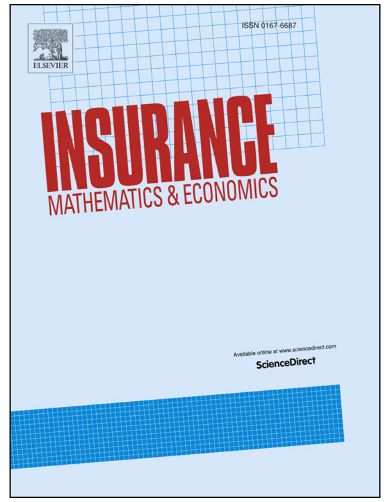
Journal Pre-proof



Optimal investment, consumption and life insurance purchase with learning about return predictability

Xingchun Peng and Baihui Li

PII: S0167-6687(23)00064-1
DOI: https://doi.org/10.1016/j.insmatheco.2023.07.005
Reference: INSUMA 2971

To appear in: *Insurance: Mathematics and Economics*

Received date: 15 October 2022
Revised date: 26 July 2023
Accepted date: 28 July 2023

Optimal investment, consumption and life insurance purchase with learning about return predictability

Xingchun Peng*, Baihui Li

School of Science, Wuhan University of Technology, Wuhan 430070, PR China

Abstract

This paper studies the optimal investment, consumption and life insurance purchase problem for a wage earner under the condition that the return on the risky asset is predictable. We assume that the market price of risk is an affine function consisting of an observable and an unobservable factor that follow the O-U processes, while the evolution of the interest rate is described by the Vasicek model. The optimal investment, consumption and life insurance strategies and the corresponding value function are derived by adopting the filtering technique and the dynamical programming principle approach. In addition, for comparative analysis, the suboptimal strategies and the utility losses are presented when the wage earner ignores learning or the randomness of the interest rate. Finally, some numerical examples are presented to illustrate the results.

Keywords: Life insurance, Return predictability, Stochastic interest rate, Learning
JEL classification: C61, G11, G52

1. Introduction

The classical optimal consumption and investment problem studies a wage earner who aims to maximize the expected discounted utility of consumption in the continuous-time model, see Merton (1969). In the framework of general asset allocation problem, the role of life insurance has been investigated since Yaari (1965), who considers the fact that the lifetime of the wage earner is uncertain. The demand for life insurance purchase is proposed to protect the beneficiary from the premature death. Moreover, the life insurance has also been considered as a hedge against the loss of the present value of future income when the family is at the risk of losing the income source, for example, see Huang et al. (2008). Therefore, it is necessary to extend the classical asset allocation models to incorporate the life insurance purchase.

A large number of works have extended the classical consumption and investment optimization problem by incorporating the life insurance purchase to highlight the mortality risk faced by the wage earner. Richard (1975) combines the optimal investment and consumption problem with life insurance purchase for a wage earner whose lifetime is bounded and fixed. Pliska and Ye (2007) study an optimal consumption and life insurance problem for a wage earner with random and unbounded lifetime. They derive the explicit solutions by converting the optimization problem with a random horizon to a problem with a fixed horizon. Ye (2007) considers an optimal investment, consumption and life insurance purchase problem with uncertain lifetime, and combines the dynamic programming principle approach and the martingale method to obtain the closed-form

*Corresponding author
Email addresses: pxch@whut.edu.cn (Xingchun Peng), 305881@whut.edu.cn (Baihui Li)

strategies. Kwak et al. (2011) also use the martingale method to derive the optimal investment-consumption-insurance strategies of a family with two generations. It is assumed that the objective of the family is to maximize the weighted average utility of the parents and children. Pirvu and Zhang (2012) derive the closed-form solution for the optimal investment-consumption-insurance problem faced by a constant relative risk aversion (CRRA) wage earner who focuses on shocks to the market price of risk. Shen and Sherris (2018) further consider the randomness of the mortality, the interest rate and the labor income in the asset allocation modelling framework, and express the optimal strategies in terms of the closed-form solutions to several Riccati equations. Shen and Wei (2016) allow for multiple risky assets in the financial market with random parameters. They obtain the explicit expressions of the optimal strategies by combining the backward stochastic differential equation (BSDE) and the Hamilton-Jacobi-Bellman (HJB) equation.

There are a number of empirical evidence indicating that the returns on risky assets are predictable, see, for example, Campbell and Shiller (1986), Fama and French (1988, 1989), Campbell and Viceira (1999). In the assumption of return predictability, many researchers study the portfolio optimization problem with completely information in the financial market, which implies that all the predictors are observable. Kim and Omberg (1996) assume that the stock price follows a mean reverting process and derive the optimal portfolio choice under the assumption of no parameter uncertainty. Wachter (2002) and Liu (2007) extend the analysis of Kim and Omberg (1996) to the case incorporating consumption. Xia (2001) introduces the linear relation between the observable predictor and the expected stock returns to an optimal investment problem. He confirms that the wage earner will suffer the utility losses when ignoring the predictability of expected returns. Ma et al. (2019) derive a closed-form solution to the optimal investment problem with transaction costs and return predictability. Ma et al. (2020) incorporate consumption into the model investigated in Ma et al. (2019).

The optimal asset allocation strategies derived in the literature mentioned above are based on the assumption that the expected asset returns are observable with known parameters. However, the expected asset returns cannot be captured by the observable predictors perfectly in the real financial market. Therefore, it is more realistic to take the unobservability of the predictors into account. Brennan (1998) first considers the uncertainty of parameters of the probability distribution and assumes that there is an "estimation risk" when estimating the risk premium. Fouque et al. (2015) study the portfolio optimization problem incorporating the unobservability of the predictor, which is estimated based on the observations of the stock price through the Kalman filter. Wang et al. (2021b) investigate a dynamic mean-variance investment problem for a DC pension plan with learning about an unobservable predictor. These works assume that the expected asset returns are completely unobservable and use the stochastic filter technique to transform the asset allocation problems with partial observations into the problems with complete observations. In a more general framework, Van Binsbergen and Koijen (2010) suppose that the expected stock returns are predicted by an observable and an unobservable factor. With a similar model for the expected return rate, Branger et al. (2013) employ the Kalman filtering technique to estimate the unobservable component and obtain the optimal investment strategy and the value function by the dynamic programming principle approach. They also conclude that the utility losses occur when the wage earner ignores the learning about the unobservable factor. This implies that the assumption of learning demonstrates the demand for the investor to hedge against unfavorable changes in the predictors. Escobar et al. (2016) incorporate stochastic interest rate into the optimal investment problem besides the assumption that the expected stock returns are predictable with observed and unobserved factors. As far as we know, in the literature, no works on the optimal investment and consumption problems capture the life insurance purchase and return predictability with an observable and an unobservable factor. This paper aims to concentrate on this problem.

Moreover, since the decision period for the wage earner who considers investment, consumption and life insurance purchase lasts long, it is essential for the wage earner to consider the stochastic interest rate risk. Sørensen (1999) obtains the mean-variance optimal portfolio strategies in a complete market, where the stochastic interest rate follows the Vasicek model in Vasicek (1977) and the market price of interest risk is assumed to be a constant. Munk and Sørensen (2004) consider the optimal consumption and investment problem with stochastic interest rate, and the wage earner hedges against changes in the interest rate by investing in a coupon bond. Han and Hung (2017) analyze the impact of the stochastic interest rate and inflation on the optimal investment, consumption and life insurance purchase policies by employing the stochastic differential utility.

This paper is an attempt to study the optimal investment, consumption and life insurance purchase problem for a CRRA wage earner with stock returns predictability and the risk of stochastic interest rate. We model the expected stock returns as an affine function, which consists of an observable and an unobservable factor. The dynamics of the predictors are formulated by O-U processes and the correlations among the unobservable factor and the risky asset are allowed. It is supposed that the wage earner tries to gather as much information about the observable processes as possible to estimate the unobservable process by using the Kalman filter. The objective for the wage earner is to maximize the utility of consumption, bequest and terminal wealth over an uncertain lifetime horizon. By using the dynamic programming principle, the closed-form solutions and the corresponding value function are derived in terms of the solutions to a system of ordinary differential equations. Particularly, under the assumption that the utility is defined over the terminal wealth, we derive the explicit expressions of the optimal investment strategies and the interest rate sensitivity of the human capital. Furthermore, we present the suboptimal strategies and measure the utility losses when the wage earner ignores the learning about the unobservable predictor or the randomness of the interest rate. Finally, the numerical examples illustrate the impacts of the predictive powers, the mortality rate and the risk aversion on the optimal investment, consumption and life insurance purchase strategies in different patterns. Since the utility losses are significant when making suboptimal decisions, it is necessary for the wage earner to take both learning and the randomness of the interest rate into account.

The rest of this paper is organized as follows. Section 2 introduces the life insurance and the financial assets, and formulates the optimization problem. Section 3 derives the optimal strategies by adopting the dynamic programming principle. Section 4 discusses the utility losses associated with ignoring learning or the randomness of the interest rate. Section 5 presents the numerical examples and illustrates the sensitivities of some main parameters. Section 6 concludes the paper.

2. Model formulation

Let $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathcal{T}}, \mathbb{P}\right)$ be a filtered complete probability space , where $\mathcal{T} = [0, T]$ is a finite-time horizon and $\{\mathcal{F}_t\}_{t\in\mathcal{T}}$ is a right continuous, $\mathbb{P}$-complete filtration with $\mathcal{F}_t$ denoting the information in the market up to time t. The finite time point T is supposed to be fixed and positive.

2.1. Insurance market

Let τ be a non-negative random variable denoting the death time of a wage earner who is alive at time $t = 0$. It is assumed that τ is defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and independent of the filtration $\{\mathcal{F}_t\}_{t\in\mathcal{T}}$. Suppose that the mortality rate $\{\lambda(t)\,|t \in \mathcal{T}\}$ is an $\mathbb{R}^+$-valued, deterministic and continuous function, which is defined by:

$$\lambda(t) = \lim_{\Delta t \to 0} \frac{P(t \leq \Delta t < t + \Delta t \,|\tau \geq t)}{\Delta t}. \tag{1}$$

Let $F(s,t)$ be the conditional survival probability of a wage earner who survives from time t to time s with $t \leq s$. Then, from (1), we have

$$F(s,t) = P(\tau > s \,|\tau > t) = \exp\left(-\int_t^s \lambda(u)du\right). \tag{2}$$

Let $f(s,t)$ represent the conditional probability density for the death of a wage earner at time s conditional upon being alive at time t with $t \leq s$. Then

$$f(s,t) = \lambda(s)\exp\left(-\int_t^s \lambda(u)du\right). \tag{3}$$

In our model, we suppose that the wage earner can purchase a life insurance or an annuity continuously by paying insurance premium at rate $p(t)$ until time $T \wedge \tau$. When $p(t)$ is positive, the insurance company needs to pay an amount $\frac{p(t)}{\eta(t)}$ to the beneficiary at death time t. Here $\eta(t)$ is called the premium-insurance ratio. However, the amount $\left|\frac{p(t)}{\eta(t)}\right|$ should be paid by the wage earner's family if $p(t)$ is negative, which means that the wage earner purchases a special term pension annuity. Note that both $\eta(t)$ and $p(t)$ are continuous and deterministic functions with respect to $t \in [0, \tau \wedge T)$, and $\frac{1}{\eta(t)}$ is referred to as loading factor. In general, it holds that $\eta(t) \geq \lambda(t)$ due to commission fees. In order to simplify the analysis, we assume that $\eta(t) = \lambda(t)$ in the frictionless market considered in this paper.

2.2. Financial market

The financial market consists of three tradable assets: a risk-free asset, a zero-coupon bond and a stock. The wage earner is supposed to receive a determined income continuously on $[0, T \wedge \tau]$. The price process of the risk-free asset $\{S_0(t)\,|t \in [0,T]\}$ is described by the following ordinary differential equation (ODE):

$$\begin{cases} dS_0(t) = r(t)S_0(t)dt, \\ S_0(0) = s_0 > 0. \end{cases} \tag{4}$$

Here, $r(t)$ is the instantaneous nominal interest rate with the following dynamics

$$\begin{cases} dr(t) = \kappa_r(\bar{r} - r(t))dt - \sigma_r dW_r(t), \\ r(0) = r_0 > 0, \end{cases} \tag{5}$$

where κ_r is the mean-reversion coefficient, $\bar{r}$ is the long-run mean of the interest rate, σ_r is the volatility and $\{W_r(t)\}$ is a standard Brownian motion.

Let $B(t,T)$ be the time t price of a nominal zero-coupon bond that delivers a payment of one dollar at maturity T. The diffusion equation of $B(t,T)$ is

$$\frac{dB(t,T)}{B(t,T)} = (r(t) + \sigma_B(T-t)q_r)dt + \sigma_B(T-t)dW_r(t), \tag{6}$$

where $\sigma_B(T-t) = \sigma_r \frac{1}{\kappa_r}(1-e^{-\kappa_r(T-t)})$ denotes the volatility of the zero-coupon bond, $\sigma_B(T-t)q_r$ denotes the expected excess return and q_r is the market price of interest rate risk. Moreover, the explicit expression for $B(t,T)$ is given by (see for example Han and Hung (2017) and Munk and Sørensen (2010))

$$B(t,T) = e^{-a(T-t)-b(T-t)r(t)}, \tag{7}$$

where

$$\begin{cases} b(T-t) = \frac{1}{\kappa_r}(1-e^{-\kappa_r(T-t)}), \\ a(T-t) = \left(\bar{r} + \frac{\sigma_r q_r}{\kappa_r} - \frac{\sigma_r^2}{2\kappa_r^2}\right)[T-t-b(T-t)] + \frac{\sigma_r^2}{4\kappa_r} b^2(T-t). \end{cases} \tag{8}$$

Note that it is unrealistic to find all the zero-coupon bonds corresponding to the specified maturing dates in the financial market. Therefore, we introduce a rolling bond with a constant maturity I, whose price process is governed by

$$\frac{dB_I(t)}{B_I(t)} = (r(t) + \sigma_B q_r)dt + \sigma_B dW_r(t), \tag{9}$$

where $\sigma_B = \sigma_r \frac{1}{\kappa_r}(1-e^{-\kappa_r I})$.

In fact, the zero-coupon bond with any maturity can be appropriately replicated by the risk-free asset and the rolling bond. The relationship between $S_0(t)$, $B(t,T)$ and $B_I(t)$ is as follows

$$\frac{dB(t,T)}{B(t,T)} = \left(1 - \frac{\sigma_B(T-t)}{\sigma_B}\right)\frac{dS_0(t)}{S_0(t)} + \frac{\sigma_B(T-t)}{\sigma_B}\frac{dB_I(t)}{B_I(t)}. \tag{10}$$

The third asset in the financial market is a stock whose price process follows

$$\begin{cases} \frac{dS(t)}{S(t)} = \mu_S(t)dt + \sigma_S dW_S(t), \\ S(0) = s > 0, \end{cases} \tag{11}$$

where $\mu_S(t)$ denotes the expected return rate, σ_S is constant volatility, $\{W_S(t)\}$ is another standard Brownian motion, correlated with $\{W_r(t)\}$ and $Cov(W_S(t), W_r(t)) = \rho_{Sr} t$, where $\rho_{Sr} \in (-1,1)$ is the correlation coefficient.

We assume that the market price of risk $\frac{\mu_S(t)-r(t)}{\sigma_S}$ is defined by an affine function:

$$\frac{\mu_S(t)-r(t)}{\sigma_S} = \phi + \phi_y y(t) + \phi_z z(t), \tag{12}$$

where $y(t)$ is an observable stochastic factor, $z(t)$ is an unobservable stochastic factor, the constants ϕ_y and ϕ_z are the predictive powers of $y(t)$ and $z(t)$ respectively. The dynamics of the factors are modeled by the O-U processes:

$$\begin{cases} dy(t) = \kappa_y(\bar{y} - y(t))dt + \sigma_y dW_r(t), \\ dz(t) = \kappa_z(\bar{z} - z(t))dt + \sigma_z dW_z(t), \end{cases} \tag{13}$$

where κ_y and κ_z are the mean-reversion coefficients, $\bar{y}$ and $\bar{z}$ denote the long-run means of the factors, σ_y and σ_z are the volatilities, $\{W_z(t)\}$ is a standard Brownian motion correlated with $\{W_S(t)\}$ and $\{W_r(t)\}$, $Cov\left(W_z(t), W_S(t)\right) = \rho_{Sz}t$ and $Cov\left(W_z(t), W_r(t)\right) = \rho_{zr}t$.

From (11) and (12), the dynamical equation for the stock price can be rewritten as

$$\frac{dS(t)}{S(t)} = \left[r(t) + \sigma_S\left(\phi + \phi_y y(t) + \phi_z z(t)\right)\right]dt + \sigma_S dW_S(t). \tag{14}$$

This model for $S(t)$ is similar to that in Van Binsbergen and Koijen (2010) and Branger et al. (2013). They also assume that the expected return rate depends on an observable and an unobservable factor. The expected stock return cannot be predicted directly due to the unobservability of the predictor $z(t)$ when $\phi_z \neq 0$. However, the wage earner tries to learn from the observable processes $S(t)$, $r(t)$ and $y(t)$ to estimate the unobservable predictor $z(t)$ by Bayesian learning. Let $\left\{\mathcal{F}_t^{S,r}\right\}_{t\in[0,T]}$ be the natural filtration generated by the observable processes $S(t)$ and $r(t)$. By (5) and (13), we can see that $y(t)$ is $\left\{\mathcal{F}_t^{S,r}\right\}$ adapted process. Thus, $\left\{\mathcal{F}_t^{S,r}\right\}$ coincides with the natural filtration generated by all the observable processes $S(t)$, $r(t)$ and $y(t)$, and can be regarded as the observable information flow. The filtered estimate of $z(t)$ is defined as

$$\hat{z}(t) = \mathbb{E}\left[z(t)\,\middle|\,\mathcal{F}_t^{S,r}\right]. \tag{15}$$

Based on Theorem 12.7 of Liptser and Shiryaev (2001), we can derive the following result.

Proposition 1. *The price processes of the risky assets can be rewritten as*

$$\begin{bmatrix} \frac{dS(t)}{S(t)} \\ \frac{dB_I(t)}{B_I(t)} \end{bmatrix} = \underbrace{\begin{bmatrix} r(t) + \sigma_S[\phi + \phi_y y(t) + \phi_z \hat{z}(t)] \\ r(t) + \sigma_B q_r \end{bmatrix}}_{\boldsymbol{\mu}} dt + \underbrace{\begin{bmatrix} \sigma_S & 0 \\ \sigma_B\rho_{Sr} & \sigma_B\hat{\rho}_r \end{bmatrix}}_{\boldsymbol{\Sigma}} \underbrace{\begin{bmatrix} dW_1(t) \\ dW_2(t) \end{bmatrix}}_{d\mathbf{W}(t)}, \tag{16}$$

and the dynamics for the factor processes $r(t)$, $y(t)$ and $\hat{z}(t)$ are given by

$$\underbrace{\begin{bmatrix} dr(t) \\ dy(t) \\ d\hat{z}(t) \end{bmatrix}}_{d\mathbf{k}(t)} = \underbrace{\begin{bmatrix} \kappa_r(\bar{r} - r(t)) \\ \kappa_y(\bar{y} - y(t)) \\ \kappa_z(\bar{z} - \hat{z}(t)) \end{bmatrix}}_{\boldsymbol{\mu}^{\mathbf{k}}} dt + \underbrace{\begin{bmatrix} -\sigma_r\rho_{Sr} & -\sigma_r\hat{\rho}_r \\ \sigma_y\rho_{Sr} & \sigma_y\hat{\rho}_r \\ H_1 & H_2 \end{bmatrix}}_{\boldsymbol{\Sigma}^{\mathbf{k}}} \underbrace{\begin{bmatrix} dW_1(t) \\ dW_2(t) \end{bmatrix}}_{d\mathbf{W}(t)}, \tag{17}$$

where $\mathbf{k}(t) = (r(t), y(t), \hat{z}(t))^T$, $\mathbf{W}(t) = (W_1(t), W_2(t))^T$, defined by (A.7), is a $\left\{\mathcal{F}_t^{S,r}\right\}$-adapted two-dimensional standard Brownian motion, H_1, H_2 and $\hat{\rho}_r$ are presented in Appendix A.

