, \ \eta = 0.5.$$

Fig. 3 depicts the optimal terminal wealth as a function of ρ when α_1 is set to 0.5, 0.3. As is shown in Fig. 3, the optimal terminal wealth has four different regions in dependence of ρ when $\alpha_1 = 0.5$, while it has only three regions for a smaller α_1.

Fig. 4 shows the optimal terminal wealth as a function of ρ when β_1 is set to 1, 1.2. The optimal solution for a smaller β_1 is four-region type. When β_1 increases, the optimal solution reduces to a three-region one.

4.3. Portfolio selection problem for a RDEU investor with both VaR and PI constraints

In the preceding sections, we looked at the portfolio selection problem (2.3) for a RDEU investor with VaR constraint. VaR, as is widely known, is concerned with the chance of loss rather than the magnitude of loss, and hence fails to capture tail risk. Portfolio insurance is

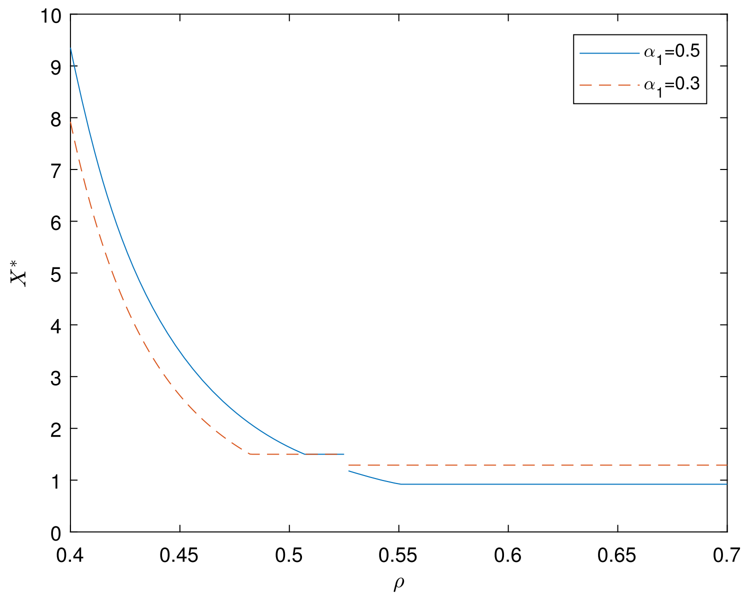


Fig. 3. The effect of α_1 on the optimal terminal wealth X^*.

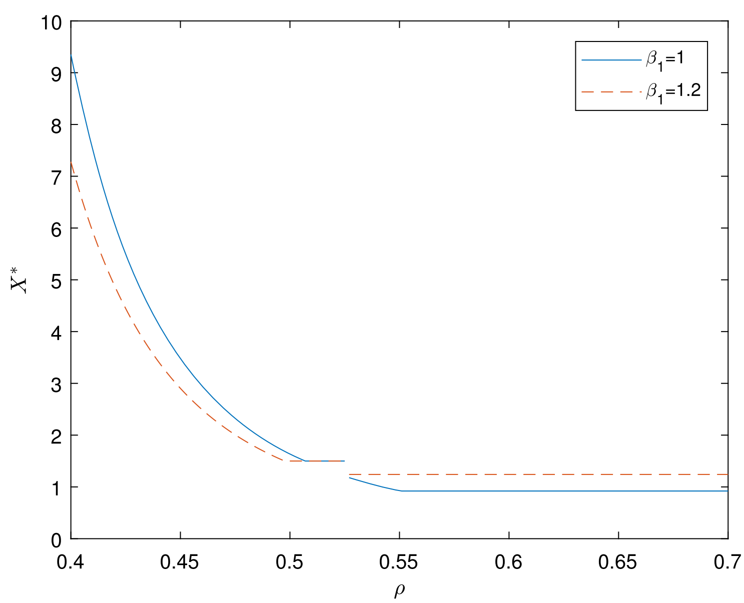


Fig. 4. The effect of β_1 on the optimal terminal wealth X^*.

a viable technique to further limit risk by ensuring the minimum value. It demands the investor to retain his money above or below a certain level. Actually, the PI constraint is a subset of the VaR constraint in which the confidence level is set to 1.

