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Optimal investment and consumption strategies for pooled annuity with partial information

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ABSTRACT

This paper considers the optimal investment and consumption problem for the pooled annuity funds, in which both the financial market and the mortality hazard rate of participants in the pool are partially observable. We manage to achieve the explicit expressions for optimal consumption and investment strategies employing filtering techniques and Hamilton-Jacobi-Bellman (HJB) equation. What is more, we also discuss the models where both the instantaneous rate of return of financial market and mortality of plan members are observable and obtain the optimal investment strategies accordingly. In addition, we look into this optimization problem under different exit mechanism including infinite exit time for the plan members. Last, but not the least, we carry out numerical analysis demonstrating the impact of observability of information.

1. Introduction

As a natural channel to hedge against longevity risk, pooled annuity funds have drawn a lot of attention from people in academia and practice. Compared with conventional life annuities, pooled annuities have some unique potential including better financial outcomes to alive plan members and it is likely that it would become a popular retirement plan that healthy people are thinking to utilize. The concept of Group Self-Annuity (GSA) is introduced by Martineau (2001). In the plan of GSA, a group is formed to pool idiosyncratic risk and this plan allows participants to pool together certain amount of money, following specific rights and obligations, to protect themselves from longevity risk.

There have been quite a little research work studying GSA from different perspectives. Some scholars study the pooled annuity fund or tontine with the purpose of obtaining actuarial fairness among plan members; some scholars focus on the payout design while different welfare criteria are under consideration, and some researchers focus on the study of this field from the route of information availability. We first recap the research work according to whether plan member is allowed to exit the plan before death and then comments on the articles based on if the market information is fully observable to plan members. The first work of GSA in which there is not specified time period for plan member to opt out can be traced back to Piggott et al. (2005). In this work, Piggott et al. (2005) give the fundamental analysis of the theoretical foundation of GSA and determine payout adjustments from a longevity-risk-pooling fund. Valdez et al. (2006) examine the anti-selection problem involved in GSA and conclude that a pooled annuity fund is a more cost-effective alternative to the conventional private annuity funds. Stamos (2008) studies the optimal continuous time dynamic consumption and investment problem for

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pooled annuity funds. Qiao and Sherris (2013) present a procedure to evaluate GSA payout assuming Makeham's law of mortality. Donnelly et al. (2014) propose a different approach to calculate the longevity credits and also prove that the fund is actuarially fair, under this assumption that the expected instantaneous actuarial gains of any individual is zero at all times. However, Donnelly (2015) investigates the pooled annuity funds using different methods of pooling mortality risk and claims that the group self-annuitization scheme is not actuarially fair. Sabin (2010) proposes a fair tontine annuity, which provides a lifetime payment stream whose expected present value matches that of a fair annuity. Milevsky and Salisbury (2015) derive the tontine structure by maximizing lifetime utility and discuss the properties of an optimized tontine payout structure. Milevsky and Salisbury (2016) generalize the natural tontine by introducing heterogeneous cohorts into one pool. Bernhardt and Donnelly (2019) consider a tontine product with bequest in the framework of power utility function and obtain the optimal proportion of total pension savings invested in the tontine account by maximizing consumption and their bequest under a power utility function.

As far as whether there is an exit mechanism is concerned, the common assumption in the literature above is that retirees stay in the plan till death. Donnelly and Young (2017) propose finite exit time to reduce selection risks and consider a kind of product for enhanced retirement income. For the tontine, the assumption that alive participants are eligible to distribute remaining amount of funds in the plan after a fixed period of time is also implemented by scholars. For instance, Sabin and Forman (2016) give the analysis of a single-period tontine. A single-period tontine is an arrangement in which a group of members contribute to an investment pool, and after a fixed period of time, the pool is distributed to those members who are still alive. Forman and Sabin (2017) study survivor funds, which is similar to a single-period tontine. Denuit and Vernic (2018) consider bivariate Bernoulli weighted sums and distribution of single-period tontine benefits.

Note that the above-mentioned papers assume that the asset price processes are completely observable while an individual makes the investment decisions. However, the appreciation rate of the risky asset and the underlying Brownian motion cannot be actually observed directly in reality. In terms of access to the market information, there have been a series of research work discussing the optimization problem with partial information in the financial market accordingly. For example, Lakner (1995) studies the optimization problem under partial information and gives explicit solutions to the maximization problem with terminal wealth. Lakner (1998) studies the optimization problem under partial information and applies the similar method to the case in which the drift process is modelled by a multi-dimensional mean-reverting Ornstein-Uhlenbeck process. Besides, Sass and Haussmann (2004) utilize Malliavin calculus to investigate an optimal investment problem in which the terminal wealth is maximized and they derive an explicit expression for the optimal strategy. Rieder and Bäuerle (2005) consider the portfolio selection problem where the drift of the stock is Markov-modulated but unknown for an investor. Callegaro et al. (2006) obtain the optimal investment strategy assuming that risky assets prices follow partially observable pure-jump processes. Bäuerle and Rieder (2007) study a similar portfolio optimization problem but under a Markov-modulated compound Poisson process and obtain the optimal investment strategy to maximize the expected utility of the terminal wealth. Liang and Bayraktar (2014) extend Bäuerle and Rieder (2007)'s models to the optimal reinsurance and investment problem. Liang and Song (2015) also consider an optimal investment and reinsurance problem for an insurer with partial information but from the perspective of mean-variance criterion. They derive the equilibrium strategy within a game theory approach using the filtering theory. For more literature, see Rieder and Bäuerle (2005), Zhang et al. (2012), Bäuerle and Leimcke (2020) and reference therein. On the other hand, researchers also have studied the case that health risk is incorporated into the model. Partially observable force of intensity is also under the consideration in Ceci et al. (2015), Ceci et al. (2017) and Ceci et al. (2020). Ceci et al. (2015) investigate hedging of unit-linked life insurance contracts with local risk-minimization method. Ceci et al. (2017) consider a similar problem in which an exogenous unobservable stochastic factor drives stock price process dynamics and the mortality rate. Ceci et al. (2020) employ BSDEs to study the pricing problem of a pure endowment contract.

Inspired by Stamos (2008), the unique contributions in our work are multiple-folds, we not only consider pool annuities with different exit mechanisms, but also incorporate partial information as well as full information into our work. To be more specific, we investigate the model in which neither the drift rate of the stock market nor the mortality rate of the pool members is completely observable in the pooled annuity funds and we assume they both are driven by an unobservable Markov chain. Secondly, for the pooled annuity fund with exit mechanism, we consider optimization problems by maximizing the logarithmic utility of cumulative consumption and terminal wealth under partial information and full information, respectively. What is more, we also study optimization problems for the pooled annuity funds without exit mechanism, in other words, we also consider the case when there is no predetermined end of time window, and this is considered as a special case of finite exit time. By the filtering theory and martingale method, we transform partial information to one with complete information, establish the Hamilton-Jacobi-Bellman (HJB) equation, and get optimal portfolio strategies and the corresponding value function under all different cases of our study. Last but not least, we carry out numerical analysis, analyze the influence of different parameters on optimal investment and consumption strategies, and compare the impact of partial information and full information on the corresponding optimal portfolio strategies.

The paper is organized as follows. In Section 2, we introduce the setting and formulate the optimization problem with partial information. In Section 3, we focus on the case that the driven Markov chain is not observable and employ the filtering theory to transform optimization problem with partial observations into the one with complete observations. In Section 4, we solve the HJB equations and achieve the optimal consumption and investment strategies together with the corresponding value function. Furthermore, we study observable Markov chain and derive the optimal consumption and investment strategies accordingly. Numerical examples are given in Section 5 and Section 6 concludes the paper.

2. Model formulation

Throughout this paper, we assume that all processes and random variables are defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}$ is a real-world probability and $\mathcal{F} := \{\mathcal{F}(t) | t \geq 0\}$ denotes the full information filtration.

Let $\mathbf{X} := \{\mathbf{X}(t) | t \in [0, T]\}$ be a continuous-time, finite-state, hidden Markov chain on $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in a finite state space $\Psi := \{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\} \subset R^n$, where the jth component of $\mathbf{e}_i$ is the Kronecker delta δ_{ij}, for each $i = 1, 2, \cdots, n$. Let $\mathbf{A} := [a_{ij}]_{i,j=1,2,\cdots,n}$ be a rate matrix of the chain $\mathbf{X}$ under $\mathbb{P}$, where a_{ij} is the constant intensity of transition of the chain $\mathbf{X}$ from state $\mathbf{e}_i$ to state $\mathbf{e}_j$. Assume that $\mathcal{F}^{\mathbf{X}} := \{\mathcal{F}^{\mathbf{X}}(t) | t \in [0, T]\}$ is the right-continuous, $\mathbb{P}$-completed, natural filtration generated by the chain $\mathbf{X}$. Clearly, $\mathcal{F}^{\mathbf{X}} \subseteq \mathcal{F}$. The following semi-martingale dynamics for the chain $\mathbf{X}$ under $\mathbb{P}$ holds:

$$\mathbf{X}(t)=\mathbf{X}(0)+\int_0^t \mathbf{A}\mathbf{X}(u)du+\mathbf{M}(t), t\in[0,T], \tag{2.1}$$

where $\mathbf{M}(t), t\in[0,T]$ is an R^n-valued, $(\mathcal{F}^X,\mathbb{P})$-martingale. Here, note that the state of Markov chain $\mathbf{X}$ is unobservable over time.

Let $L(0)\in\{1,2,\cdots\}$ be the initial number of members in the pool. Each pool member's time of death $\tau_i, i\in\{1,\cdots,L(0)\}$ is determined by the first jump time of a point process $N_i=\{N_i(t)|t\geq 0\}$ with time-dependent intensity $\lambda_i(t):=\langle\boldsymbol{\lambda}_i,\mathbf{X}(t)\rangle=\sum_{k=1}^n\langle\mathbf{X}(t),\mathbf{e}_k\rangle\lambda_k^i$, where $\boldsymbol{\lambda}_i=(\lambda_1^i,\lambda_2^i,\cdots,\lambda_n^i)^\top\in R^n$, $\lambda_k^i>0$ for each $k=1,2,\cdots,n$, and $\langle\cdot,\cdot\rangle$ denotes the standard inner product in R^n. In other words,

$$\tau_i=\min\{t:N_i(t)=1\}.$$

To model homogeneous, it is assumed that the members of all groups in the pool are homogeneous, that is, all people are of the same gender and age, then the jump intensity does not depend on i, i.e., $\lambda_i(t)\equiv\lambda(t):=\langle\boldsymbol{\lambda},\mathbf{X}(t)\rangle=\sum_{k=1}^n\langle\mathbf{X}(t),\mathbf{e}_k\rangle\lambda_k$, where $\boldsymbol{\lambda}=(\lambda_1,\lambda_2,\cdots,\lambda_n)^\top$ and $\lambda_k>0$ for each $k=1,2,\cdots,n$. Furthermore, each individual's death time τ_i has the same distribution. Besides, as in Frey and Schmidt (2012) and Ceci et al. (2015), we introduce conditionally independent doubly stochastic random times. Note that the unobservable finite state Markov chain usually describes bull and bear markets in finance. For an individual, the unobservable finite state Markov chain can represent the level of mortality, i.e., high and low. Due to interdependencies between the financial market and the mortality intensity, for instance, the severe acute respiratory syndrome (SARS) and the COVID-19 pandemic are events with effects on the both financial market and mortality intensity, the unobservable finite state Markov chain could affect them at the same time, that is, the risky asset's price and mortality intensity are modulated by a continuous-time, finite-state, hidden Markov chain.

Assumption 2.1. Assume that death times are conditionally independent doubly stochastic random times, that is

$$\mathbb{P}\left(\tau_1>t_1,\ldots,\tau_{L(0)}>t_{L(0)}\mid\mathcal{F}^X(T)\right)=\prod_{i=1}^{L(0)}e^{-\int_0^{t_i}\lambda(u)du}.$$

Note that under Assumption 2.1, it indicates that $\tau_i\neq\tau_j$ $\mathbb{P}$-a.s. for all $i\neq j$, $i,j\in\{1,\cdots,L(0)\}$.

At time t, the number of living members is thus $L(t)=L(0)-\sum_{i=1}^{L(0)}N_i(t\wedge\tau_i)$. The dynamic equation of the number of surviving investors in the pool is

$$dL(t)=-\sum_{i=1}^{L(0)}dN_i(t\wedge\tau_i).$$

Let

$$N^1(t):=\sum_{i=1}^{L(0)}N_i(t\wedge\tau_i)=\sum_{i=1}^{L(0)}I_{\{\tau_i\leq t\}},$$

where $I_{\{\cdot\}}$ is the indicator function. Therefore,

$$dL(t)=-dN^1(t).$$

Due to Assumption 2.1, the process $N^1(t)=\sum_{i=1}^{L(0)}I_{\{\tau_i\leq t\}}$ has $\mathcal{F}$-predictable intensity $(L(0)-N^1(t-))\lambda(t-)=L(t-)\lambda(t-)$, that is, for any $\mathcal{F}$-predictable process H the following equality holds

$$E\left[\int_0^t H(u)dN^1(u)\right]=E\left[\int_0^t H(u)L(u-)\lambda(u-)du\right]=E\left[\int_0^t H(u)L(u)\lambda(u)du\right]$$

or equivalently

$$N^1(t)-\int_0^t L(u)\lambda(u)du$$

is a $\mathcal{F}$-martingale (for details see Brémaud (1981)). Here, E is the expectation under $\mathbb{P}$. In fact, it is known that $I_{\{\tau_i\leq t\}}-\int_0^{t\wedge\tau_i}\lambda(u)du$ is a $\mathcal{F}$-martingale and summing over i, $N^1(t)=\sum_{i=1}^{L(0)}I_{\{\tau_i\leq t\}}-\int_0^t\left(L(0)-N^1(u-)\right)\lambda(u)du$ is a $\mathcal{F}$-martingale.

In this paper, we consider two kinds of assets in the financial market, namely, a risky asset with price process $R=\{R(t)|t\in[0,T]\}$ and a risk-free asset described by $B=\{B(t)|t\in[0,T]\}$. The price process of the risky asset is given by the $\mathbb{P}$-dynamic

$$\frac{dR(t)}{R(t-)}=\mu(t)dt+\sigma dZ(t),$$

and the price process of the risk-free asset is given by the $\mathbb{P}$-dynamic

$$dB(t)=rB(t)dt,$$

where $\mu(t) > 0$ denotes the instantaneous rate of return of the risky asset with $\mu(t) := \langle \boldsymbol{\mu}, \mathbf{X}(t) \rangle = \sum_{k=1}^{n} \langle \mathbf{X}(t), \mathbf{e}_k \rangle \mu_k$ and $\boldsymbol{\mu} = (\mu_1, \mu_2, \cdots, \mu_n)^\top$, $\sigma > 0$ is the volatility of the risky asset, and $\{Z(t)|t \in [0,T]\}$ is a standard $\mathbb{P}$-Brownian motion and is independent of $\{N_i(t)|t \geq 0\}$. Note that the volatility is assumed to be an observable constant in our model, and this is in line with the work of Capponi et al. (2015) and Nagai and Runggaldier (2008) due to the fact that the quadratic variation of Brownian motion converges almost surely to the integrated volatility, and observable stock prices lead to accessible volatility.

For the convenience of calculation, let $Y(t) := \ln(R(t)/R(0))$. By applying Itô's lemma, we can obtain

$$dY(t) = (\mu(t) - \frac{1}{2}\sigma^2)dt + \sigma dZ(t).$$

Suppose that $\mathcal{F}^Y := \{\mathcal{F}^Y(t)|t \in [0,T]\}$ is the right-continuous, $\mathbb{P}$-completed, natural filtration generated by the history of Y up to and including time t. Note that $\mathcal{F}^Y(t)$ can be observable up to time t.

At the initial time $t = 0$, each investor $m \in \{1, \cdots, L(0)\}$ contributes $W_m(0)$ amount of wealth to the annuity fund, the total initial value of the annuity fund is thus given by

$$W_{vf}(0) = \sum_{i=1}^{L(0)} W_i(0).$$

Denote the proportion of the fund invested in the risky asset by $\pi(t)$, and the remaining part is invested in the risk-free asset. When $\pi(t) < 0$, it means that the stock is sold short; while $\pi(t) > 1$ corresponds to a credit. The dynamic equation of pool annuity is

$$\frac{dW_{vf}(t)}{W_{vf}(t)} = \Big[r + \pi(t)(\mu(t) - r) - c(t)\Big]dt + \pi(t)\sigma dZ(t),$$

where $c(t)$ is the rate withdrawn from the fund at time t.