Proof *See Appendix A.*

Remark 1. *From the dynamics for the price processes and the factor processes, we can find that the financial market considered is complete. This is partially owed to the assumption that the driven noises for the observable stochastic factor $y(t)$ and the interest rate $r(t)$ are the same. If $y(t)$ is driven by $\{W_r(t)\}$ and $\{W_S(t)\}$, i.e.,*

$$dy(t) = k_y\left(\bar{y} - y(t)\right)dt + \sigma_{y_1}dW_r(t) + \sigma_{y_2}dW_S(t),$$

with k_y, $\bar{y}$, σ_{y_1} and σ_{y_2} being constants, then the financial market may also be complete. However, this assumption for $y(t)$ is not reasonable. In fact, by this assumption for $y(t)$ and the dynamical equations for the observable processes $S(t)$ and $r(t)$, we can deduce that

$$\mathcal{F}_t^{W_S} \subset \mathcal{F}_t^{S,r,y},$$

for $t \in [0,T]$, where $\mathcal{F}_t^{W_S} = \sigma\left(W_S(u), u \le t\right)$ and $\mathcal{F}_t^{S,r,y} = \sigma\left(S(u), r(u), y(u); u \le t\right)$. This means that the natural filtration generated by $\{W_S(t)\}$ is contained in the observable information flow. So the noise process $\{W_S(t)\}$ is observable. From equation (11), we have

$$S(t) = s \exp\left\{ \int_0^t \mu_S(u)du + \sigma_S W_S(t) - \frac{\sigma_S^2 t}{2} \right\}.$$

So $\int_0^t \mu_S(u)du = \ln S(t) - \ln s - \sigma_S W_S(t) + \frac{\sigma_S^2 t}{2}$ is $\mathcal{F}_t^{S,r,y}$ measurable for any $t \in [0,T]$. Furthermore, for any $t \in (0,T]$ and $\varepsilon \in (0,t)$, $\int_{t-\varepsilon}^t \mu_S(u)du = \int_0^t \mu_S(u)du - \int_0^{t-\varepsilon} \mu_S(u)du$ is $\mathcal{F}_t^{S,r,y}$ measurable. Note that the process $\{\mu_S(u)\}$ is continuous in u. Then $\mu_S(t) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{t-\varepsilon}^t \mu_S(u)du$ is $\mathcal{F}_t^{S,r,y}$ measurable for any $t \in (0,T]$. This implies that $\{\mu_S(t)\}$ is observable, which contradicts with our assumption for $\{\mu_S(t)\}$.

It is more reasonable and interesting to introduce other driven noises into the dynamical equation for $y(t)$ so that the risk of $y(t)$ cannot be completely hedged by investing in the financial market. For example, as considered in Branger et al. (2013) and Escobar et al. (2016), $y(t)$ is driven by a Brownian motion that is correlated with $W_S(t)$ with nonzero correlation coefficient. In this case, the financial market is incomplete. Under the criterion of maximizing the expected utility of terminal wealth, the analytical expressions for optimal investment strategies are derived in Branger et al. (2013) and Escobar et al. (2016). However, compared to these two works, the optimization objective in this paper includes not only the utility of terminal wealth, but also the utilities of consumption and life insurance (see (19) or (21)). This makes the analytical solution for the optimization problem in this paper (see (22)) cannot be derived in general. So we do not investigate the more general model for $y(t)$ in this paper and leave it to our future research. See also the comments after Proposition 5.

2.3. Optimization problem

Let x_0 be the initial wealth, $c(t)$ and $p(t)$ be respectively the consumption rate and insurance premium rate, $\pi^S(t)$ and $\pi^B(t)$ denote the amounts invested in the risky asset and zero-coupon bond respectively. Define $\boldsymbol{\pi}(t) = (\pi^S(t), \pi^B(t))^T$ as the investment strategy. The triplet of the strategy $\boldsymbol{\psi}(t) = (\boldsymbol{\pi}(t), c(t), p(t))^T$ represents the investment, consumption, and insurance purchase demand at time t. Assume that the wage earner receives labor income continuously at a deterministic rate $i(t)$. Then the wealth process $X(t)$ of the wage earner associated with $\boldsymbol{\psi}(t)$ is governed by the following stochastic differential equation

$$\begin{cases} dX(t) = \left[X(t)r(t) + \boldsymbol{\pi}^T(t)\left(\boldsymbol{\mu} - \mathbf{r}\right) - c(t) - p(t) + i(t)\right]dt + \boldsymbol{\pi}^T(t)\boldsymbol{\Sigma}d\mathbf{W}(t), \\ X(0) = x_0 > 0, \end{cases} \tag{18}$$

where $\mathbf{r} = (r(t), r(t))^T$.

Definition 1. *An investment-consumption-insurance strategy $\boldsymbol{\psi} = (\boldsymbol{\pi}, c, p)$ is said to be admissible if the following conditions hold:*

(1) $\boldsymbol{\psi} = (\boldsymbol{\pi}, c, p)$ *is a* $\left\{\mathcal{F}_t^{S,r}\right\}_{t\in[0,T]}$ *progressively measurable process with values in* $\mathbb{R}^2 \times \mathbb{R}^+ \times \mathbb{R}$*;*

(2)

$$\int_0^T |\boldsymbol{\pi}(t)|^2 ds < \infty, \int_0^T c(t)dt < \infty, \int_0^T p(t)dt < \infty, \quad a.s.;$$

(3) for any given initial value $(t_0, x_0, r_0, y_0, z_0) \in [0, T] \times \mathbb{R}^4$ *with* $x_0 > 0$*, the stochastic differential equation (18) associated with* $(\boldsymbol{\pi}, c, p)$ *has a unique strong solution* $X(t)$ *such that for* $t \in [0, T]$

$$X(t) + \frac{p(t)}{\eta(t)} \geq 0, \quad X(t) + h(t) \geq 0, \quad a.s.$$

where

$$h(t) = \mathbb{E}^{\mathbb{Q}}_{t,\mathbf{k}} \left\{ \int_t^T i(s) \exp\left[- \int_t^s (\lambda(u) + r(u))du \right] ds \right\}$$

can be interpreted as the human capital which is the actuarial present value of the future income.

The set of all admissible strategies $(\boldsymbol{\pi}, c, p)$ *is denoted by* $\mathcal{A}$*.*

The wage earner aims to maximize the expected utility from the consumption, the legacy left to the family and the terminal wealth. Then, the performance functional is defined by

$$\begin{aligned}\widetilde{\mathcal{J}}(t, x, \mathbf{k}; \boldsymbol{\psi}) =& \mathbb{E}_{t,x,\mathbf{k}} \left[\alpha \int_t^{\tau\wedge T} e^{-\omega(s-t)} U\left(c(s)\right) ds + \beta e^{-\omega(\tau-t)} U\left(Z(\tau)\right) 1_{\{\tau \leq T\}} \right. \\ &\left. + e^{-\omega(T-t)} U\left(X(T)\right) 1_{\{\tau > T\}} \right],\end{aligned} \tag{19}$$

where

$$Z(\tau) = X(\tau) + \frac{p(\tau)}{\eta(\tau)},$$

and $\mathbb{E}_{t,x,\mathbf{k}}[\cdot]$ is the conditional expectation $\mathbb{E}[\cdot\,|X(t) = x, \mathbf{k}(t) = \mathbf{k}]$ taken under $\mathbb{P}$, $\omega > 0$ is the subjective discount rate, α and β are positive constants, denoting the relative utility weights for the preferences toward the consumption and the bequest motive, respectively. The utility function $U(\cdot)$ is assumed to be the following power function:

$$U(x) = \begin{cases} \dfrac{x^\gamma}{\gamma}, & \text{if x} > 0, \\ -\infty, & \text{if x} \leq 0, \end{cases} \tag{20}$$

where $1 - \gamma$ is the relative risk aversion parameter and $\gamma \in (-\infty, 0)$.

Next, we transform the performance functional with random time horizon into the following one with deterministic time horizon $[0, T]$ by the results in Pliska and Ye (2007):

$$\begin{aligned}\mathcal{J}(t, x, \mathbf{k}; \boldsymbol{\psi}) =& \mathbb{E}_{t,x,\mathbf{k}} \left[\alpha \int_t^T F(s,t) e^{-\omega(s-t)} U(c(s)) ds + \beta \int_t^T f(s,t) e^{-\omega(s-t)} U(Z(s)) ds \right. \\ &\left. + F(T,t) e^{-\omega(T-t)} U(X(T)) \right].\end{aligned} \tag{21}$$

Now, we formulate the optimization problem in this paper as follows

$$\begin{cases} V(t,x,\mathbf{k}) = \sup\limits_{\boldsymbol{\psi}\in\mathcal{A}} \mathcal{J}(t,x,\mathbf{k};\boldsymbol{\psi}), \\ \text{subject to } X(t) \text{ and } \mathbf{k}(\mathbf{t}) \text{ satisfy (18) and (17).} \end{cases} \tag{22}$$

Here $V(t,x,\mathbf{k})$ is called the value function which maximizes the performance functional $\mathcal{J}(t,x,\mathbf{k};\boldsymbol{\psi})$ given by (21).

3. Solution to the optimization problem

In this section, we derive the solution to the optimization problem (22) by employing the dynamic programming principle approach.

3.1. HJB equation and verification theorem

We define the infinitesimal generator $\mathcal{L}^{\boldsymbol{\psi}}$ acting on a function $\varphi(t,x,\mathbf{k}) \in \mathcal{C}^{1,2,2,2,2}([0,T]\times\mathbb{R}^4)$ by

$$\begin{aligned} \mathcal{L}^{\boldsymbol{\psi}}[\varphi(t,x,\mathbf{k})] =& \varphi_t - (\lambda+\omega)\varphi + [rx + \boldsymbol{\pi}^T(\boldsymbol{\mu}-\mathbf{r}) + i - c - p]\varphi_x + \boldsymbol{\mu}^{\mathbf{k}T}\varphi_{\mathbf{k}} \\ &+ \frac{1}{2}\boldsymbol{\pi}^T\boldsymbol{\Sigma}\boldsymbol{\Sigma}^T\boldsymbol{\pi}\varphi_{xx} + \frac{1}{2}tr(\boldsymbol{\Sigma}^{\mathbf{k}}\boldsymbol{\Sigma}^{\mathbf{k}T}\varphi_{\mathbf{kk}}) + \boldsymbol{\pi}^T\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{\mathbf{k}T}\varphi_{x\mathbf{k}}. \end{aligned} \tag{23}$$

Here $tr(\cdot)$ is the trace operator, $\varphi_t, \varphi_x, \varphi_{\mathbf{k}}$ denote the first-order partial derivatives of φ with respect to $t, x, \mathbf{k}$, $\varphi_{xx}, \varphi_{x\mathbf{k}}, \varphi_{\mathbf{kk}}$ represent the second-order partial derivatives of φ with respect to x and $\mathbf{k}$, i.e.

$$\varphi_{\mathbf{k}} = \begin{bmatrix} \varphi_r \\ \varphi_y \\ \varphi_{\hat{z}} \end{bmatrix}, \varphi_{x\mathbf{k}} = \begin{bmatrix} \varphi_{xr} \\ \varphi_{xy} \\ \varphi_{x\hat{z}} \end{bmatrix}, \varphi_{\mathbf{kk}} = \begin{bmatrix} \varphi_{rr} & \varphi_{ry} & \varphi_{r\hat{z}} \\ \varphi_{ry} & \varphi_{yy} & \varphi_{y\hat{z}} \\ \varphi_{r\hat{z}} & \varphi_{y\hat{z}} & \varphi_{\hat{z}\hat{z}} \end{bmatrix}. \tag{24}$$

By the dynamic programming principle (refer to Yong and Zhou (1999)), it can be derived that the value function satisfies the following HJB equation

$$\begin{cases} \sup\limits_{\boldsymbol{\psi}\in\mathcal{A}} \left\{ \mathcal{L}^{\boldsymbol{\psi}}\left[V(t,x,\mathbf{k})\right] + \alpha U(c) + \beta\lambda(t) U\left(x + \dfrac{p}{\eta(t)}\right) \right\} = 0, \\ V(T,x,\mathbf{k}) = \dfrac{x^\gamma}{\gamma}, \end{cases} \tag{25}$$

where $\mathcal{L}^{\boldsymbol{\psi}}\left[V(t,x,\mathbf{k})\right]$ is defined by (23). Moreover, the verification theorem for the optimization problem (22) is presented as follows.

Theorem 1. *Suppose that there exists a function $V(t,x,\mathbf{k}) \in \mathcal{C}^{1,2,2,2,2}([0,T]\times\mathbb{R}^4)$ and an admissible control $\boldsymbol{\psi}^* = (\boldsymbol{\pi}^*, c^*, p^*) \in \mathcal{A}$ such that*

(1) $\mathcal{L}^{\boldsymbol{\psi}}\left[V(t,x,\mathbf{k})\right] + \alpha U(c) + \beta\lambda U\left(x + \dfrac{p}{\eta(t)}\right) \le 0$, *for all* $(\boldsymbol{\pi}, c, p) \in \mathcal{A}$, $t \in [0,T]$*;*

(2) $\mathcal{L}^{\boldsymbol{\psi}^*}\left[V(t,x,\mathbf{k})\right] + \alpha U(c^*) + \beta\lambda U\left(x + \dfrac{p^*}{\eta(t)}\right) = 0$, *for all* $t \in [0,T]$*;*

(3) $V(T,x,\mathbf{k}) = U(x)$*;*

(4) for fixed $t \in [0,T]$,

$$M^{\boldsymbol{\psi}^*}(s) \hat{=} \int_t^s e^{-w(u-t)} F(u,t) \left[V_x\left(u, X^{\boldsymbol{\psi}^*}(u), \mathbf{k}(u)\right) \boldsymbol{\pi}^T(u)\boldsymbol{\Sigma} + V_{\mathbf{k}}\big(u, X^{\boldsymbol{\psi}^*}(u), \mathbf{k}(u)\big)^T \boldsymbol{\Sigma}^{\mathbf{k}} \right] d\mathbf{W}(u)$$

is a martingale.

Then

$$V(t,x,\mathbf{k})=\sup_{\boldsymbol{\psi}\in\mathcal{A}}\mathcal{J}(t,x,\mathbf{k};\boldsymbol{\psi})=\mathcal{J}(t,x,\mathbf{k};\boldsymbol{\psi}^*),$$

and $\boldsymbol{\psi}^*=(\boldsymbol{\pi}^*,c^*,p^*)$ *is an optimal strategy.*
Proof *See Appendix B.*

3.2. The optimal strategies

In this subsection, we first derive the analytical expressions for the value function and the optimal strategy by solving the HJB equation (25). Then, we provide a verification result for the admissibility and optimality of the strategy derived.

Proposition 2. *The value function has the following form*

$$V(t,x,\mathbf{k})=\frac{(x+h(t,\mathbf{k}))^{\gamma}}{\gamma}f^{1-\gamma}(t,\mathbf{k}),\tag{26}$$

where $h(t,\mathbf{k})$ *and* $f(t,\mathbf{k})$ *are continuously differentiable with respect to* t *and* $\mathbf{k}$*, and satisfy the following two partial differential equations (PDEs) respectively:*

$$h_t-(r+\lambda)h+\left[\boldsymbol{\mu}^{\mathbf{k}}-\boldsymbol{\Sigma}^{\mathbf{k}}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}-\mathbf{r})\right]^T h_{\mathbf{k}}+\frac{1}{2}tr(\boldsymbol{\Sigma}^{\mathbf{k}}\boldsymbol{\Sigma}^{\mathbf{k}T}h_{\mathbf{kk}})+i=0,\tag{27}$$

and

$$\begin{aligned}&f_t+\left[\boldsymbol{\mu}^{\mathbf{k}}+\frac{\gamma}{1-\gamma}\boldsymbol{\Sigma}^{\mathbf{k}}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}-\mathbf{r})\right]^T f_{\mathbf{k}}+\frac{1}{2}tr(\boldsymbol{\Sigma}^{\mathbf{k}}\boldsymbol{\Sigma}^{\mathbf{k}T}f_{\mathbf{kk}})\\&+\left[\frac{\gamma}{1-\gamma}r-\frac{1}{1-\gamma}\omega-\lambda+\frac{\gamma}{2(1-\gamma)^2}(\boldsymbol{\mu}-\mathbf{r})^T\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}^T\right)^{-1}(\boldsymbol{\mu}-\mathbf{r})\right]f\\&+\alpha^{\frac{1}{1-\gamma}}+\beta^{\frac{1}{1-\gamma}}\lambda=0,\end{aligned}\tag{28}$$

with the terminal conditions $h(T,\mathbf{k})=0$ *and* $f(T,\mathbf{k})=1$*. The optimal investment-consumption-insurance strategy* $(\boldsymbol{\pi}^*,c^*,p^*)$*, for all* $t\in[0,T]$*, is given by*

$$\boldsymbol{\pi}^*(t)=\frac{x+h(t,\mathbf{k})}{1-\gamma}\left[\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}^T\right)^{-1}(\boldsymbol{\mu}-\mathbf{r})+(1-\gamma)\left(\boldsymbol{\Sigma}^T\right)^{-1}\boldsymbol{\Sigma}^{\mathbf{k}T}\frac{f_{\mathbf{k}}(t,\mathbf{k})}{f(t,\mathbf{k})}\right]-(\boldsymbol{\Sigma}^T)^{-1}\boldsymbol{\Sigma}^{\mathbf{k}T}h_{\mathbf{k}}(t,\mathbf{k}),\tag{29}$$

$$c^*(t)=\alpha^{\frac{1}{1-\gamma}}\frac{x+h(t,\mathbf{k})}{f(t,\mathbf{k})},\tag{30}$$

$$p^*(t)=\lambda(t)\left[\beta^{\frac{1}{1-\gamma}}\frac{x+h(t,\mathbf{k})}{f(t,\mathbf{k})}-x\right].\tag{31}$$

Proof *See Appendix C.*

Define probability measures $\mathbb{Q}$ and $\widetilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ on $\mathcal{F}_T$ with the Radon-Nikodym derivatives being given by

$$\left.\frac{d\mathbb{Q}}{d\mathbb{P}}\right|_{\mathcal{F}_T}=\exp\left\{-\frac{1}{2}\int_0^T|\boldsymbol{\theta}(t)|^2dt-\int_0^T\boldsymbol{\theta}(t)^T dW(t)\right\},\tag{32}$$

and

$$\left.\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}\right|_{\mathcal{F}_T} = \exp\left\{-\frac{\gamma^2}{2(1-\gamma)^2}\int_0^T |\boldsymbol{\theta}(t)|^2 dt + \frac{\gamma}{1-\gamma}\int_0^T \boldsymbol{\theta}(t)^T dW(t)\right\}, \tag{33}$$

where $\boldsymbol{\theta} = (\theta_1, \theta_2)^T = \boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{r})$ is the Sharpe ratio. Let $\mathbb{E}^{\mathbb{Q}}_{t,\mathbf{k}}[\cdot]$ and $\widetilde{\mathbb{E}}_{t,\mathbf{k}}[\cdot]$ represent the expectations under $\mathbb{Q}$ and $\widetilde{\mathbb{P}}$, respectively. By using the Feynman-Kac formulas for the solutions to PDEs (27) and (28), the functions $h(t,\mathbf{k})$ and $f(t,\mathbf{k})$ admit the following two expectation representations:

$$h(t,\mathbf{k}) = \mathbb{E}^{\mathbb{Q}}_{t,\mathbf{k}}\left\{\int_t^T i(s)\exp\left[-\int_t^s (\lambda(u) + r(u))du\right]ds\right\}, \tag{34}$$

and

$$f(t,\mathbf{k}) = \widetilde{\mathbb{E}}_{t,\mathbf{k}}\left[\Gamma(t,T) + \int_t^T K(s)\cdot\Gamma(t,s)ds\right], \tag{35}$$

for $t \in [0,T]$, where

$$K(s) = \alpha^{\frac{1}{1-\gamma}} + \beta^{\frac{1}{1-\gamma}}\lambda(s), \tag{36}$$

and

$$\Gamma(t,s) = \exp\left\{\int_t^s \left[\frac{\gamma}{2(1-\gamma)^2}|\boldsymbol{\theta}(u)|^2 + \frac{\gamma}{1-\gamma}r(u) - \frac{\omega}{1-\gamma} - \lambda(u)\right]du\right\}. \tag{37}$$

As mentioned in Zvi Bodie and Samuelson (1992), Munk and Sørensen (2010) and Pirvu and Zhang (2012), $h(t,\mathbf{k})$ in (34) can be interpreted as the human captital, which is the actuarial present value of the future income and incorporates the mortality risk and the interest risk. When the wage earner receives a spanned income with no investment constraints, she can replicate it by dynamic trading strategies of the traded assets. Therefore, we can suppose that the wage earner does not receive the income, however, she has an initial wealth $x + h(t,\mathbf{k})$ instead of just having an initial wealth x.