We formulate the (reduced) portfolio selection problem for a RDEU investor with both VaR and PI constraints as follows:

$$\begin{aligned} \sup_X \quad & \int_0^\infty u(x)d(1-w(1-F_X(x))) \\ \text{subject to} \quad & \mathbb{E}[\rho X] \leqslant x_0,\ X \geqslant 0, \\ & \mathbb{P}(X \geqslant A) \geqslant \alpha, \\ & X \geqslant a. \end{aligned} \tag{4.5}$$

In this problem, regardless of whether the market is good or bad, the agent's outcome must be greater than a, hence controlling the risk. When the VaR constraint is removed from (4.5), the model will reduce to the (reduced) portfolio selection problem for a RDEU investor with a PI constraint.

Clearly problem (4.5) is equivalent to

$$\begin{aligned}
&\sup_X \int_0^\infty u(x)d(1-w(1-F_X(x)))\\
&\text{subject to}\quad \mathbb{E}[\rho X]\leqslant x_0,\ X\geqslant \max\{0,a\},\\
&\qquad\qquad\quad \mathbb{P}(X\geqslant A)\geqslant \alpha.
\end{aligned}$$

Let $Y=X-\max\{0,a\}$ and $\overline{A}=A-\max\{0,a\}>0$. Then the problem becomes

$$\begin{aligned}
&\sup_Y \int_0^\infty \bar{u}(x)d(1-w(1-F_Y(x)))\\
&\text{subject to}\quad \mathbb{E}[\rho Y]\leqslant \bar{x}_0,\ Y\geqslant 0,\\
&\qquad\qquad\quad \mathbb{P}(Y\geqslant \overline{A})\geqslant \alpha,
\end{aligned}$$

where $\bar{u}(x)=u(x+\max\{0,a\})$ and $\bar{x}_0=x_0-\max\{0,a\}\mathbb{E}[\rho]$. Clearly this is noting but problem (2.4) with different parameters. So all the theoretical results obtained thus far can be applied to problem (4.5) immediately.

According the above analysis and Theorem 3.10, we have the following theorem.

Theorem 4.5. *Suppose*

$$\int_0^1 F_\rho^{-1}(z)(u')^{-1}\left(\lambda_{\min}\delta'(1-w(z))\right)dz > x_0 > A\int_0^\alpha F_\rho^{-1}(z)dz.$$

Then the optimal solution to problem (4.5) *is given by*

$$X^*=\begin{cases}(u')^{-1}(\lambda^*\delta'(1-w(F_\rho(\rho)))) & \text{if } \rho\leqslant \frac{u'(A)}{\lambda^*}w'\left(w^{-1}\left(1-(\delta')^{-1}\left(\frac{u'(A)}{\lambda^*}\right)\right)\right),\\ A & \text{if } \frac{u'(A)}{\lambda^*}w'\left(w^{-1}\left(1-(\delta')^{-1}\left(\frac{u'(A)}{\lambda^*}\right)\right)\right)<\rho\leqslant F_\rho^{-1}(\alpha),\\ (u')^{-1}(\lambda^*\delta'(1-w(F_\rho(\rho)))) & \text{if } F_\rho^{-1}(\alpha)<\rho\leqslant \frac{u'(a)}{\lambda^*}w'\left(w^{-1}\left(1-(\delta')^{-1}\left(\frac{u'(a)}{\lambda^*}\right)\right)\right),\\ a & \text{if } \rho>\frac{u'(a)}{\lambda^*}w'\left(w^{-1}\left(1-(\delta')^{-1}\left(\frac{u'(a)}{\lambda^*}\right)\right)\right),\end{cases}$$

if $(\delta')^{-1}\left(\frac{u'(a)}{\lambda^*}\right)<1-w(\alpha)$, *and is given by*