Let $w_m(t) = W_m(t)/W_{vf}(t)$, which is the fraction of fund wealth for investor m. When a pool member j, $j \neq m$ dies at time t, remaining survivors will reallocate member j's wealth $W_j(t-) = w_j(t-)W_{vf}(t-)$. For the sake of simplicity and to diminish the curse of dimensionality we assume that all individuals have the same initial wealth $W(0)$ so that each investor has the same fraction of the wealth, i.e., $w_m(t) = 1/L(t)$. Therefore, the wealth $W_m(t)$ of a representative member m follows

$$\begin{aligned} W_m(t) &= W_m(t-) + \frac{W_m(t-)}{W_{vf}(t-) - W_j(t-)} W_j(t-) \\ &= W_m(t-)\left(1 + \frac{w_j(t-)}{1 - w_j(t-)}\right) \\ &= W_m(t-)\left(1 + \frac{1}{L(t-) - 1}\right). \end{aligned}$$

Correspondingly, when $\tau_m > t$ and $L(t-) > 1$, the wealth dynamic $W_m(t)$ for any member m in the pool is

$$\begin{aligned} \frac{dW_m(t)}{W_m(t-)} &= \Big[r + \pi(t)(\mu(t) - r) - c(t)\Big]dt + \pi(t)\sigma dZ(t) + \sum_{i=1, i\neq m}^{L(0)} \frac{w_i(t-)}{1 - w_i(t-)} dN_i(t \wedge \tau_i) \\ &= \Big[r + \pi(t)(\mu(t) - r) - c(t)\Big]dt + \pi(t)\sigma dZ(t) + \frac{1}{L(t-) - 1} dN(t), \end{aligned} \tag{2.2}$$

where $N(t) := \sum_{i=1, i\neq m}^{L(0)} N_i(t \wedge \tau_i) = \sum_{i=1, i\neq m}^{L(0)} I_{\{\tau_i \leq t\}}$ has $\mathcal{F}$-predictable intensity $(L(t-) - 1)\lambda(t-)$. Note that n refers to the assumption that there are n states in the Markov chain and $\lambda(t)$ is unobservable.

Remark 2.1. For any $t \in [0, T]$, there exists a unique solution to equation (2.2) under the assumption $(\pi(.), c(.))$ is $\mathcal{G}$-progressively measurable such that $E(\int_0^T \pi^2(t)dt) < \infty$, $E(\int_0^T c^2(t)dt) < \infty$. The explicit expression of W_m by applying Doléans-Dade formula is as follows:

$$\begin{aligned} \frac{W_m(s)}{W_m(t)} &= \exp\left\{ \int_t^s \left(r + \pi(u)(\mu(u) - r) - c(u) - \frac{1}{2}\pi^2(u)\sigma^2\right)du + \int_t^s \pi(u)\sigma dZ(u) \right\} \\ &\quad \times \prod_{s \leq u \leq t} \left(1 + \frac{1}{L(u-) - 1}\right) \Delta N(u). \end{aligned}$$

When there is only one member living in the pool at time t, i.e., $L(t) = 1$, the dynamics of the total fund is described by

$$\frac{dW_m(t)}{W_m(t-)} = \Big[r + \pi(t)(\mu(t) - r) - c(t)\Big]dt + \pi(t)\sigma dZ(t), \quad t < \tau_m. \tag{2.3}$$

When we consider an infinitely large pool at time t, i.e., $L(t) = \infty$, we can rewrite (2.2) as follows

$$\frac{dW_m(t)}{W_m(t-)} = \Big[r + \pi(t)(\mu(t) - r) + \lambda(t) - c(t)\Big]dt + \pi(t)\sigma dZ(t), \quad t < \tau_m. \tag{2.4}$$

For the specific details of the derivation of equation (2.4), please refer to equation (17) of Stamos (2008).

Remark 2.2. Note that both $\mu(t)$ and $\lambda(t)$ are deterministic functions of time in Stamos (2008). However, $\mu(t)$ and $\lambda(t)$ in equations (2.2)-(2.4) above depend on the unobservable Markov chain, and thus are unobservable.

Let $y_i(0) := \mathbb{P}(\mathbf{X}(0) = \mathbf{e}_i)$, $i = 1, \cdots, n$, $\underline{\mathbf{y}}(0) := (y_1(0), \cdots, y_n(0))^\top$. The optimization problem we consider is to find such optimal strategies that the welfare of accumulative consumptions and terminal wealth can be maximized, i.e.,

$$\begin{aligned}\bar{J}(w, \ell, \underline{\mathbf{y}}) :&= E\Bigg[\int_0^{\tau_m \wedge T} e^{-\int_0^s \rho(u)du} U(C(s))ds + \alpha e^{-\int_0^T \rho(u)du} U(W_m(T)) I_{\{\tau_m > T\}} \\ &\qquad \Bigg| W_m(0) = w, L(0) = \ell, \underline{\mathbf{y}}(0) = \underline{\mathbf{y}}\Bigg] \\ &= E\Bigg[\int_0^T e^{-\int_0^s (\rho(u)+\lambda(u))du} U(C(s))ds + \alpha e^{-\int_0^T (\rho(u)+\lambda(u))du} U(W_m(T)) \\ &\qquad \Bigg| W_m(0) = w, L(0) = \ell, \underline{\mathbf{y}}(0) = \underline{\mathbf{y}}\Bigg],\end{aligned} \tag{2.5}$$

where τ_m is the stochastic time of death, $\rho(t)$ is the time preference and $C(t) = c(t)W_m(t)$ denotes the consumption at time t. The degree of the preference for the terminal wealth relative to the desire for consumption is denoted by the constant $\alpha \geq 0$. Similar to Bernhardt and Donnelly (2019), the larger the value of α, the stronger the motivation to retain more terminal wealth. For the derivation of equation (2.5), see Appendix Lemma A.3.

Suppose that $\mathcal{F}^N := \{\mathcal{F}^N(t) | t \in [0, T]\}$ is the right-continuous, $\mathbb{P}$-completed, natural filtration generated by the history of N up to and including time t. Note that we can observe stock prices and the total number of deaths in the pool under partial information. Therefore, the events of $\mathcal{F}^Y$ and $\mathcal{F}^N$ are observable and the admissible strategy adapts to $\mathcal{F}^Y$ and $\mathcal{F}^N$. Observation filtration $\mathcal{G} := \{\mathcal{G}(t) | t \in [0, T]\}$ denotes the minimal filtration generated by both $\mathcal{F}^Y(t)$ and $\mathcal{F}^N(t)$, where $\mathcal{G}(t) := \mathcal{F}^Y(t) \vee \mathcal{F}^N(t)$ for each $t \in [0, T]$. With that clarified, we proceed with the definition of admissible strategy as below.

Definition 2.1. An investment-consumption strategy $(\pi(\cdot), c(\cdot))$ for $0 \leq t \leq T$ is said to be admissible, if it satisfies the following conditions.

- $(\pi(\cdot), c(\cdot))$ is $\mathcal{G}$-progressively measurable such that
$$E\left(\int_0^T \pi^2(t)dt\right) < \infty, \qquad E\left(\int_0^T c^2(t)dt\right) < \infty;$$
- Under the investment strategy $(\pi(\cdot), c(\cdot))$, equation (2.2) has a unique strong solution satisfying $W_m(t) \geq 0$;
- $E\left[\int_0^T e^{-\int_0^s (\rho(u)+\lambda(u))du} |U(C(s))| ds + e^{-\int_0^T (\rho(u)+\lambda(u))du} |U(W_m(T))|\right] < \infty$,
  where $U(\cdot)$ is the utility function.

We denote the set of all admissible strategies by $\mathcal{M}$.

Therefore, the objective function of the pool member m is as follows

$$\bar{V}(w, \ell, \underline{\mathbf{y}}) := \sup_{(\pi,c)\in\mathcal{M}} \bar{J}(w, \ell, \underline{\mathbf{y}}).$$

As Callegaro et al. (2020) point out that in current formulation, the optimal problem with unobservable information is not Markovian and the dynamic programming approach via the HJB equation is not applicable. Therefore, in the next section, we apply filtering theory to introduce an equivalent problem with full information, the so-called separated problem, which is a Markovian structure.

3. Equivalent observable problem with full information

In this section, we obtain the separated problem by applying the 'certainty equivalence' approach or the 'separation principle' approach. First, we use the filtering theory to introduce the best estimate of the hidden Markov chain given the observations. Furthermore, we get equivalent observable control problem with full information.

3.1. The filtering problem

For the filtering problem, we mainly find the best mean-squared estimate of the hidden Markov chain based on current information. Let $\mathcal{H}(t) := \mathcal{G}(t) \vee \mathcal{F}^{\mathbf{X}}(t)$ for each $t \in [0, T]$, which is the minimal filtration generated by both $\mathcal{G}(t)$ and $\mathcal{F}^{\mathbf{X}}(t)$, $\mathcal{H} := \{\mathcal{H}(t) | t \in [0, T]\}$. Clearly, $\mathcal{G} \subseteq \mathcal{H} \subseteq \mathcal{F}$. Then for any $\mathcal{H}$-adapted, real-valued process $\varphi := \{\varphi(t) | t \in [0, T]\}$, let $\hat{\varphi}$ be the $\mathcal{G}$-optional projection of φ under the probability measure $\mathbb{P}$, i.e.,

$$\hat{\varphi}(t) = E(\varphi(t)|\mathcal{G}(t)), \quad t \in [0, T], \quad \mathbb{P} - as,$$

where E is the expectation under $\mathbb{P}$. It indicates that based on $\mathcal{G}(t)$, we can estimate $\varphi(t)$ by its least-square estimate $E(\varphi(t)|\mathcal{G}(t))$.

As Elliott et al. (2009) point out that to facilitate the derivation of the Zakai forms of the filtering equation governing the evolution of the filtered estimates of the hidden state of the chain over time, we need to introduce the reference probability measure $\hat{\mathbb{P}}$. This method based on the reference probability measure in some literature is discussed, such as Zhu et al. (2016), Shen and Siu (2017) and so on.

Firstly, we consider the following $\mathcal{H}$-adapted, real-valued process $\Lambda_1 := \{\Lambda_1(t)|t \in [0, T]\}$ and $\Lambda_2 := \{\Lambda_2(t)|t \in [0, T]\}$ on $(\Omega, \mathcal{F}, \mathbb{P})$,

$$\Lambda_1(t) := \exp\left\{ \int_0^t \frac{\langle \mathbf{h}(u), \mathbf{X}(u)\rangle}{\sigma^2} dY(u) - \frac{1}{2}\int_0^t \frac{\langle \mathbf{h}(u), \mathbf{X}(u)\rangle^2}{\sigma^2} du \right\}, \tag{3.1}$$

and

$$\begin{aligned}\Lambda_2(t) := \exp\Bigg\{ &\int_0^t \sum_{i=1}^n \langle \mathbf{X}(u-), \mathbf{e}_i\rangle [1 - (L(u-) - 1)\lambda_i] du \\ &+ \int_0^t \sum_{i=1}^n \langle \mathbf{X}(u-), \mathbf{e}_i\rangle \ln((L(u-) - 1)\lambda_i) dN(u) \Bigg\}.\end{aligned} \tag{3.2}$$

Furthermore, we define the $\mathcal{H}$-adapted, real-valued process $\Lambda := \{\Lambda(t)|t \in [0, T]\}$ and the equivalent probability measure $\hat{\mathbb{P}} \sim \mathbb{P}$ on $(\Omega, \mathcal{F}, \mathbb{P})$,

$$\Lambda(t) := \Lambda_1(t)\Lambda_2(t), \qquad \frac{d\mathbb{P}}{d\hat{\mathbb{P}}} := \Lambda(T).$$

We can prove that Λ is a $(\mathcal{H}, \hat{\mathbb{P}})$-martingale. For any $\mathcal{H}$-adapted, real-valued process $\varphi := \{\varphi(t)|t \in [0, T]\}$, let $\bar{\sigma}(\varphi)$ be the $\mathcal{G}$-optional projection of $\Lambda\varphi$ under the probability measure $\hat{\mathbb{P}}$, i.e.,

$$\bar{\sigma}(\varphi(t)) = \hat{E}(\Lambda(t)\varphi(t)|\mathcal{G}(t)), \quad t \in [0, T], \quad \hat{\mathbb{P}} - as,$$

where $\hat{E}$ is the expectation under $\hat{\mathbb{P}}$. Due to the Bayes' rule, we have

$$E(\varphi(t)|\mathcal{G}(t)) = \frac{\hat{E}(\Lambda(t)\varphi(t)|\mathcal{G}(t))}{\hat{E}(\Lambda(t)|\mathcal{G}(t))}, \tag{3.3}$$

which indicates we can get the estimate of φ under $\mathbb{P}$ by estimating $\Lambda\varphi$ under $\hat{\mathbb{P}}$. For $t \in [0, T]$, we define $G_i(t) := (L(t) - 1)\lambda_i$, $\mathbf{G}(t) := (G_1(t), G_2(t), \cdots, G_n(t))^\top$ and $\mathbf{1} := (1, 1, \cdots, 1)^\top$. Let $h(t) := \mu(t) - \frac{1}{2}\sigma^2$, then

$$dY(t) = h(t)dt + \sigma dZ(t).$$

When $\mathbf{X}(t) = \mathbf{e}_i$ at time t, $\mu(t) = \mu_i$. Therefore, $h_i(t) := h(\mathbf{X}(t) = \mathbf{e}_i) = \mu(t) - \frac{1}{2}\sigma^2 = \mu_i - \frac{1}{2}\sigma^2$ for $i = 1, 2, \cdots, n$, $t \in [0, T]$, and $\mathbf{h}(t) := (h_1(t), h_2(t), \cdots, h_n(t))^\top$. Let $\mathbf{B}(t) := \mathbf{diag}(\mathbf{h}(t))$, a diagonal matrix in which the entries outside the main diagonal are all zero and elements of the main diagonal are $\mathbf{h}(t) = (h_1(t), h_2(t), \cdots, h_n(t))^\top$.

Proposition 3.1. *Let $\mathbf{q}(t) := \bar{\sigma}(\mathbf{X}(t))$ and $\hat{\mathbf{X}}(t) := E(\mathbf{X}(t)|\mathcal{G}(t))$, then $\mathbf{q}(t)$ satisfies the following stochastic differential equation*

$$\begin{aligned}\mathbf{q}(t) = \mathbf{q}(0) &+ \int_0^t \mathbf{A}\mathbf{q}(u)du - \int_0^t \mathbf{diag}(\mathbf{G}(u) - \mathbf{1})\mathbf{q}(u)du \\ &+ \int_0^t \mathbf{diag}(\mathbf{G}(u-) - \mathbf{1})\mathbf{q}(u-)dN(u) + \int_0^t \sigma^{-2}\mathbf{B}(u)\mathbf{q}(u)dY(u)\end{aligned} \tag{3.4}$$

and

$$\hat{\mathbf{X}}(t) = \frac{\mathbf{q}(t)}{\langle \mathbf{q}(t), \mathbf{1}\rangle},$$

where $\mathbf{q}(0) := E(\mathbf{X}(0))$ is the initial distribution of the chain $\mathbf{X}$, which is assumed to be known.

Proof. The proof is given in Appendix A.1. □

Remark 3.1. For any $t \in [0, T]$, $\hat{\mathbf{X}}(t)$ is the filtered estimate of hidden Markov chain $\mathbf{X}(t)$ given $\mathcal{G}(t)$. Because all the elements of all matrices $\mathbf{A}$, $\mathbf{diag}(\mathbf{G}(t) - \mathbf{1})$ and $\mathbf{B}(t)$ are bounded, by using Theorem 2.1 of Xi and Zhu (2017), equation (3.4) admits a unique strong solution.