$f(t,\mathbf{k})$ can be interpreted as the capital consumption ratio, which is affected by the predictors $y(t)$ and $z(t)$. From (31), the optimal life insurance purchase strategy can be rewritten as

$$p^*(t) = \lambda(t)\left[\left(\frac{\beta^{\frac{1}{1-\gamma}}}{f(t,\mathbf{k})} - 1\right)x + \frac{\beta^{\frac{1}{1-\gamma}}}{f(t,\mathbf{k})}h(t,\mathbf{k})\right]. \tag{38}$$

When $0 < \frac{\beta^{\frac{1}{1-\gamma}}}{f(t,\mathbf{k})} < 1$, similar to the insurance principle pointed out in Pliska and Ye (2007) and Shen and Sherris (2018), the current wealth of the wage earner has a negative effect on the life insurance purchase strategy $p^*(t)$, while the human capital $h(t,\mathbf{k})$ has a positive effect on $p^*(t)$. Meanwhile, a greater capital consumption ratio $f(t,\mathbf{k})$ leads to a smaller life insurance purchase strategy. The impacts of the predictive powers ϕ_y and ϕ_z on $p^*(t)$ cannot be observed directly from the expression (38). The specific numerical analysis about the impacts of ϕ_y and ϕ_z on the life insurance purchase strategy will be presented in Section 5.

According to the particular structures of the processes for stochastic factors, we can derive the explicit expressions for the two expectations (34) and (35) in the following two propositions.

Proposition 3. *The analytical expression for $h(t,\mathbf{k})$ is given by*

$$h(t,\mathbf{k}) = \int_t^T i(s)e^{g(t,s)+a(s-t)}B(t,s)ds, \quad t \in [0,T], \tag{39}$$

where

$$g(t,s) = -\frac{\chi}{\kappa_r}(s-t) + \frac{\chi}{\kappa_r}b(s-t) + \int_t^s \left(\frac{1}{2}\sigma_r^2 b^2(s-u) - \lambda(u)\right)du, \tag{40}$$

with the functions $B(t,s)$, $a(s-t)$ and $b(s-t)$ being given by (7) and (8).
Proof *See Appendix D.*

Proposition 4. *The closed-form expression for $f(t,\mathbf{k})$ is given by*

$$f(t,\mathbf{k}) = e^{-\int_t^T \left(\frac{1}{1-\gamma}\omega + \lambda(u)\right)du}\widetilde{f}(t,\mathbf{k}) + \int_t^T K(s)e^{-\int_t^s \left(\frac{1}{1-\gamma}\omega+\lambda(u)\right)du}\widetilde{f}(s,\mathbf{k})ds, \tag{41}$$

where

$$\widetilde{f}(t,\mathbf{k}) = \widetilde{\mathbb{E}}_{t,\mathbf{k}}\left\{\exp\left[\int_t^T \left(\frac{\gamma}{2(1-\gamma)^2}(\boldsymbol{\mu}-\mathbf{r})^T\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}^T\right)^{-1}(\boldsymbol{\mu}-\mathbf{r}) + \frac{\gamma}{1-\gamma}r(u)\right)du\right]\right\}. \tag{42}$$

Furthermore, $\widetilde{f}(t,\mathbf{k})$ can be rewritten as the exponential affine form:

$$\widetilde{f}(t,\mathbf{k}) = \exp\left(a(t) + \mathbf{b}^T(t)\mathbf{k} + \frac{1}{2}\mathbf{k}^T\mathbf{Q}(t)\mathbf{k}\right), \tag{43}$$

with the boundary condition $\widetilde{f}(T,\mathbf{k}) = 1$. The real-valued function $a(t)$, the vector-valued function $\mathbf{b}(t) = (b_1(t), b_2(t), b_3(t))^T$ and the matrix-valued function $\mathbf{Q}(t) = (q_{ij}(t))_{i,j=1,2,3}$ satisfy the following system of equations

$$\begin{cases} \dfrac{da(t)}{dt} + \left(\boldsymbol{\delta}_0 + \dfrac{\gamma}{1-\gamma}\mathbf{g}_0\right)^T\mathbf{b}(t) + \dfrac{1}{2}\mathbf{b}^T(t)\mathbf{l}_0\mathbf{b}(t) + \dfrac{1}{2}tr(\mathbf{l}_0\mathbf{Q}(t)) + \dfrac{\gamma}{2(1-\gamma)^2}h_0 = 0, \\ \dfrac{d\mathbf{b}(t)}{dt} + \left(-\boldsymbol{\delta}_1 + \dfrac{\gamma}{1-\gamma}\mathbf{g}_1\right)^T\mathbf{b}(t) + \mathbf{Q}(t)\mathbf{l}_0\mathbf{b}(t) + \mathbf{Q}(t)\left(\boldsymbol{\delta}_0 + \dfrac{\gamma}{1-\gamma}\mathbf{g}_0\right) + \dfrac{\gamma}{2(1-\gamma)^2}\mathbf{h}_1^T \\ \qquad + \dfrac{\gamma}{1-\gamma}\mathbf{d}^T = \mathbf{0}_{3\times1}, \\ \dfrac{d\mathbf{Q}(t)}{dt} + \left(-\boldsymbol{\delta}_1{}^T + \dfrac{\gamma}{1-\gamma}\mathbf{g}_1{}^T\right)\mathbf{Q}(t) + \mathbf{Q}(t)\left(-\boldsymbol{\delta}_1 + \dfrac{\gamma}{1-\gamma}\mathbf{g}_1\right) + \mathbf{Q}(t)\mathbf{l}_0\mathbf{Q}(t) + \dfrac{\gamma}{2(1-\gamma)^2}\mathbf{h}_2 = \mathbf{0}_{3\times3}, \end{cases} \tag{44}$$

with terminal conditions $a(T) = 0, \mathbf{b}(T) = \mathbf{0}_{3\times1}$ and $\mathbf{Q}(T) = \mathbf{0}_{3\times3}$. Here $\boldsymbol{\delta}_0, \boldsymbol{\delta}_1, \mathbf{g}_0, \mathbf{g}_1, \mathbf{l}_0, h_0, \mathbf{h}_1, \mathbf{h}_2, \mathbf{d}$ are given in Appendix E.
Proof *See Appendix E.*

We now at the position to verify the admissibility and optimality of $\boldsymbol{\psi}^* = (\boldsymbol{\pi}^*, c^*, p^*)$ given by (29)-(31).

Proposition 5. *The strategy $\boldsymbol{\psi}^* = (\boldsymbol{\pi}^*, c^*, p^*)$ given by (29)-(31) is admissible and is the optimal strategy to problem (22).*

Proof. From (35)-(37), we can deduce that

$$\begin{aligned} f(t,\mathbf{k}) &\geq c_0 \left[e^{-\int_t^T \lambda(u)du} + \int_t^T \left(\alpha^{\frac{1}{1-\gamma}} + \beta^{\frac{1}{1-\gamma}} \lambda(s) \right) e^{-\int_t^s \lambda(u)du} ds \right] \\ &\geq c_0 \left[P\left(\tau \geq T \,|\tau > t\right) + \beta^{\frac{1}{1-\gamma}} P\left(\tau \leq T \,|\tau > t\right) \right] \\ &\geq c_0 \min(1, \beta^{\frac{1}{1-\gamma}}), \end{aligned}$$

where c_0 is a positive constant depending only on T and w. Therefore, f is uniformly bounded above zero. For $t \in [0,T]$, let $Y(t) = X^{\psi^*}(t) + h(t,\mathbf{k}(t))$, with X^{ψ^*} being the wealth process associated with $\psi^* = (\pi^*, c^*, p^*)$. By (18), (27) and the Itô formula, we can derive that

$$dY(t) = Y(t)\left[g_1(t,\mathbf{k}(t))dt + g_2(t,\mathbf{k}(t))d\mathbf{W}(t)\right], \tag{45}$$

where

$$g_1(t,\mathbf{k}(t)) = r(t) + \lambda(t) + \frac{|\boldsymbol{\theta}(t)|^2}{1-\gamma} + \boldsymbol{\theta}^T(t)\boldsymbol{\Sigma}^{\mathbf{k}T}\frac{f_{\mathbf{k}}(t,\mathbf{k}(t))}{f(t,\mathbf{k}(t))} - \frac{\alpha^{\frac{1}{1-\gamma}} + \beta^{\frac{1}{1-\gamma}}\lambda(t)}{f(t,\mathbf{k}(t))},$$

$$g_2(t,\mathbf{k}(t)) = \frac{\boldsymbol{\theta}^T(t)}{1-\gamma} + \frac{f_{\mathbf{k}}^T(t,\mathbf{k}(t))}{f(t,\mathbf{k}(t))}\boldsymbol{\Sigma}^{\mathbf{k}}.$$

From (41) and (43), we can see that $f(t,\mathbf{k})$ is continuously differentiable. Moreover, since the process $\mathbf{k}(t)$ is continuous and $f(t,\mathbf{k})$ is uniformly bounded above zero, the processes $g_1(t,\mathbf{k}(t))$ and $g_2(t,\mathbf{k}(t))$ are continuous. Then $\int_0^T |g_1(t,\mathbf{k}(t))|dt < \infty$, $\int_0^T |g_2(t,\mathbf{k}(t))|^2 dt < \infty$ a.s.. Notice that $Y(0) = x_0 + h(0,\mathbf{k}(0)) > x_0 > 0$. Therefore, the unique solution $Y(t)$ to (45) satisfies

$$Y(t) = Y(0)\exp\left\{\int_0^t g_1(s,\mathbf{k}(s))ds + \int_0^t g_2(s,\mathbf{k}(s))d\mathbf{W}(s) - \frac{1}{2}\int_0^t |g_2(s,\mathbf{k}(s))|^2 ds\right\} \geq 0 \quad a.s.$$

Furthermore, from (30) and (31), it can be derived that both $c^*(t)$ and $X^{\psi^*}(t) + \dfrac{p(t)}{\eta(t)}$ are nonnegative. So $\psi^* = (\pi^*, c^*, p^*)$ satisfies condition (3) of Definition 1.

Due to the continuity of the processes $\mathbf{k}(t)$ and $Y(t)$, the optimal strategy ψ^* is also continuous. Consequently, ψ^* satisfies conditions (1) and (2) of Definition 1 and it is admissible.

We turn to verify the optimality of the strategy ψ^*. By Theorem 1, it remains to verify that

$$M^{\psi^*}(s) = \int_t^s e^{-w(u-t)}F(u,t)\left[V_x\left(u, X^{\psi^*}(u), \mathbf{k}(u)\right)\boldsymbol{\pi}^T(u)\boldsymbol{\Sigma} + V_{\mathbf{k}}\big(u, X^{\psi^*}(u), \mathbf{k}(u)\big)^T\boldsymbol{\Sigma}^{\mathbf{k}}\right]d\mathbf{W}(u)$$

is a martingale. From (B.9), (26) and (29), we obtain

$$\begin{aligned} dD^{\psi^*}(s) =& dM^{\psi^*}(s) = e^{-w(s-t)}F(s,t)\left[V_x\left(s, X^{\psi^*}(s), \mathbf{k}(s)\right)\boldsymbol{\pi}^T(s)\boldsymbol{\Sigma} + V_{\mathbf{k}}\big(s, X^{\psi^*}(s), \mathbf{k}(s)\big)^T\boldsymbol{\Sigma}^{\mathbf{k}}\right]d\mathbf{W}(s) \\ =& M^{\psi^*}(s)E(s)d\mathbf{W}(s), \end{aligned}$$

where

$$E(s) = \frac{e^{-w(s-t)}F(s,t)V\left(s, X^{\psi^*}(s), \mathbf{k}(s)\right)}{D^{\psi^*}(s)}\left[\frac{\gamma}{1-\gamma}\boldsymbol{\theta}^T(s) + \frac{f_{\mathbf{k}}^T(s,\mathbf{k}(s))}{f(s,\mathbf{k}(s))}\boldsymbol{\Sigma}^{\mathbf{k}}\right].$$

From the discussions in Appendix E, $\mathbf{Q}(t)$ is bounded in the matrix norm uniformly in t (see

(E.7)). Then, there exists a constant $c_1 > 0$ such that

$$\sup_{t\in[0,T]} |\mathbf{b}(t) + \mathbf{Q}(t)\mathbf{k}| \le c_1(1 + |\mathbf{k}|),$$

for $\mathbf{k} \in \mathbb{R}^3$. Therefore,

$$\begin{aligned}
|f_{\mathbf{k}}(t,\mathbf{k})| =& \Big| e^{-\int_t^T \left(\frac{w}{1-\gamma}+\lambda(u)\right)du} \tilde{f}(t,\mathbf{k}) \left(\mathbf{b}(t) + \mathbf{Q}(t)\mathbf{k}\right) \\
&+ \int_t^T \mathbf{k}(s) e^{-\int_t^s \left(\frac{w}{1-\gamma}+\lambda(u)\right)du} \tilde{f}(s,\mathbf{k}) \left(\mathbf{b}(s) + \mathbf{Q}(s)\mathbf{k}\right) ds \Big| \\
\le& e^{-\int_t^T \left(\frac{w}{1-\gamma}+\lambda(u)\right)du} \tilde{f}(t,\mathbf{k}) \left|\mathbf{b}(t) + \mathbf{Q}(t)\mathbf{k}\right| \\
&+ \int_t^T \mathbf{k}(s) e^{-\int_t^s \left(\frac{w}{1-\gamma}+\lambda(u)\right)du} \tilde{f}(s,\mathbf{k}) \left|\mathbf{b}(s) + \mathbf{Q}(s)\mathbf{k}\right| ds \\
\le& c_1 \left(1 + |\mathbf{k}|\right) f(t,\mathbf{k}),
\end{aligned}$$

for $t \in [0,T]$. Moreover, it is evident that

$$\frac{e^{-w(s-t)} F(s,t) V\left(s, X^{\boldsymbol{\psi}^*}(s), \mathbf{k}(s)\right)}{D^{\boldsymbol{\psi}^*}(s)} \le 1.$$

Hence, we have $|E(s)| \le c_2(1 + |\mathbf{k}(s)|)$ for some constant $c_2 > 0$, $s \in [0,T]$. Notice that the components of $\mathbf{k}(s)$ are Gaussian processes. Then for some small enough $\delta > 0$, we have

$$\sup_{0\le s\le T} \mathbb{E}\left(e^{\delta|E(s)|^2}\right) \le \sup_{0\le s\le T} \mathbb{E}\left(e^{2\delta c_2^2\left(1+|\mathbf{k}(s)|^2\right)}\right) < \infty.$$

By Corollary 12.1 in Baldi (2017), $D^{\boldsymbol{\psi}^*}(s)$ is a martingale, and hence $M^{\boldsymbol{\psi}^*}$ is a martingale. □

In fact, it is owing to the completeness of the financial market that we can derive the explicit expressions for the value function and the optimal strategies. If the market is incomplete, the explicit expressions can be obtained only if the utility for the wage earner is defined over the terminal wealth. For detailed discussion on this issue, one can refer to Liu (2007), which studies utility maximization problem without life insurance purchase.

When α=β=0, the utility is defined over the terminal wealth and $K(s) \equiv 0$. Then (41) reduces to

$$f(t,\mathbf{k}) = \exp\left\{-\int_t^T \left(\frac{1}{1-\gamma}\omega + \lambda(u)\right) du\right\} \tilde{f}(t,\mathbf{k}). \tag{46}$$

From (29) and (43), we have the following result.

Corollary 1. *The optimal investment strategy $\boldsymbol{\pi}_1^*(t)$ in the case α=β=0 is given by*

$$\begin{aligned}
\boldsymbol{\pi}_1^*(t) =& \frac{x + h(t,\mathbf{k})}{1-\gamma} (\boldsymbol{\Sigma}\boldsymbol{\Sigma}^T)^{-1}(\boldsymbol{\mu} - \mathbf{r}) + (x + h(t,\mathbf{k}))\,(\boldsymbol{\Sigma}^T)^{-1}\boldsymbol{\Sigma}^{\mathbf{k}T}(\mathbf{b}(t) + \mathbf{Q}(t)\mathbf{k}) - (\boldsymbol{\Sigma}^T)^{-1}\boldsymbol{\Sigma}^{\mathbf{k}T} h_{\mathbf{k}}(t,\mathbf{k}) \\
=& \frac{x + h(t,\mathbf{k})}{1-\gamma} \begin{bmatrix} \frac{1}{\sigma_S^2(1-\rho_{Sr}^2)} & -\frac{\rho_{Sr}}{\sigma_S\sigma_B(1-\rho_{Sr}^2)} \\ -\frac{\rho_{Sr}}{\sigma_S\sigma_B(1-\rho_{Sr}^2)} & \frac{1}{\sigma_B^2(1-\rho_{Sr}^2)} \end{bmatrix} \begin{bmatrix} \sigma_S(\phi + \phi_y y(t) + \phi_z \hat{z}(t)) \\ \sigma_B q_r \end{bmatrix}
\end{aligned} \tag{47}$$

Journal Pre-proof

$$
+\begin{bmatrix} 0 & 0 & \dfrac{H_1}{\sigma_S}-\dfrac{H_2\rho_{Sr}}{\sigma_S\hat{\rho}_r} \\ -\dfrac{\sigma_r}{\sigma_B} & \dfrac{\sigma_y}{\sigma_B} & \dfrac{H_2}{\sigma_B\hat{\rho}_r} \end{bmatrix}\left[(x+h(t,\mathbf{k}))\left(\mathbf{b}(t)+\mathbf{Q}(t)\mathbf{k}\right)-h_{\mathbf{k}}(t,\mathbf{k})\right].
$$

Remark 2. *The explicit expression for $\pi^*_{1S}(t)$ can be decomposed as*

$$
\begin{aligned}
\pi^*_{1S}(t) &= \pi^*_{1S-spec}(t)+\pi^*_{1S-unobs}(t)\\
&= \underbrace{\frac{x+h(t,\mathbf{k})}{(1-\gamma)(1-\rho^2_{Sr})}\left(\frac{\phi+\phi_y y(t)+\phi_z\hat{z}(t)}{\sigma_S}-\frac{q_r\rho_{Sr}}{\sigma_S}\right)}_{\pi^*_{1S-spec}(t)}\\
&+\underbrace{(x+h(t,\mathbf{k}))\left(\frac{\sigma_z(\rho_{Sz}-\rho_{Sr}\rho_{rz})+m\phi_z}{\sigma_S(1-\rho^2_{Sr})}\right)(b_3(t)\ +q_{31}(t)r(t)+q_{32}(t)y(t)+q_{33}\hat{z}(t))}_{\pi^*_{1S-unobs}(t)}.
\end{aligned}
\tag{48}
$$

The optimal stock investment strategy $\pi^*_{1S}(t)$ is divided into two terms. The first term $\pi^*_{1S-spec}(t)$ is the speculative demand. It increases with the market price of the stock risk $\phi+\phi_y y(t)+\phi_z\hat{z}(t)$ and decreases with the market price of the interest rate risk q_r if the stock price and the interest rate are positively correlated, that is, $\rho_{Sr}>0$. The second term $\pi^*_{1S-unobs}(t)$ hedges against adverse changes in the unobservable factor $z(t)$ and this term will not disappear for the unobservability of $z(t)$ $(m\neq 0)$ even if it is deterministic, i.e. $\sigma_z=0$.

