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Optimal entry decision of unemployment insurance under partial information

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ABSTRACT

The aim of this paper is to study the optimal time for the individual to join an unemployment insurance scheme which is intended to protect workers against the consequences of job loss and to encourage the unemployed workers to find a new job as early as possible. The wage dynamic is described by a geometric Brownian motion model under drift uncertainty and the problem is a kind of two-dimensional degenerate optimal stopping problems which is hard to analyze. The optimal time of decision for the workers is given by the first time at which the wage process hits the free boundary which therefore plays a key role in solving the problem. This paper analyzes the monotonicity and continuity of the free boundary and derives a nonlinear integral equation for it. For a particular case the closed-form formula of free boundary is obtained and for the general case the free boundary is solved by the numerical solution of the nonlinear integral equation. The key in the analysis is to convert the degenerate problem into the non-degenerate one using the probability approach.

1. Introduction

Due to the COVID-19 pandemic, unemployment poses great challenge to society. The decision for the individual to enter an unemployment insurance (UI) contract is crucial to help cushion the financial blow of loss of job. Various UI systems are available in many countries and funded by the governments or insurance company (Holmlund (1998)). Mortensen (1977) has been a basic reference in the literature on UI. The design of UI contract has received considerable attention from economic literature (see e.g., Hopenhayn and Nicolini (2009), Biagini and Widenmann (2012), Biagini et al. (2013), Barnichon and Zylberberg (2022), and the recent book Potestio (2022) and references therein).

A particular type of UI products is designed to help cushion the financial blow of loss of job and to encourage unemployed workers to find a new job as early as possible in view of the continued reduction of benefits. The protection is normally provided in the form of regular financial benefits payable after the insured individual becomes unemployed and until a new job is found, but often only up to a certain maximum duration and with payments gradually decreasing over time. The optimal entry time to join the scheme is the main concern of this product. More precisely, the individual should choose an optimal entry time to maximize the expected net present value

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of the UI scheme. Anquandah and Bogachev (2019) formulate this problem as a perpetual optimal stopping problem under the assumption that the wage process follows geometric Brownian motion and give a closed-form solution to this problem.

To be noted that the UI scheme in Anquandah and Bogachev (2019) is designed on the complete information of the wage process. However in practice incomplete information is often inevitable as one needs a very long time series to estimate the drift which is rarely available. The incomplete models have been widely used in the financial market, for example, optimal liquidation problems (Ekström and Lu (2011), Lu (2013), Ekström and Vannestål (2016), Ekström and Vaicenavicius (2016) and Vaicenavicius (2020)); optimal investment problems (Björk and Davis (2010), Hata and Sheu (2018), Bäuerle and Chen (2019), and Xiong et al. (2021)); American option pricing (Gapeev (2012) and Ekström and Vannestål (2019)); optimal time to invest in an indivisible project (Décamps et al. (2005) and Klein (2009)); optimal redeeming problem of stock loan (Xu and Yi (2020)); perpetual commodity equities (Gapeev (2021)); optimal insurance (Huang et al. (2010), Wei et al. (2012), Ceci et al. (2017), and Brachetta and Ceci (2020)).

But to the best of our knowledge, the incomplete information model in the UI scheme has not received sufficient attention. Therefore, this paper attempts to fill in the gap by studying the optimal time to join the UI scheme under incomplete information. In particular it is assumed in this paper that the individual does not know the drift of the wage process, so he or she has to make a timing decision to enter an UI scheme based on the incomplete information. The unknown drift of the wage makes the corresponding two-dimensional optimal stopping problem degenerate which is hard to analyze and not possible to solve using the argument in Anquandah and Bogachev (2019). To overcome the difficulty, this paper introduces an auxiliary non-degenerate optimal stopping problem. The monotonicity and continuity of the free boundary to the auxiliary problem are proved and the nonlinear integral equation is derived. In turn, the free boundary to the auxiliary problem is solved analytically for a particular case or numerically by the numerical methods for the nonlinear integral equation. With these theoretical and numerical results, the original two-dimensional optimal stopping problem is then solved by the inverse transformation.