3.2. The separated problem

In the section, based on filtered estimate of hidden Markov chain and innovations process, we find the best estimate for the risky asset price and wealth process. Then we can transfer the partially observable system to a completely observable system. Define the compensated random measure

$$d\hat{N}(t) = dN(t) - \sum_{i=1}^{n} \langle \hat{\mathbf{X}}(t), \mathbf{e}_i \rangle [(L(t-) - 1)\lambda_i] dt.$$

Define the $\mathcal{G}$-adapted process $\hat{Z} := \{\hat{Z}(t) | t \in [0, T]\}$ as follows

$$\begin{aligned} \hat{Z}(t) &:= \frac{Y(t)}{\sigma} - \int_0^t \frac{\hat{h}(u)}{\sigma} du \\ &= Z(t) + \int_0^t \frac{h(u) - \hat{h}(u)}{\sigma} du. \end{aligned} \tag{3.5}$$

Because $\{\hat{Z}(t), t \in [0, T]\}$ on $(\mathcal{G}, \mathbb{P})$ is innovations process, we can prove that $\hat{Z}$ is a $(\mathcal{G}, \mathbb{P})$-Brownian motion by filtering theory. For more detailed discussions, we refer readers to, for example, Karatzas and Zhao (2001) and Liptser and Shiryaev (1977). Due to (3.5), we can obtain

$$\sigma d\hat{Z}(t) + \hat{h}(t) dt = \sigma dZ(t) + h(t) dt,$$

where $\hat{h}(t) = \hat{\mu}(t) - \frac{1}{2}\sigma^2$. Therefore, for the unobservable model (2.2), the equivalent completely observable model of the total funds is

$$\begin{aligned} \frac{dW_m(t)}{W_m(t-)} &= \left[r + \pi(t)(\hat{\mu}(t) - r) - c(t) + \sum_{i=1}^{n} \langle \hat{\mathbf{X}}(t), \mathbf{e}_i \rangle \lambda_i \right] dt + \pi(t) \sigma d\hat{Z}(t) \\ &\quad + \frac{1}{L(t-) - 1} d\hat{N}(t), \quad t < \tau_m, 1 < L(t-), \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} d\mathbf{q}(t) &= \mathbf{K}(t)\mathbf{q}(t)dt + \sum_{i=1}^{n} \langle \hat{\mathbf{X}}(t), \mathbf{e}_i \rangle \mathbf{diag}(\mathbf{G}(t) - \mathbf{1})(L(t-) - 1)\lambda_i dt + \sigma^{-1}\mathbf{B}(t)\mathbf{q}(t) d\hat{Z}(t) \\ &\quad + \mathbf{diag}(\mathbf{G}(t-) - \mathbf{1})\mathbf{q}(t-) d\hat{N}(t), \end{aligned} \tag{3.7}$$

where $\hat{\mathbf{X}}(t) = E(\mathbf{X}(t)|\mathcal{G}(t)) = \frac{\mathbf{q}(t)}{\langle \mathbf{q}(t), \mathbf{1} \rangle}$, $\hat{\mu}(t) = \frac{\langle \boldsymbol{\mu}, \mathbf{q}(t) \rangle}{\langle \mathbf{q}(t), \mathbf{1} \rangle}$, $\mathbf{K}(t) = \mathbf{A} - \mathbf{diag}(\mathbf{G}(t) - \mathbf{1}) + \sigma^{-2}\mathbf{B}(t)\hat{h}(t)$.

Remark 3.2. For any $t \in [0, T]$, there exists a unique solution to equation (3.6). According to Itô's lemma for jump diffusions, the growth rate of wealth between t and s $(t < s)$ is

$$\begin{aligned} \frac{W_m(s)}{W_m(t)} &= \exp\left\{ \int_t^s \left(r + \pi(u)(\hat{\mu}(u) - r) - c(u) - \frac{1}{2}\pi^2(u)\sigma^2 \right) du + \int_t^s \pi(u) \sigma d\hat{Z}(u) \right\} \\ &\quad \times \prod_{s \le u \le t} \left(1 + \frac{1}{L(u-) - 1} \right) \Delta N(u). \end{aligned}$$

Moreover, similar to equation (3.4), (3.7) admits a unique strong solution.

Furthermore, considering only one investor, i.e., $L(t) = 1$ in the pooled annuity funds and the perfect pool, i.e., $L(t) = \infty$, we reduce the partially observable model of the total funds to an equivalent model with complete observations. When $L(t) = 1$, for the unobservable model (2.3), the equivalent completely observable model of the total fund asset is

$$\frac{dW_m(t)}{W_m(t-)} = \left[r + \pi(t)(\hat{\mu}(t) - r) - c(t) \right] dt + \pi(t) \sigma d\hat{Z}(t), \quad t < \tau_m, \tag{3.8}$$

and

$$d\mathbf{q}(t) = \mathbf{K}(t)\mathbf{q}(t)dt + \sigma^{-1}\mathbf{B}(t)\mathbf{q}(t) d\hat{Z}(t), \tag{3.9}$$

where

$$\mathbf{K}(t) = \mathbf{A} + \sigma^{-2}\mathbf{B}(t)\hat{h}(t), \quad \hat{\mu}(t) = \frac{\langle \boldsymbol{\mu}, \mathbf{q}(t) \rangle}{\langle \mathbf{q}(t), \mathbf{1} \rangle}.$$

When $L(t)=\infty$, for the partially observable model (2.4), the equivalent completely observable model of the total funds is as follows

$$\frac{dW_m(t)}{W_m(t-)}=\left[r+\pi(t)(\hat{\mu}(t)-r)-c(t)+\sum_{i=1}^{n}\langle\hat{\mathbf{X}}(t),\mathbf{e}_i\rangle\lambda_i\right]dt+\pi(t)\sigma d\hat{Z}(t),\quad t<\tau_m, \tag{3.10}$$

and

$$d\mathbf{q}(t)=\mathbf{K}(t)\mathbf{q}(t)dt+\sigma^{-1}\mathbf{B}(t)\mathbf{q}(t)d\hat{Z}(t), \tag{3.11}$$

where

$$\mathbf{K}(t)=\mathbf{A}+\sigma^{-2}\mathbf{B}(t)\hat{h}(t),\quad \hat{\mathbf{X}}(t)=E(\mathbf{X}(t)|\mathcal{G}(t))=\frac{\mathbf{q}(t)}{\langle\mathbf{q}(t),\mathbf{1}\rangle},$$
$$\hat{\mu}(t)=\frac{\langle\boldsymbol{\mu},\mathbf{q}(t)\rangle}{\langle\mathbf{q}(t),\mathbf{1}\rangle}.$$

Remark 3.3. For the proof of the equations (3.8)-(3.11), they are similar to the derivation of (3.6) and (3.7), we omit it here.

Now, we apply the filtering theory to transform the model into one with complete observations. Besides, we can prove that the following equation holds:

$$\begin{aligned}&E\left[\int_0^T e^{-\int_0^s(\rho(u)+\lambda(u))du}U(C(s))ds+\alpha e^{-\int_0^T(\rho(u)+\lambda(u))du}U(W_m(T))\right.\\&\qquad\left.\Big|W_m(0)=w,L(0)=\ell,\underline{\mathbf{y}}(0)=\underline{\mathbf{y}}\right]\\&=E\left[\int_0^T e^{-\int_0^s(\rho(u)+\hat{\lambda}(u))du}U(C(s))ds+\alpha e^{-\int_0^T(\rho(u)+\hat{\lambda}(u))du}U(W_m(T))\right.\\&\qquad\left.\Big|W_m(0)=w,L(0)=\ell,\mathbf{q}(0)=\mathbf{q}\right],\end{aligned}$$

see Appendix Lemma A.4. Hence, equation (2.5) can be rewritten in terms of observable quantities as

$$\begin{aligned}J(w,\ell,\mathbf{q}):=&E\left[\int_0^T e^{-\int_0^s(\rho(u)+\hat{\lambda}(u))du}U(C(s))ds+\alpha e^{-\int_0^T(\rho(u)+\hat{\lambda}(u))du}U(W_m(T))\right.\\&\qquad\left.\Big|W_m(0)=w,L(0)=\ell,\mathbf{q}(0)=\mathbf{q}\right]\\=&\bar{J}(w,\ell,\underline{\mathbf{y}}).\end{aligned}$$

Furthermore, we give the definition of admissible strategy and value function under full information, respectively.

Definition 3.1. An investment-consumption strategy $(\pi(\cdot),c(\cdot))$ for $0\le t\le T$ is said to be admissible, if it satisfies the following conditions.

- $(\pi(\cdot),c(\cdot))$ is $\mathcal{G}$-progressively measurable such that
$$E\left(\int_0^T\pi^2(t)dt\right)<\infty,\qquad E\left(\int_0^T c^2(t)dt\right)<\infty;$$
- Under the investment strategy $(\pi(\cdot),c(\cdot))$, equation (3.6) has a unique strong solution satisfying $W_m(t)\ge 0$; $E\left[\int_0^T e^{-\int_0^s(\rho(u)+\hat{\lambda}(u))du}|U(C(s))|ds+e^{-\int_0^T(\rho(u)+\hat{\lambda}(u))du}|U(W_m(T))|\right]<\infty$,
where $U(\cdot)$ is the utility function, and $\hat{\lambda}(u)=\sum_{i=1}^n\langle\hat{\mathbf{X}}(u),\mathbf{e}_i\rangle\lambda_i$ for each $u\in[0,T]$.

We denote the set of all admissible strategies by $\mathcal{A}$.

Given $(\pi,c)\in\mathcal{A}$ and for $t\in[0,T]$, $W_m(t)$, $\mathbf{q}(t)$ solve (3.6) and (3.7), respectively, and $L(t)$ is measurable with respect to $\{\mathcal{G}(t)|t\in[0,T]\}$. Because the triple $\{(W_m(t),L(t),\mathbf{q}(t))|t\in[0,T]\}$ is Markov with respect to the information flow $\{\mathcal{G}(t)|t\in[0,T]\}$, the problem under full information has a Markovian structure. Therefore, the objective function of the pool member becomes

$$V(w,\ell,\mathbf{q}):=\sup_{(\pi,c)\in\mathcal{A}}J(w,\ell,\mathbf{q}). \tag{3.12}$$

Note that because all the processes are $\mathcal{G}$-adapted, this is now a stochastic problem under full information so that we can use traditional method to deal with it. Furthermore, to proceed with specific derivations, we utilize the logarithmic utility function to describe the welfare of a typical plan member, i.e., $U(x) = \ln(x)$.

Due to the strong uniqueness of the solutions to (3.6) and (3.7), and the construction of the separated problem, we can obtain the following proposition.

Proposition 3.2. *Let $(w, \ell, \underline{\mathbf{y}}) \in R \times R \times [0,1]^n$ be the initial values of the process $\{(W_m(t), L(t), \underline{\mathbf{y}}(t))|t \in [0,T]\}$ in the problem (2.5) under partial information. Then*

$$\bar{V}(w, \ell, \underline{\mathbf{y}}) = V(w, \ell, \mathbf{q})$$

Furthermore, $\mathcal{M} = \mathcal{A}$ and (π^*, c^*) is an optimal control for the separated problem (3.12) if and only if it is optimal for the original problem (2.5) under partial information.

3.3. Verification theorem

In this subsection, we provide the verification theorem to the optimization problem with full information. Define the value function of the pool member m as follows

$$\Phi(t, w, \ell, \mathbf{q}) := \sup_{(\pi,c)\in\mathcal{A}} E\Bigg[\int_t^T e^{-\int_t^s (\rho(u)+\hat{\lambda}(u))du} U(C(s))ds + \alpha e^{-\int_t^T (\rho(u)+\hat{\lambda}(u))du} U(W_m(T)) \Bigg| W_m(t) = w, L(t) = \ell, \mathbf{q}(t) = \mathbf{q}\Bigg].$$

Let $\mathcal{O} := \mathcal{C}^1([0,T]) \times \mathcal{C}^2(R) \times \mathcal{B}(R) \times \mathcal{C}^2(R^n)$ be the space of real-valued functions on a set $[0,T] \times R \times R \times R^n$ and $\bar{\mathcal{O}}$ is the closure of $\mathcal{O}$. A function $\phi(t, w, \ell, q) : [0,T] \times R \times R \times R^n \to R$ satisfies $\phi(t, w, \ell, q) \in \mathcal{O}$. Then we denote the partial derivatives of ϕ with respect to t, w and the second order partial derivatives of ϕ with respect to w by ϕ_t, ϕ_w and ϕ_{ww}, respectively. The infinitesimal generator $\mathcal{L}^{\pi,c}$ is given by

$$\begin{aligned}\mathcal{L}^{\pi,c}[\phi(t, w, \ell, \mathbf{q})] = {} & \phi_t + \phi_w w\Big[r + \pi(\hat{\mu}(t) - r) - c\Big] + \frac{1}{2}\phi_{ww} w^2 \pi^2 \sigma^2 \\ & + \langle \mathbf{K}(t)\mathbf{q}, \mathbf{D_q}\phi\rangle + \frac{1}{2}\sigma^{-2}\langle \mathbf{B}(t)\mathbf{q}, (\mathbf{D^2_q}\phi)\mathbf{B}(t)\mathbf{q}\rangle + \langle \mathbf{B}(t)\mathbf{q}, \mathbf{D_q}\phi_w\rangle \pi w \\ & + \sum_{i=1}^n \langle \hat{\mathbf{X}}(t), \mathbf{e}_i\rangle \lambda_i(\ell - 1)\Big[\phi\Big(t, w(1 + \frac{1}{\ell - 1}), \ell - 1, \mathbf{diag}(\mathbf{G}(t))\mathbf{q}\Big) \\ & - \phi(t, w, \ell, \mathbf{q})\Big],\end{aligned}$$

where

$$\begin{aligned}&\mathbf{D_q}\phi := \left(\frac{\partial\phi}{\partial q_1}, \frac{\partial\phi}{\partial q_2}, \cdots, \frac{\partial\phi}{\partial q_n}\right)^\top, \quad \mathbf{D^2_q}\phi := \left[\frac{\partial^2\phi}{\partial q_i \partial q_j}\right]_{i,j=1,2,\cdots,n}, \\ &\mathbf{D_q}\phi_w := \left(\frac{\partial^2\phi}{\partial w\partial q_1}, \frac{\partial^2\phi}{\partial w\partial q_2}, \cdots, \frac{\partial^2\phi}{\partial w\partial q_n}\right)^\top,\end{aligned}$$

$\mathbf{D_q}\phi$ and $\mathbf{D^2_q}\phi$ are then the gradient vector and Hessian matrix of ϕ with respect to $\mathbf{q}$, respectively. $\mathbf{D_q}\phi_w$ is the gradient vector of the derivative ϕ_w with respect to $\mathbf{q}$.

Theorem 3.1 *(Verification theorem). Let ϕ be a function in $\mathcal{O} \cap \bar{\mathcal{O}}$ and an admissible control $(\pi^*, c^*) \in \mathcal{A}$ such that*

(1) $\mathcal{L}^{\pi,c}\phi(t, w, \ell, \mathbf{q}) - \Big(\rho(t) + \sum_{i=1}^n \langle \hat{\mathbf{X}}(t), \mathbf{e}_i\rangle\lambda_i\Big)\phi + U(cw) \le 0, \mathbb{P}-a.s., for\ all\ (\pi, c) \in \mathcal{A}$ *and* $(t, w, \ell, q) \in [0,T] \times R \times R \times R^n$*;*

(2) $\mathcal{L}^{\pi^*,c^*}\phi(t, w, \ell, \mathbf{q}) - \Big(\rho(t) + \sum_{i=1}^n \langle \hat{\mathbf{X}}(t), \mathbf{e}_i\rangle\lambda_i\Big)\phi + U(c^*w) = 0, \mathbb{P}-a.s., for\ all\ (t, w, \ell, q) \in [0,T] \times R \times R \times R^n$*;*

(3) $for\ all (\pi, c) \in \mathcal{A}$,

$$\lim_{t\to T^-} \phi(t, w, \ell, \mathbf{q}) = U(w), \mathbb{P}-a.s.;$$

(4) *let $\mathcal{K}$ denote the set of stopping times $\kappa \le T$. The family $\{\phi(\kappa, W(\kappa), L(\kappa), \mathbf{q}(\kappa))|\kappa \in \mathcal{K}\}$ is uniformly integrable. Then,*

$$\phi(t, w, \ell, \mathbf{q}) = \Phi(t, w, \ell, \mathbf{q}),$$

and (π^*, c^*) is an optimal control.

Proof. The proof is deferred to Appendix A.2. □

4. The optimal strategy of investment and consumption

4.1. The optimal strategy with partial information

In this section, we give the results to the unobservable model. We consider three cases, the first is that the number of person in the pool is more than one but finite ($1 < \ell < \infty$), the second is that there is only one person in the pool ($\ell = 1$), and the last is the perfect pool ($\ell = \infty$).