Remark 3. *The explicit expression for $\pi^*_{1B}(t)$ can be rewritten as*

$$
\begin{aligned}
\pi^*_{1B}(t) &= \pi^*_{1B-spec}(t)+\pi^*_{1B-r}(t)+\pi^*_{1B-obs}(t)+\pi^*_{1B-unobs}(t)\\
&= \underbrace{\frac{x+h(t,\mathbf{k})}{(1-\gamma)(1-\rho^2_{Sr})}\left(\frac{q_r}{\sigma_B}-\frac{\rho_{Sr}\left(\phi+\phi_y y(t)+\phi_z\hat{z}(t)\right)}{\sigma_B}\right)}_{\pi^*_{1B-spec}(t)}\\
&\quad \underbrace{-(x+h(t,\mathbf{k}))\frac{\sigma_r}{\sigma_B}\left(b_1(t)+q_{11}(t)r(t)+q_{12}(t)y(t)+q_{13}\hat{z}(t)\right)+\frac{\sigma_r}{\sigma_B}h_r(t)}_{\pi^*_{1B-r}(t)}\\
&\quad +\underbrace{(x+h(t,\mathbf{k}))\frac{\sigma_y}{\sigma_B}\left(b_2(t)+q_{21}(t)r(t)+q_{22}(t)y(t)+q_{23}\hat{z}(t)\right)}_{\pi^*_{1B-obs}(t)}\\
&\quad +\underbrace{(x+h(t,\mathbf{k}))\frac{\sigma_z(\rho_{rz}-\rho_{Sr}\rho_{Sz})-m\rho_{Sr}\phi_z}{\sigma_B(1-\rho^2_{Sr})}\left(b_3(t)+q_{31}(t)r(t)+q_{32}(t)y(t)+q_{33}\hat{z}(t)\right)}_{\pi^*_{1B-unobs}(t)},
\end{aligned}
\tag{49}
$$

where $h_r(t)$ represents the interest rate sensitivity of the human capital:

$$
h_r(t)=\int_t^T i(s)\left[-e^{g(t,s)+a(s-t)}b(s-t)B(t,s)\right]ds,\quad t\in[0,T].
\tag{50}
$$

The optimal bond investment strategy $\pi^*_{1B}(t)$ is divided into five components. The speculative component $\pi^*_{1B-spec}(t)$ decreases with the market price of the stock risk $\phi+\phi_y y(t)+\phi_z\hat{z}(t)$ and increases with the market price of the interest rate risk q_r if $\rho_{Sr}>0$. $\pi^*_{1B-r}(t)$ and $\pi^*_{1B-obs}(t)$ are *the components that hedge against the risk of the interest rate and the risk of the observed variable $y(t)$, respectively. In general, both $\pi^*_{1B-r}(t)$ and $\pi^*_{1B-obs}(t)$ vanish when $r(t)$ and $y(t)$ degenerate to deterministic functions, i.e., $\sigma_r = 0$ and $\sigma_y = 0$. $\pi^*_{1B-unobs}(t)$ is the component that hedges against the risk of the unobserved variable $z(t)$, which disappears when $\rho_{rz} = \rho_{Sr}\rho_{Sz}$ or there is no correlation between the stock and the interest rate ($\rho_{Sr} = 0$).*

4. Suboptimal strategies

In this section, we investigate two kinds of suboptimal strategies which lead to utility losses. These two suboptimal strategies respectively correspond to the situations that the wage earner ignores learning about the unobservable factor $z(t)$ or the randomness of the interest rate. Then, the wage earner adopts the admissible strategies $\widetilde{\boldsymbol{\psi}}^{(i)}(i = 1, 2)$ which are optimal for the cases of ignoring learning about $z(t)$ or the randomness of the interest rate in the optimization problem (22). Compared to the optimal strategy $\boldsymbol{\psi}^*$ generating the expected utility $V(t, x, \mathbf{k}) = \sup_{\boldsymbol{\psi}\in\mathcal{A}} \mathcal{J}(t, x, \mathbf{k}; \boldsymbol{\psi})$, the admissible strategies $\widetilde{\boldsymbol{\psi}}^{(i)}(i = 1, 2)$ generate lower expected utilities $\widetilde{V}^{(i)}(t, x, \mathbf{k}) \hat{=} \mathcal{J}(t, x, \mathbf{k}; \widetilde{\boldsymbol{\psi}}^{(i)})$ $(i = 1, 2)$. Then $\widetilde{\boldsymbol{\psi}}^{(i)}(i = 1, 2)$ are called the suboptimal strategies of the optimization problem (22).

To measure the utility losses arising from adopting suboptimal strategies. We have to derive the expected utilities $\widetilde{V}^{(i)}(t, x, \mathbf{k})(i = 1, 2)$ associated with suboptimal strategies $\widetilde{\boldsymbol{\psi}}^{(i)}(i = 1, 2)$. This can be achieved by solving the partial differential equation (25) without the supremum over $\boldsymbol{\psi} \in \mathcal{A}$. To facilitate the comparison analysis, we only derive the suboptimal strategies and the associated expected utilities under the situation that α=β=0.

4.1. Ignoring learning

Ignoring learning means that the wage earner ignores the fact that she can learn about the unobservable predictor from realized asset price. Instead of using the filtered estimate $\hat{z}(t)$, she chooses the long-run average level $\bar{z}$ to predict the expected return rate of the stock. Under the assumption that the wage earner ignores learning, the dynamic processes of the traded assets and the stochastic factors evolve as

$$\begin{bmatrix} \dfrac{dS(t)}{S(t)} \\ \dfrac{dB_I(t)}{B_I(t)} \end{bmatrix} = \underbrace{\begin{bmatrix} r(t) + \sigma_S[\phi + \phi_y y(t) + \phi_z \bar{z}] \\ r(t) + \sigma_B q_r \end{bmatrix}}_{\widetilde{\boldsymbol{\mu}}^{(1)}} dt + \underbrace{\begin{bmatrix} \sigma_S & 0 \\ \sigma_B\rho_{Sr} & \sigma_B\hat{\rho}_r \end{bmatrix}}_{\boldsymbol{\Sigma}} \underbrace{\begin{bmatrix} dW_1(t) \\ dW_2(t) \end{bmatrix}}_{d\mathbf{W}(t)}, \tag{51}$$

and

$$\underbrace{\begin{bmatrix} dr(t) \\ dy(t) \\ d\bar{z}(t) \end{bmatrix}}_{d\mathbf{k}(t)} = \underbrace{\begin{bmatrix} \kappa_r(\bar{r} - r(t)) \\ \kappa_y(\bar{y} - y(t)) \\ 0 \end{bmatrix}}_{\widetilde{\boldsymbol{\mu}}^{\mathbf{k}(1)}} dt + \underbrace{\begin{bmatrix} -\sigma_r\rho_{Sr} & -\sigma_r\hat{\rho}_r \\ \sigma_y\rho_{Sr} & \sigma_y\hat{\rho}_r \\ 0 & 0 \end{bmatrix}}_{\widetilde{\boldsymbol{\Sigma}}^{\mathbf{k}(1)}} \underbrace{\begin{bmatrix} dW_1(t) \\ dW_2(t) \end{bmatrix}}_{d\mathbf{W}(t)}. \tag{52}$$

Here $\bar{z}(t) \equiv \bar{z}$ is nonrandom in fact. From the results of optimal strategy $\boldsymbol{\psi}^*$ in Proposition 2, we can derive that the suboptimal strategy $\widetilde{\boldsymbol{\psi}}^{(1)} = (\widetilde{\boldsymbol{\pi}}^{(1)}, \widetilde{c}^{(1)}, \widetilde{p}^{(1)})$ is given by

$$\begin{cases} \widetilde{\boldsymbol{\pi}}^{(1)}(t) = \dfrac{x + h^{(1)}(t,\mathbf{k})}{1-\gamma} \left[(\boldsymbol{\Sigma}\boldsymbol{\Sigma}^T)^{-1} (\widetilde{\boldsymbol{\mu}}^{(1)} - \mathbf{r}) + (1-\gamma)(\boldsymbol{\Sigma}^T)^{-1} \widetilde{\boldsymbol{\Sigma}}^{\mathbf{k}(1)T} \left(\bar{\mathbf{b}}^{(1)}(t) + \bar{\mathbf{Q}}^{(1)}(t)\mathbf{k} \right) \right] \\ \qquad - (\boldsymbol{\Sigma}^T)^{-1} \widetilde{\boldsymbol{\Sigma}}^{\mathbf{k}(1)T} h^{(1)}_{\mathbf{k}}(t,\mathbf{k}), \\ \widetilde{c}^{(1)}(t) = \alpha^{\frac{1}{1-\gamma}} \dfrac{x + h^{(1)}(t,\mathbf{k})}{\bar{f}^{(1)}(t,\mathbf{k})}, \\ \widetilde{p}^{(1)}(t) = \lambda(t) \left[\beta^{\frac{1}{1-\gamma}} \dfrac{x + h^{(1)}(t,\mathbf{k})}{\bar{f}^{(1)}(t,\mathbf{k})} - x \right], \end{cases} \tag{53}$$

where the functions $\bar{f}^{(1)}(t,\mathbf{k})$, $\bar{\mathbf{b}}^{(1)}(t)$ and $\bar{\mathbf{Q}}^{(1)}(t)$ are derived by solving (44) with parameters $\widetilde{\boldsymbol{\delta}}_0, \widetilde{\boldsymbol{\delta}}_1, \widetilde{\mathbf{l}}_0, \mathbf{d}, \widetilde{h}_0, \widetilde{\mathbf{h}}_1, \widetilde{\mathbf{h}}_2, \widetilde{\mathbf{g}}_0, \widetilde{\mathbf{g}}_1$ being given in Appendix F.

Proposition 6. *The expected utility $\widetilde{V}^{(1)}(t,x,\mathbf{k})$ corresponding to $\widetilde{\boldsymbol{\psi}}^{(1)}$ is given by*

$$\widetilde{V}^{(1)}(t,x,\mathbf{k}) = \frac{\left(x + h^{(1)}(t,\mathbf{k})\right)^{\gamma}}{\gamma} \left(f^{(1)}(t,\mathbf{k})\right)^{1-\gamma}. \tag{54}$$

Here $h^{(1)}(t,\mathbf{k})$ is the human capital given by (32) and (34), and $f^{(1)}(t,\mathbf{k})$ can be written as

$$f^{(1)}(t,\mathbf{k}) = e^{-\int_t^T \left(\frac{1}{1-\gamma}\omega + \lambda(u)\right)du} \widetilde{f}^{(1)}(t,\mathbf{k}), \tag{55}$$

where $\widetilde{f}^{(1)}(t,\mathbf{k})$ has the form

$$\widetilde{f}^{(1)}(t,\mathbf{k}) = \exp\left(\widetilde{a}^{(1)}(t) + \widetilde{\mathbf{b}}^{(1)T}(t)\mathbf{k} + \frac{1}{2}\mathbf{k}^T \widetilde{\mathbf{Q}}^{(1)}(t)\mathbf{k} \right), \tag{56}$$

with the functions $\widetilde{a}^{(1)}(t)$, $\widetilde{\mathbf{b}}^{(1)}(t)$, $\widetilde{\mathbf{Q}}^{(1)}(t)$ being the solutions to (F.3) with $i = 1$ in Appendix F.

Proof See Appendix F.

4.2. Ignoring stochastic interest rate

We now consider that the wage earner ignores the randomness of the interest rate and use the long-run value $\bar{r}$ instead of the stochastic interest rate $r(t)$. Under this assumption, the wage earner does not trade in the zero-coupon bond and replace the observed factor $y(t)$ with the long-run average level $\bar{y}$. The dynamics of the stock price process and the filtered estimate $\hat{z}(t)$ are

$$\frac{dS(t)}{S(t)} = \underbrace{\left[\bar{r} + \sigma_S\left(\phi + \phi_y \bar{y} + \phi_z \hat{z}(t)\right)\right]}_{\widetilde{\mu}^{(2)}} dt + \underbrace{\sigma_S}_{\widetilde{\Sigma}^{(2)}} dW_1(t) \tag{57}$$

and

$$d\hat{z}(t) = \underbrace{\kappa_z(\bar{z} - \hat{z}(t))}_{\widetilde{\mu}^{\hat{z}(2)}} dt + \underbrace{\widetilde{H}_1}_{\widetilde{\Sigma}^{\hat{z}(2)}} dW_1(t), \tag{58}$$

where $\widetilde{H}_1 = \sigma_z \rho_{Sz} + \bar{m}\phi_z$.

The suboptimal strategy $\widetilde{\boldsymbol{\psi}}^{(2)} = (\widetilde{\pi}_S^{(2)}, \widetilde{c}^{(2)}, \widetilde{p}^{(2)})$ with the constant interest rate is given by

$$\begin{cases} \widetilde{\pi}_S^{(2)}(t) = \dfrac{x + h^{(2)}(t,\mathbf{k})}{\sigma_S} \left[\dfrac{\phi + \phi_y \bar{y} + \phi_z \hat{z}(t)}{1-\gamma} + \widetilde{\Sigma}^{\hat{z}(2)} \left(\bar{b}^{(2)}(t) + \bar{Q}^{(2)}(t)\hat{z}(t) \right) \right], \\ \widetilde{c}^{(2)}(t) = \alpha^{\frac{1}{1-\gamma}} \dfrac{x + h^{(2)}(t,\mathbf{k})}{\bar{f}^{(2)}(t,\mathbf{k})}, \\ \widetilde{p}^{(2)}(t) = \lambda(t) \left(\beta^{\frac{1}{1-\gamma}} \dfrac{x + h^{(2)}(t,\mathbf{k})}{\bar{f}^{(2)}(t,\mathbf{k})} - x \right), \end{cases} \tag{59}$$

where the functions $\bar{f}^{(2)}(t,\mathbf{k})$, $\bar{b}^{(2)}(t)$ and $\bar{Q}^{(2)}(t)$ are derived by solving (44) with parameters $\widetilde{\delta}_0$, $\widetilde{\delta}_1$, $\widetilde{l}_0$, $\widetilde{h}_0$, $\widetilde{h}_1$, $\widetilde{h}_2$, $\widetilde{g}_0$, $\widetilde{g}_1$ being given in Appendix F.

Proposition 7. *The expected utility $\widetilde{V}^{(2)}(t,x,\mathbf{k})$ associated with $\widetilde{\boldsymbol{\psi}}^{(2)}$ is expressed by*

$$\widetilde{V}^{(2)}(t,x,\mathbf{k}) = \frac{\left(x + h^{(2)}(t,\mathbf{k})\right)^{\gamma}}{\gamma} \left(f^{(2)}(t,\mathbf{k}) \right)^{1-\gamma}, \tag{60}$$

where the human capital simplifies to

$$h^{(2)}(t,\mathbf{k}) = i(t) \int_t^T \exp\left[- \int_t^s \left(\lambda(u) + \bar{r} \right) du \right] ds, \tag{61}$$

$f^{(2)}(t,\mathbf{k})$ satisfies

$$f^{(2)}(t,\mathbf{k}) = e^{-\int_t^T \left(\frac{1}{1-\gamma}\omega + \lambda(u) \right) du} \widetilde{f}^{(2)}(t,\mathbf{k}), \tag{62}$$

and $\widetilde{f}^{(2)}(t,\mathbf{k})$ is given by

$$\widetilde{f}^{(2)}(t,\mathbf{k}) = \exp\left(\widetilde{a}^{(2)}(t) + \widetilde{\mathbf{b}}^{(2)T}(t)\mathbf{k} + \frac{1}{2}\mathbf{k}^T \widetilde{\mathbf{Q}}^{(2)}(t)\mathbf{k} \right), \tag{63}$$

with the functions $\widetilde{a}^{(2)}(t)$, $\widetilde{\mathbf{b}}^{(2)}(t)$, $\widetilde{\mathbf{Q}}^{(2)}(t)$ being the solutions to (F.3) with $i = 2$ in Appendix F.

Proof See Appendix F.

260 *4.3. Utility losses*

In what follows, we measure the utility losses $L^i (i = 1, 2)$ by the percentage of the initial wealth. As discussed in Branger et al. (2013) and Escobar et al. (2016), the wealth-equivalent utility losses $L^i (i = 1, 2)$, representing the wealth losses over the entire lifetime of the wage earner, satisfy the following equations:

$$V\left(t, x(1-L^i), \mathbf{k}\right) = \widetilde{V}^{(i)}(t,x,\mathbf{k}), \tag{64}$$

for $i = 1, 2$. From Propositions 2, 6 and 7, the utility losses are expressed by

$$\begin{aligned} L^i = & 1 - \frac{x + h^{(i)}(t,\mathbf{k})}{x} \exp\left[\frac{1-\gamma}{\gamma} \left(\widetilde{a}^{(i)}(t) - a(t) + \left(\widetilde{\mathbf{b}}^{(i)}(t) - \mathbf{b}(t) \right)^T \mathbf{k} \right.\right. \\ & \left.\left. + \frac{1}{2}\mathbf{k}^T \left(\widetilde{\mathbf{Q}}^{(i)}(t) - \mathbf{Q}(t) \right) \mathbf{k} \right) \right] + \frac{h(t,\mathbf{k})}{x}, \end{aligned} \tag{65}$$

for $i = 1, 2$.

5. Numerical illustration

Some numerical examples concerning the effects of the main parameters on the optimal strategies are presented in this section. We also compare the optimal and the suboptimal investment-consumption-insurance strategies and study the utility losses. Boudoukh et al. (2007) provide empirical evidence that the net payout yield (dividend plus equity repurchases less equity issuances) can predict expected stock returns, and can be used as the observable predictor $y(t)$. With this specification, the model parameters can be estimated by the Kalman filtering technique and the maximum likelihood estimation. For more details about the estimation for the model based on practical data, one can refer to Branger et al. (2013). For simplicity, the values of $y(t)$ and $z(t)$ are substituted by the long-run average levels $\bar{y}$ and $\bar{z}$, and the conditional variance $m(t)$ converges to its long-run level m, which is determined by (A.13). The mortality rate $\lambda(t)$ is described by Gompertz law according to Zeng et al. (2015) and is given by

$$\lambda(t) = \frac{1}{9.5} \exp\left(\frac{w_0 + t - 86.3}{9.5}\right), \tag{66}$$

where w_0 represents the initial age of the wage earner and is assumed to be 35 in this section. Referring to Han and Hung (2017), the labor income rate $i(t)$ of the wage earner expressed as

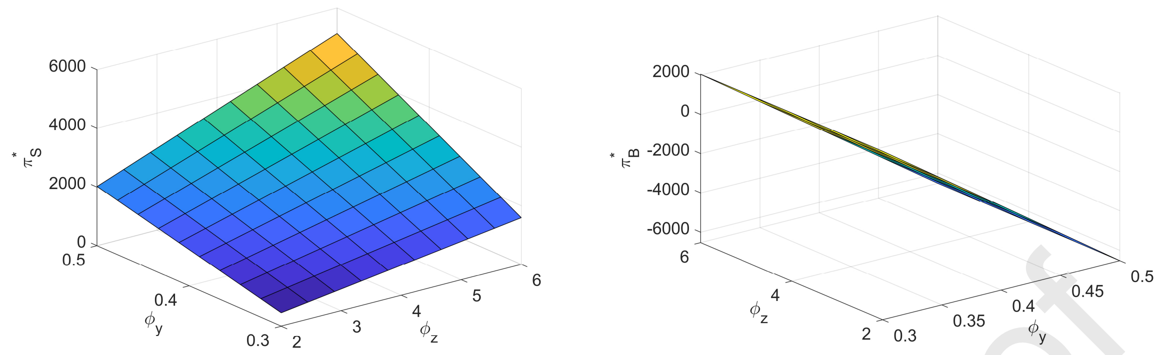
$$i(t) = 40 \exp(0.0043t). \tag{67}$$

Table 1: Default values of model parameters

κ_r	κ_y	κ_z	$\bar{r}$	$\bar{y}$	$\bar{z}$	σ_r	σ_y	σ_z
0.50	0.30	4.00	0.02	-2.15	0.00	0.03	0.15	0.26
σ_S	ρ_{Sr}, q_r	ρ_{rz}	ρ_{Sz}	ϕ_y	ϕ_z	ϕ	ω	γ
0.20	0.00	0.03	-0.02	0.32	2.68	1.00	0.03	-3.00

The default values of the parameters are listed in Table 1, which come from Branger et al. (2013) and Wang et al. (2021a). The wage earner starts to purchase the life insurance with initial wealth 100, the expected stock return rate is set as 5.5% and the investment horizon $T - t$ is equal to 20. The following illustrations for the portfolio strategies are conducted under the assumption that α=β=0 for the sake of simplifying the numerical examples.

Figure 1 explores the impacts of the predictive powers ϕ_y and ϕ_z on the optimal investment strategy $\boldsymbol{\pi}^* = (\pi_S^*, \pi_B^*)$. We can see that as the predictive power ϕ_y increases, the wage earner will invest more in stock, while less in the zero-coupon bond. Due to the fact that a greater ϕ_y leads to a better estimate for the expected stock return rate, the wage earner tends to invest more in the stock. The decrease of investment in the zero-coupon bond is resulted from the correlation between the dynamics of the stock and the zero-coupon bond. On the other hand, the optimal stock investment strategy π_S^* increases with the predictive power ϕ_z. Since the predictability of the expected stock return increases when ϕ_z is larger, the wage earner prefers to invest more in the stock. Moreover, the impact of ϕ_z on the optimal zero-coupon bond investment strategy π_B^* is not significant when ϕ_y is fixed. This observation may be reasonable owing to the correlation between the two predictors $y(t)$ and $z(t)$. The similar result has been reported in Escobar et al. (2016).



(a) Effects of ϕ_y and ϕ_z on π^*_S

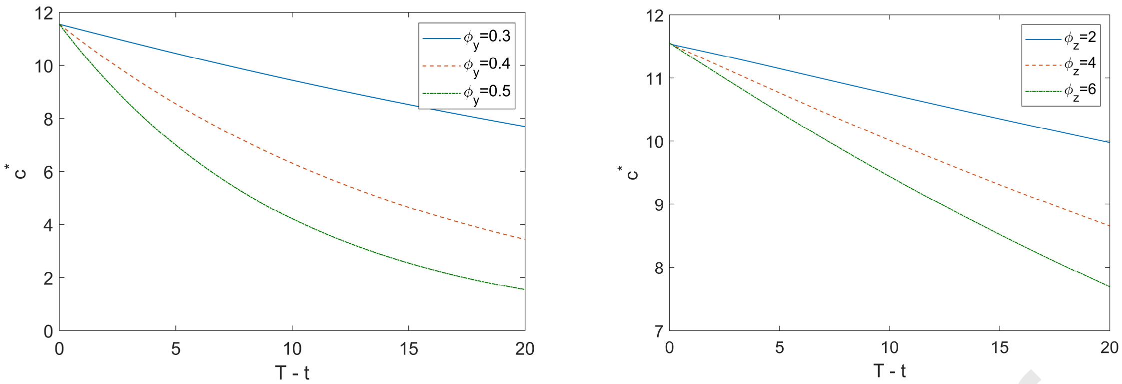
(b) Effects of ϕ_y and ϕ_z on π^*_B

Figure 1: Effects of ϕ_y and ϕ_z on π^*_S and π^*_B at t=0

In Figure 2, we illustrate the effects of the predictive powers ϕ_y and ϕ_z on the optimal consumption strategy c^* with the investment horizon $T - t$ increasing. As the investment horizon increases, the wage earner will consume less. Indeed, a longer investment horizon implies that the stock is more predictable, and the stock investment seems more attractive to the wage earner. Thus, she even gives up part of the consumption. From Figure 2(a) and 2(b), we can see that the greater ϕ_y and ϕ_z drive the wage earner to consume less. As shown by Figure 1, with greater ϕ_y and ϕ_z, the wage earner is willing to invest more in stock, so less room is left to the consumption.