$$X^*=\begin{cases}(u')^{-1}(\lambda^*\delta'(1-w(F_\rho(\rho)))) & \text{if } \rho\leqslant \frac{u'(A)}{\lambda^*}w'\left(w^{-1}\left(1-(\delta')^{-1}\left(\frac{u'(A)}{\lambda^*}\right)\right)\right),\\ A & \text{if } \frac{u'(A)}{\lambda^*}w'\left(w^{-1}\left(1-(\delta')^{-1}\left(\frac{u'(A)}{\lambda^*}\right)\right)\right)<\rho\leqslant F_\rho^{-1}(\alpha),\\ a & \text{if } \rho>F_\rho^{-1}(\alpha),\end{cases}$$

if $(\delta')^{-1}\left(\frac{u'(a)}{\lambda^*}\right)\geqslant 1-w(\alpha)$. *In both cases,* λ^* *is the unique constant such that* $\mathbb{E}[\rho X^*]=x_0$.

Remark 4.6. When w is the identify function, the optimal outcome for problem (4.5) is

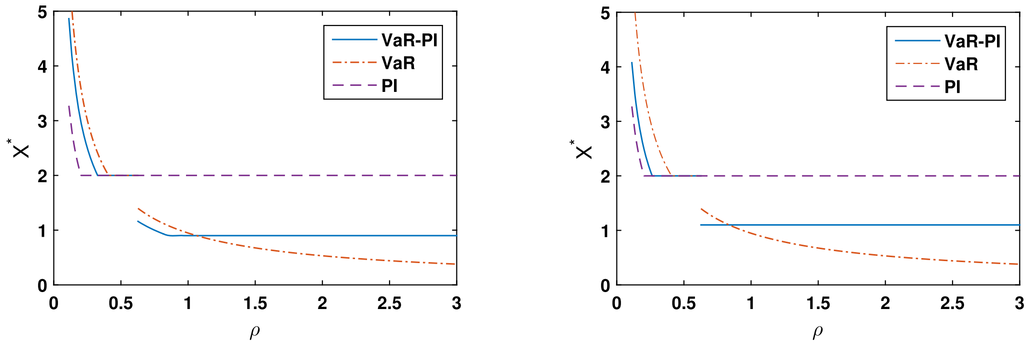
$$X^*=\begin{cases}(u')^{-1}(\lambda^*\rho) & \text{if } \rho\leqslant\frac{u'(A)}{\lambda^*},\\ A & \text{if } \frac{u'(A)}{\lambda^*}<\rho\leqslant F_\rho^{-1}(\alpha),\\ (u')^{-1}(\lambda^*\rho) & \text{if } F_\rho^{-1}(\alpha)<\rho\leqslant\frac{u'(a)}{\lambda^*},\\ a & \text{if } \rho>\frac{u'(a)}{\lambda^*},\end{cases}\tag{4.6}$$

if $\frac{u'(a)}{\lambda^*}>F_\rho^{-1}(\alpha)$, and

$$X^*=\begin{cases}(u')^{-1}(\lambda^*\rho) & \text{if } \rho\leqslant\frac{u'(A)}{\lambda^*},\\ A & \text{if } \frac{u'(A)}{\lambda^*}<\rho\leqslant F_\rho^{-1}(\alpha),\\ a & \text{if } \rho>F_\rho^{-1}(\alpha),\end{cases}\tag{4.7}$$

if $\frac{u'(a)}{\lambda^*}\leqslant F_\rho^{-1}(\alpha)$. This is the same as Theorem (2.1) in Chen et al. (2018). Hence, our result generalizes their results by allowing nonlinear probability distortion function.