Theorem 4.1. *Let*

$$\theta(t) = \alpha \exp\left(-\int_t^T \left(\rho(u) + \frac{\langle \boldsymbol{\lambda}, \mathbf{q}(u)\rangle}{\langle \mathbf{q}(u), \mathbf{1}\rangle}\right) du\right) + \int_t^T \exp\left(-\int_t^s \left(\rho(u) + \frac{\langle \boldsymbol{\lambda}, \mathbf{q}(u)\rangle}{\langle \mathbf{q}(u), \mathbf{1}\rangle}\right) du\right) ds,$$

and

$$f(t, \ell, \mathbf{q}) = A_0(t, \ell) + \sum_{i=1}^{n} A_i(t, \ell) q_i + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} B_{ij}(t, \ell) q_i q_j + \sum_{i=1}^{n} C_i(t, \ell) q_i^2$$

with terminal conditions $\lim_{t\to T} f(t, \ell, \mathbf{q}) = 0$ *and* $f(t, 0, \mathbf{q}) = 0$. *When* $1 < \ell < \infty$, *the optimal investment and consumption strategies for problem (3.12) are given by*

$$\pi^*(t) = \frac{\frac{\langle \boldsymbol{\mu}, \mathbf{q}(t)\rangle}{\langle \mathbf{q}(t), \mathbf{1}\rangle} - r}{\sigma^2}, \quad c^*(t) = (\theta(t))^{-1}, \tag{4.1}$$

and the value function is

$$\Phi(t, w, \ell, \mathbf{q}) = \theta(t) \ln(w) + f(t, \ell, \mathbf{q}), \tag{4.2}$$

where $\mathbf{q}(t)$ *satisfies (3.7), coefficients* $A_i(t, \ell) = B_{ij}(t, \ell) = C_i(t, \ell) = 0$, $A_0(t, \ell)$ *satisfies (A.18) with boundary conditions* $\lim_{t\to T} A_0(t, \ell) = 0$.

When $\ell = 1$ *or* $\ell = \infty$, *the optimal investment and consumption strategies, and the value function satisfy (4.1) and (4.2), respectively. Note that for* $\ell = 1$, $\mathbf{q}(t)$ *satisfies (3.9), coefficients* $A_i(t, 1) = B_{ij}(t, 1) = C_i(t, 1) = 0$, $A_0(t, 1)$ *satisfies (A.19) with boundary conditions* $\lim_{t\to T} A_0(t, 1) = 0$. *For* $\ell = \infty$, $\mathbf{q}(t)$ *satisfies (3.11), coefficients* $A_i(t, \infty) = B_{ij}(t, \infty) = C_i(t, \infty) = 0$, $A_0(t, \infty)$ *satisfies (A.20) with boundary conditions* $\lim_{t\to T} A_0(t, \infty) = 0$.

Proof. The proof is given in Appendix A.3. □

When considering the pooled annuity funds without exit mechanism, i.e., $T \to \infty$, we can easily obtain the following corollary from Theorem 4.1.

Corollary 4.1. *When* $T \to \infty$, *let*

$$\theta(t) = \int_t^\infty \exp\left(-\int_t^s \left(\rho(u) + \frac{\langle \boldsymbol{\lambda}, \mathbf{q}(u)\rangle}{\langle \mathbf{q}(u), \mathbf{1}\rangle}\right) du\right) ds,$$

and

$$f(t, \ell, \mathbf{q}) = A_0(t, \ell) + \sum_{i=1}^{n} A_i(t, \ell) q_i + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} B_{ij}(t, \ell) q_i q_j + \sum_{i=1}^{n} C_i(t, \ell) q_i^2$$

with terminal conditions $\lim_{t\to\infty} f(t, \ell, \mathbf{q}) = 0$ *and* $f(t, 0, \mathbf{q}) = 0$. *When* $1 < \ell < \infty$, *the optimal investment and consumption strategies, the value function and* $\mathbf{q}(t)$ *satisfy (4.1), (4.2) and (3.7), respectively. Coefficients* $A_i(t, \ell) = B_{ij}(t, \ell) = C_i(t, \ell) = 0$, $A_0(t, \ell)$ *satisfies (A.18) with boundary conditions* $\lim_{t\to\infty} A_0(t, \ell) = 0$.

When $\ell = 1$ *or* $\ell = \infty$, *the optimal investment and consumption strategies, and the value function satisfy (4.1) and (4.2), respectively. Note that for* $\ell = 1$, $\mathbf{q}(t)$ *satisfies (3.9), coefficients* $A_i(t, 1) = B_{ij}(t, 1) = C_i(t, 1) = 0$, $A_0(t, 1)$ *satisfies (A.19) with boundary conditions* $\lim_{t\to\infty} A_0(t, 1) = 0$. *For* $\ell = \infty$, $\mathbf{q}(t)$ *satisfies (3.11), coefficients* $A_i(t, 1) = B_{ij}(t, 1) = C_i(t, 1) = 0$, $A_0(t, 1)$ *satisfies (A.20) with boundary conditions* $\lim_{t\to\infty} A_0(t, \infty) = 0$.

4.2. The optimal strategy with full information

In the section, we consider the state of Markov chain $\mathbf{X}$ is observable over time, which means that all parameters of equations (2.2), (2.3) and (2.4) are known. Stamos (2008) discusses the pooled annuity funds without exit mechanism, i.e., $T \to \infty$ assuming the instantaneous rate of return, $\mu(t)$ and intensity, $\lambda(t)$ of Poisson process are both given functions. However, $\mu(t)$ and $\lambda(t)$ are dependent on the Markov chain in our model and we are dedicated to finding the optimal strategies under this setting. The utility function is chosen by $U(x) = \ln(x)$ as well, and for each $k = 1, 2, \cdots, n$, the objective function of the pool member is given by

$$\begin{aligned}
V(w,\ell,\mathbf{e}_k) :=& \sup_{(\pi,c)\in\mathcal{M}_1} E\Bigg[\int_0^{\tau_m\wedge T} e^{-\int_0^s \rho(u)du} U(C(s))ds + \alpha e^{-\int_0^T \rho(u)du} U(W_m(T)) I_{\{\tau_m>T\}} \\
& \Bigg| W_m(0)=w, L(0)=\ell, \mathbf{X}(0)=\mathbf{e}_k\Bigg] \\
=& \sup_{(\pi,c)\in\mathcal{M}_1} E\Bigg[\int_0^{T} e^{-\int_0^s (\rho(u)+\lambda(u))du} U(C(s))ds + \alpha e^{-\int_0^T (\rho(u)+\lambda(u))du} U(W_m(T)) \\
& \Bigg| W_m(0)=w, L(0)=\ell, \mathbf{X}(0)=\mathbf{e}_k\Bigg],
\end{aligned} \tag{4.3}$$

where $\lambda(u)=\sum_{i=1}^n \langle \mathbf{X}(u),\mathbf{e}_i\rangle\lambda_i$, $\mathcal{M}_1$ is the set of admissible strategies that satisfies the assumption $(\pi(.),c(.))$ is $\mathcal{F}$-progressively measurable such that $E(\int_0^T \pi^2(t)dt)<\infty$, $E(\int_0^T c^2(t)dt)<\infty$, and conditions 2 and 3 of Definition 2.1.

Theorem 4.2. *The optimal investment and consumption strategies under the observable model are*

$$\pi^*(t)=\frac{\mu(t)-r}{\sigma^2}, \quad c^*(t)=(\theta(t))^{-1}, \tag{4.4}$$

where

$$\begin{aligned}
\theta(t) =& \alpha \exp\left(-\int_t^T \left(\rho(u)+\sum_{i=1}^n \langle \mathbf{X}(u),\mathbf{e}_i\rangle\lambda_i\right)du\right) \\
&+\int_t^T \exp\left(-\int_t^s \left(\rho(u)+\sum_{i=1}^n \langle \mathbf{X}(u),\mathbf{e}_i\rangle\lambda_i\right)du\right)ds.
\end{aligned}$$

The value function is

$$\Phi(t,w,\ell,\mathbf{e}_k)=\begin{cases}\theta(t)\ln(w)+f(t,\ell,\mathbf{e}_k),\\ \theta(t)\ln(w)+f(t,1,\mathbf{e}_k),\\ \theta(t)\ln(w)+f(t,\infty,\mathbf{e}_k),\end{cases}$$

where $f(t,\ell,\mathbf{e}_k)$, $f(t,1,\mathbf{e}_k)$ *and* $f(t,\infty,\mathbf{e}_k)$ *satisfy*

$$\begin{aligned}
\frac{\partial f(t,\ell,\mathbf{e}_k)}{\partial t} =& \ln(\theta(t)) + \left(\rho(t)+\sum_{i=1}^n \langle \mathbf{X}(t),\mathbf{e}_i\rangle\lambda_i\right) f(t,\ell,\mathbf{e}_k) - \theta(t)r - \frac{1}{2}\frac{(\mu(t)-r)^2\theta(t)}{\sigma^2}+1 \\
& - \sum_{j=1}^n a_{kj} f(t,\ell,\mathbf{e}_j) - \sum_{i=1}^n \langle \mathbf{X}(t),\mathbf{e}_i\rangle\lambda_i(\ell-1)\Bigg[\theta(t)\ln(\frac{\ell}{\ell-1}) \\
& + f(t,\ell-1,\mathbf{e}_k) - f(t,\ell,\mathbf{e}_k)\Bigg], \quad \lim_{t\to T} f(t,\ell,\mathbf{e}_k)=0, \quad f(t,0,\mathbf{e}_k)=0, \\
\frac{\partial f(t,1,\mathbf{e}_k)}{\partial t} =& \ln(\theta(t)) + \left(\rho(t)+\sum_{i=1}^n \langle \mathbf{X}(t),\mathbf{e}_i\rangle\lambda_i\right) f(t,1,\mathbf{e}_k) - \theta(t)r - \frac{1}{2}\frac{(\mu(t)-r)^2\theta(t)}{\sigma^2}+1 \\
& - \sum_{j=1}^n a_{kj} f(t,1,\mathbf{e}_j), \quad \lim_{t\to T} f(t,1,\mathbf{e}_k)=0, \quad f(t,0,\mathbf{e}_k)=0, \\
\frac{\partial f(t,\infty,\mathbf{e}_k)}{\partial t} =& \ln(\theta(t)) + \left(\rho(t)+\sum_{i=1}^n \langle \mathbf{X}(t),\mathbf{e}_i\rangle\lambda_i\right) f(t,\infty,\mathbf{e}_k) - \theta(t)r - \frac{1}{2}\frac{(\mu(t)-r)^2\theta(t)}{\sigma^2}+1 \\
& - \sum_{i=1}^n \langle \mathbf{X}(t),\mathbf{e}_i\rangle\lambda_i - \sum_{j=1}^n a_{kj} f(t,\infty,\mathbf{e}_j), \quad \lim_{t\to T} f(t,\infty,\mathbf{e}_k)=0, \quad f(t,0,\mathbf{e}_k)=0.
\end{aligned}$$

The proof of Theorem 4.2 is similar to that of Theorem 4.1, so we omit it here.

Corollary 4.2. *When considering the infinite time horizon, i.e.,* $T\to\infty$, *the optimal investment and consumption strategies under the observable model are*

$$\pi^*(t)=\frac{\mu(t)-r}{\sigma^2}, \quad c^*(t)=(\theta(t))^{-1},$$

Table 1
Model parameters.

ρ	r	σ	L_0
0.03	0.03	0.2	100
T	α	μ_1	μ_2
15	1	0.1	0.05
λ_1	λ_2	$q_1(0)$	$q_2(0)$
0.03	0.3	0.9	0.1
a_{11}	a_{12}	a_{21}	a_{22}
-5	5	10	-10

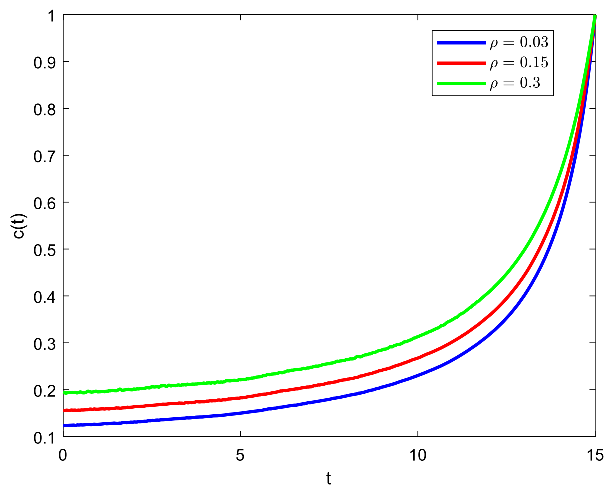

Fig. 1. The influence of time t on consumption fraction under different ρ. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

where

$$\theta(t) = \int_t^\infty \exp\left(-\int_t^s \left(\rho(u) + \sum_{i=1}^n \langle \mathbf{X}(u), \mathbf{e}_i\rangle \lambda_i\right) du\right) ds.$$

The corresponding value function under the case of $T \to \infty$ is similar to that under the case of finite time horizon. The main difference is the terminal conditions $\lim_{t\to\infty} f(t, \ell, \mathbf{e}_k) = 0$ and $f(t, 0, \mathbf{e}_k) = 0$, for $1 \le \ell < \infty$ and $\ell = \infty$, respectively.

Remark 4.1. From Theorem 4.2 and Corollary 4.2, we find that the optimal strategies are the same under three cases $1 < \ell < \infty$, $\ell = 1$ and $\ell = \infty$. The main reason may be the logarithmic utility that we choose.

5. Numerical analysis

We illustrate our results numerically in this section. Our results are demonstrated with graphs organized into the following way: we first introduce some baseline parameters below for the sake of convenience. The parameters take the values assumed in Table 1 through the whole section unless otherwise specified. Then we study the impact of key parameters on the optimal investment and consumption strategies. We conclude this section by comparing the optimal strategies with complete information with that in partially observable setting.

Note that we demonstrate our results with a two-state ($n = 2$) Markov chain in this section and thus in the table above the subscripts i, j of transition intensities a_{ij} take values 1 or 2 only. Also we assume that the discount rate function $\rho(t)$ is a constant ρ for convenience of analysis. Last, but not least, recall that α, L_0, and $\mathbf{q}(0) = (q_1(0), q_2(0)) = E[\mathbf{X}(0)]$ denote the relative importance of terminal wealth, the size of the pooled annuity and the expectation of the Markov chain at the initial moment $t = 0$, respectively.

Fig. 1 shows the trend of consumption fraction, in other words, the proportion of wealth on consumption over time. As we can see from the graph, the consumption fraction increases steadily over the time window of our interest. Given that one either consumes or invests in our model, an ordinary person tends to spend more, for example, on medical bills and invest less, in particular, in risky asset like equities, while aging. It is therefore reasonable to see the increasing pattern. The upward trend is slow due to the fact that the market is not fully observable. Consequently, the participant chooses to increase the percentage on consumption little by little and we thus do not witness distinct increment of consumption over any time frame even when market might be very prospectus at a certain period of time. Another observation is that one has higher percentage of consumption as discount rate goes up. The reason behind is that the discount rate reveals the time preference of a retiree. Higher value of discount rate demonstrates more emphasis present, and thus implies that more consumption is expected while other things being equal.

Fig. 2 delineates the initial behavior of consumption with respect to terminal time and it shows that the corresponding consumption is a decreasing function of time T. It fits our intuition as a retiree spends more now while his/her objective is to maximum the welfare

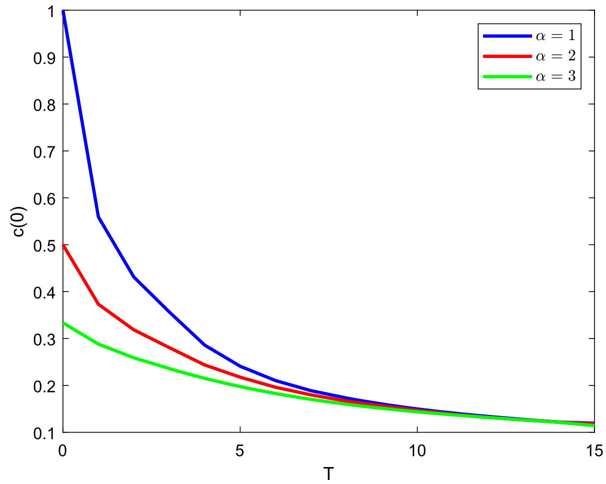

Fig. 2. The influence of terminal time T on consumption fraction under different α.