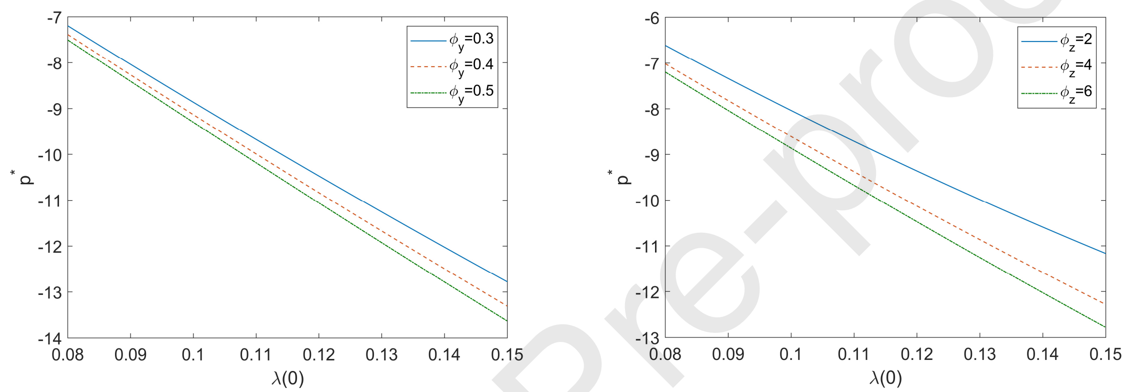
Figure 3 displays the effects of the predictive powers ϕ_y and ϕ_z on the optimal life insurance purchase strategy p^* with the initial force of mortality rate $\lambda(0)$ ranging from 0.08 to 0.15. We find that the wage earner receives higher annuity payments as the initial force of mortality rate $\lambda(0)$ increases. Clearly, the wage earner tries to seek for more protection against the death when the trend of mortality risk increases. She is willing to leave more bequest to the beneficiary for the reason that the reduced life expectancy gives the wage earner less time to achieve an investment objective. This appears to be consistent with the results in Shen and Sherris (2018). In addition, the optimal life insurance strategy p^* decreases as the values of the predictive powers ϕ_y and ϕ_z increase. The intuition behind this trend may be that the wage earner keeps investing more in stock when the values of the predictive powers ϕ_y and ϕ_z are larger, which makes the demand for life insurance purchase reduced.

Figure 4(a) shows that the optimal consumption strategy c^* decreases as the investment horizon $T - t$ increases, and the risk aversion $1 - \gamma$ has a positive effect on the optimal consumption rate. In other words, the more risk averse the wage earner is, the more aggressive consuming behavior she keeps. Figure 4(b) displays the effects of the risk aversion $1 - \gamma$ and the investment horizon $T - t$ on the optimal life insurance purchase strategy p^*. As mentioned in Figure 2, the longer the investment horizon $T - t$ is, the more capital to be invested in stock. Thus, the bequest left to the beneficiary shrinks, and the wage earner tends to receive higher annuity payments as the investment horizon $T - t$ increases. On the other hand, the variations of the optimal life insurance purchase strategies p^* are not significant for several different levels of $1 - \gamma$, which means that the life insurance purchase is a necessary expense in our life and cannot be influenced significantly by the risk attitudes of the wage earner.



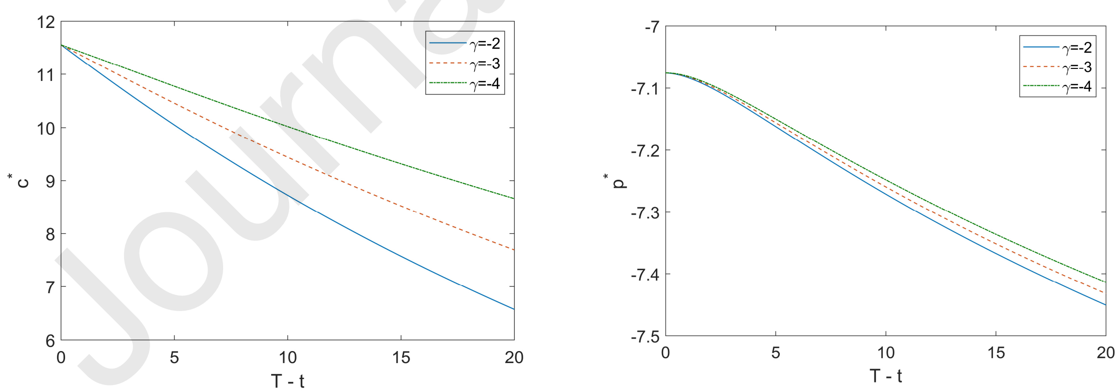
(a) Effects of ϕ_y and $T-t$ on c^* (b) Effects of ϕ_z and $T-t$ on c^*

Figure 2: Effect of ϕ_y and ϕ_z on the optimal consumption strategy



(a) Effects of ϕ_y and $\lambda(0)$ on p^* at t=0 (b) Effects of ϕ_z and $\lambda(0)$ on p^* at t=0

Figure 3: Effect of ϕ_y and ϕ_z on the optimal life insurance purchase strategy p^*



(a) Effects of γ and $T-t$ on c^* (b) Effects of γ and $T-t$ on p^*

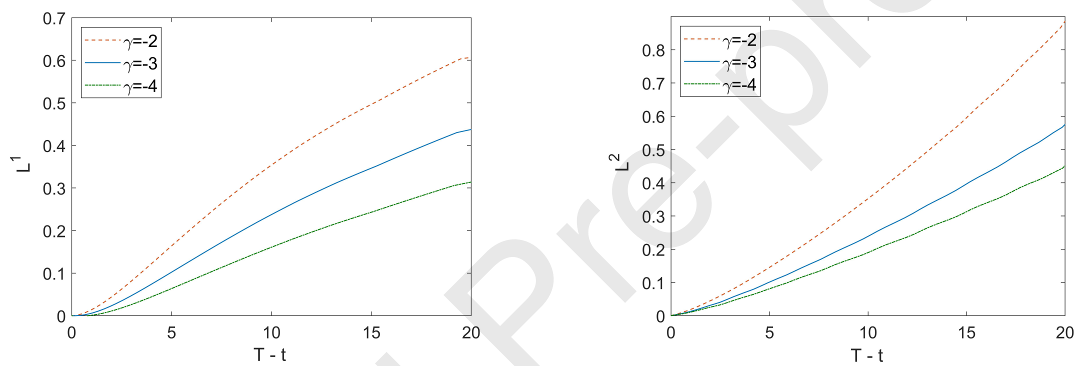
Figure 4: Effects of γ and $T-t$ on the optimal consumption and life insurance strategies

Next, we illustrate the utility losses from ignoring learning or the randomness of the interest rate for the wage earner with different risk aversions. From Figure 5, we can derive the following three insights:

(1) The risk aversion parameter $1-\gamma$ has a negative influence on the welfare loss. In other words, the more risk averse the wage earner is, the less losses are incurred from ignoring learning or the randomness of the interest rate, which are consistent with the findings in Branger et al. (2013) and Escobar et al. (2016).

(2) Comparing Figure 5(a) with 5(b), we can find that the utility loss from ignoring the randomness of interest rate is greater than that from ignoring learning. This is owing to the fact that the interest rate is perfectly correlated with the zero-coupon bond and the observable stochastic factor.

(3) The utility losses from ignoring learning is approximately 30% for the wage earner with risk aversion $1-\gamma=5$ and can reach 60% when she has a risk aversion $1-\gamma=3$. In addition, the wage earner with $1-\gamma=5$ suffers a utility loss from the constant interest rate as much as 45% of the initial wealth, and this loss increases to 90% when $1-\gamma=3$. To sum up, it is essential to point out that the wage earner should take learning about the unobserved stochastic factor $z(t)$ and the stochastic interest rate $r(t)$ into account when making portfolio, consumption and life insurance purchase decisions.

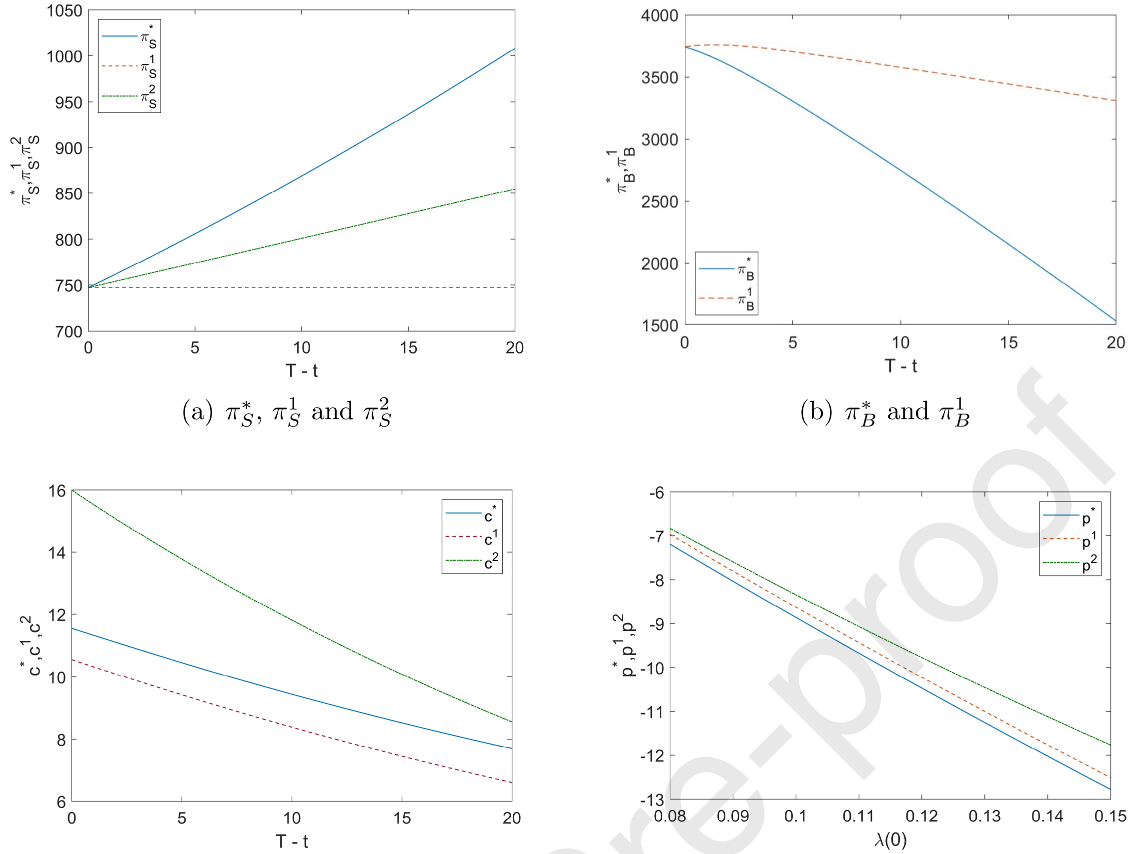


(a) Utility loss when ignoring learning (b) Utility loss when $r(t)$ is a constant

Figure 5: Utility loss L^1 and L^2

Figure 6 compares the optimal strategies with two kinds of suboptimal strategies for the wage earner. It can be observed from Figure 6(a) and 6(b) that the suboptimal stock investment strategy π_S^1 in the absence of learning is smaller than π_S^*, while the suboptimal zero-coupon bond investment strategy π_B^1 is larger than π_B^*. This is due to the absence of learning leads to a worse estimate for the expected stock return. Hence, the wage earner tends to reduce the stock investment, which is diverted to the zero-coupon bond. Meanwhile, the suboptimal consumption strategy c^1 when ignoring learning is smaller than c^*, and the suboptimal life insurance purchase strategy p^1 is slightly larger than p^*. In addition, Figure 6 shows that in the case of ignoring the randomness of the interest rate, the suboptimal stock investment strategy π_S^2 is smaller than π_S^* while the suboptimal consumption strategy c^2 and suboptimal life insurance purchase strategy p^2 are larger than the optimal strategies c^* and p^* respectively. Indeed, when the wage earner ignores the randomness of the interest rate $r(t)$, the observed predictor $y(t)$ is replaced by the long-run average level $\bar{y}$. In this case, the estimated accuracy of the expected stock return decreases. Consequently, the wage earner tends to invest less in stock, more room is left to the consumption and life insurance purchase.

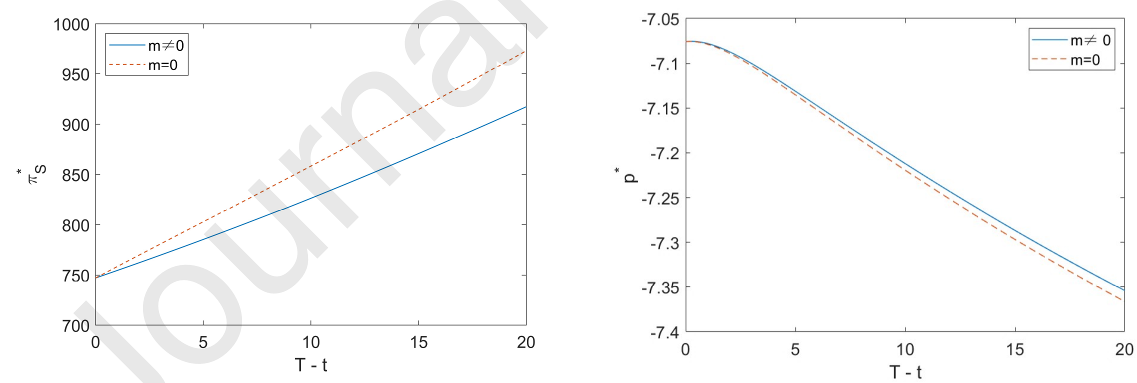
Figure 7 depicts the optimal stock investment strategies and life insurance purchase strategies for the partial and complete information cases corresponding to $m \neq 0$ and $m = 0$ respectively.



(a) π_S^*, π_S^1 and π_S^2 (b) π_B^* and π_B^1

(c) c^*, c^1 and c^2 (d) p^*, p^1 and p^2

Figure 6: The investment, consumption and insurance strategies for various cases



(a) The optimal stock investment strategies for $m \neq 0$ and $m = 0$ (b) The optimal life insurance purchase strategies for $m \neq 0$ and $m = 0$

Figure 7: The optimal stock investment and life insurance purchase strategies for $m \neq 0$ and $m = 0$

Figure 7(a) demonstrates that the optimal stock investment strategy π_S^* for the complete information case is larger than that for the partial information case. In fact, $m = 0$ implies that the estimate for the unobservable factor is accurate and no information is lost. Thus, the wage earner is willing to invest more in stock. Figure 7(b) indicates that the life insurance purchase strategy

p^* is relatively small in the complete information case. This might due to the increase of the investment in the stock under the situation of complete information.

6. Conclusion

This paper investigates an optimal investment, consumption and life insurance purchase problem when the market price of risk is an affine function of an observable and an unobservable factor. The unobservable factor is estimated by the filtering technique based on the observable processes. We derive the closed-form expressions of the optimal strategies and the corresponding value function by employing the dynamic programming principle and the HJB equation. Besides, we also obtain the utility losses from ignoring learning and the randomness of the interest rate. Numerical examples reveal the impacts of the predictive powers, the mortality rate and the risk aversion on the optimal investment, consumption and life insurance purchase strategies. In numerical illustration, we find that both ignoring learning and the randomness of the interest rate will lead to significant utility losses.

Acknowledgement

The authors are very grateful to the Editor and two anonymous referees for their valuable comments and suggestions, which leads to a significant improvement of this paper. This work is supported by the National Natural Science Foundation of China (No.11701436), Humanities and Social Science Research Planning Foundation of Ministry of Education of China (No.22YJAZH087), and the Fundamental Research Funds for the Central Universities (WUT:3120621545).

References

Abou-Kandil, H., Freiling, G., Ionescu, V., Jank, G., 2012. Matrix Riccati equations in control and systems theory. Birkhäuser.

Baldi, P., 2017. Stochastic calculus: An introduction through theory and exercises. Springer, Berlin.

Boudoukh, J., Michaely, R., Richardson, M., Roberts, M.R., 2007. On the importance of measuring payout yield: Implications for empirical asset pricing. The Journal of Finance 62, 877–915.

Branger, N., Larsen, L.S., Claus, M., 2013. Robust portfolio choice with ambiguity and learning about return predictability. Journal of Banking and Finance 37, 1397–1411.

Brennan, M.J., 1998. The role of learning in dynamic portfolio decisions. Review of Finance 1, 295–306.

Brockett, R.W., 2015. Finite dimensional linear systems. Society for Industrial and Applied Mathematics.

Campbell, J., Viceira, L., 1999. Consumption and portfolio decisions when expected returns are time varying. The Quarterly Journal of Economics 114, 433–495.

Campbell, J.Y., Shiller, R.J., 1986. The dividend-price ratio and expectations of future dividends and discount factors. The Review of Financial Studies 1, 195–228.

Escobar, M., Ferrando, S., Rubtsov, A., 2016. Portfolio choice with stochastic interest rates and learning about stock return predictability. International Review of Economics and Finance 41, 347–370.

Fama, E.F., French, K.R., 1988. Permanent and temporary components of stock prices. Journal of Political Economy 96, 246–273.

Fama, E.F., French, K.R., 1989. Business conditions and expected returns on stocks and bonds. Journal of Financial Economics 25, 23–49.

Fouque, J.P., Papanicolaou, A., Sircar, R., 2015. Filtering and portfolio optimization with stochastic unobserved drift in asset returns. Communications In Mathematical Science 13, 935–953.

Han, N.W., Hung, M.W., 2017. Optimal consumption, portfolio, and life insurance policies under interest rate and inflation risks. Insurance: Mathematics and Economics 73, 54–67.

Huang, H., Milevsky, M.A., Wang, J., 2008. Portfolio choice and life insurance: The crra case. Journal of Risk and Insurance 75, 847–872.

Kim, T.S., Omberg, E., 1996. Dynamic nonmyopic portfolio behavior. The Review of Financial Studies 9, 141–161.

Kwak, M., Shin, Y.H., Choi, U.J., 2011. Optimal investment and consumption decision of a family with life insurance. Insurance: Mathematics and Economics 48, 176–188.

Liptser, R.S., Shiryaev, A.N., 2001. Statistics of random processes. Springer-Verlag, Berlin.

Liu, J., 2007. Portfolio selection in stochastic environments. The Review of Financial Studies 20, 1–39.

Ma, G., Siu, C.C., Zhu, S.P., 2019. Dynamic portfolio choice with return predictability and transaction costs. European Journal of Operational Research 278, 976–988.

Ma, G., Siu, C.C., Zhu, S.P., 2020. Optimal investment and consumption with return predictability and execution costs. Economic Modelling 88, 408–419.

Merton, R.C., 1969. Lifetime portfolio selection under uncertainty: The continuous-time case. The review of Economics and Statistics 51, 247–257.

Munk, C., Sørensen, C., 2004. Optimal consumption and investment strategies with stochastic interest rates. Journal of Banking and Finance 28, 1987–2013.

Munk, C., Sørensen, C., 2010. Dynamic asset allocation with stochastic income and interest rates. Journal of Financial Economics 96, 433–462.

Pirvu, T.A., Zhang, H., 2012. Optimal investment, consumption and life insurance under mean-reverting returns: The complete market solution. Insurance: Mathematics and Economics 51, 303–309.

Pliska, S.R., Ye, J., 2007. Optimal life insurance purchase and consumption/investment under uncertain lifetime. Journal of Banking and Finance 31, 1307–1319.

Richard, S.F., 1975. Optimal consumption, portfolio and life insurance rules for an uncertain lived individual in a continuous time model. Journal of Financial Economics 2, 187–203.

Shen, Y., Sherris, M., 2018. Lifetime asset allocation with idiosyncratic and systematic mortality risks. Scandinavian Actuarial Journal 2018, 294–327.

Shen, Y., Wei, J., 2016. Optimal investment-consumption-insurance with random parameters. Scandinavian Actuarial Journal 2016, 37–62.