The rest of this paper is arranged as follows. In Section 2, we set up the model and then use the techniques from filtering theory to reformulate it as an auxiliary non-degenerate optimal stopping problem. In Section 3, we analyze the auxiliary problem and obtain the main results for the original problem. In Section 4, we give the numerical results. In Section 5, we give the conclusions.

2. Optimal stopping problems of unemployment insurance

In this section, we model the problem of unemployment insurance. On a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ where $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ satisfies the usual conditions, we assume that the dynamic of the individual's wage X_t follows

$$dX_t = \mu X_t dt + \sigma X_t d\bar{W}_t, \quad X_0 = x, \tag{2.1}$$

where $\bar{W}$ is a standard Brownian motion, $\sigma > 0$ is the volatility and μ, which is uncertain and independent of the Brownian motion $\bar{W}$, takes only two possible values μ_h and μ_l with $\mu_h > \mu_l$. Let τ_0 be the time that the individual gets unemployment and the individual finds a new job again after the unemployment spell of duration τ_1. Furthermore, the random times τ_0 and τ_1, which are both independent of X_t, have independent exponential distribution (with parameters λ_0 and λ_1, respectively). We assume that

$$\mathbb{E}\Big[\int_0^\infty e^{-\tilde{r}t} X_t dt\Big] < \infty, \tag{2.2}$$

where $\tilde{r} := r + \lambda_0$ and $r > 0$ is the inflation rate. The dynamic benefit $h(s)X_{\tau_0}$ will be paid by the insurance company during the period of unemployment, where X_{τ_0} is final wage and $h(s)$ is a decreasing function in order to encourage the individual to look for a new job. Using Fubini's theorem, the expected future benefit to be received during the period of unemployment is given by

$$X_{\tau_0}\mathbb{E}\Big[\int_0^{\tau_1} e^{-rs}h(s)ds\Big] = X_{\tau_0}\int_0^\infty e^{-(r+\lambda_1)s}h(s)ds = \gamma X_{\tau_0},$$

where

$$\gamma := \int_0^\infty e^{-(r+\lambda_1)s}h(s)ds. \tag{2.3}$$

Next, we consider a delayed entry time $\tau \in \mathcal{T}$ and the payment of premium is P, where P is a positive constant and $\mathcal{T}$ is the set of $\mathbb{F}^X$–stopping time. Here $\mathbb{F}^X = \{\mathcal{F}_t^X\}_{t\geq 0}$ is the natural filtration generated by the wage process X. Then the individual's problem is to maximize the gain by choosing an optimal entry time $\tau \in \mathcal{T}$, i.e.,

$$\mathbb{V}(x,\pi) := \sup_{\tau\in\mathcal{T}} \mathbb{E}\Big[e^{-r\tau}(e^{-r(\tau_0-\tau)}\gamma X_{\tau_0} - P)I(\tau < \tau_0)\Big], \tag{2.4}$$

where $I(\cdot)$ is the indicator function of some set.

To determine the optimal stopping time, the policyholder has to estimate the current trend of the wage first. To this end, we introduce a posteriori probability process $\Pi = (\Pi_t)_{t\geq 0}$ as

$$\Pi_t := \mathbb{P}(\mu = \mu_h | \mathcal{F}_t^X),$$

and the innovation process

$$\tilde{W}_t := \bar{W}_t + \int_0^t \frac{\mu - (1-\Pi_s)\mu_l - \Pi_s \mu_h}{\sigma} ds.$$

By Lévy's characterization theorem, $\tilde{W}$ is a $\mathbb{P}-$standard Brownian motion. The innovation approach (see (Liptser and Shiryaev, 2001, Chapter 7.9)) gives that

$$d\Pi_t = \omega \Pi_t (1-\Pi_t) d\tilde{W}_t, \tag{2.5}$$

where

$$\omega := \frac{\mu_h - \mu_l}{\sigma} > 0. \tag{2.6}$$

Applying Itô's formula, the wage dynamic (2.1) is re-written as

$$\frac{dX_t}{X_t} = [\mu_l + \Pi_t(\mu_h - \mu_l)]dt + \sigma d\tilde{W}_t. \tag{2.7}$$

We then simplify the individual's problem (2.4) as the following proposition.