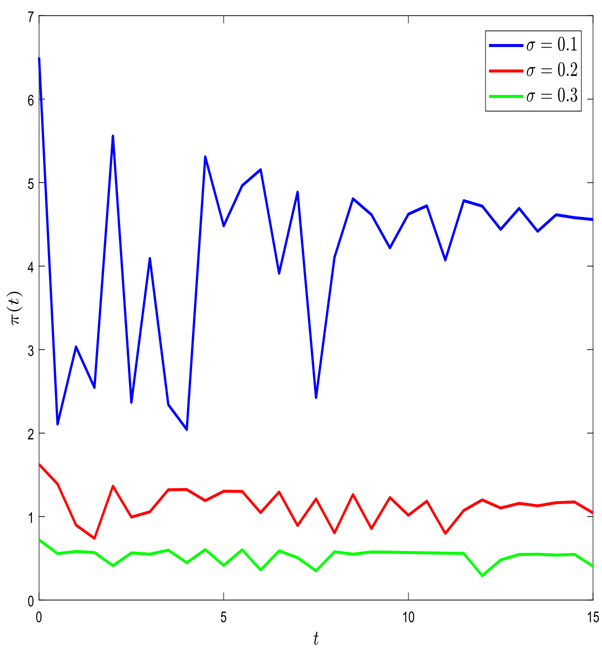

Fig. 3. The influence of time t on investment strategy under different σ.

over shorter time horizon and consumes less now while planning for longer time period. What is more, the three lines in this figure also demonstrate the impact of α on the consumption. Recall that α measures how a plan participant weighs in the terminal wealth, one thus consumes the least initially if he/she cares the terminal wealth most. It is also worth to point out that we spot that this impact of α can be dismissed if the terminal time is long enough with our choice of parameters.

We switch our gears and study of investment strategies instead in Figs. 3 and 4. Both Figs. 3 and 4 describe the trajectory of investment strategies over the course of time and we notice the fluctuations of $\pi(t)$ in both Figures due to the jumps of hidden Markov chains. We will comment Fig. 3 before going to the discussion of Fig. 4 in below. Considering the fact that our model is based on a natural log utility function and the risk aversion level is $\gamma = 1$ in the framework of power utility function, our assumed plan member is thus not a risk preferred investor. We are not too surprised to see that, in Fig. 3, it is optimal for a retiree to invest less in a more volatile market environment. That being said, we want to claim that it is not too trivial to study the relation between $\pi(t)$ and σ. As we can see from the optimal investment strategy expression in Equation (4.1), σ appears in $\mathbf{q}(t)$, the top of the expression for optimal $\pi(t)$ (see equation (3.7) for details) and bottom of that at the same time.

Different from Fig. 3 in which the investment strategy with partial information is considered, we make a comparison of investment strategies under different models in Fig. 4. For the case of observable Markov chain, with specified values of parameters, the optimal investment behavior should be a constant regardless of time according to its expression in equation (4.4). It is clear from the optimal investment expression that higher return results in more risky asset investments. The investment strategy with partial information is between the two strategies with full information. When the market is not fully observable, investor uses the average return (between bull market and bear market) as the criterion to make investment decision and thus the resulting strategies are capped by the best scenario case and worse scenario case of the contrasting model in which full access to information is assumed, respectively.

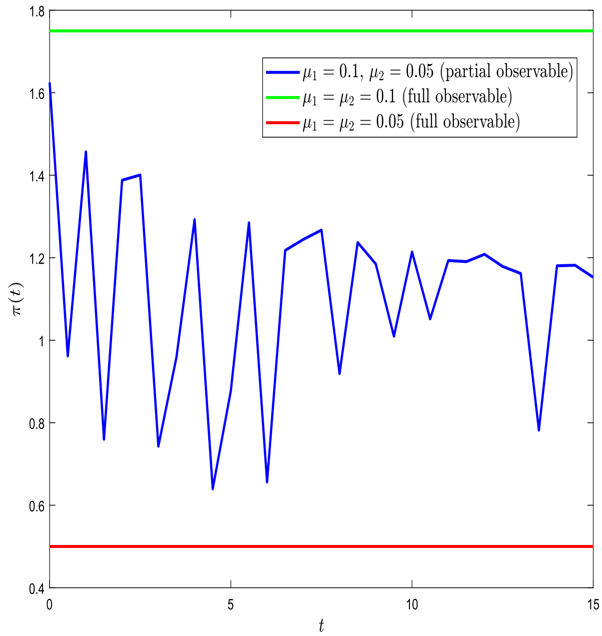

Fig. 4. The comparison of partially observable and fully observable instantaneous rate of return.

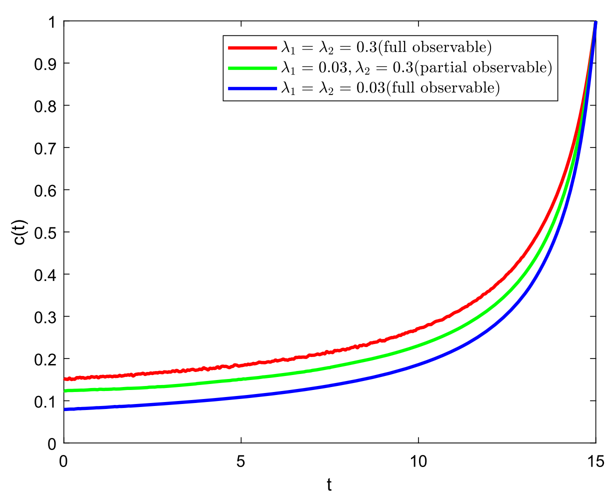

Fig. 5. The comparison of consumption strategies with partially observable and fully observable information.

Similar to Fig. 4, Fig. 5 also examines the impact of information availability. That said, we focus on dynamics of consumption according to the force of mortality in Fig. 5. We make the following observations. First of all, $c(t)$ is an slow-increasing function of time and this is in line with our comments in Fig. 1. Second, most optimal consumption is obtained while force of mortality is the largest. The driving force behind this observation is the idea of pooled annuity in which plan members essentially pool longevity risk and alive people are taking advantage of the money left the pool. Higher mortality probability leads to more consumption for a living retiree is thus a natural end of product to expect and observe. Along the line of what we commented in Fig. 4, when the market is not fully observable, plan members take the approach of averaging out the different scenario and it is optimal to maintain a middle level of consumption.

6. Conclusion and future remark

In this paper, we study the optimal consumption and investment problem under partial information for pooled annuity with different exit mechanisms. By using martingale method and the classical filtering theory, we reduce partial information to one with complete information. Furthermore, applying stochastic control theory and the HJB equation, we obtain the explicit expressions for the optimal investment and consumption strategies accordingly. Simultaneously, we also investigate this optimization problem in the setting of complete information and present the closed form expressions for consumption and investment respectively. Finally, we use numerical simulations to illustrate the conclusions and compare the results. One direction of our future work is to extend our methods to different utility functions while risk aversion levels among participants can be modelled in an appropriate way. It is likely that explicit solutions may not be available for the case power utility function or exponential utility function, just for example. Therefore, approximation method is expected to be implemented and more careful thoughts will be deserved.

Declaration of competing interest

The authors declared no potential conflicts of interest with respect to the research, authorship, and publication of this article.

Data availability

No data was used for the research described in the article.

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Appendix A

A.1. Proof of Proposition 3.1

Proof. Following the proof of Theorem 4.1 of Elliott and Siu (2012), we give the proof of Proposition 3.1. Due to (3.1) and (3.2), we can obtain

$$d\Lambda_1(t) = \Lambda_1(t)\frac{\langle \mathbf{h}(t), \mathbf{X}(t)\rangle}{\sigma^2}dY(t), \tag{A.1}$$

and

$$\begin{aligned}
d\Lambda_2(t) = \Lambda_2(t)&\sum_{i=1}^{n}\langle \mathbf{X}(t-), \mathbf{e}_i\rangle[1-(L(t-)-1)\lambda_i]dt\\
&+\Lambda_2(t-)\sum_{i=1}^{n}\langle \mathbf{X}(t-), \mathbf{e}_i\rangle[(L(t-)-1)\lambda_i-1]dN(t).
\end{aligned}\tag{A.2}$$

Combining (A.1) and (A.2) yields

$$\begin{aligned}
d\Lambda(t) &= \Lambda_1(t-)d\Lambda_2(t) + \Lambda_2(t-)d\Lambda_1(t) + d[\Lambda_1, \Lambda_2](t)\\
&= \Lambda(t)\sum_{i=1}^{n}\langle \mathbf{X}(t-), \mathbf{e}_i\rangle[1-(L(t-)-1)\lambda_i]dt\\
&\quad+\Lambda(t-)\sum_{i=1}^{n}\langle \mathbf{X}(t-), \mathbf{e}_i\rangle[(L(t-)-1)\lambda_i-1]dN(t)\\
&\quad+\Lambda(t)\frac{\langle \mathbf{h}(t), \mathbf{X}(t)\rangle}{\sigma^2}dY(t).
\end{aligned}\tag{A.3}$$

Because of (2.1) and (A.3), we have

$$\begin{aligned}
d\Lambda(t)\mathbf{X}(t) &= \Lambda(t-)d\mathbf{X}(t) + \mathbf{X}(t-)d\Lambda(t) + d[\Lambda, \mathbf{X}](t)\\
&= \Lambda(t-)\mathbf{A}\mathbf{X}(t)dt + \Lambda(t-)d\mathbf{M}(t) + \Lambda(t)\mathbf{X}(t-)\sum_{i=1}^{n}\langle \mathbf{X}(t-), \mathbf{e}_i\rangle[1-(L(t-)-1)\lambda_i]dt\\
&\quad+\Lambda(t-)\mathbf{X}(t-)\sum_{i=1}^{n}\langle \mathbf{X}(t-), \mathbf{e}_i\rangle[(L(t-)-1)\lambda_i-1]dN(t)\\
&\quad+\Lambda(t)\mathbf{X}(t-)\frac{\langle \mathbf{h}(t), \mathbf{X}(t)\rangle}{\sigma^2}dY(t).
\end{aligned}\tag{A.4}$$

Integrating both sides of (A.4) implies

$$\begin{aligned}
\Lambda(t)\mathbf{X}(t) = \Lambda(0)\mathbf{X}(0) &+ \int_0^t \Lambda(u-)\mathbf{A}\mathbf{X}(u)du + \int_0^t \Lambda(u-)d\mathbf{M}(u)\\
&+\int_0^t \Lambda(u)\mathbf{X}(u-)\sum_{i=1}^{n}\langle \mathbf{X}(u-), \mathbf{e}_i\rangle[1-(L(u-)-1)\lambda_i]du
\end{aligned}$$

$$
+\int_0^t \Lambda(u-)\mathbf{X}(u-)\sum_{i=1}^n \langle \mathbf{X}(u-), \mathbf{e}_i\rangle[(L(u-)-1)\lambda_i - 1]dN(u)
$$

$$
+\int_0^t \Lambda(u)\mathbf{X}(u-)\frac{\langle \mathbf{h}(u), \mathbf{X}(u)\rangle}{\sigma^2}dY(u). \tag{A.5}
$$

On the one hand,

$$
\begin{aligned}
&\hat{E}\left[\Lambda(u-)\mathbf{X}(u-)\sum_{i=1}^n \langle \mathbf{X}(u-), \mathbf{e}_i\rangle[1-(L(u-)-1)\lambda_i]\middle|\mathcal{G}(u)\right]\\
&=\sum_{i=1}^n \langle \bar{\sigma}(\mathbf{X}(u-)), \mathbf{e}_i\rangle[1-(L(u-)-1)\lambda_i]\mathbf{e}_i.
\end{aligned}
$$

On the other hand, due to $\sum_{i=1}^n \langle \mathbf{X}(u), \mathbf{e}_i\rangle = 1$, we obtain

$$
\begin{aligned}
&\hat{E}\left[\Lambda(u)\mathbf{X}(u)\frac{\langle \mathbf{h}(u), \mathbf{X}(u)\rangle}{\sigma^2}\middle|\mathcal{G}(u)\right]\\
&=\hat{E}\left[\Lambda(u)\mathbf{X}(u)\frac{\langle \mathbf{h}(u), \mathbf{X}(u)\rangle}{\sigma^2}\sum_{i=1}^n \langle \mathbf{X}(u), \mathbf{e}_i\rangle\middle|\mathcal{G}(u)\right]\\
&=\sum_{i=1}^n \langle \bar{\sigma}(\mathbf{X}(u)), \mathbf{e}_i\rangle\langle \mathbf{h}(u), \mathbf{e}_i\rangle\mathbf{e}_i\\
&=\mathbf{diag}(\mathbf{h}(u))\bar{\sigma}(\mathbf{X}(u))\\
&=\mathbf{B}(u)\bar{\sigma}(\mathbf{X}(u)).
\end{aligned}
$$

Taking expectation for both sides of (A.5) on $\mathcal{G}(t)$ under measure $\hat{\mathbb{P}}$, we have

$$
\begin{aligned}
\bar{\sigma}(\mathbf{X}(t)) = \bar{\sigma}(\mathbf{X}(0)) &+ \int_0^t \mathbf{A}\bar{\sigma}(\mathbf{X}(u))du - \int_0^t \mathbf{diag}(\mathbf{G}(u)-\mathbf{1})\bar{\sigma}(\mathbf{X}(u))du\\
&+\int_0^t \mathbf{diag}(\mathbf{G}(u-)-\mathbf{1})\bar{\sigma}(\mathbf{X}(u-))dN(u) + \int_0^t \frac{\mathbf{B}(u)\bar{\sigma}(\mathbf{X}(u))}{\sigma^2}dY(u).
\end{aligned} \tag{A.6}
$$

Because of (A.6), we have

$$
\begin{aligned}
\mathbf{q}(t) = \mathbf{q}(0) &+ \int_0^t \mathbf{A}\mathbf{q}(u)du - \int_0^t \mathbf{diag}(\mathbf{G}(u)-\mathbf{1})\mathbf{q}(u)du\\
&+\int_0^t \mathbf{diag}(\mathbf{G}(u-)-\mathbf{1})\mathbf{q}(u-)dN(u) + \int_0^t \sigma^{-2}\mathbf{B}(u)\mathbf{q}(u)dY(u).
\end{aligned}
$$

Due to $\langle \mathbf{X}(t), \mathbf{1}\rangle = 1$, then

$$
\begin{aligned}
\langle \hat{E}(\Lambda(t)\mathbf{X}(t)|\mathcal{G}(t)), \mathbf{1}\rangle &= \hat{E}(\Lambda(t)\langle \mathbf{X}(t), \mathbf{1}\rangle|\mathcal{G}(t))\\
&=\hat{E}(\Lambda(t)|\mathcal{G}(t)).
\end{aligned}
$$

By using (3.3), we obtain

$$
\hat{\mathbf{X}}(t) = \frac{\mathbf{q}(t)}{\langle \mathbf{q}(t), \mathbf{1}\rangle}. \quad \square
$$

Due to a form of the Bayes' rule, we know

$$
\begin{aligned}
E(e^{-\int_0^t \lambda(u)du}|\mathcal{G}(t)) &= \frac{\hat{E}(e^{-\int_0^t \lambda(u)du}\Lambda(T)|\mathcal{G}(t))}{\hat{E}(\Lambda(T)|\mathcal{G}(t))}\\
&=\frac{\hat{E}(e^{-\int_0^t \lambda(u)du}\Lambda(t)|\mathcal{G}(t)}{\hat{E}(\Lambda(t)|\mathcal{G}(t))}.
\end{aligned}
$$

Let $p_i(t) := \mathbb{P}(\mathbf{X}(t) = \mathbf{e}_i|\mathcal{G}(t)),\ i = 1, \cdots, n,\ \mathbf{p}(t) := (p_1(t), \cdots, p_n(t))^\top,\ \hat{\lambda}(t) = E(\lambda(t)|\mathcal{G}(t)),\ \hat{G}(\mathbf{p}(t-)) := (L(t-) - 1)\hat{\lambda}(t),\ G(\mathbf{p}(t-)) := (L(t-) - 1)\lambda(t)$, and $\hat{h}(t) = E(h(t)|\mathcal{G}(t))$, then

$$\begin{aligned} dp_i(t) &= \sum_{l=1}^{n} a_{li} p_l(t)dt + p_i(t-)\sigma^{-2}(h_i(t) - \hat{h}(t))(dY(t) - \hat{h}(t))dt \\ &\quad + p_i(t-)\frac{G_i(t-) - \hat{G}(\mathbf{p}(t-))}{\hat{G}(\mathbf{p}(t-))}(dN(t) - \hat{G}(\mathbf{p}(t-))dt). \end{aligned}$$

Let $M(t) := \Lambda(t)e^{-\int_0^t \lambda(u)du}$, then

$$dM(t) = M(t-)\left(\frac{h(t)}{\sigma^2}dY(t) + (G(\mathbf{p}(t-)) - 1)dN(t) - (G(\mathbf{p}(t-)) - 1)dt - \lambda(t)dt\right).$$

Define

$$d\hat{\Lambda}(t) = \hat{\Lambda}(t)(\hat{G}(\mathbf{p}(t-)) - 1)dt + \hat{\Lambda}(t-)(\hat{G}(\mathbf{p}(t-)) - 1)dN(t) + \hat{\Lambda}(t)\frac{\hat{h}(t)}{\sigma^2}dY(t),$$

and

$$\hat{M}(t) := \hat{\Lambda}(t)e^{-\int_0^t \hat{\lambda}(u)du},$$

we have

$$d\hat{M}(t) = \hat{M}(t-)\left(\frac{\hat{h}(t)}{\sigma^2}dY(t) + (\hat{G}(\mathbf{p}(t-)) - 1)dN(t) - (\hat{G}(\mathbf{p}(t-)) - 1)dt - \hat{\lambda}(t)dt\right),$$

furthermore,

$$\begin{aligned} \hat{M}(t) = \hat{M}(0)\exp\Bigg\{ &\int_0^t \frac{\hat{h}(s)}{\sigma^2}dY(s) - \int_0^t (\hat{G}(\mathbf{p}(s-)) - 1)ds - \int_0^t (\hat{\lambda}(s) + \frac{\hat{h}(s)}{2})ds \\ &+ \int_0^t \ln(\hat{G}(\mathbf{p}(s-)))dN(s)\Bigg\}. \end{aligned}$$

Lemma A.1. *Let*

$$q_i(t) = \hat{E}(M(t)I_{\{\mathbf{X}(t)=\mathbf{e}_i\}}|\mathcal{G}(t)). \tag{A.7}$$

Then the following propositions are sustained.