Sørensen, C., 1999. Dynamic asset allocation and fixed income management. The Journal of Financial and Quantitative Analysis 34, 513–531.

Van Binsbergen, J., Koijen, R., 2010. Predictive regressions: A present-value approach. The Journal of Finance 65, 1439–1471.

Vasicek, O., 1977. An equilibrium characterization of the term structure. Journal of Financial Economics 5, 177–188.

Wachter, J.A., 2002. Portfolio and consumption decisions under mean-reverting returns: An exact solution for complete markets. The Journal of Financial and Quantitative Analysis 37, 63–91.

Wang, N., Jin, Z., Siu, T.K., Qiu, M., 2021a. Household consumption-investment-insurance decisions with uncertain income and market ambiguity. Scandinavian Actuarial Journal 2021, 832–865.

Wang, P., Shen, Y., Zhang, L., Kang, Y., 2021b. Equilibrium investment strategy for a dc pension plan with learning about stock return predictability. Insurance: Mathematics and Economics 100, 384–407.

Xia, Y., 2001. Learning about predictability: The effects of parameter uncertainty on dynamic asset allocation. The Journal of Finance 56, 205–246.

Yaari, M.E., 1965. Uncertain lifetime, life insurance, and the theory of the consumer. The Review of Economic Studies 32, 137–150.

Ye, J., 2007. Optimal life insurance, consumption and portfolio under uncertainty: martingale methods, in: American Control Conference, pp. 1103–1109.

Yong, J., Zhou, X., 1999. Stochastic controls: Hamiltonian systems and HJB equations. Springer, New York.

Zeng, X., Wang, Y., Carson, J.M., 2015. Dynamic portfolio choice with stochastic wage and life insurance. North American Actuarial Journal 19, 256–272.

Zvi Bodie, R.M., Samuelson, W.F., 1992. Labor supply flexibility and portfolio choice in a life cycle model. Journal of Economic Dynamics and Control 16, 427–449.

Appendix A.

Poof of Proposition 1 The dynamical processes of the stock, the interest rate and the unobservable factor can be rewritten as

$$\begin{bmatrix} \frac{dS(t)}{S(t)} \\ dr(t) \\ dz(t) \end{bmatrix} = \begin{bmatrix} r(t) + \sigma_S\left(\phi + \phi_y y(t) + \phi_z z(t)\right) \\ \kappa_r(\bar{r} - r(t)) \\ \kappa_z(\bar{z} - z(t)) \end{bmatrix} dt + \begin{bmatrix} \sigma_S & 0 & 0 \\ 0 & -\sigma_r & 0 \\ 0 & 0 & \sigma_z \end{bmatrix} \begin{bmatrix} dW_S(t) \\ dW_r(t) \\ dW_z(t) \end{bmatrix}. \tag{A.1}$$

For a three-dimensional Brownian motion $\left(W_{(1)}(t), W_{(2)}(t), W_{(3)}(t)\right)^T$, we obtain the following dynamics

$$\begin{aligned} \begin{bmatrix} \frac{dS(t)}{S(t)} \\ dr(t) \\ dz(t) \end{bmatrix} &= \begin{bmatrix} r(t) + \sigma_S\left(\phi + \phi_y y(t) + \phi_z z(t)\right) \\ \kappa_r(\bar{r} - r(t)) \\ \kappa_z(\bar{z} - z(t)) \end{bmatrix} dt \\ &+ \begin{bmatrix} \sigma_S & 0 & 0 \\ 0 & -\sigma_r & 0 \\ 0 & 0 & \sigma_z \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ \rho_{Sr} & \hat{\rho}_r & 0 \\ \rho_{Sz} & \hat{\rho}_{rz} & \hat{\rho}_z \end{bmatrix} \begin{bmatrix} dW_{(1)}(t) \\ dW_{(2)}(t) \\ dW_{(3)}(t) \end{bmatrix}, \end{aligned} \tag{A.2}$$

where

$$\hat{\rho}_r = \sqrt{1 - \rho_{Sr}^2}, \hat{\rho}_{rz} = \frac{\rho_{rz} - \rho_{Sr}\rho_{Sz}}{\sqrt{1 - \rho_{Sr}^2}}, \hat{\rho}_z = \sqrt{1 - \rho_{Sz}^2 - \hat{\rho}_{rz}^2}. \tag{A.3}$$

Then, following the notations in Theorem 12.7 in Liptser and Shiryaev (2001), we separate the observed processes $S(t)$ and $r(t)$ from the unobservable process $z(t)$. The observable processes can be rewritten as follows

$$\begin{bmatrix} \frac{dS(t)}{S(t)} \\ dr(t) \end{bmatrix} = \left[\underbrace{\begin{pmatrix} r(t) + \sigma_S(\phi + \phi_y y(t)) \\ \kappa_r(\overline{r} - r(t)) \end{pmatrix}}_{\mathbf{A_0}} + \underbrace{\begin{pmatrix} \sigma_S \phi_z \\ 0 \end{pmatrix}}_{\mathbf{A_1}} z(t) \right] dt + \underbrace{0}_{\mathbf{B}_1} dW_{(3)}(t) + \underbrace{\begin{bmatrix} \sigma_S & 0 \\ -\sigma_r \rho_{Sr} & -\sigma_r \widehat{\rho}_r \end{bmatrix}}_{\mathbf{B_2}} \begin{bmatrix} dW_{(1)}(t) \\ dW_{(2)}(t) \end{bmatrix}, \tag{A.4}$$

and the unobservable process is governed by

$$dz(t) = \left(\underbrace{\kappa_z \bar{z}}_{a_0} + \underbrace{(-\kappa_z)}_{a_1} z(t) \right) dt + \underbrace{\sigma_z \hat{\rho}_z}_{\mathbf{c}_1} dW_{(3)}(t) + \underbrace{\begin{bmatrix} \sigma_z \rho_{Sz} & \sigma_z \hat{\rho}_{rz} \end{bmatrix}}_{\mathbf{c}_2} \begin{bmatrix} dW_{(1)}(t) \\ dW_{(2)}(t) \end{bmatrix}. \tag{A.5}$$

Denote $\mathbf{B} = (\mathbf{B}_1, \mathbf{B}_2)$ and $\mathbf{c} = (\mathbf{c}_1, \mathbf{c}_2)$. Since $\mathbf{B}_1$=0, we have $\mathbf{B} \circ \mathbf{B} = \mathbf{B}_2 \mathbf{B}_2^T$, $(\mathbf{B} \circ \mathbf{B})^{-1} = (\mathbf{B}_2^T)^{-1} \mathbf{B}_2^{-1}$, $\mathbf{c} \circ \mathbf{B} = \mathbf{c}_2 \mathbf{B}_2^T$ and $\mathbf{c} \circ \mathbf{c} = \mathbf{c}_1 \mathbf{c}_1^T + \mathbf{c}_2 \mathbf{c}_2^T$, where

$$\left(\mathbf{B}_2^T\right)^{-1} = \begin{bmatrix} \frac{1}{\sigma_S} & -\frac{\rho_{Sr}}{\sigma_S \hat{\rho}_r} \\ 0 & -\frac{1}{\sigma_r \hat{\rho}_r} \end{bmatrix}. \tag{A.6}$$

We define $(W_1(t), W_2(t))^T$ as follows:

$$\begin{bmatrix} dW_1(t) \\ dW_2(t) \end{bmatrix} = \mathbf{B}_2^{-1} \times \left\{ \begin{bmatrix} \frac{dS(t)}{S(t)} \\ dr(t) \end{bmatrix} - \begin{bmatrix} r(t) + \sigma_S \left(\phi + \phi_y y(t) + \phi_z \hat{z}(t)\right) \\ \kappa_r \left(\bar{r} - r(t)\right) \end{bmatrix} \right\} dt, \tag{A.7}$$

where $\hat{z}(t)$ is the filtered estimate defined by (15). According to Liptser and Shiryaev (2001), $(dW_1(t), dW_2(t))^T$ is a two-dimensional independent Brownian motion adapted to the filtration $\left\{\mathcal{F}_t^{S,r}\right\}_{t\in[0,T]}$. Then, the dynamics of observable variables are

$$\begin{bmatrix} \frac{dS(t)}{S(t)} \\ dr(t) \end{bmatrix} = \begin{bmatrix} r(t) + \sigma_S \left(\phi + \phi_y y(t) + \phi_z \hat{z}(t)\right) \\ \kappa_r \left(\bar{r} - r(t)\right) \end{bmatrix} dt + \begin{bmatrix} \sigma_S & 0 \\ -\sigma_r \rho_{Sr} & -\sigma_r \hat{\rho}_r \end{bmatrix} \begin{bmatrix} dW_1(t) \\ dW_2(t) \end{bmatrix}, \tag{A.8}$$

and the filtered estimate $\hat{z}(t)$ satisfies

$$d\hat{z}(t) = [a_0 + a_1 \hat{z}(t)]dt + [(\mathbf{c} \circ \mathbf{B}) + m(t) A_1^T] \times (\mathbf{B} \circ \mathbf{B})^{-1} \times \left\{ \begin{pmatrix} \frac{dS(t)}{S(t)} \\ dr(t) \end{pmatrix} - (\mathbf{A}_0 + \mathbf{A}_1 \hat{z}(t)) dt \right\}. \tag{A.9}$$

Simplifying the above equation (A.9) yields

$$d\hat{z}(t) = \kappa_z \left(\bar{z} - \hat{z}(t)\right) dt + H_1 dW_1(t) + H_2 dW_2(t), \tag{A.10}$$

where

$$\begin{cases} H_1 = \sigma_z\rho_{Sz} + m(t)\phi_z, \\ H_2 = \dfrac{\sigma_z\hat{\rho}_{rz}\hat{\rho}_r - m(t)\rho_{Sr}\phi_z}{\hat{\rho}_r}. \end{cases} \tag{A.11}$$

Moreover, the dynamics of the conditional variance $m(t) = \mathbb{E}\left[(z(t) - \hat{z}(t))^2 \middle| \mathcal{F}_t^{S,r}\right]$ follows

$$\begin{aligned} \frac{dm(t)}{dt} &= a_1 m(t) + m(t)a_1^T + (\mathbf{c}\circ\mathbf{c}) - \left((\mathbf{c}\circ\mathbf{B}) + m(t)\mathbf{A}_1^T\right)(\mathbf{B}\circ\mathbf{B})^{-1}\left((\mathbf{c}\circ\mathbf{B}) + m(t)\mathbf{A}_1^T\right)^T \\ &= -2\kappa_z m(t) + \sigma_z^2 - H_1^2 - H_2^2. \end{aligned} \tag{A.12}$$

The conditional variance converges to the long-run value m determined by

$$-2\kappa_z m + \sigma_z^2 - (\sigma_z\rho_{Sz} + m\phi_z)^2 - \left(\frac{\sigma_z\hat{\rho}_{rz}\hat{\rho}_r - m\rho_{Sr}\phi_z}{\hat{\rho}_r}\right)^2 = 0. \tag{A.13}$$

Furthermore, the observable factor can be rewritten as

$$dy(t) = \kappa_y(\bar{y} - y(t)) + \sigma_y\rho_{Sr}dW_1(t) + \sigma_y\hat{\rho}_r dW_2(t), \tag{A.14}$$

with the relationship between $\{W_r(t)\}$, $\{W_1(t)\}$ and $\{W_2(t)\}$ being

$$dW_r(t) = \rho_{Sr}dW_1(t) + \hat{\rho}_r dW_2(t). \tag{A.15}$$

In the end, we can obtain the dynamical process of the zero-coupon bond

$$\frac{dB_I(t)}{B_I(t)} = (r(t) + \sigma_B q_r)\,dt + \sigma_B\rho_{Sr}dW_1(t) + \sigma_B\hat{\rho}_r dW_2(t). \tag{A.16}$$

Appendix B.

Poof of Theorem 1 Let $X^{\boldsymbol{\psi}}$ be the wealth process corresponding to $\boldsymbol{\psi} \in \mathcal{A}$. For fixed $t \in [0, T]$, we define

$$\begin{aligned} D^{\boldsymbol{\psi}}(s) =& F(s,t)e^{-w(s-t)}V\left(s, X^{\boldsymbol{\psi}}(s), \mathbf{k}(s)\right) + \alpha\int_t^s F(u,t)e^{-w(u-t)}U\left(c(u)\right)du \\ &+ \beta\int_t^s f(u,t)e^{-w(u-t)}U\left(Z(u)\right)du. \end{aligned} \tag{B.1}$$

By the Itô formula, we obtain

$$\begin{aligned} dD^{\boldsymbol{\psi}}(s) = e^{-w(s-t)}F(s,t)&\left[\mathcal{L}^{\boldsymbol{\psi}}V\left(s, X^{\boldsymbol{\psi}}(s), \mathbf{k}(s)\right) + \alpha U\left(c(s)\right)\right. \\ &\left. + \beta\lambda(s)U\left(X^{\boldsymbol{\psi}}(s) + \frac{p(s)}{\eta(s)}\right)\right]ds + dM^{\boldsymbol{\psi}}(s), \end{aligned} \tag{B.2}$$

where

$$dM^{\boldsymbol{\psi}}(s) = e^{-w(s-t)}F(s,t)\left[V_x\left(s, X^{\boldsymbol{\psi}^*}(s), \mathbf{k}(s)\right)\boldsymbol{\pi}^{\mathrm{T}}(s)\boldsymbol{\Sigma} + V_{\mathbf{k}}\left(s, X^{\boldsymbol{\psi}}(s), \mathbf{k}(s)\right)^{\mathrm{T}}\boldsymbol{\Sigma}^{\mathbf{k}}\right]d\mathbf{W}(s). \tag{B.3}$$

Let $\{\tau_n\}$ be a localizing sequence of the local martingale $M^{\boldsymbol{\psi}}$. Since $\tau_n \uparrow \infty$ a.s., $\tau_n > t$ a.s. for n large enough. Therefore,

$$\begin{aligned}&\mathbb{E}\left(D^{\boldsymbol{\psi}}(s\wedge\tau_n)\,|\mathcal{F}_t\right) = D^{\boldsymbol{\psi}}(t)\\&+\mathbb{E}\left[\int_t^{s\wedge\tau_n} e^{-w(u-t)}F(u,t)\left[\mathcal{L}^{\boldsymbol{\psi}}V\left(u,X^{\boldsymbol{\psi}}(u),\mathbf{k}(u)\right)+\alpha U\left(c(u)\right)+\beta\lambda(u)U\left(X^{\boldsymbol{\psi}}(u)+\frac{p(u)}{\eta(u)}\right)\right]du\right].\end{aligned}\tag{B.4}$$

By condition (1) in Theorem 1,

$$\mathbb{E}\left(D^{\boldsymbol{\psi}}(s\wedge\tau_n)\,|\mathcal{F}_t\right) \leq D^{\boldsymbol{\psi}}(t).\tag{B.5}$$

Applying the Fatou Lemma to (B.5) yields

$$\mathbb{E}\left(D^{\boldsymbol{\psi}}(s)\,|\mathcal{F}_t\right) \leq \lim_{n\to\infty}\mathbb{E}\left(D^{\boldsymbol{\psi}}(s\wedge\tau_n)\,|\mathcal{F}_t\right) \leq D^{\boldsymbol{\psi}}(t).\tag{B.6}$$

Setting $s = T$, $X(t) = x$ and $\mathbf{k}(t) = \mathbf{k}$, we get

$$\mathbb{E}_{t,x,\mathbf{k}}\left(D^{\boldsymbol{\psi}}(T)\right) \leq D^{\boldsymbol{\psi}}(t).\tag{B.7}$$

From (B.1), (B.7) and condition (3), we have

$$\mathcal{J}(t,x,\mathbf{k};\boldsymbol{\psi}) \leq V(t,x,\mathbf{k})\tag{B.8}$$

for any $\boldsymbol{\psi} \in \mathcal{A}$.

On the other hand, when $\boldsymbol{\psi} = \boldsymbol{\psi}^* \in \mathcal{A}$, by (B.2) and condition (2), we have

$$dD^{\boldsymbol{\psi}^*}(s) = dM^{\boldsymbol{\psi}^*}(s).\tag{B.9}$$

This equality together with condition (4) deduces that $D^{\boldsymbol{\psi}^*}$ is a martingale. Consequently,

$$\mathbb{E}_{t,x,k}\left(D^{\boldsymbol{\psi}^*}(T)\right) = D^{\boldsymbol{\psi}^*}(t).\tag{B.10}$$

This implies that

$$\mathcal{J}(t,x,\mathbf{k};\boldsymbol{\psi}^*) = V(t,x,\mathbf{k}).\tag{B.11}$$

Combining (B.8) and (B.11) leads to the conclusion that $\boldsymbol{\psi}^*$ is an optimal strategy and $V(t,x,\mathbf{k})$ is the associated value function.

455 **Appendix C.**

Poof of Proposition 2 The value function is assumed to be the form given by (26). By (23), HJB equation (25) can be rewritten as

$$\begin{aligned}&\sup_{\boldsymbol{\psi}\in\mathcal{A}}\{-(\lambda+\omega)V+V_t+\left[rx+\boldsymbol{\pi}^T(\boldsymbol{\mu}-\mathbf{r})+i-c-p\right]V_x+\boldsymbol{\mu}^{\mathbf{k}T}V_{\mathbf{k}}\\&+\frac{1}{2}\boldsymbol{\pi}^T\boldsymbol{\Sigma}\boldsymbol{\Sigma}^T\boldsymbol{\pi}V_{xx}+\frac{1}{2}tr(\boldsymbol{\Sigma}^{\mathbf{k}}\boldsymbol{\Sigma}^{\mathbf{k}T}V_{\mathbf{kk}})+\boldsymbol{\pi}^T\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{\mathbf{k}T}V_{x\mathbf{k}}+\alpha\frac{c^\gamma}{\gamma}+\beta\frac{\lambda}{\gamma}\Big(x+\frac{p}{\lambda}\Big)^\gamma\Big\}=0.\end{aligned}\tag{C.1}$$

According to the first-order conditions, we obtain

$$\begin{cases} \boldsymbol{\pi}^*(t) = -\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}^T\right)^{-1}\left[(\boldsymbol{\mu}-\mathbf{r})\frac{V_x}{V_{xx}} + \boldsymbol{\Sigma}\boldsymbol{\Sigma}^{\mathbf{k}T}\frac{V_{x\mathbf{k}}}{V_{xx}}\right], \\ c^*(t) = \left(\frac{V_x}{\alpha}\right)^{\frac{1}{\gamma-1}}, \\ p^*(t) = \lambda(t)\left[\left(\frac{V_x}{\beta}\right)^{\frac{1}{\gamma-1}} - x\right]. \end{cases} \tag{C.2}$$

Substituting the above expressions for $(\boldsymbol{\pi}^*, c^*, p^*)$ into HJB equation (C.1) yields

$$\begin{aligned} &-(\lambda(t)+\omega)V + V_t + \left[(r(t)+\lambda(t))x + i\right]V_x + \boldsymbol{\mu}^{\mathbf{k}T}V_{\mathbf{k}} + \frac{1}{2}tr\left(\boldsymbol{\Sigma}^{\mathbf{k}}\boldsymbol{\Sigma}^{\mathbf{k}T}V_{\mathbf{kk}}\right) \\ &-\frac{1}{2}(\mu-\mathbf{r})^T(\boldsymbol{\Sigma}\boldsymbol{\Sigma}^T)^{-1}(\boldsymbol{\mu}-\mathbf{r})\frac{V_x^2}{V_{xx}} - \left[\boldsymbol{\Sigma}^{\mathbf{k}}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}-\mathbf{r})\right]^T\frac{V_xV_{x\mathbf{k}}}{V_{xx}} \\ &-\frac{1}{2}\frac{V_{x\mathbf{k}}^T\boldsymbol{\Sigma}^{\mathbf{k}}\boldsymbol{\Sigma}^{\mathbf{k}T}V_{x\mathbf{k}}}{V_{xx}} + \frac{1-\gamma}{\gamma}\left(\alpha^{\frac{1}{1-\gamma}} + \beta^{\frac{1}{1-\gamma}}\right)V_x^{\frac{\gamma}{1-\gamma}} = 0. \end{aligned} \tag{C.3}$$

From (26), we have the following partial derivatives

$$\begin{aligned} V_t =&(x+h)^{\gamma-1}h_tf^{1-\gamma} + \frac{1-\gamma}{\gamma}(x+h)^\gamma f^{-\gamma}f_t, V_x = (x+h)^{\gamma-1}f^{1-\gamma}, \\ V_{\mathbf{k}} =&\frac{1-\gamma}{\gamma}(x+h)^\gamma f^{-\gamma}f_{\mathbf{k}} + (x+h)^{\gamma-1}f^{1-\gamma}h_{\mathbf{k}}, V_{xx} = (\gamma-1)(x+h)^{\gamma-2}f^{1-\gamma}, \\ V_{x\mathbf{k}} =&(\gamma-1)(x+h)^{\gamma-2}f^{1-\gamma}h_{\mathbf{k}} + (1-\gamma)(x+h)^{\gamma-1}f^{-\gamma}f_{\mathbf{k}}, \\ V_{\mathbf{kk}} =&(\gamma-1)(x+h)^{\gamma-2}f^{1-\gamma}h_{\mathbf{k}}^Th_{\mathbf{k}} + (x+h)^{\gamma-1}f^{1-\gamma}h_{\mathbf{kk}} + 2(1-\gamma)(x+h)^{\gamma-1}f^{-\gamma}f_{\mathbf{k}}h_{\mathbf{k}} \\ &+(\gamma-1)(x+h)^\gamma f^{-\gamma-1}f_{\mathbf{k}}^Tf_{\mathbf{k}} + \frac{1-\gamma}{\gamma}(x+h)^\gamma f^{-\gamma}f_{\mathbf{kk}}. \end{aligned} \tag{C.4}$$

Substituting (C.4) into (C.2) leads to the optimal strategies presented by (29)-(31). Moreover, substituting (C.4) into (C.3) yields

$$\begin{aligned} &(x+h)^{\gamma-1}f^{1-\gamma}\left\{h_t - (r+\lambda)h + \left[\boldsymbol{\mu}^{\mathbf{k}} - \boldsymbol{\Sigma}^{\mathbf{k}}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}-\mathbf{r})\right]^Th_{\mathbf{k}} + \frac{1}{2}tr(\boldsymbol{\Sigma}^{\mathbf{k}}\boldsymbol{\Sigma}^{\mathbf{k}T}h_{\mathbf{kk}}) + i\right\} \\ &+\frac{1-\gamma}{\gamma}(x+h)^\gamma f^{-\gamma}\left\{f_t + \left[\boldsymbol{\mu}^{\mathbf{k}} + \frac{\gamma}{1-\gamma}\boldsymbol{\Sigma}^{\mathbf{k}}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}-\mathbf{r})\right]^Tf_{\mathbf{k}} + \frac{1}{2}tr(\boldsymbol{\Sigma}^{\mathbf{k}}\boldsymbol{\Sigma}^{\mathbf{k}T}f_{\mathbf{kk}})\right. \\ &+\left[\frac{\gamma}{1-\gamma}r - \frac{1}{1-\gamma}\omega - \lambda + \frac{\gamma}{2(1-\gamma)^2}(\boldsymbol{\mu}-\mathbf{r})^T\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}^T\right)^{-1}(\boldsymbol{\mu}-\mathbf{r})\right]f \\ &\left.+\alpha^{\frac{1}{1-\gamma}} + \beta^{\frac{1}{1-\gamma}}\lambda\right\} = 0. \end{aligned} \tag{C.5}$$

Setting the coefficients of $(x+h)^{\gamma-1}f^{1-\gamma}$ and $\frac{1-\gamma}{\gamma}(x+h)^\gamma f^{-\gamma}$ to be zeros gives rise to PDEs (27) and (28).