We now compare the performances of three agents: a VaR agent using model (2.3); a VaR-PI agent using model (4.5); and a PI agent using model (4.5) but without the VaR constraint. The optimal terminal wealths for them are depicted in Fig. 5 with parameters $a=0.9$ and $a=1.1$, $A=2$, $\alpha=0.2$. In comparison to the VaR agent, the VaR-PI agent sacrifices some of his earnings in good market situations



(a) $a = 0.9$ (b) $a = 1.1$

Fig. 5. The optimal terminal wealths for different models.

but guarantees a minimum level of wealth in bad market states, whereas the VaR agent may have extreme low results in bad market states. When compared to the PI agent, the VaR-PI agent achieves better results in good market conditions and acceptable results in bad market states. When a is changed to 1.1 in Fig. 5(b), the conclusion remains the same. Overall, we may conclude that the model with both PI and VaR constraints is a good compromise between the model with only PI constraint and the model with only VaR constraint. This, of course, corresponds to our financial intuition.

5. Conclusions

In this research, we first investigate the portfolio selection problem for a RDEU investor with a VaR restriction. After translating the problem into its quantile formulation, the relaxation method is used to find the solution. We address the problem with both VaR and PI constraints to further control the risk. Economically, we find that the model with both PI and VaR constraints is a good middle ground between models with solely VaR or PI constraint.

In our study, the market is assumed to be complete so that the martingale approach can be applied. It is extremely important to study the incomplete market case in which the martingale approach may fail. Also, it is worth to study the case with an expected shortfall (ES) constraint. The ES risk measure has many advantages compared to the VaR risk measure.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Data availability

No data was used for the research described in the article.

Appendix A. Proof of Corollary 3.13

Let $c = \inf\{z > 0 : \varphi(z) = \delta(z)\}$. Because φ is S-shaped, $\varphi(c) - \varphi(0) - \varphi'(c)c = 0$ and

$$\delta'(z) = \begin{cases} \varphi'(c), & z < c \\ \varphi'(z), & z \geqslant c. \end{cases}$$

Substituting the analytical expression of $\delta'(z)$ into (3.22), we obtain

- if $c < z_{\min}$,

$$Q_\lambda^*(z) = \begin{cases} (u')^{-1}(\lambda\varphi'(c)), & 0 < z < c; \\ (u')^{-1}(\lambda\varphi'(z)), & c \leqslant z < z_{\min}; \\ A, & z_{\min} \leqslant z < z_{\max}(\lambda); \\ (u')^{-1}(\lambda\varphi'(z)), & z_{\max}(\lambda) \leqslant z < 1; \end{cases}$$

- if $z_{\min} \leqslant c < z_{\max}(\lambda)$,

$$Q_\lambda^*(z) = \begin{cases} (u')^{-1}(\lambda\varphi'(c)), & 0 < z < z_{\min}; \\ A, & z_{\min} \leqslant z < z_{\max}(\lambda); \\ (u')^{-1}(\lambda\varphi'(z)), & z_{\max}(\lambda) \leqslant z < 1; \end{cases}$$

where $z_{\min} = 1 - w(\alpha)$ and $z_{\max}(\lambda) = (\delta')^{-1}(\frac{u'(A)}{\lambda}) = (\varphi')^{-1}(\frac{u'(A)}{\lambda})$.

The proof is complete by showing that $c < z_{\max}(\lambda)$. Suppose on the contrary that $c \geqslant z_{\max}(\lambda)$. Because $\delta'(z) = \delta'(c)$ for $z \in (0, c]$ and $\delta'(z) < \delta'(c)$ for $z > c$, we have

$$\delta'(c) = \delta'(z_{\max}(\lambda)) = \delta'(z_{\min}) = \delta'(1 - w(\alpha)).$$

On the other hand, $z_{\max}(\lambda) = (\delta')^{-1}(\frac{u'(A)}{\lambda})$, so

$$\delta'(z_{\max}(\lambda)) = \frac{u'(A)}{\lambda}.$$

Comparing above equations, we get

$$\delta'(1 - w(\alpha)) = \frac{u'(A)}{\lambda}.$$

But this contradicts (3.11) and $\lambda > \lambda_{\min}$.