(1) $dq_i(t) = \sum_{l=1}^{n} a_{li}(t)q_l(t)dt + q_i(t-)\frac{h_i(t)}{\sigma^2}dY(t) - q_i(t-)\lambda_i dt + q_i(t-)(G_i(t-) - 1)d\tilde{N}(t)$, *where* $d\tilde{N}(t) = dN(t) - dt$;
(2) $q_i(t) = \hat{M}(t)p_i(t)$;
(3) $p_i(t) = \frac{q_i(t)}{\sum_{j=1}^{n} q_j(t)}$.

Proof. Following the proof of Lemma A.1 and Lemma A.2 of Capponi et al. (2015), we can easily prove Lemma A.1, so we omit it here. □

Lemma A.2. *The following result is established.*

$$\begin{aligned} E(e^{-\int_0^t \lambda(u)du}|\mathcal{G}(t)) &= \frac{\hat{E}(e^{-\int_0^t \lambda(u)du}\Lambda(t)|\mathcal{G}(t))}{\hat{E}(\Lambda(t)|\mathcal{G}(t))} \\ &= \exp\left\{-\int_0^t \hat{\lambda}(u)du\right\}. \end{aligned}$$

Proof. Because of Lemma A.1, we know that

$$\begin{aligned} \hat{E}(e^{-\int_0^t \lambda(u)du}\Lambda(t)|\mathcal{G}(t)) &= \hat{E}(M(t)|\mathcal{G}(t)) \\ &= \hat{E}(\sum_{i=1}^{n} M(t)I_{\{\mathbf{X}(t)=\mathbf{e}_i\}}|\mathcal{G}(t)) \\ &= \sum_{i=1}^{n} \hat{E}(M(t)I_{\{\mathbf{X}(t)=\mathbf{e}_i\}}|\mathcal{G}(t)) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} q_i(t) = \sum_{i=1}^{n} \hat{M}(t) p_i(t) = \hat{M}(t) \\
&= \hat{M}(0) \exp\left\{ \int_0^t \frac{\hat{h}(s)}{\sigma^2} dY(s) - \int_0^t (\hat{G}(\mathbf{p}(s-)) - 1) ds - \int_0^t (\hat{\lambda}(s) + \frac{\hat{h}(s)}{2}) ds \right. \\
&\quad \left. + \int_0^t \ln(\hat{G}(\mathbf{p}(s-))) dN(s) \right\} \\
&= \hat{\Lambda}(t) e^{-\int_0^t \hat{\lambda}(u) du}.
\end{aligned}
$$

For $\forall t \in [0, T]$, let $\lambda(t) = 0$, we have

$$
\begin{aligned}
\hat{E}(\Lambda(t)|\mathcal{G}(t)) &= \hat{M}(0) \exp\left\{ \int_0^t \frac{\hat{h}(s)}{\sigma^2} dY(s) - \int_0^t (\hat{G}(\mathbf{p}(s-)) - 1) ds + \int_0^t \frac{\hat{h}(s)}{2} ds \right. \\
&\quad \left. + \int_0^t \ln(\hat{G}(\mathbf{p}(s-))) dN(s) \right\} \\
&= \hat{\Lambda}(t).
\end{aligned}
$$

Therefore, we obtain

$$
\begin{aligned}
E(e^{-\int_0^t \lambda(u) du} | \mathcal{G}(t)) &= \frac{\hat{E}(e^{-\int_0^t \lambda(u) du} \Lambda(t) | \mathcal{G}(t))}{\hat{E}(\Lambda(t)|\mathcal{G}(t))} \\
&= \frac{\hat{\Lambda}(t) e^{-\int_0^t \hat{\lambda}(u) du}}{\hat{\Lambda}(t)} \\
&= \exp\left\{ -\int_0^t \hat{\lambda}(u) du \right\}. \quad \square
\end{aligned}
$$

Lemma A.3. *For the objective function with partially observable problem, the following equation holds:*

$$
\begin{aligned}
&E\Big[\int_0^{\tau_m \wedge T} e^{-\int_0^s \rho(u) du} U(C(s)) ds + \alpha e^{-\int_0^T \rho(u) du} U(W_m(T)) I_{\{\tau_m > T\}} \\
&\qquad \Big| W_m(0) = w, L(0) = \ell, \underline{\mathbf{y}}(0) = \underline{\mathbf{y}} \Big] \\
&= E\Big[\int_0^{T} e^{-\int_0^s (\rho(u) + \lambda(u)) du} U(C(s)) ds + \alpha e^{-\int_0^T (\rho(u) + \lambda(u)) du} U(W_m(T)) \\
&\qquad \Big| W_m(0) = w, L(0) = \ell, \underline{\mathbf{y}}(0) = \underline{\mathbf{y}} \Big].
\end{aligned}
$$

Proof. The welfare of accumulative consumptions and terminal wealth of the pool member m is as follows:

$$
\begin{aligned}
\bar{J}(w, \ell, \underline{\mathbf{y}}) := E\Big[&\int_0^{\tau_m \wedge T} e^{-\int_0^s \rho(u) du} U(C(s)) ds + \alpha e^{-\int_0^T \rho(u) du} U(W_m(T)) \\
&\times I_{\{\tau_m > T\}} \Big| W_m(0) = w, L(0) = \ell, \underline{\mathbf{y}}(0) = \underline{\mathbf{y}} \Big].
\end{aligned}
$$

For $E(I_{\{\tau_m > s\}} | \mathcal{F}(s))$, we have

$$
E(I_{\{\tau_m > s\}} | \mathcal{F}(s)) = \exp\left\{ -\int_0^s \lambda(u) du \right\}.
$$

Therefore,

$$\begin{aligned}\bar{J}(w,\ell,\underline{\mathbf{y}}) &= E\Big[\int_0^{\tau_m\wedge T} e^{-\int_0^s \rho(u)du}U(C(s))ds + \alpha e^{-\int_0^T \rho(u)du}U(W_m(T)) \\ &\quad \times I_{\{\tau_m>T\}}\Big| W_m(0)=w, L(0)=\ell, \underline{\mathbf{y}}(0)=\underline{\mathbf{y}}\Big] \\ &= E\Big[\int_0^T e^{-\int_0^s \rho(u)du}U(C(s))E(I_{\{\tau_m>s\}}|\mathcal{F}(s))ds + \alpha e^{-\int_0^T \rho(u)du} \\ &\quad \times U(W_m(T))E(I_{\{\tau_m>T\}}|\mathcal{F}(T))\Big| W_m(0)=w, L(0)=\ell, \underline{\mathbf{y}}(0)=\underline{\mathbf{y}}\Big] \\ &= E\Big[\int_0^T e^{-\int_0^s \rho(u)du}U(C(s))e^{-\int_0^s \lambda(u)du}ds + \alpha e^{-\int_0^T \rho(u)du} \\ &\quad \times U(W_m(T))e^{-\int_0^T \lambda(u)du}\Big| W_m(0)=w, L(0)=\ell, \underline{\mathbf{y}}(0)=\underline{\mathbf{y}}\Big]. \quad \square\end{aligned}$$

Lemma A.4. *When the objective function be rewritten in terms of observable quantities, the following equation is established:*

$$\begin{aligned}&E\Big[\int_0^T e^{-\int_0^s(\rho(u)+\lambda(u))du}U(C(s))ds + \alpha e^{-\int_0^T(\rho(u)+\lambda(u))du}U(W_m(T)) \\ &\qquad \Big| W_m(0)=w, L(0)=\ell, \underline{\mathbf{y}}(0)=\underline{\mathbf{y}}\Big] \\ &= E\Big[\int_0^T e^{-\int_0^s(\rho(u)+\hat{\lambda}(u))du}U(C(s))ds + \alpha e^{-\int_0^T(\rho(u)+\hat{\lambda}(u))du}U(W_m(T)) \\ &\qquad \Big| W_m(0)=w, L(0)=\ell, \mathbf{q}(0)=\mathbf{q}\Big].\end{aligned}$$

Proof. The welfare of accumulative consumptions and terminal wealth of the pool member m in terms of observable quantities is as follows:

$$\begin{aligned}J(w,\ell,\mathbf{q}) := E\Big[&\int_0^T e^{-\int_0^s(\rho(u)+\hat{\lambda}(u))du}U(C(s))ds + \alpha e^{-\int_0^T(\rho(u)+\hat{\lambda}(u))du}U(W_m(T)) \\ &\Big| W_m(0)=w, L(0)=\ell, \mathbf{q}(0)=\mathbf{q}\Big].\end{aligned}$$

Due to

$$\underline{\mathbf{y}}(0) = E[\mathbf{X}(0)] = \mathbf{q}(0),$$

we have

$$\begin{aligned}\bar{J}(w,\ell,\underline{\mathbf{y}}) = E\Big[&\int_0^{\tau_m\wedge T} e^{-\int_0^s \rho(u)du}U(C(s))ds + \alpha e^{-\int_0^T \rho(u)du}U(W_m(T))I_{\{\tau_m>T\}} \\ &\Big| W_m(0)=w, L(0)=\ell, \mathbf{q}(0)=\mathbf{q}\Big].\end{aligned}$$

By applying Lemma A.3, we know

$$\begin{aligned}\bar{J}(w,\ell,\underline{\mathbf{y}}) &= E\Big[\int_0^T e^{-\int_0^s \rho(u)du}U(C(s))e^{-\int_0^s \lambda(u)du}ds + \alpha e^{-\int_0^T \rho(u)du} \\ &\quad \times U(W_m(T))e^{-\int_0^T \lambda(u)du}\Big| W_m(0)=w, L(0)=\ell, \mathbf{q}(0)=\mathbf{q}\Big], \\ &= E\Big[\int_0^T e^{-\int_0^s \rho(u)du}U(C(s))E(e^{-\int_0^s \lambda(u)du}|\mathcal{G}(s))ds\end{aligned}$$

$$+\alpha e^{-\int_0^T \rho(u)du}U(W_m(T))E(e^{-\int_0^T \lambda(u)du}|\mathcal{G}(T))\bigg|W_m(0)=w, L(0)=\ell, \mathbf{q}(0)=\mathbf{q}\bigg].$$

By using Lemma A.2, we can get

$$E(e^{-\int_0^s \lambda(u)du}|\mathcal{G}(s)) = \exp\left\{-\int_0^s \hat{\lambda}(u)du\right\}$$

and

$$E(e^{-\int_0^T \lambda(u)du}|\mathcal{G}(T)) = \exp\left\{-\int_0^T \hat{\lambda}(u)du\right\}.$$

Therefore, we obtain

$$\bar{J}(w,\ell,\underline{\mathbf{y}}) = E\bigg[\int_0^T e^{-\int_0^s(\rho(u)+\hat{\lambda}(u))du}U(C(s))ds + \alpha e^{-\int_0^T(\rho(u)+\hat{\lambda}(u))du}$$
$$\times U(W_m(T))\bigg|W_m(0)=w, L(0)=\ell, \mathbf{q}(0)=\mathbf{q}\bigg],$$

where $\hat{\lambda}(u) = \sum_{i=1}^n \langle \hat{\mathbf{X}}(u), \mathbf{e}_i\rangle\lambda_i$. Hence, we have

$$\bar{J}(w,\ell,\underline{\mathbf{y}}) = J(w,\ell,\mathbf{q}). \quad \square$$

A.2. Proof of Theorem 3.1

Proof. Since $\phi \in \mathcal{O}\cap\bar{\mathcal{O}}$, choosing

$$\bar{\tau}_n = \inf\left\{s \ge t : \int_t^s |e^{-\int_t^u(\rho(v)+\hat{\lambda}(v))dv}(\phi_w W_m(u)\pi(u)\sigma + \langle\sigma^{-1}\mathbf{B}(u)\mathbf{q}, \mathbf{D}_\mathbf{q}\phi\rangle)|^2du \ge n\right\},$$

we have that $\forall (t,w,\ell,q) \in [0,T]\times R\times R\times R^n$, $(\pi,c)\in\mathcal{A}$, $s\in[t,T]$ and $\bar{\tau}_n\in[t,\infty)$, the stopped process $\left\{\int_t^{s\wedge\bar{\tau}_n}(\phi_w W_m(u)\pi(u)\sigma + \langle\sigma^{-1}\mathbf{B}(u)\mathbf{q}, \mathbf{D}_\mathbf{q}\phi\rangle)d\hat{Z}(u)|s\in[t,T]\right\}$ is a martingale. By using Itô's formula, we have

$$e^{-\int_t^{s\wedge\bar{\tau}_n}(\rho(v)+\hat{\lambda}(v))dv}\phi(s\wedge\bar{\tau}_n, W_m(s\wedge\bar{\tau}_n), L(s\wedge\bar{\tau}_n), \mathbf{q}(s\wedge\bar{\tau}_n))$$
$$=\phi(t,w,\ell,\mathbf{q}) + \int_t^{s\wedge\bar{\tau}_n} e^{-\int_t^u(\rho(v)+\hat{\lambda}(v))dv}\mathcal{L}^{\pi,c}\phi(u, W_m(u), L(u), \mathbf{q}(u))du$$
$$-\int_t^{s\wedge\bar{\tau}_n} e^{-\int_t^u(\rho(v)+\hat{\lambda}(v))dv}(\rho(u)+\hat{\lambda}(u))\phi(u, W_m(u), L(u), \mathbf{q}(u))du$$
$$+\int_t^{s\wedge\bar{\tau}_n} e^{-\int_t^u(\rho(v)+\hat{\lambda}(v))dv}(\phi_w W_m(u)\pi(u)\sigma + \langle\sigma^{-1}\mathbf{B}(u)\mathbf{q}, \mathbf{D}_\mathbf{q}\phi\rangle)d\hat{Z}(u).$$