Appendix D.

Poof of Proposition 3 From (34), we have

$$\begin{aligned} h(t,\mathbf{k}) &= \mathbb{E}^{\mathbb{Q}}_{t,\mathbf{k}}\left\{\int_t^T i(s)\exp\left[-\int_t^s(\lambda(u)+r(u))du\right]ds\right\} \\ &= \int_t^T i(s)e^{-\int_t^s\lambda(u)du}\mathbb{E}^{\mathbb{Q}}_{t,\mathbf{k}}\left[e^{-\int_t^s r(u)du}\right]ds. \end{aligned} \tag{D.1}$$

It follows from the Girsanov theorem that

$$\mathbf{W}^{\mathbb{Q}}(t) = \mathbf{W}(t) + \int_0^t \boldsymbol{\theta}(s)ds \tag{D.2}$$

is a standard Brownian motion under $\mathbb{Q}$, where $\mathbf{W}^{\mathbb{Q}}(t) = (W_1^{\mathbb{Q}}(t), W_2^{\mathbb{Q}}(t))^T$, $\mathbf{W}(t) = (W_1(t), W_2(t))^T$ and $\boldsymbol{\theta}(s) = (\theta_1(s), \theta_2(s))^T$. Recall that

$$\begin{aligned} \boldsymbol{\theta} =& \boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}-\mathbf{r}) \\ =& \left[\phi+\phi_y y(t)+\phi_z\hat{z}(t), \frac{q_r-\rho_{Sr}(\phi+\phi_y y(t)+\phi_z\hat{z}(t))}{\hat{\rho}_r}\right]^T. \end{aligned} \tag{D.3}$$

Then, the interest rate process can be rewritten as

$$dr(t) = (\kappa_r\bar{r}+\sigma_r q_r-\kappa_r r(t))\,dt - \sigma_r\left[\rho_{Sr}dW_1^{\mathbb{Q}}(t)+\hat{\rho}_r dW_2^{\mathbb{Q}}(t)\right]. \tag{D.4}$$

The solution to (D.4) is

$$\begin{aligned} r(u) =& e^{-\kappa_r(u-t)}r(t) + \frac{\chi}{\kappa_r}(1-e^{-\kappa_r(u-t)}) - \int_t^u \sigma_r\rho_{Sr}e^{-\kappa_r(u-v)}dW_1^{\mathbb{Q}}(v) \\ & - \int_t^u \sigma_r\hat{\rho}_r e^{-\kappa_r(u-v)}dW_2^{\mathbb{Q}}(v), \end{aligned} \tag{D.5}$$

for $u \geq t$, where $\chi{=}\kappa_r\bar{r}+\sigma_r q_r$. By virtue of the stochastic Fubini Theorem, we have

$$\begin{aligned} &\int_t^s\int_t^u \sigma_r\rho_{Sr}e^{-\kappa_r(u-v)}dW_1^{\mathbb{Q}}(v)du = \int_t^s\int_v^s \sigma_r\rho_{Sr}e^{-\kappa_r(u-v)}dudW_1^{\mathbb{Q}}(v) \\ =& \int_t^s \frac{\sigma_r\rho_{Sr}}{\kappa_r}(1-e^{-\kappa_r(s-v)})dW_1^{\mathbb{Q}}(v), \end{aligned} \tag{D.6}$$

and

$$\int_t^s\int_t^u \sigma_r\hat{\rho}_r e^{-\kappa_r(u-v)}dW_2^{\mathbb{Q}}(v)du = \int_t^s \frac{\sigma_r\hat{\rho}_r}{\kappa_r}(1-e^{-\kappa_r(s-v)})dW_2^{\mathbb{Q}}(v). \tag{D.7}$$

Consequently,

$$\begin{aligned} e^{-\int_t^s r(u)du} =& \exp\left\{\left(\frac{\chi}{\kappa_r^2}-\frac{r(t)}{\kappa_r}\right)(1-e^{-\kappa_r(s-t)}) - \frac{\chi}{\kappa_r}(s-t)\right. \\ & \left.+\frac{\sigma_r\rho_{Sr}}{\kappa_r}\int_t^s(1-e^{-\kappa_r(s-u)})dW_1^{\mathbb{Q}}(u) + \frac{\sigma_r\hat{\rho}_r}{\kappa_r}\int_t^s(1-e^{-\kappa_r(s-u)})dW_2^{\mathbb{Q}}(u)\right\}. \end{aligned} \tag{D.8}$$

By substituting (D.8) into (D.1), we obtain

$$
\begin{aligned}
h(t,\mathbf{k}) = \int_t^T i(s)\exp\Bigg\{ & -\int_t^s \lambda(v)dv + \left[\frac{\chi}{\kappa_r} - r(t)\right] b(s-t) - \frac{\chi}{\kappa_r}(s-t) \\
& + \frac{1}{2}\sigma_r^2 \int_t^s b^2(s-u)du \Bigg\} ds,
\end{aligned}
\tag{D.9}
$$

where $b(s-t)$ is given by (8).

Appendix E.

Poof of Proposition 4 From (35), $f(t,\mathbf{k})$ can be rewritten as

$$
\begin{aligned}
f(t,\mathbf{k}) = & e^{-\int_t^T \left(\frac{1}{1-\gamma}\omega+\lambda(u)\right)du}\widetilde{\mathbb{E}}_{t,\mathbf{k}}\left[\exp\left(\int_t^T \frac{\gamma}{2(1-\gamma)^2}(\boldsymbol{\mu}-\mathbf{r})^T\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}^T\right)^{-1}(\boldsymbol{\mu}-\mathbf{r}) + \frac{\gamma}{1-\gamma}r(u)du\right)\right] \\
& + \int_t^T K(s)e^{-\int_t^s \left(\frac{1}{1-\gamma}\omega+\lambda(u)\right)du}\widetilde{\mathbb{E}}_{t,\mathbf{k}}\left[\exp\left(\int_s^T \frac{\gamma}{2(1-\gamma)^2}(\boldsymbol{\mu}-\mathbf{r})^T\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}^T\right)^{-1}(\boldsymbol{\mu}-\mathbf{r})\right.\right. \\
& \left.\left. + \frac{\gamma}{1-\gamma}r(u)du\right)\right] ds.
\end{aligned}
\tag{E.1}
$$

Define

$$
\widetilde{f}(t,\mathbf{k}) = \widetilde{\mathbb{E}}_{t,\mathbf{k}}\left\{\exp\left[\int_t^T \left(\frac{\gamma}{2(1-\gamma)^2}(\boldsymbol{\mu}-\mathbf{r})^T\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}^T\right)^{-1}(\boldsymbol{\mu}-\mathbf{r}) + \frac{\gamma}{1-\gamma}r(u)\right)du\right]\right\}. \tag{E.2}
$$

By the Feynman-Kac formula, $\widetilde{f}(t,\mathbf{k})$ is the solution to the following PDE

$$
\begin{aligned}
& \widetilde{f}_t + \left[\frac{\gamma}{2(1-\gamma)^2}(\boldsymbol{\mu}-\mathbf{r})^T\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}^T\right)^{-1}(\boldsymbol{\mu}-\mathbf{r}) + \frac{\gamma}{1-\gamma}r\right]\widetilde{f} + \frac{1}{2}tr\left(\boldsymbol{\Sigma}^{\mathbf{k}}\boldsymbol{\Sigma}^{\mathbf{k}T}\widetilde{f}_{\mathbf{kk}}\right) \\
& + \left[\boldsymbol{\mu}^{\mathbf{k}} + \frac{\gamma}{1-\gamma}\boldsymbol{\Sigma}^{\mathbf{k}}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}-\mathbf{r})\right]^T\widetilde{f}_{\mathbf{k}} = 0,
\end{aligned}
\tag{E.3}
$$

with terminal condition $\widetilde{f}(T,\mathbf{k}) = 1$. Inspired by Liu (2007), we assume that $\widetilde{f}(t,\mathbf{k})$ has the exponential affine form given by (43). Then, the partial derivatives of $\widetilde{f}(t,\mathbf{k})$ are

$$
\begin{aligned}
& \widetilde{f}_t = \widetilde{f}(a_t + \mathbf{b}_t^T\mathbf{k} + \frac{1}{2}\mathbf{k}^T\mathbf{Q}_t\mathbf{k}),\ \widetilde{f}_{\mathbf{k}} = \widetilde{f}(\mathbf{b}+\mathbf{Q}\mathbf{k}), \\
& \widetilde{f}_{\mathbf{kk}} = \widetilde{f}(\mathbf{b}+\mathbf{Q}\mathbf{k})(\mathbf{b}^T + \mathbf{k}^T\mathbf{Q}) + \widetilde{f}\mathbf{Q},
\end{aligned}
\tag{E.4}
$$

where $\widetilde{f}_{\mathbf{kk}}$ denotes the Hessian matrix of $\widetilde{f}$ with respect to $\mathbf{k}$. Substituting (E.4) into equation (E.3) yields

$$
\begin{aligned}
& a_t + \mathbf{b}_t^T\mathbf{k} + \frac{1}{2}\mathbf{k}^T\mathbf{Q}_t\mathbf{k} + \frac{\gamma}{2(1-\gamma)^2}(\boldsymbol{\mu}-\mathbf{r})^T\left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}^T\right)^{-1}(\boldsymbol{\mu}-\mathbf{r}) + \frac{\gamma}{1-\gamma}r + \frac{1}{2}tr\left(\boldsymbol{\Sigma}^{\mathbf{k}}\boldsymbol{\Sigma}^{\mathbf{k}T}\mathbf{Q}\right) \\
& + \frac{1}{2}\left(\mathbf{b}^T + \mathbf{k}^T\mathbf{Q}\right)\boldsymbol{\Sigma}^{\mathbf{k}}\boldsymbol{\Sigma}^{\mathbf{k}T}\left(\mathbf{b}+\mathbf{Q}\mathbf{k}\right) + \left(\boldsymbol{\mu}^{\mathbf{k}} + \frac{\gamma}{1-\gamma}\boldsymbol{\Sigma}^{\mathbf{k}}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}-\mathbf{r})\right)^T(\mathbf{b}+\mathbf{Q}\mathbf{k}) = 0.
\end{aligned}
\tag{E.5}
$$

In what follows, we rewrite the coefficients as linear or quadratic functions of the state vector $\mathbf{k}$.

(1) The drift vector $\boldsymbol{\mu}^{\mathbf{k}}$ and the diffusion matrix $\boldsymbol{\Sigma}^{\mathbf{k}}\boldsymbol{\Sigma}^{\mathbf{k}T}$ are

$$\boldsymbol{\mu}^{\mathbf{k}} = \underbrace{\begin{bmatrix} \kappa_r \bar{r} \\ \kappa_y \bar{y} \\ \kappa_z \bar{z} \end{bmatrix}}_{\boldsymbol{\delta_0}} - \underbrace{\begin{bmatrix} \kappa_r & 0 & 0 \\ 0 & \kappa_y & 0 \\ 0 & 0 & \kappa_z \end{bmatrix}}_{\boldsymbol{\delta_1}} \mathbf{k},$$

and

$$\boldsymbol{\Sigma}^{\mathbf{k}}\boldsymbol{\Sigma}^{\mathbf{k}T} = \underbrace{\begin{bmatrix} \sigma_r^2 & -\sigma_r\sigma_y & -\sigma_r\rho_{Sr}H_1 - \sigma_r\widehat{\rho}_r H_2 \\ -\sigma_r\sigma_y & \sigma_y^2 & \sigma_y\rho_{Sr}H_1 + \sigma_y\widehat{\rho}_r H_2 \\ -\sigma_r\rho_{Sr}H_1 - \sigma_r\widehat{\rho}_r H_2 & \sigma_y\rho_{Sr}H_1 + \sigma_y\widehat{\rho}_r H_2 & H_1^2 + H_2^2 \end{bmatrix}}_{\mathbf{l_0}}.$$

(2) The interest rate r is

$$r = \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{\mathbf{d}} \mathbf{k}.$$

(3) The squared-Sharpe ratio $(\boldsymbol{\mu} - \mathbf{r})^T \left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}^T\right)^{-1} (\boldsymbol{\mu} - \mathbf{r})$ is

$$\begin{aligned} & \frac{(\phi + \phi_y y + \phi_z \hat{z})^2 - 2q_r\rho_{Sr}(\phi + \phi_y y + \phi_z \hat{z}) + q_r^2}{1 - \rho_{Sr}^2} \\ &= \underbrace{\frac{\phi^2 - 2\phi q_r \rho_{Sr} + q_r^2}{1 - \rho_{Sr}^2}}_{h_0} + \underbrace{\frac{2(\phi - q_r\rho_{Sr})}{1 - \rho_{Sr}^2} \begin{bmatrix} 0 & \phi_y & \phi_z \end{bmatrix}}_{\mathbf{h_1}} \mathbf{k} \\ &+ \frac{1}{2}\mathbf{k}^T \underbrace{\frac{2}{1 - \rho_{Sr}^2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \phi_y^2 & \phi_y\phi_z \\ 0 & \phi_y\phi_z & \phi_z^2 \end{bmatrix}}_{\mathbf{h_2}} \mathbf{k}. \end{aligned}$$

(4) The hedging covariance vector $\boldsymbol{\Sigma}^{\mathbf{k}}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu} - \mathbf{r})$ is

$$\underbrace{\begin{bmatrix} -\sigma_r q_r \\ \sigma_y q_r \\ L_1 \end{bmatrix}}_{\mathbf{g_0}} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & L_2 & L_3 \end{bmatrix}}_{\mathbf{g_1}} \mathbf{k},$$

where

$$\begin{cases} L_1 = H_1\phi - \dfrac{\rho_{Sr}H_2}{\hat{\rho}_r}\phi + \dfrac{q_r H_2}{\hat{\rho}_r}, \\ L_2 = H_1\phi_y - \dfrac{\rho_{Sr}H_2}{\hat{\rho}_r}\phi_y, \\ L_3 = H_1\phi_z - \dfrac{\rho_{Sr}H_2}{\hat{\rho}_r}\phi_z, \end{cases}$$

with H_1 and H_2 being presented in (A.11).

Then, equation (E.5) is transformed into the following ODE:

$$a_t + \left(\boldsymbol{\delta}_0 + \frac{\gamma}{1-\gamma}\mathbf{g}_0\right)^T \mathbf{b}(t) + \frac{1}{2}tr\left(\mathbf{l}_0\mathbf{Q}(t)\right) + \frac{1}{2}\mathbf{b}^T(t)\mathbf{l}_0\mathbf{b}(t) + \frac{\gamma}{2(1-\gamma)^2}h_0$$
$$+\left\{\mathbf{b}_t + \frac{\gamma}{2(1-\gamma)^2}\mathbf{h}_1^T + \frac{\gamma}{1-\gamma}\mathbf{d}^T + \mathbf{Q}(t)\mathbf{l}_0\mathbf{b}(t) + \left(-\boldsymbol{\delta}_1 + \tfrac{\gamma}{1-\gamma}\mathbf{g}_1\right)^T\mathbf{b}(t) + \mathbf{Q}(t)\left(\boldsymbol{\delta}_0 + \frac{\gamma}{1-\gamma}\mathbf{g}_0\right)\right\}^T\mathbf{k}$$
$$+\frac{1}{2}\mathbf{k}^T\left\{\mathbf{Q}_t + \mathbf{Q}(t)\left(-\boldsymbol{\delta}_1 + \frac{\gamma}{1-\gamma}\mathbf{g}_1\right) + \left(-\boldsymbol{\delta}_1 + \frac{\gamma}{1-\gamma}\mathbf{g}_1\right)^T\mathbf{Q}(t) + \mathbf{Q}(t)\mathbf{l}_0\mathbf{Q}(t) + \frac{\gamma}{2(1-\gamma)^2}\mathbf{h}_2\right\}\mathbf{k}$$
$$= 0. \tag{E.6}$$