Proposition 2.1. *Finding the solution to problem (2.4) is equivalent to finding the solution to the following optimal stopping problem:*

$$\inf_{\tau\in\mathcal{T}} \mathbb{E}\Big[\int_0^\tau \gamma\lambda_0 e^{-\tilde{r}\xi} X_\xi d\xi + Pe^{-\tilde{r}\tau}\Big| X_0 = x, \Pi_0 = \pi\Big], \tag{2.8}$$

where $\tilde{r} := r + \lambda_0$, which is further equivalent to finding the solution to

$$\hat{V}(x,\pi) := \inf_{\tau\in\mathcal{T}} \mathbb{E}\Big[\int_0^\tau e^{-\tilde{r}t}(\gamma\lambda_0 X_t - \tilde{r}P)dt \Big| X_0 = x, \Pi_0 = \pi\Big]. \tag{2.9}$$

Proof. Due to law of total expectation, Fubini's theorem, and the fact that τ_0 is independent of X and τ, a direct computation shows that

$$\begin{aligned}
&\mathbb{E}[e^{-r\tau}(e^{-r(\tau_0-\tau)}\gamma X_{\tau_0} - P)I(\tau<\tau_0)] \\
&= \mathbb{E}[\mathbb{E}[e^{-r\tau}(e^{-r(\tau_0-\tau)}\gamma X_{\tau_0} - P)I(\tau<\tau_0)|\tau_0]] \\
&= \int_0^\infty \mathbb{E}[e^{-r\tau}(e^{-r(\tau_0-\tau)}\gamma X_{\tau_0} - P)I(\tau<\tau_0)|\tau_0=\xi]\lambda_0 e^{-\lambda_0\xi}d\xi \\
&= \int_0^\infty \mathbb{E}[e^{-r\tau}(e^{-r(\xi-\tau)}\gamma X_\xi - P)I(\tau<\xi)]\lambda_0 e^{-\lambda_0\xi}d\xi \\
&= \mathbb{E}\Big[\int_\tau^\infty e^{-r\tau}(e^{-r(\xi-\tau)}\gamma X_\xi - P)\lambda_0 e^{-\lambda_0\xi}d\xi\Big] \\
&= \mathbb{E}\Big[\int_0^\infty e^{-\tilde{r}\xi}\lambda_0\gamma X_\xi d\xi\Big] - \mathbb{E}\Big[\int_0^\tau e^{-\tilde{r}\xi}\lambda_0\gamma X_\xi d\xi + Pe^{-\tilde{r}\tau}\Big].
\end{aligned} \tag{2.10}$$

Since the first term in the last equality of (2.10) is independent of stopping time τ, solving problem (2.4) is equivalent to solving problem (2.8), which is further equivalent to solving problem (2.9) using Itô's formula. □

Utilizing the dynamic programming principle, the value function $\hat{V}$ satisfies the following variational inequality

$$\min\Big\{\mathcal{L}_{X,\Pi}\hat{V} - \tilde{r}\hat{V} + \gamma\lambda_0 x - \tilde{r}P, -\hat{V}\Big\} = 0, \tag{2.11}$$

where the infinitesimal generator of (X_t, Π_t) is given by

$$\mathcal{L}_{X,\Pi}\hat{V} = \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 \hat{V}}{\partial x^2} + \frac{1}{2}\omega^2\pi^2(1-\pi)^2\frac{\partial^2\hat{V}}{\partial\pi^2} + \sigma x\pi(1-\pi)\omega\frac{\partial^2\hat{V}}{\partial x\partial\pi} + (\omega\sigma\pi + \mu_l)x\frac{\partial\hat{V}}{\partial x}.$$

Since $\mathcal{L}_{X,\Pi}$ is degenerate in the entire region $(0,+\infty)\times(0,1)$, it is hard to analyze (2.11) (or (2.9)) directly.