Furthermore, by taking the expectation yields

$$E\left[e^{-\int_t^{s\wedge\bar{\tau}_n}(\rho(v)+\hat{\lambda}(v))dv}\phi(s\wedge\bar{\tau}_n, W_m(s\wedge\bar{\tau}_n), L(s\wedge\bar{\tau}_n), \mathbf{q}(s\wedge\bar{\tau}_n))\right]$$
$$=\phi(t,w,\ell,\mathbf{q}) + E\left[\int_t^{s\wedge\bar{\tau}_n} e^{-\int_t^u(\rho(v)+\hat{\lambda}(v))dv}\mathcal{L}^{\pi,c}\phi(u, W_m(u), L(u), \mathbf{q}(u))du\right]$$
$$-E\left[\int_t^{s\wedge\bar{\tau}_n} e^{-\int_t^u(\rho(v)+\hat{\lambda}(v))dv}(\rho(u)+\hat{\lambda}(u))\phi(u, W_m(u), L(u), \mathbf{q}(u))du\right].$$

According to condition 1, we derive $\forall(\pi,c)\in\mathcal{A}$,

$$E\left[e^{-\int_t^{s\wedge\bar{\tau}_n}(\rho(v)+\hat{\lambda}(v))dv}\phi(s\wedge\bar{\tau}_n, W_m(s\wedge\bar{\tau}_n), L(s\wedge\bar{\tau}_n), \mathbf{q}(s\wedge\bar{\tau}_n))\right]$$
$$\leq \phi(t,w,\ell,\mathbf{q}) - E\left[\int_t^{s\wedge\bar{\tau}_n} e^{-\int_t^u(\rho(v)+\hat{\lambda}(v))dv}U(c(u)W_m(u))du\right]. \tag{A.8}$$

By using conditions 3-4, the Dominated Theorem and (A.8), we have for $\forall(\pi,c)\in\mathcal{A}$,

$$E\left[e^{-\int_t^{s}(\rho(v)+\hat{\lambda}(v))dv}\phi(s, W_m(s), L(s), \mathbf{q}(s))\right]$$
$$\leq \phi(t,w,\ell,\mathbf{q}) - E\left[\int_t^{s} e^{-\int_t^u(\rho(v)+\hat{\lambda}(v))dv}U(c(u)W_m(u))du\right]. \tag{A.9}$$

Similarly, by using conditions 2-4, we obtain

$$E\left[e^{-\int_t^{s}(\rho(v)+\hat{\lambda}(v))dv}\phi(s, W_m^*(s), L(s), \mathbf{q}(s))\right]$$
$$= \phi(t,w,\ell,\mathbf{q}) - E\left[\int_t^{s} e^{-\int_t^u(\rho(v)+\hat{\lambda}(v))dv}U(c^*(u)W_m^*(u))du\right].$$

Replacing s by T, we get

$$\phi(t,w,\ell,\mathbf{q}) = E\Bigg[\int_t^T e^{-\int_t^u(\rho(v)+\hat{\lambda}(v))dv}U(c^*(u)W_m^*(u))du$$
$$+\alpha e^{-\int_t^T(\rho(v)+\hat{\lambda}(v))dv}U(W_m^*(T))\Bigg]. \tag{A.10}$$

Combining (A.9) and (A.10), we have $\phi(t,w,\ell,\mathbf{q}) = \Phi(t,w,\ell,\mathbf{q})$ and (π^*,c^*) is an optimal control. $\square$

A.3. Proof of Theorem 4.1

Proof. When $1<\ell<\infty$, combining (3.6) and (3.7) yields

$$\left(\rho(t)+\sum_{i=1}^n\langle\hat{\mathbf{X}}(t),\mathbf{e}_i\rangle\lambda_i\right)\Phi = \sup_{(\pi,c)\in\mathcal{A}}\Bigg\{U(C)+\Phi_t+\Phi_w w\Big[r+\pi(\hat{\mu}(t)-r)-c\Big]+\frac{1}{2}\Phi_{ww}w^2\pi^2\sigma^2$$
$$+\langle\mathbf{K}(t)\mathbf{q},\mathbf{D_q}\Phi\rangle+\frac{1}{2}\sigma^{-2}\langle\mathbf{B}(t)\mathbf{q},(\mathbf{D_q^2}\Phi)\mathbf{B}(t)\mathbf{q}\rangle+\langle\mathbf{B}(t)\mathbf{q},\mathbf{D_q}\Phi_w\rangle\pi w$$
$$+\sum_{i=1}^n\langle\hat{\mathbf{X}}(t),\mathbf{e}_i\rangle\lambda_i(\ell-1)\Big[\Phi\Big(t,w(1+\frac{1}{\ell-1}),\ell-1,\mathbf{diag}(\mathbf{G}(t))\mathbf{q}\Big)$$
$$-\Phi(t,w,\ell,\mathbf{q})\Big]\Bigg\}, \tag{A.11}$$

where

$$\Phi_t := \frac{\partial\Phi}{\partial t},\qquad \Phi_w := \frac{\partial\Phi}{\partial w},\qquad \Phi_{ww} := \frac{\partial^2\Phi}{\partial w^2},\qquad \mathbf{D_q}\Phi := \left(\frac{\partial\Phi}{\partial q_1},\frac{\partial\Phi}{\partial q_2},\cdots,\frac{\partial\Phi}{\partial q_n}\right)^\top,$$
$$\mathbf{D_q^2}\Phi := \left[\frac{\partial^2\Phi}{\partial q_i\partial q_j}\right]_{i,j=1,2,\cdots,n},\qquad \mathbf{D_q}\Phi_w := \left(\frac{\partial^2\Phi}{\partial w\partial q_1},\frac{\partial^2\Phi}{\partial w\partial q_2},\cdots,\frac{\partial^2\Phi}{\partial w\partial q_n}\right)^\top.$$

According to (A.11) and the utility function $U(x)=\ln(x)$, we have

$$0=\ln(c^*w)+\Phi_t-\left(\rho(t)+\sum_{i=1}^n\langle\hat{\mathbf{X}}(t),\mathbf{e}_i\rangle\lambda_i\right)\Phi+\Phi_w w\Big[r+\pi^*(\hat{\mu}(t)-r)-c^*\Big]$$
$$+\frac{1}{2}\Phi_{ww}w^2(\pi^*)^2\sigma^2+\langle\mathbf{K}(t)\mathbf{q},\mathbf{D_q}\Phi\rangle+\frac{1}{2}\sigma^{-2}\langle\mathbf{B}(t)\mathbf{q},(\mathbf{D_q^2}\Phi)\mathbf{B}(t)\mathbf{q}\rangle+\langle\mathbf{B}(t)\mathbf{q},\mathbf{D_q}\Phi_w\rangle\pi^*w$$
$$+\sum_{i=1}^n\langle\hat{\mathbf{X}}(t),\mathbf{e}_i\rangle\lambda_i(\ell-1)\Big[\Phi\Big(t,w(1+\frac{1}{\ell-1}),\ell-1,\mathbf{diag}(\mathbf{G}(t))\mathbf{q}\Big)-\Phi(t,w,\ell,\mathbf{q})\Big]. \tag{A.12}$$

Due to the first-order conditions and (A.12), we know

$$\pi^*(t) = -\frac{\langle \mathbf{B}(t)\mathbf{q}, \mathbf{D_q}\Phi_w\rangle + (\hat{\mu}(t)-r)\Phi_w}{\Phi_{ww} w\sigma^2}, \tag{A.13}$$

and

$$c^*(t) = (\Phi_w w)^{-1}. \tag{A.14}$$

Let

$$\theta(t) = \alpha \exp\Big(-\int_t^T \Big(\rho(u) + \sum_{i=1}^n \langle \hat{\mathbf{X}}(u), \mathbf{e}_i\rangle\lambda_i\Big)du\Big) + \int_t^T \exp\Big(-\int_t^s \Big(\rho(u) + \sum_{i=1}^n \langle \hat{\mathbf{X}}(u), \mathbf{e}_i\rangle\lambda_i\Big)du\Big)ds.$$

Suppose that the form of the value function is

$$\Phi(t, w, \ell, \mathbf{q}) = \theta(t)\ln(w) + f(t, \ell, \mathbf{q}). \tag{A.15}$$

Substituting (A.13) and (A.14) into (A.12) yields

$$\begin{aligned} 0 = &-\ln(\Phi_w) + \Phi_t - \Big(\rho(t) + \sum_{i=1}^n \langle \hat{\mathbf{X}}(t), \mathbf{e}_i\rangle\lambda_i\Big)\Phi + \Phi_w wr - \frac{\Phi_w(\hat{\mu}(t)-r)\langle \mathbf{B}(t)\mathbf{q}, \mathbf{D_q}\Phi_w\rangle}{\Phi_{ww}\sigma^2} \\ &-\frac{1}{2}\frac{(\hat{\mu}(t)-r)^2\Phi_w^2}{\Phi_{ww}\sigma^2} - 1 + \langle \mathbf{K}(t)\mathbf{q}, \mathbf{D_q}\Phi\rangle + \frac{1}{2}\sigma^{-2}\langle \mathbf{B}(t)\mathbf{q}, (\mathbf{D_q^2}\Phi)\mathbf{B}(t)\mathbf{q}\rangle \\ &-\frac{1}{2}\frac{\langle \mathbf{B}(t)\mathbf{q}, \mathbf{D_q}\Phi_w\rangle^2}{\Phi_{ww}\sigma^2} + \sum_{i=1}^n \langle \hat{\mathbf{X}}(t), \mathbf{e}_i\rangle\lambda_i(\ell-1) \\ &\times\Big[\Phi\Big(t, w(1+\frac{1}{\ell-1}), \ell-1, \mathbf{diag}(\mathbf{G}(t))\mathbf{q}\Big) - \Phi(t, w, \ell, \mathbf{q})\Big]. \end{aligned} \tag{A.16}$$

Due to (A.15), we have

$$\begin{aligned} &\Phi_t := \theta'(t)\ln(w) + \frac{\partial f}{\partial t}, \quad \Phi_w := \frac{\theta(t)}{w}, \quad \Phi_{ww} := -\frac{\theta(t)}{w^2}, \quad \mathbf{D_q}\Phi := \Big(\frac{\partial f}{\partial q_1}, \cdots, \frac{\partial f}{\partial q_n}\Big)^\top, \\ &\mathbf{D_q^2}\Phi := \Big[\frac{\partial^2 f}{\partial q_i \partial q_j}\Big]_{i,j=1,2,\cdots,n}, \quad \mathbf{D_q}\Phi_w := (0, \cdots, 0)^\top. \end{aligned} \tag{A.17}$$

Let $f(t, \ell, \mathbf{q}) = A_0(t, \ell) + \sum_{i=1}^n A_i(t, \ell)q_i + \sum_{i=1}^{n-1}\sum_{j=i+1}^n B_{ij}(t, \ell)q_i q_j + \sum_{i=1}^n C_i(t, \ell)q_i^2$. Then

$$\begin{aligned} &\frac{\partial f}{\partial t} = A_0'(t, \ell) + \sum_{i=1}^n A_i'(t, \ell)q_i + \sum_{i=1}^{n-1}\sum_{j=i+1}^n B_{ij}'(t, \ell)q_i q_j + \sum_{i=1}^n C_i'(t, \ell)q_i^2, \\ &\frac{\partial f}{\partial q_i} = A_i(t, \ell) + \sum_{j=1}^{i-1} B_{ji}(t, \ell)q_j + \sum_{j=i+1}^n B_{ij}(t, \ell)q_j + 2C_i(t, \ell)q_i, \quad \frac{\partial^2 f}{\partial q_i^2} = 2C_i(t, \ell), \\ &\frac{\partial^2 f}{\partial q_i \partial q_j} = \frac{\partial^2 f}{\partial q_j \partial q_i} = \begin{cases} B_{ij}, & i < j, \\ B_{ji}, & i > j. \end{cases} \end{aligned}$$

Therefore, we have

$$\pi^*(t) = \frac{\hat{\mu}(t) - r}{\sigma^2},$$

and

$$c^*(t) = (\theta(t))^{-1}.$$

When $1 < \ell < \infty$, let

$$\mathbf{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ldots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}, \quad \mathbf{K}(t) = \begin{pmatrix} k_{11} & \cdots & k_{1n} \\ \vdots & \ldots & \vdots \\ k_{n1} & \cdots & k_{nn} \end{pmatrix}, \quad \mathbf{q}(t) = (q_1(t), \cdots, q_n(t))^\top,$$

where

$$-a_{ii}=\sum_{j\neq i}a_{ij}>0, k_{ij}(t)=a_{ij}(i\neq j),$$
$$k_{ii}(t)=a_{ii}+1-(L(t)-1)\lambda_i+\sigma^{-2}h_i(t)\hat{h}(t).$$

Then

$$\mathbf{K}(t)\mathbf{q}(t)=\left(\sum_{j=1}^{n}k_{1j}(t)q_j(t),\cdots,\sum_{j=1}^{n}k_{nj}(t)q_j(t)\right)^{\top},$$
$$\mathbf{B}(t)\mathbf{q}(t)=(h_1(t)q_1(t),\cdots,h_n(t)q_n(t))^{\top},\quad \mathbf{D_q}\Phi=\left(\frac{\partial f}{\partial q_1},\cdots,\frac{\partial f}{\partial q_n}\right)^{\top},$$
$$\mathbf{D_q}\Phi_w:=(0,\cdots,0)^{\top},\quad \mathbf{D}^2_{\mathbf{q}}\Phi=\begin{pmatrix}\frac{\partial^2 f}{\partial q_1^2} & \cdots & \frac{\partial^2 f}{\partial q_1\partial q_n}\\ \vdots & \cdots & \vdots\\ \frac{\partial^2 f}{\partial q_n\partial q_1} & \cdots & \frac{\partial^2 f}{\partial q_n^2}\end{pmatrix},$$
$$\langle\mathbf{K}(t)\mathbf{q}(t),\mathbf{D_q}\Phi\rangle=\sum_{i=1}^{n}\sum_{j=1}^{n}k_{ij}(t)q_j(t)\frac{\partial f}{\partial q_i},\qquad \langle\mathbf{B}(t)\mathbf{q}(t),\mathbf{D_q}\Phi_w\rangle=0.$$

Let

$$\mathbf{D}(t,\ell)=\mathbf{D}^2_{\mathbf{q}}\Phi=\begin{pmatrix}2C_1(t,\ell) & \cdots & B_{1n}(t,\ell)\\ \vdots & \cdots & \vdots\\ B_{1n}(t,\ell) & \cdots & 2C_n(t,\ell)\end{pmatrix}.$$