Next, matching the coefficients for $\mathbf{k}$ and constant, we can obtain ODEs (44). Note that $\mathbf{l}_0$ and $\mathbf{h}_2$ are nonnegative definite matrices, and $\gamma < 0$. By the comparison theorem for matrix-valued Riccati equations (see Theorem 4.1.4 in Abou-Kandil et al. (2012)), the unique solution $\mathbf{Q}$ of the matrix-valued Riccati equation in (44) satisfies

$$\hat{\mathbf{Q}}(t) \le \mathbf{Q}(t) \le \mathbf{0}_{3\times3}, \tag{E.7}$$

for $0 \le t \le T$, where $\hat{\mathbf{Q}}$ is the solution of the following linear differential equation:

$$\frac{d\hat{\mathbf{Q}}(t)}{dt} + \left(-\boldsymbol{\delta}_1^{\mathrm{T}} + \frac{\gamma}{1-\gamma}\mathbf{g}_1^{\mathrm{T}}\right)\hat{\mathbf{Q}}(t) + \hat{\mathbf{Q}}(t)\left(-\boldsymbol{\delta}_1 + \frac{\gamma}{1-\gamma}\mathbf{g}_1\right) + \frac{\gamma}{2(1-\gamma)^2}\mathbf{h}_2 = \mathbf{0}_{3\times3}, \tag{E.8}$$

with $\hat{\mathbf{Q}}(T) = \mathbf{0}_{3\times3}$. Consequently, $\mathbf{Q}(t)$ exists for all $t \in [0,T]$, since it cannot have finite escape-time. Moreover, by Radon's lemma (see Theorem 3.1.1 in Abou-Kandil et al. (2012)), the unique solution $\mathbf{Q}$ admits the following representation

$$\mathbf{Q}(t) = \mathbf{Z}(u)\mathbf{R}(u)^{-1}, \tag{E.9}$$

where $u = T - t$, $\mathbf{R}(u), \mathbf{Z}(u) \in \mathcal{C}([0,T], \mathbb{R}^{3\times3})$ and $(\mathbf{R}(u)^T, \mathbf{Z}(u)^T)^T$ are given by

$$\frac{d}{du}\begin{pmatrix}\mathbf{R}\\ \mathbf{Z}\end{pmatrix} = \begin{pmatrix}\boldsymbol{\delta}_1 - \frac{\gamma}{1-\gamma}\mathbf{g}_1 & -\mathbf{l}_0\\ \frac{\gamma}{2(1-\gamma)^2}\mathbf{h}_2 & -\boldsymbol{\delta}_1{}^T + \frac{\gamma}{1-\gamma}\mathbf{g}_1{}^T\end{pmatrix}\begin{pmatrix}\mathbf{R}\\ \mathbf{Z}\end{pmatrix}, \tag{E.10}$$

with initial conditions $\mathbf{R}(0) = \mathbf{I}_{3\times3}$ and $\mathbf{Z}(0) = \mathbf{0}_{3\times3}$. Note that $\left(-\boldsymbol{\delta}_1 + \frac{\gamma}{1-\gamma}\mathbf{g}_1\right)$, $\mathbf{h}_2$ and $\mathbf{l}_0$=$\boldsymbol{\Sigma}^{\mathbf{k}}\boldsymbol{\Sigma}^{\mathbf{k}T}$ are matrices with constant elements. Then, the solution to $(\mathbf{R}(u)^T, \mathbf{Z}(u)^T)^T$ is given by

$$\begin{pmatrix}\mathbf{R}(u)\\ \mathbf{Z}(u)\end{pmatrix} = \exp\left[\begin{pmatrix}\boldsymbol{\delta}_1 - \frac{\gamma}{1-\gamma}\mathbf{g}_1 & -\mathbf{l}_0\\ \frac{\gamma}{2(1-\gamma)^2}\mathbf{h}_2 & -\boldsymbol{\delta}_1{}^T + \frac{\gamma}{1-\gamma}\mathbf{g}_1{}^T\end{pmatrix}u\right]\begin{pmatrix}\mathbf{I}_{3\times3}\\ \mathbf{0}_{3\times3}\end{pmatrix}, \tag{E.11}$$

where $\mathbf{I}_{n\times n}$ is the n-dimensional identity matrix. Substituting (E.11) into (E.9) gives the explicit solution to $\mathbf{Q}(t)$. Hence, the linear ODE for vector-valued function $\mathbf{b}(t)$ also has an explicit solution as follows (see Brockett (2015)):

$$\mathbf{b}^T(u) = \int_0^u \left\{ \left[\mathbf{Q}(s) \left(\boldsymbol{\delta}_0 + \frac{\gamma}{1-\gamma}\mathbf{g}_0 \right) + \frac{\gamma}{2(1-\gamma)^2}\mathbf{h}_1^T + \frac{\gamma}{1-\gamma}\mathbf{d}^T \right]^T \boldsymbol{\Phi}(s,0) \right\} ds[\boldsymbol{\Phi}(u,0)]^{-1}, \tag{E.12}$$

with the martix-valued function $\boldsymbol{\Phi}(t,s)$ satisfying the following linear system

$$\frac{\partial \boldsymbol{\Phi}(t,s)}{\partial t} = \left(\boldsymbol{\delta}_1 - \frac{\gamma}{1-\gamma}\mathbf{g}_1 - \mathbf{Q}(t)\mathbf{l}_0 \right) \boldsymbol{\Phi}(t,s), \;\; \boldsymbol{\Phi}(s,s) = \mathbf{I}_{3\times 3}. \tag{E.13}$$

Therefore, the real-valued function $a(t)$ can be obtained by an integration:

$$a(u) = \int_0^u \left[\left(\delta_0 + \frac{\gamma}{1-\gamma}\mathbf{g}_0 \right)^T \mathbf{b}(s) + \frac{1}{2}\mathbf{b}^T(s)\mathbf{l}_0\mathbf{b}(s) + \frac{1}{2}tr(\mathbf{l}_0\mathbf{Q}(s)) + \frac{\gamma}{2(1-\gamma)^2}h_0 \right] ds. \tag{E.14}$$

Finally, by (41), (43) and the solutions to $\mathbf{Q}(t), \mathbf{b}(t)$ and $a(t)$ (E.9, E.12, E.14), we have the concrete expression for $f(t,\mathbf{k})$.

Appendix F.

Proof of Proposition 5 Similar to the proof of Proposition 2, we conjecture that the value functions $\widetilde{V}^{(i)}(i = 1,2)$ has the form (54) and (60) respectively. Substituting the admissible strategies $\widetilde{\boldsymbol{\psi}}^{(i)}$ given in (53), (59) and the guess for $\widetilde{V}^{(i)}$ into the HJB equation (25) without the supremum over $\boldsymbol{\psi} \in \mathcal{A}$, we obtain the following PDE:

$$\begin{aligned} &-(\lambda+\omega)\widetilde{V}^{(i)} + \widetilde{V}_t^{(i)} + \left[rx + \widetilde{\boldsymbol{\pi}}^{(i)T}(\boldsymbol{\mu} - \mathbf{r}) + i - \widetilde{c}^{(i)} - \widetilde{p}^{(i)} \right] \widetilde{V}_x^{(i)} + \boldsymbol{\mu}^{\mathbf{k}T}\widetilde{V}_{\mathbf{k}}^{(i)} + \widetilde{\boldsymbol{\pi}}^{(i)T}\boldsymbol{\Sigma}\boldsymbol{\Sigma}^{\mathbf{k}T}\widetilde{V}_{x\mathbf{k}}^{(i)} \\ &+\frac{1}{2}\widetilde{\boldsymbol{\pi}}^{(i)T}\boldsymbol{\Sigma}\boldsymbol{\Sigma}^T\widetilde{\boldsymbol{\pi}}^{(i)}\widetilde{V}_{xx}^{(i)} + \frac{1}{2}tr\left(\boldsymbol{\Sigma}^{\mathbf{k}}\boldsymbol{\Sigma}^{\mathbf{k}T}\widetilde{V}_{\mathbf{kk}}^{(i)} \right) + \alpha\frac{c^\gamma}{\gamma} + \beta\frac{\lambda}{\gamma}\left(x + \frac{\widetilde{p}^{(i)}}{\lambda} \right)^\gamma = 0. \end{aligned} \tag{F.1}$$

Substituting the partial derivatives of the value functions $\widetilde{V}^{(i)}$ into (F.1) and letting the coefficient of $\frac{1-\gamma}{\gamma}(x + h^{(i)}(t,\mathbf{k}))f^{(i)^{-\gamma}}(t,\mathbf{k})$ be zero yields

$$\begin{aligned} &f_t^{(i)} + \left(\mathbf{F}_1^{(i)} + \mathbf{F}_2^{(i)}\mathbf{k} \right)^T f_{\mathbf{k}}^{(i)} + \frac{1}{2}tr\left(\boldsymbol{\Sigma}^{\mathbf{k}}\boldsymbol{\Sigma}^{\mathbf{k}T}f_{\mathbf{kk}}^{(i)} \right) - \frac{\gamma}{2f^{(i)}}f_{\mathbf{k}}^{(i)T}\boldsymbol{\Sigma}^{\mathbf{k}}\boldsymbol{\Sigma}^{\mathbf{k}T}f_{\mathbf{k}}^{(i)} \\ &+\frac{\gamma}{(1-\gamma)^2}\left(F_3^{(i)} + \mathbf{F}_4^{(i)}\mathbf{k} + \frac{1}{2}\mathbf{k}^T\mathbf{F}_5^{(i)}\mathbf{k} \right) f^{(i)} = 0, \end{aligned} \tag{F.2}$$

where the parameters $\mathbf{F}_1^{(i)}$, $\mathbf{F}_2^{(i)}$, $F_3^{(i)}$, $\mathbf{F}_4^{(i)}$, $\mathbf{F}_5^{(i)}$ are elaborated below. We suppose that $f^{(i)}(t,\mathbf{k})(i = 1,2)$ has the form (55) and (62) respectively. Plugging the partial derivatives of $f^{(i)}(t,\mathbf{k})$ into (F.2) and letting the coefficients be zeros lead to

$$\begin{cases} \dfrac{d\widetilde{\mathbf{Q}}^{(i)}(t)}{dt}+\left(\mathbf{F}_2^{(i)}\right)^T\widetilde{\mathbf{Q}}^{(i)}(t)+\widetilde{\mathbf{Q}}^{(i)}(t)\mathbf{F}_2^{(i)}+\widetilde{\mathbf{Q}}^{(i)}(t)\boldsymbol{\Sigma}^{\mathbf{k}}\boldsymbol{\Sigma}^{\mathbf{k}T}\widetilde{\mathbf{Q}}^{(i)}(t)+\dfrac{\gamma}{2(1-\gamma)^2}\left(\mathbf{F}_5^{(i)}+\mathbf{F}_5^{(i)T}\right)=\mathbf{0}_{3\times 3},\\ \dfrac{d\widetilde{\mathbf{b}}^{(i)}(t)}{dt}+\left(\mathbf{F}_2^{(i)}\right)^T\widetilde{\mathbf{b}}^{(i)}(t)+\widetilde{\mathbf{Q}}^{(i)}(t)\boldsymbol{\Sigma}^{\mathbf{k}}\boldsymbol{\Sigma}^{\mathbf{k}T}\widetilde{\mathbf{b}}^{(i)}(t)+\widetilde{\mathbf{Q}}^{(i)}(t)\mathbf{F}_1^{(i)}+\dfrac{\gamma}{2(1-\gamma)^2}\left(\mathbf{F}_4^{(i)}\right)^T=\mathbf{0}_{3\times 1},\\ \dfrac{d\widetilde{a}^{(i)}(t)}{dt}+\left(\mathbf{F}_1^{(i)}\right)^T\widetilde{\mathbf{b}}^{(i)}(t)+\dfrac{1}{2}\widetilde{\mathbf{b}}^{(i)T}(t)\boldsymbol{\Sigma}^{\mathbf{k}}\boldsymbol{\Sigma}^{\mathbf{k}T}\widetilde{\mathbf{b}}^{(i)}(t)+\dfrac{1}{2}tr\left(\boldsymbol{\Sigma}^{\mathbf{k}}\boldsymbol{\Sigma}^{\mathbf{k}T}\widetilde{\mathbf{Q}}^{(i)}(t)\right)+\dfrac{\gamma}{2(1-\gamma)^2}F_3^{(i)}=0. \end{cases} \tag{F.3}$$

1. Ignore learning. The coefficients $\mathbf{F}_1^{(1)}$, $\mathbf{F}_2^{(1)}$, $F_3^{(1)}$, $\mathbf{F}_4^{(1)}$, $\mathbf{F}_5^{(1)}$ are given by

$$\mathbf{F}_1^{(1)}=\boldsymbol{\delta}_0+\frac{\gamma}{1-\gamma}\begin{bmatrix}-\sigma_r q_r\\ \sigma_y q_r\\ L_1\end{bmatrix}+\gamma\boldsymbol{\Sigma}^{\mathbf{k}}\widetilde{\boldsymbol{\Sigma}}^{\mathbf{k}(1)}\bar{\mathbf{b}}^{(1)}(t),$$

$$\mathbf{F}_2^{(1)}=-\boldsymbol{\delta}_1+\frac{\gamma}{1-\gamma}\begin{bmatrix}0&0&0\\0&0&0\\0&L_2&0\end{bmatrix}+\gamma\boldsymbol{\Sigma}^{\mathbf{k}}\widetilde{\boldsymbol{\Sigma}}^{\mathbf{k}(1)}\bar{\mathbf{Q}}^{(1)}(t),$$

$$L_1=\left(H_1-\frac{\rho_{Sr}H_2}{\hat{\rho}_r}\right)(\phi+\phi_z\bar{z})+\frac{q_r H_2}{\hat{\rho}_r},$$

$$L_2=H_1\phi_y-\frac{\rho_{Sr}H_2}{\hat{\rho}_r}\phi_y,$$

$$\begin{aligned} F_3^{(1)}=&-\frac{(\phi+\phi_z\bar{z})^2-2(\phi+\phi_z\bar{z})q_r\rho_{Sr}+q_r^2}{2(1-\rho_{Sr}^2)}-(1-\gamma)\bar{\mathbf{b}}^{(1)T}(t)\widetilde{\mathbf{g}}_0\\ &+\frac{\phi(\phi+\phi_z\bar{z})-(2\phi+\phi_z\bar{z})q_r\rho_{Sr}+q_r^2}{1-\rho_{Sr}^2}-\frac{(1-\gamma)^2}{2}\bar{\mathbf{b}}^{(1)T}\widetilde{\mathbf{l}}_0\bar{\mathbf{b}}^{(1)}+(1-\gamma)\bar{\mathbf{b}}^{(1)T}\mathbf{K}, \end{aligned}$$

$$\begin{aligned} \mathbf{F}_4^{(1)}=&-\frac{\phi+\phi_z\bar{z}-q_r\rho_{Sr}}{1-\rho_{Sr}^2}\begin{bmatrix}0&\phi_y&0\end{bmatrix}-(1-\gamma)\widetilde{\mathbf{g}}_0^T\bar{\mathbf{Q}}^{(1)T}(t)\\ &+\frac{1}{1-\rho_{Sr}^2}\begin{bmatrix}2\phi\phi_y-2\phi_y q_r\rho_{Sr}+\phi_y\phi_z\bar{z}&0&\phi\phi_z+\phi_z^2\bar{z}-\phi_z q_r\rho_{Sr}\end{bmatrix}\\ &-(1-\gamma)^2\bar{\mathbf{b}}^{(1)T}\widetilde{\mathbf{l}}_0\bar{\mathbf{Q}}^{(1)}+(1-\gamma)K^T\bar{\mathbf{Q}}^{(1)T}+(1-\gamma)\mathbf{d}, \end{aligned}$$

$$\mathbf{F}_5^{(1)}=-\frac{1}{1-\rho_{Sr}^2}\begin{bmatrix}0&0&0\\0&\phi_y^2&0\\0&0&0\end{bmatrix}+\frac{2}{1-\rho_{Sr}^2}\begin{bmatrix}0&0&0\\0&\phi_y^2&0.5\phi_y\phi_z\\0&0.5\phi_y\phi_z&\phi_z^2\end{bmatrix}-(1-\gamma)^2\bar{\mathbf{Q}}^{(1)T}\widetilde{\mathbf{l}}_0\bar{\mathbf{Q}}^{(1)},$$

$$\mathbf{K}=\begin{bmatrix}-\sigma_r q_r&\sigma_y q_r&0\end{bmatrix}^T.$$

Here, the coefficients $\widetilde{\boldsymbol{\delta}}_0, \widetilde{\boldsymbol{\delta}}_1, \widetilde{\mathbf{l}}_0, \mathbf{d}, \widetilde{h}_0, \widetilde{\mathbf{h}}_1, \widetilde{\mathbf{h}}_2, \widetilde{\mathbf{g}}_0, \widetilde{\mathbf{g}}_1$ are elaborated as follows:

$$\widetilde{\boldsymbol{\mu}}^{\mathbf{k}(1)}=\underbrace{\begin{bmatrix}\kappa_r\bar{r}\\ \kappa_y\bar{y}\\ 0\end{bmatrix}}_{\widetilde{\boldsymbol{\delta}}_0}-\underbrace{\begin{bmatrix}\kappa_r&0&0\\0&\kappa_y&0\\0&0&0\end{bmatrix}}_{\widetilde{\boldsymbol{\delta}}_1}\mathbf{k},$$

$$\widetilde{\boldsymbol{\Sigma}}^{\mathbf{k}(1)}\widetilde{\boldsymbol{\Sigma}}^{\mathbf{k}(1)T} = \underbrace{\begin{bmatrix} \sigma_r^2 & -\sigma_r\sigma_y & 0 \\ -\sigma_r\sigma_y & \sigma_y^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\widetilde{\mathbf{l}}_0},$$

$$r = \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{\mathbf{d}}\mathbf{k},$$

$$\begin{aligned} &\left(\widetilde{\boldsymbol{\mu}}^{(1)} - \mathbf{r}\right)^T \left(\boldsymbol{\Sigma}\boldsymbol{\Sigma}^T\right)^{-1} \left(\widetilde{\boldsymbol{\mu}}^{(1)} - \mathbf{r}\right) \\ &= \underbrace{\frac{(\phi + \phi_z\bar{z})^2 - 2(\phi + \phi_z\bar{z})q_r\rho_{Sr} + q_r^2}{1 - \rho_{Sr}^2}}_{\widetilde{h}_0} + \underbrace{\frac{2\left(\phi + \phi_z\bar{z} - q_r\rho_{Sr}\right)}{1 - {\rho_{Sr}}^2}\begin{bmatrix} 0 & \phi_y & 0 \end{bmatrix}}_{\widetilde{\mathbf{h}}_\mathbf{1}}\mathbf{k} \\ &+ \frac{1}{2}\mathbf{k}^T \underbrace{\frac{2}{1 - \rho_{Sr}^2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & \phi_y^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\widetilde{\mathbf{h}}_\mathbf{2}}\mathbf{k}, \end{aligned}$$

$$\widetilde{\boldsymbol{\Sigma}}^{\mathbf{k}(1)}\boldsymbol{\Sigma}^{-1}\left(\widetilde{\boldsymbol{\mu}}^{(1)} - \mathbf{r}\right) = \underbrace{\begin{bmatrix} -\sigma_r q_r & \sigma_y q_r & 0 \end{bmatrix}^T}_{\widetilde{\mathbf{g}}_\mathbf{0}} + \underbrace{\mathbf{0}}_{\widetilde{\mathbf{g}}_\mathbf{1}} \cdot \mathbf{k}.$$

2. Ignore the stochastic interest rate. The coefficients $\mathbf{F}_1^{(2)}, \mathbf{F}_2^{(2)}, F_3^{(2)}, \mathbf{F}_4^{(2)}, \mathbf{F}_5^{(2)}$ are given by

$$\mathbf{F}_1^{(2)} = \boldsymbol{\delta}_0 + \gamma\boldsymbol{\Sigma}^{\mathbf{k}}[m_1, 0]^T,$$

$$\mathbf{F}_2^{(2)} = -\boldsymbol{\delta}_1 + \gamma\boldsymbol{\Sigma}^{\mathbf{k}}\begin{bmatrix} 0 & 0 & m_2 \\ 0 & 0 & 0 \end{bmatrix},$$

$$F_3^{(2)} = (1-\gamma)\left(\phi + \phi_y\bar{y}\right)m_1 - \frac{1}{2}(1-\gamma)^2 m_1^2,$$

$$\mathbf{F}_4^{(2)} = \begin{bmatrix} (1-\gamma)\phi_y m_1, & (1-\gamma), & (1-\gamma)m_1\phi_z + (1-\gamma)\left(\phi + \phi_y\bar{y}\right)m_2 - (1-\gamma)^2 m_1 m_2 \end{bmatrix},$$

$$\mathbf{F}_5^{(2)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2(1-\gamma)\phi_y m_2 & 2(1-\gamma)\phi_z m_2 - (1-\gamma)^2 m_2^2 \end{bmatrix},$$

$$m_1 = \frac{\phi + \phi_y\bar{y}}{1-\gamma} + \frac{\widetilde{H}_1}{\sigma_S}\bar{\mathbf{b}}^{(2)}(t), \quad m_2 = \frac{\phi_z}{1-\gamma} + \frac{\widetilde{H}_1}{\sigma_S}\bar{\mathbf{Q}}^{(2)}(t).$$

Here, the coefficients $\widetilde{\delta}_0$, $\widetilde{\delta}_1$, $\widetilde{l}_0$, $\widetilde{h}_0$, $\widetilde{h}_1$, $\widetilde{h}_2$, $\widetilde{g}_0$, $\widetilde{g}_1$ are elaborated as follows:

$$\mu^{\hat{z}(2)} = \underbrace{\kappa_z\bar{z}}_{\widetilde{\delta}_0} - \underbrace{\kappa_z}_{\widetilde{\delta}_1}\hat{z},$$

$$\left(\widetilde{\Sigma}^{\hat{z}(2)}\right)^2 = \underbrace{\left(\sigma_z\rho_{Sz} + \bar{m}\phi_z\right)^2}_{\widetilde{l}_0},$$

$$\bar{r} = d,$$

$$\left(\frac{\widetilde{\mu}^{(2)}-\bar{r}}{\widetilde{\Sigma}^{(2)}}\right)^2 = \underbrace{(\phi+\phi_y\bar{y})^2}_{\widetilde{h}_0} + \underbrace{2(\phi+\phi_y\bar{y})\phi_z}_{\widetilde{h}_1}\hat{z} + \frac{1}{2}\underbrace{\left(2\phi_z^2\right)}_{\widetilde{h}_2}\hat{z}^2,$$

$$\frac{\widetilde{\Sigma}^{\hat{z}(2)}}{\widetilde{\Sigma}^{(2)}}(\widetilde{\mu}^{(2)}-\bar{r}) = \underbrace{(\widetilde{H}_1+\phi+\phi_y\bar{y})}_{\widetilde{g}_0} + \underbrace{\phi_z}_{\widetilde{g}_1}\hat{z}.$$