To analyze the problem, we transform the degenerate optimal stopping problems (2.9) into the non-degenerate one using change of measure. Following Klein (2009), we introduce a new process

$$W_t := \omega \int_0^t \Pi_s ds + \tilde{W}_s,$$

and a new measure $\mathbb{Q}$ by Radon-Nikodym derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{F}_t^X} = \exp\Big(-\frac{1}{2}\int_0^t \omega^2(\Pi_s)^2 ds - \int_0^t \omega \Pi_s d\tilde{W}_s\Big)$$
$$= \exp\Big(\frac{1}{2}\int_0^t \omega^2(\Pi_s)^2 ds - \int_0^t \omega \Pi_s dW_s\Big), \quad \text{for any } t > 0. \tag{2.12}$$

By Girsanov's theorem, W is a $\mathbb{Q}$-standard Brownian motion. Define the likelihood ratio process $\Phi_t := \frac{\Pi_t}{1-\Pi_t}$. Due to Itô's formula, we have Φ satisfies

$$d\Phi_t = \omega \Phi_t dW_t$$

with

$$\Phi_0 = \varphi := \frac{\pi}{1-\pi},$$

which gives that

$$\Phi_t = \varphi Z_t, \tag{2.13}$$

where

$$Z_t := \exp\Big(\omega W_t - \frac{1}{2}\omega^2 t\Big). \tag{2.14}$$

Then we can express the dynamics in terms of W by

$$\begin{pmatrix} dX_t/X_t \\ d\Phi_t/\Phi_t \end{pmatrix} = \begin{pmatrix} \mu_l \\ 0 \end{pmatrix} dt + \begin{pmatrix} \sigma \\ \omega \end{pmatrix} dW_t.$$

Eliminating W yields

$$X_t = x e^{\epsilon t}\Big(\frac{\Phi_t}{\varphi}\Big)^{\beta}, \tag{2.15}$$

where

$$\epsilon := \frac{1}{2}(\mu_h + \mu_l - \sigma^2) \text{ and } \beta := \frac{\sigma^2}{\mu_h - \mu_l} = \frac{\sigma}{\omega} > 0, \tag{2.16}$$

ω is defined in (2.6). Let

$$F_t = \frac{1+\Phi_t}{1+\varphi}.$$

Then

$$\frac{dF_t}{F_t} = \frac{d\Phi_t}{1+\Phi_t} = \frac{\omega \Phi_t dW_t}{1+\Phi_t} = \omega \Pi_t dW_t$$

with $F_0 = 1$. By (2.12), it turns out that

$$\frac{d\mathbb{P}}{d\mathbb{Q}}\Big|_{\mathcal{F}_t^X} = F_t = \exp\Big(-\frac{1}{2}\int_0^t \omega^2(\Pi_s)^2 ds + \int_0^t \omega \Pi_s dW_s\Big), \quad t > 0. \tag{2.17}$$

Now, we will study the optimal stopping problem (2.9) under measure $\mathbb{Q}$.

Proposition 2.2. *With the measure $\mathbb{Q}$ given by (2.17), finding the solution to the optimal stopping problem (2.9) is equivalent to finding the solution to*

$$V(\zeta, \varphi) := \inf_{\tau \in \mathcal{T}} \mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\tau} e^{-\tilde{r}t} H(\zeta + t, \Phi_t) dt\Big], \tag{2.18}$$

for $\epsilon \neq 0$, where ϵ is defined by (2.16), Φ_t by (2.13) and

$$H(\zeta,\varphi):=(1+\varphi)(e^{\epsilon\zeta}\varphi^{\beta}-K),\quad \zeta:=\frac{1}{\epsilon}\log\frac{\gamma\lambda_0 x}{\varphi^{\beta}},\quad K:=\tilde{r}P, \tag{2.19}$$

with assumption that

$$\mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\infty} e^{-\tilde{r}\zeta}|H(\zeta,\Phi_{\zeta})|d\zeta\Big]<\infty \quad or \quad \tilde{r}-\epsilon>\frac{1}{2}\omega^2\beta(\beta+1), \tag{2.20}$$

and finding the solution to (2.9) is equivalent to finding the solution to

$$V(\varphi):=\inf_{\tau\in\mathcal{T}}\mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\tau} e^{-\tilde{r}t}\hat{H}(\Phi_t)dt\Big], \tag{2.21}$$

for $\epsilon=0$, *where*

$$\hat{H}(\varphi):=(1+\varphi)(\varphi^{\beta}-\hat{K}),\quad \hat{K}:=\frac{\tilde{r}P\varphi^{\beta}}{\gamma\lambda_0 x}, \tag{2.22}$$

with assumption that

$$\mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\infty} e^{-\tilde{r}t}|\hat{H}(\Phi_t)|dt\Big]<\infty \quad or \quad \tilde{r}>\frac{1}{2}\omega^2\beta(\beta+1). \tag{2.23}$$