Due to

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{m=1}^{n}k_{ij}(t)D_{im}(t,\ell)q_j(t)q_m=\sum_{m=1}^{n}\sum_{j=1}^{n}\sum_{i=1}^{n}k_{mj}(t)D_{mi}(t,\ell)q_j(t)q_i,$$
$$\sum_{i=1}^{n}\sum_{j=1}^{n}k_{ij}(t)A_i(t,\ell)q_j(t)=\sum_{j=1}^{n}\sum_{i=1}^{n}k_{ji}(t)A_j(t,\ell)q_i(t),$$

and

$$\begin{aligned}&\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\sum_{m=1}^{n}k_{mj}(t)D_{mi}(t,\ell)\right)q_jq_i\\&=\sum_{i=1}^{n}\left(\sum_{m=1}^{n}k_{mi}(t)D_{mi}(t,\ell)\right)q_i^2\\&\quad+\sum_{i=1}^{n-1}\sum_{j=i}^{n}\left(\sum_{m=1}^{n}k_{mj}(t)D_{mi}(t,\ell)+\sum_{m=1}^{n}k_{mi}(t)D_{mj}(t,\ell)\right)q_jq_i,\end{aligned}$$

we have

$$\begin{aligned}&\sum_{i=1}^{n}\left(\sum_{m=1}^{n}k_{mi}(t)D_{mi}(t,\ell)\right)q_i^2\\&=\sum_{i=1}^{n}\left(\sum_{m=1}^{i-1}k_{mi}(t)B_{mi}(t,\ell)+2k_{ii}(t)C_i(t,\ell)+\sum_{m=i+1}^{n}k_{mi}(t)B_{im}(t,\ell)\right)q_i^2,\\&\sum_{i=1}^{n-1}\sum_{j=i}^{n}\left(\sum_{m=1}^{n}k_{mj}(t)D_{mi}(t,\ell)+\sum_{m=1}^{n}k_{mi}(t)D_{mj}(t,\ell)\right)q_jq_i\\&=\sum_{i=1}^{n-1}\sum_{j=i}^{n}\left(\sum_{m=1}^{n}\Big(k_{mj}(t)D_{mi}(t,\ell)+k_{mi}(t)D_{mj}(t,\ell)\Big)\right)q_jq_i\\&=\sum_{i=1}^{n-1}\sum_{j=i}^{n}\left(\sum_{m=1}^{i-1}\Big(k_{mi}(t)B_{mj}(t,\ell)+k_{mj}(t)B_{mi}(t,\ell)\Big)+k_{ii}(t)B_{ij}(t,\ell)+2k_{ij}(t)C_i(t,\ell)\right.\end{aligned}$$

$$+\sum_{m=i+1}^{j-1}\Big(k_{mi}(t)B_{mj}(t,\ell)+k_{mj}(t)B_{im}(t,\ell)\Big)+2k_{ji}(t)C_j(t,\ell)+k_{jj}(t)B_{ij}(t,\ell)$$
$$+\sum_{m=i+1}^{j-1}\Big(k_{mi}(t)B_{jm}(t,\ell)+k_{mj}(t)B_{im}(t,\ell)\Big)\Big)q_jq_i,$$

$$\begin{aligned}
&\langle \mathbf{K}(t)\mathbf{q}(t),\mathbf{D_q}\Phi\rangle\\
&=\sum_{i=1}^{n}\sum_{j=1}^{n}k_{ij}(t)q_j\frac{\partial f}{\partial q_i}\\
&=\sum_{i=1}^{n}\sum_{j=1}^{n}k_{ij}(t)q_j\Big(A_i(t,\ell)+\sum_{m=1}^{i-1}B_{mi}(t,\ell)q_m+\sum_{m=i+1}^{n}B_{im}(t,\ell)q_m+2C_i(t,\ell)q_i\Big)\\
&=\sum_{i=1}^{n}\Big(\sum_{j=1}^{n}k_{ji}(t)A_j(t,\ell)\Big)q_i+\sum_{i=1}^{n}\Big(\sum_{m=1}^{i-1}k_{mi}(t)B_{mi}(t,\ell)+2k_{ii}(t)C_i(t,\ell)\\
&\quad+\sum_{m=i+1}^{n}k_{mi}(t)B_{im}(t,\ell)\Big)q_i^2+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\Big(\sum_{m=1}^{i-1}\Big(k_{mi}(t)B_{mj}(t,\ell)+k_{mj}(t)B_{mi}(t,\ell)\Big)\\
&\quad+k_{ii}(t)B_{ij}(t,\ell)+2k_{ij}(t)C_i(t,\ell)+\sum_{m=i+1}^{j-1}\Big(k_{mi}(t)B_{mj}(t,\ell)+k_{mj}(t)B_{im}(t,\ell)\Big)\\
&\quad+2k_{ji}(t)C_j(t,\ell)+k_{jj}(t)B_{ij}(t,\ell)+\sum_{m=i+1}^{j-1}\Big(k_{mi}(t)B_{jm}(t,\ell)+k_{mj}(t)B_{im}(t,\ell)\Big)\Big)q_jq_i,
\end{aligned}$$

$$\mathbf{D}^2_{\mathbf{q}}\Phi\mathbf{B}(t)\mathbf{q}(t)=\Big(\sum_{j=1}^{n}h_j(t)q_j(t)\frac{\partial^2 f}{\partial q_1\partial q_j},\cdots,\sum_{j=1}^{n}h_j(t)q_j(t)\frac{\partial^2 f}{\partial q_n\partial q_j}\Big)^{\top},$$

and

$$\begin{aligned}
\langle \mathbf{B}(t)\mathbf{q}(t),\mathbf{D}^2_{\mathbf{q}}\Phi\mathbf{B}(t)\mathbf{q}(t)\rangle&=\sum_{i=1}^{n}\sum_{j=1}^{n}h_i(t)h_j(t)q_i(t)q_j(t)\frac{\partial^2 f}{\partial q_i\partial q_j}\\
&=\sum_{i=1}^{n}\sum_{j=1}^{n}h_i(t)h_j(t)D_{ij}(t,\ell)q_i(t)q_j(t)\\
&=\sum_{i=1}^{n}h_i^2(t)D_{ii}(t,\ell)q_i^2(t)+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}2h_i(t)h_j(t)D_{ij}(t,\ell)q_i(t)q_j(t)\\
&=\sum_{i=1}^{n}2h_i^2(t)C_i(t,\ell)q_i^2(t)+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}2h_i(t)h_j(t)B_{ij}(t,\ell)q_i(t)q_j(t).
\end{aligned}$$

Due to

$$\frac{\Phi_w(\hat{\mu}(t)-r)\langle \mathbf{B}(t)\mathbf{q},\mathbf{D_q}\Phi_w\rangle}{\Phi_{ww}\sigma^2}=0,$$
$$\frac{\langle \mathbf{B}(t)\mathbf{q},\mathbf{D_q}\Phi_w\rangle^2}{\Phi_{ww}\sigma^2}=0,$$

then

$$\begin{aligned}
0=&-\ln(\theta(t))+A_0'(t,\ell)+\sum_{i=1}^{n}A_i'(t,\ell)q_i+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}B_{ij}'(t,\ell)q_iq_j+\sum_{i=1}^{n}C_i'(t,\ell)q_i^2\\
&-\Big(\rho(t)+\sum_{i=1}^{n}\langle\hat{\mathbf{X}}(t),\mathbf{e}_i\rangle\lambda_i\Big)\Big(A_0(t,\ell)+\sum_{i=1}^{n}A_i(t,\ell)q_i+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}B_{ij}(t,\ell)q_iq_j\\
&+\sum_{i=1}^{n}C_i(t,\ell)q_i^2\Big)+\theta(t)r+\frac{1}{2\sigma^2}(\hat{\mu}(t)-r)^2\theta(t)-1
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{i=1}^{n}\Bigg(\sum_{j=1}^{n}k_{ji}(t)A_j(t,\ell)\Bigg)q_i+\sum_{i=1}^{n}\Bigg(\sum_{m=1}^{i-1}k_{mi}(t)B_{mi}(t,\ell)+2k_{ii}(t)C_i(t,\ell)\\
&+\sum_{m=i+1}^{n}k_{mi}(t)B_{im}(t,\ell)\Bigg)q_i^2+\sum_{i=1}^{n-1}\sum_{j=i}^{n}\Bigg(\sum_{m=1}^{i-1}\Big(k_{mi}(t)B_{mj}(t,\ell)+k_{mj}(t)B_{mi}(t,\ell)\Big)\\
&+k_{ii}(t)B_{ij}(t,\ell)+2k_{ij}(t)C_i(t,\ell)+\sum_{m=i+1}^{j-1}\Big(k_{mi}(t)B_{mj}(t,\ell)+k_{mj}(t)B_{im}(t,\ell)\Big)\\
&+2k_{ji}(t)C_j(t,\ell)+k_{jj}(t)B_{ij}(t,\ell)+\sum_{m=i+1}^{j-1}\Big(k_{mi}(t)B_{jm}(t,\ell)+k_{mj}(t)B_{im}(t,\ell)\Big)\Bigg)q_jq_i\\
&+\sum_{i=1}^{n}\sigma^{-2}h_i^2(t)C_i(t,\ell)q_i^2(t)+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\sigma^{-2}h_i(t)h_j(t)B_{ij}(t,\ell)q_i(t)q_j(t)\\
&+\Bigg(\sum_{i=1}^{n}\langle\hat{\mathbf{X}}(t),\mathbf{e}_i\rangle\lambda_i(\ell-1)\Bigg)\Bigg(\theta(t)\ln\Big(\frac{\ell}{\ell-1}\Big)+A_0(t,\ell-1)-A_0(t,\ell)\\
&+\sum_{i=1}^{n}\Big(A_i(t,\ell-1)(\ell-1)\lambda_i-A_i(t,\ell)\Big)q_i\\
&+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\Big(B_{ij}(t,\ell-1)(\ell-1)^2\lambda_i\lambda_j-B_{ij}(t,\ell)\Big)q_iq_j\\
&+\sum_{i=1}^{n}\Big(C_i(t,\ell-1)(\ell-1)^2\lambda_i^2-C_i(t,\ell)\Big)q_i^2\Bigg).
\end{aligned}
$$

Therefore, we can obtain

$$
\begin{aligned}
A_0'(t,\ell)&=\ln(\theta(t))+\Bigg(\rho(t)+\sum_{i=1}^{n}\langle\hat{\mathbf{X}}(t),\mathbf{e}_i\rangle\lambda_i\Bigg)A_0(t,\ell)-\theta(t)r+1-\frac{1}{2\sigma^2}(\hat{\mu}(t)-r)^2\theta(t)\\
&\quad-\Bigg(\sum_{i=1}^{n}\langle\hat{\mathbf{X}}(t),\mathbf{e}_i\rangle\lambda_i(\ell-1)\Bigg)\Bigg(\theta(t)\ln\Big(\frac{\ell}{\ell-1}\Big)+A_0(t,\ell-1)-A_0(t,\ell)\Bigg),\\
A_i'(t,\ell)&=\Bigg(\rho(t)+\sum_{i=1}^{n}\langle\hat{\mathbf{X}}(t),\mathbf{e}_i\rangle\lambda_i\Bigg)A_i(t,\ell)-\sum_{j=1}^{n}k_{ji}(t)A_j(t,\ell)\\
&\quad-\Bigg(\sum_{i=1}^{n}\langle\hat{\mathbf{X}}(t),\mathbf{e}_i\rangle\lambda_i(\ell-1)\Bigg)\Bigg(A_i(t,\ell-1)(\ell-1)\lambda_i-A_i(t,\ell)\Bigg),\\
C_i'(t,\ell)&=\Bigg(\rho(t)+\sum_{i=1}^{n}\langle\hat{\mathbf{X}}(t),\mathbf{e}_i\rangle\lambda_i\Bigg)C_i(t,\ell)-\Bigg(\sum_{m=1}^{i-1}k_{mi}(t)B_{mi}(t,\ell)+2k_{ii}(t)C_i(t,\ell)\\
&\quad+\sum_{m=i+1}^{n}k_{mi}(t)B_{im}(t,\ell)\Bigg)-\sigma^{-2}h_i^2(t)C_i(t,\ell)\\
&\quad-\Bigg(\sum_{i=1}^{n}\langle\hat{\mathbf{X}}(t),\mathbf{e}_i\rangle\lambda_i(\ell-1)\Bigg)\Bigg(C_i(t,\ell-1)(\ell-1)^2\lambda_i^2-C_i(t,\ell)\Bigg),\\
B_{ij}'(t,\ell)&=\Bigg(\rho(t)+\sum_{i=1}^{n}\langle\hat{\mathbf{X}}(t),\mathbf{e}_i\rangle\lambda_i\Bigg)B_{ij}(t,\ell)-\Bigg(\sum_{m=1}^{i-1}\Big(k_{mi}(t)B_{mj}(t,\ell)+k_{mj}(t)B_{mi}(t,\ell)\Big)\\
&\quad+k_{ii}(t)B_{ij}(t,\ell)+2k_{ij}(t)C_i(t,\ell)+\sum_{m=i+1}^{j-1}\Big(k_{mi}(t)B_{mj}(t,\ell)+k_{mj}(t)B_{im}(t,\ell)\Big)\\
&\quad+2k_{ji}(t)C_j(t,\ell)+k_{jj}(t)B_{ij}(t,\ell)+\sum_{m=i+1}^{j-1}\Big(k_{mi}(t)B_{jm}(t,\ell)+k_{mj}(t)B_{im}(t,\ell)\Big)\Bigg)
\end{aligned}
$$

$$-\sigma^{-2}h_i(t)h_j(t)B_{ij}(t,\ell)-\left(\sum_{i=1}^{n}\langle\hat{\mathbf{X}}(t),\mathbf{e}_i\rangle\lambda_i(\ell-1)\right)$$
$$\times\left(B_{ij}(t,\ell-1)(\ell-1)^2\lambda_i\lambda_j-B_{ij}(t,\ell)\right).$$

For $\forall t\in[0,T]$, $1\le i<j\le n$ and $1<\ell<\infty$, due to terminal conditions $\lim_{t\to T} f(t,\ell,\mathbf{q})=0$, we know that $\lim_{t\to T}A_0(t,\ell)=0$, $\lim_{t\to T}A_i(t,\ell)=0$, $\lim_{t\to T}C_i(t,\ell)=0$ and $\lim_{t\to T}B_{ij}(t,\ell)=0$. Therefore, $A_i(t,\ell)=C_i(t,\ell)=B_{ij}(t,\ell)=0$, $A_0(t,\ell)$ satisfies the corresponding equation

$$A_0'(t,\ell)=\ln(\theta(t))+\left(\rho(t)+\sum_{i=1}^{n}\langle\hat{\mathbf{X}}(t),\mathbf{e}_i\rangle\lambda_i\right)A_0(t,\ell)-\theta(t)r+1$$
$$-\frac{1}{2\sigma^2}(\hat{\mu}(t)-r)^2\theta(t)-\left(\sum_{i=1}^{n}\langle\hat{\mathbf{X}}(t),\mathbf{e}_i\rangle\lambda_i(\ell-1)\right)\left(\theta(t)\ln\left(\frac{\ell}{\ell-1}\right)\right.$$
$$\left.+A_0(t,\ell-1)-A_0(t,\ell)\right). \tag{A.18}$$

The proofs under the cases of $\ell=1$ and $\ell=\infty$ are similar to that under the case of $1<\ell<\infty$, so we omit them here. When considering only one individual ($\ell=1$) and an infinitely large pool of annuity funds ($\ell=\infty$), we have the following conclusions.

(1) $\forall t\in[0,T]$ and $\ell=1$,

$$f(t,1,\mathbf{q})=A_0(t,1)+\sum_{i=1}^{n}A_i(t,1)q_i+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}B_{ij}(t,1)q_iq_j+\sum_{i=1}^{n}C_i(t,1)q_i^2,$$

where for $1\le i<j\le n$, coefficients $A_i(t,1)=B_{ij}(t,1)=C_i(t,1)=0$, $A_0(t,1)$ satisfies the corresponding equation

$$A_0(t,1)=A_0(T,1)\exp\left\{-\int_t^T(\rho(s)+\sum_{i=1}^{n}\langle\hat{\mathbf{X}}(s),\mathbf{e}_i\rangle\lambda_i)ds\right\}$$
$$-\int_t^T\exp\left\{-\int_t^s(\rho(u)+\sum_{i=1}^{n}\langle\hat{\mathbf{X}}(u),\mathbf{e}_i\rangle\lambda_i)du\right\}\left(\ln(\theta(s))-\theta(s)r+1\right.$$
$$\left.-\frac{1}{2\sigma^2}(\hat{\mu}(s)-r)^2\theta(s)\right)ds, \tag{A.19}$$

where $\lim_{t\to T}A_0(t,1)=0$.

(2) $\forall t\in[0,T]$ and $\ell=\infty$,

$$f(t,\infty,\mathbf{q})=A_0(t,\infty)+\sum_{i=1}^{n}A_i(t,\infty)q_i+\sum_{i=1}^{n-1}\sum_{j=i+1}^{n}B_{ij}(t,\infty)q_iq_j+\sum_{i=1}^{n}C_i(t,\infty)q_i^2,$$

where for $1\le i<j\le n$, coefficients $A_i(t,\infty)=B_{ij}(t,\infty)=C_i(t,\infty)=0$, $A_0(t,\infty)$ satisfies the corresponding equation

$$A_0(t,\infty)=A_0(T,\infty)\exp\left\{-\int_t^T(\rho(s)+\sum_{i=1}^{n}\langle\hat{\mathbf{X}}(s),\mathbf{e}_i\rangle\lambda_i)ds\right\}-\int_t^T\exp\left\{-\int_t^s(\rho(u)\right.$$
$$\left.+\sum_{i=1}^{n}\langle\hat{\mathbf{X}}(u),\mathbf{e}_i\rangle\lambda_i)du\right\}\left(\ln(\theta(s))-\theta(s)r+1-\frac{1}{2\sigma^2}(\hat{\mu}(s)-r)^2\theta(s)-\theta(s)\right.$$
$$\left.\times\sum_{i=1}^{n}\langle\hat{\mathbf{X}}(s),\mathbf{e}_i\rangle\lambda_i\right)ds, \tag{A.20}$$

where $\lim_{t\to T}A_0(t,\infty)=0$.

The calculations are similar to that under the case of $1<\ell<\infty$, we omit them here. □

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