Appendix B. Proof of Theorem 4.2

We first notice

$$S(T) = S(0)e^{(b-\frac{1}{2}\sigma^2)T+\sigma W(T)}, \quad \rho = \rho(T) = e^{-(r+\frac{1}{2}\theta^2)T-\theta W(T)},$$

so

$$\rho = (\varpi S(T))^{-\theta/\sigma}.$$

Because

$$\ln \rho \sim N\left(\mu_\rho, \sigma_\rho^2\right),$$

where

$$\mu_\rho = -(r + \frac{1}{2}\theta^2)T, \quad \sigma_\rho = \theta\sqrt{T},$$

we have

$$F_\rho(\rho) = \Phi\left(\frac{\ln \rho - \mu_\rho}{\sigma_\rho}\right).$$

Because $w(x) = \Phi(\Phi^{-1}(x) + \beta)$, we see

$$w'(\Phi(x)) = e^{-\beta x - \frac{\beta^2}{2}}.$$

Therefore,

$$w'\left(F_\rho(\rho)\right) = e^{-\beta\frac{\ln\rho-\mu_\rho}{\sigma_\rho} - \frac{\beta^2}{2}} = \rho^{-\frac{\beta}{\sigma_\rho}} e^{\frac{\beta\mu_\rho}{\sigma_\rho} - \frac{\beta^2}{2}}.$$

By virtue of this, we have

$$\begin{aligned} X^*(T) = g(S(T)) &= C_1 S(T)^{\kappa_1} 1_{S(T)<C_2} + A 1_{C_2 \leqslant S(T) < C_3} + C_1 S(T)^{\kappa_1} 1_{S(T)\geqslant C_3} \\ &= \left(\lambda^* e^{\frac{\beta^2}{2} - \frac{\beta\mu_\rho}{\sigma_\rho}}\right)^{-\frac{1}{\eta}} \rho^{-\frac{1}{\eta}\left(1+\frac{\beta}{\sigma_\rho}\right)} 1_{\{\rho \leqslant q_1\}} + A 1_{\{q_1 < \rho \leqslant q_2\}} \\ &\quad + \left(\lambda^* e^{\frac{\beta^2}{2} - \frac{\beta\mu_\rho}{\sigma_\rho}}\right)^{-\frac{1}{\eta}} \rho^{-\frac{1}{\eta}\left(1+\frac{\beta}{\sigma_\rho}\right)} 1_{\{\rho > q_2\}}, \end{aligned}$$

where

$$q_1 = F_\rho^{-1}\left(\Phi\left(\Phi^{-1}\left(1 - (\varphi')^{-1}\left((\lambda^*)^{-1}A^{-\eta}\right)\right) - \beta\right)\right), \quad q_2 = F_\rho^{-1}(\alpha).$$

The right hand side is nothing but the optimal terminal wealth X^* given in (3.31). So $X^*(T)$ is an optimal solution to problem (2.4).

To show $\pi^*(t, S(t)))$ is an optimal strategy to problem (2.3), suffices to verify that $(X^*(t), \pi^*(t, S(t)))$ satisfies the wealth process (2.2). By the Feynman-Kac Formula, we have

$$rf(t,x) = f_t(t,x) + \frac{1}{2}f_{xx}(t,x)\sigma^2 x^2 + rxf_x(t,x), \quad f(T,x) = g(x).$$

So by Ito's lemma,

$$\begin{aligned} dX^*(t) &= df(t, S(t)) \\ &= f_t(t, S(t))dt + f_x(t, S(t))S(t)(bdt + \sigma dW(t)) + \frac{1}{2}f_{xx}(t, S(t))\sigma^2 S(t)^2 dt \\ &= rX^*(t)dt + \pi^*(t, S(t))\sigma[\theta dt + dW(t)], \end{aligned}$$

which completes the proof.