Proof. By monotone convergence theorem and change of measure, we see that

$$\begin{aligned}\mathbb{E}\Big[\int_0^{\tau} e^{-\tilde{r}t}X_t dt\Big]&=\lim_{n\to\infty}\lim_{m\to\infty}\mathbb{E}\Big[\int_0^{\tau\wedge m} e^{-\tilde{r}t}(X_t\wedge n)dt\Big]\\&=\lim_{n\to\infty}\lim_{m\to\infty}\mathbb{E}^{\mathbb{Q}}\Big[F_{\tau\wedge m}\int_0^{\tau\wedge m} e^{-\tilde{r}t}(X_t\wedge n)dt\Big],\end{aligned} \tag{2.24}$$

where $a\wedge b:=\min\{a,b\}$. Furthermore, Itô's formula yields

$$F_s\int_0^s e^{-\tilde{r}t}(X_t\wedge n)dt=\int_0^s F_t e^{-\tilde{r}t}(X_t\wedge n)dt+\int_0^s\int_0^{\xi} e^{-\tilde{r}t}(X_t\wedge n)dtdF_{\xi}.$$

Since F is a martingale under $\mathbb{Q}$, the optional sampling theorem gives that

$$\mathbb{E}^{\mathbb{Q}}\Big[F_{\tau\wedge m}\int_0^{\tau\wedge m} e^{-\tilde{r}t}(X_t\wedge n)dt\Big]=\mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\tau\wedge m} F_t e^{-\tilde{r}t}(X_t\wedge n)dt\Big]. \tag{2.25}$$

Combining (2.24) and (2.25) and using monotone convergence theorem, we deduce that

$$\begin{aligned}\mathbb{E}\Big[\int_0^{\tau} e^{-\tilde{r}t}X_t dt\Big]&=\lim_{n\to\infty}\lim_{m\to\infty}\mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\tau\wedge m} F_t e^{-\tilde{r}t}(X_t\wedge n)dt\Big]\\&=\mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\tau} e^{-\tilde{r}t}F_t X_t dt\Big].\end{aligned} \tag{2.26}$$

By a similar argument, it is easy to see that

$$\mathbb{E}\Big[\int_0^{\tau} e^{-\tilde{r}t}dt\Big]=\mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\tau} e^{-\tilde{r}t}F_t dt\Big]. \tag{2.27}$$

Using (2.26), (2.27) and (2.15) gives that

$$\begin{aligned}\mathbb{E}\Big[\int_0^{\tau} e^{-\tilde{r}t}(\gamma\lambda_0 X_t-\tilde{r}P)dt\Big]&=\mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\tau} e^{-\tilde{r}t}F_t(\gamma\lambda_0 X_t-\tilde{r}P)dt\Big]\\&=\frac{1}{1+\varphi}\mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\tau} e^{-\tilde{r}t}(1+\Phi_t)\Big(\gamma\lambda_0 x e^{\epsilon t}\Big(\frac{\Phi_t}{\varphi}\Big)^{\beta}-\tilde{r}P\Big)dt\Big]\end{aligned} \tag{2.28}$$

for any stopping time τ.

If $\epsilon \neq 0$, using the transformation $\zeta = \frac{1}{\epsilon}\log\frac{\gamma\lambda_0 x}{\varphi^\beta}$ inspired by Johnson and Peskir (2017a) and Johnson and Peskir (2017b), (2.28) is re-written as

$$\mathbb{E}\Big[\int_0^\tau e^{-\tilde{r}t}(\gamma\lambda_0 X_t - \tilde{r}P)dt\Big] = \frac{1}{1+\varphi}\mathbb{E}^{\mathbb{Q}}\Big[\int_0^\tau e^{-\tilde{r}t}(1+\Phi_t)\Big(e^{\epsilon(t+\zeta)}(\Phi_t)^\beta - K\Big)dt\Big], \tag{2.29}$$

where $K = \tilde{r}P > 0$. Therefore, solving optimal stopping problem (2.9) is equivalent to solving (2.18). Moreover simple calculation gives that

$$\mathbb{E}^{\mathbb{Q}}\Big[\int_0^\infty e^{-\tilde{r}\zeta}|H(\zeta,\Phi_\zeta)|d\zeta\Big] < \infty,$$

which is equivalent to

$$\tilde{r} - \epsilon > \frac{1}{2}\omega^2\beta(\beta+1).$$

If $\epsilon = 0$, we deduce that

$$\begin{aligned}\mathbb{E}\Big[\int_0^\tau e^{-\tilde{r}t}(\gamma\lambda_0 X_t - \tilde{r}P)dt\Big] &= \frac{1}{1+\varphi}\mathbb{E}^{\mathbb{Q}}\Big[\int_0^\tau e^{-\tilde{r}t}(1+\Phi_t)\Big(\gamma\lambda_0 x\Big(\frac{\Phi_t}{\varphi}\Big)^\beta - \tilde{r}P\Big)dt\Big] \\ &= \frac{\gamma\lambda_0 x}{(1+\varphi)\varphi^\beta}\mathbb{E}^{\mathbb{Q}}\Big[\int_0^\tau e^{-\tilde{r}t}(1+\Phi_t)\Big((\Phi_t)^\beta - \hat{K}\Big)dt\Big],\end{aligned} \tag{2.30}$$

where $\hat{K} = \frac{\tilde{r}P\varphi^\beta}{\gamma\lambda_0 x}$. In this case, solving optimal stopping problem (2.9) is equivalent to solving (2.21) and

$$\mathbb{E}^{\mathbb{Q}}\Big[\int_0^\infty e^{-\tilde{r}t}|\hat{H}(\Phi_t)|dt\Big] < \infty,$$

is equivalent to

$$\tilde{r} > \frac{1}{2}\omega^2\beta(\beta+1). \quad \square$$

3. Analysis and solution of the optimal stopping problems

We first consider the case $\epsilon \neq 0$. Applying dynamic programming principle, we deduce that V in (2.18) satisfies the following variational inequality

$$\min\Big\{V_\zeta + \frac{1}{2}\omega^2\varphi^2 V_{\varphi\varphi} - \tilde{r}V + H(\zeta,\varphi),\ -V\Big\} = 0, \tag{3.1}$$

where H is defined in (2.19). As usual in optimal stopping theory, we define the stopping region and the continuation region by

$$\mathcal{S} := \{(\zeta,\varphi):\ V(\zeta,\varphi) = 0\}, \tag{3.2}$$

$$\mathcal{C} := \{(\zeta,\varphi):\ V(\zeta,\varphi) < 0\}. \tag{3.3}$$

From Peskir and Shiryaev (2006), we see that the optimal stopping time for problem (2.18) is given by

$$\tau^*(\zeta,\varphi) = \inf\{s \geq 0:\ (s+\zeta,\Phi_s) \in \mathcal{S}\}.$$

We first study the smoothness of the value function defined in (2.18) in the following proposition.

Proposition 3.1. *The value function V defined in (2.18) satisfies that*

(i) *It is locally Lipschitz continuous in $\mathbb{R}\times\mathbb{R}_+$ and satisfies for a.e. $(\zeta,\varphi) \in \mathbb{R}\times\mathbb{R}_+$ that*

$$\frac{\partial V}{\partial\varphi}(\zeta,\varphi) = \mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\tau^*} e^{-\tilde{r}s}H_\varphi(\zeta+s,\Phi_s)Z_s ds\Big], \tag{3.4}$$

where $Z_s = \exp\Big(\omega W_s - \frac{1}{2}\omega^2 s\Big)$ defined in (2.14) and

$$\frac{\partial V}{\partial\zeta}(\zeta,\varphi) = \epsilon e^{\epsilon\zeta}\mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\tau^*} e^{-(\tilde{r}-\epsilon)s}(1+\Phi_s)(\Phi_s)^\beta ds\Big]. \tag{3.5}$$

(ii) *It belongs to $C^{1,2}$ in the continuation region and satisfies*

$$V_\zeta + \frac{1}{2}\omega^2\varphi^2 V_{\varphi\varphi} - \tilde{r}V + H(\zeta,\varphi) = 0, \quad (\zeta,\varphi) \in \mathcal{C},$$
$$V(\zeta,\varphi) = 0, \quad (\zeta,\varphi) \in \partial\mathcal{C}.$$