Appendix C. Proof of (4.4)

Suppose $\log\xi \sim \Phi(\bar{\mu}, \bar{\sigma}^2)$. Then

$$\mathbb{E}\left[\xi^{\bar{a}} 1_{\xi<\bar{b}}\right] = e^{\bar{a}\bar{\mu}}\,\mathbb{E}\left[e^{\bar{a}\bar{\sigma}\frac{\log\xi-\bar{\mu}}{\bar{\sigma}}} 1_{\frac{\log\xi-\bar{\mu}}{\bar{\sigma}}<\frac{\log\bar{b}-\bar{\mu}}{\bar{\sigma}}}\right] = e^{\bar{a}\bar{\mu}} \int_{-\infty}^{\frac{\log\bar{b}-\bar{\mu}}{\bar{\sigma}}} e^{\bar{a}\bar{\sigma}x} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$= e^{\bar{a}\bar{\mu}+\frac{\bar{a}^2\bar{\sigma}^2}{2}} \int_{-\infty}^{\frac{\log\bar{b}-\bar{\mu}}{\bar{\sigma}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\bar{a}\bar{\sigma})^2}{2}} dx = e^{\bar{a}\bar{\mu}+\frac{\bar{a}^2\bar{\sigma}^2}{2}} \Phi\left(\frac{\log\bar{b}-\bar{\mu}-\bar{a}\bar{\sigma}^2}{\bar{\sigma}}\right).$$

Similarly, we have

$$\mathbb{E}\left[\xi^{\bar{a}} 1_{\xi>\bar{b}}\right] = e^{\bar{a}\bar{\mu}+\frac{\bar{a}^2\bar{\sigma}^2}{2}} \int_{\frac{\log\bar{b}-\bar{\mu}}{\bar{\sigma}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{(x-\bar{a}\bar{\sigma})^2}{2}} dx = e^{\bar{a}\bar{\mu}+\frac{\bar{a}^2\bar{\sigma}^2}{2}} \Phi\left(-\frac{\log\bar{b}-\bar{\mu}-\bar{a}\bar{\sigma}^2}{\bar{\sigma}}\right),$$

and

$$\mathbb{E}\left[1_{\bar{a}<\xi<\bar{b}}\right] = \mathbb{E}\left[1_{\frac{\log\bar{a}-\bar{\mu}}{\bar{\sigma}}<\frac{\log\xi-\bar{\mu}}{\bar{\sigma}}<\frac{\log\bar{b}-\bar{\mu}}{\bar{\sigma}}}\right] = \Phi\left(\frac{\log\bar{b}-\bar{\mu}}{\bar{\sigma}}\right) - \Phi\left(\frac{\log\bar{a}-\bar{\mu}}{\bar{\sigma}}\right).$$

Write

$$f(t,x) = e^{-r(T-t)}\,\mathbb{E}\left[g\left(xe^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(W(T)-W(t))}\right)\right] = e^{-r(T-t)}\,\mathbb{E}\left[g(\xi)\right],$$

where $\log\xi \sim \Phi\left(\log x + (r-\frac{1}{2}\sigma^2)(T-t), \sigma^2(T-t)\right)$. Then

$$\begin{aligned}
\mathbb{E}[g(\xi)] &= C_1\,\mathbb{E}\left[\xi^{\kappa_1} 1_{\xi<C_2}\right] + A\,\mathbb{E}\left[1_{C_2\leqslant\xi<C_3}\right] + C_1\,\mathbb{E}\left[\xi^{\kappa_1} 1_{\xi\geqslant C_3}\right], \\
&= C_1 x^{\kappa_1} e^{(\kappa_1 r+\frac{\kappa_1(\kappa_1-1)\sigma^2}{2})(T-t)} \Phi\left(\frac{\log C_2 - \log x - (r-\frac{1}{2}\sigma^2+\kappa_1\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&\quad + A\Phi\left(\frac{\log C_3 - \log x - (r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&\quad - A\Phi\left(\frac{\log C_2 - \log x - (r-\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&\quad + C_1 x^{\kappa_1} e^{(\kappa_1 r+\frac{\kappa_1(\kappa_1-1)\sigma^2}{2})(T-t)} \Phi\left(-\frac{\log C_3 - \log x - (r-\frac{1}{2}\sigma^2+\kappa_1\sigma^2)(T-t)}{\sigma\sqrt{T-t}}\right).
\end{aligned}$$

This gives (4.4).

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