Proof. Following the techniques of Gapeev (2021), we prove this proposition. We first prove (i). For any $\delta > 0$, we choose $\tau^*(\zeta,\varphi)$ as an optimal stopping time for $V(\zeta,\varphi)$. Then using the mean value theorem and (2.13) gives

$$\begin{aligned} V(\zeta,\varphi+\delta) - V(\zeta,\varphi) &\le \mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\tau^*} e^{-\tilde{r}s} H(\zeta+s,\Phi_s^{\varphi+\delta})ds\Big] - \mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\tau^*} e^{-\tilde{r}s} H(\zeta+s,\Phi_s^{\varphi})ds\Big] \\ &= \mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\tau^*} e^{-\tilde{r}s} H_\varphi(\zeta+s,\xi_1 Z_s)(\Phi_s^{\varphi+\delta} - \Phi_s^{\varphi})ds\Big] \\ &= \delta\mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\tau^*} e^{-\tilde{r}s}\left(e^{\epsilon(\zeta+s)}\beta(\xi_1 Z_s)^{\beta-1} + e^{\epsilon(\zeta+s)}(\beta+1)(\xi_1 Z_s)^{\beta} - K\right) Z_s ds\Big], \end{aligned} \tag{3.6}$$

for some $\xi_1 \in (\varphi,\varphi+\delta)$, where Φ^φ denotes the process Φ starting at φ. The assumption (2.20) implies that

$$\int_0^\infty \mathbb{E}^{\mathbb{Q}}\Big[e^{-(\tilde{r}-\epsilon)s}(Z_s)^\beta\Big]ds < \infty, \quad \int_0^\infty \mathbb{E}^{\mathbb{Q}}\Big[e^{-(\tilde{r}-\epsilon)s}(Z_s)^{\beta+1}\Big]ds < \infty. \tag{3.7}$$

Denote $m(\varphi) := \max\{(\varphi/2)^{\beta-1}, (\varphi+1)^{\beta-1}\}$. Then for small $0 < \delta < 1$ using (3.7) and Fubini's theorem gives that

$$\begin{aligned} &V(\zeta,\varphi+\delta) - V(\zeta,\varphi) \\ &\le \delta\mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\tau^*} e^{-(\tilde{r}-\epsilon)s}\left(e^{\epsilon\zeta} m(\varphi)\beta(Z_s)^\beta + e^{\epsilon\zeta}(\beta+1)(\varphi+1)^\beta (Z_s)^{\beta+1}\right)ds\Big] + \frac{K\delta}{\tilde{r}} \\ &\le \delta\mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\infty} e^{-(\tilde{r}-\epsilon)s}\left(e^{\epsilon\zeta} m(\varphi)\beta(Z_s)^\beta + e^{\epsilon\zeta}(\beta+1)(\varphi+1)^\beta (Z_s)^{\beta+1}\right)ds\Big] + \frac{K\delta}{\tilde{r}} \\ &\le L(\zeta,\varphi)\delta, \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} L(\zeta,\varphi) = \beta e^{\epsilon\zeta} m(\varphi) \int_0^\infty \mathbb{E}^{\mathbb{Q}}\Big[e^{-(\tilde{r}-\epsilon)s}(Z_s)^\beta\Big]ds + e^{\epsilon\zeta}(\beta+1)(\varphi+1)^\beta \\ \cdot \int_0^\infty \mathbb{E}^{\mathbb{Q}}\Big[e^{-(\tilde{r}-\epsilon)s}(Z_s)^{\beta+1}\Big]ds + \frac{K}{\tilde{r}} < +\infty \end{aligned} \tag{3.9}$$

by (3.7). A symmetric argument shows that for small $0 < \delta < \min\{1, \varphi/2\}$

$$\begin{aligned} &V(\zeta,\varphi) - V(\zeta,\varphi-\delta) \\ &\ge \delta\mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\tau^*} e^{-\tilde{r}s}\left(e^{\epsilon(\zeta+s)}\beta(\xi_2 Z_s)^{\beta-1} + e^{\epsilon(\zeta+s)}(\beta+1)(\xi_2 Z_s)^{\beta} - K\right) Z_s ds\Big], \end{aligned} \tag{3.10}$$

for some $\xi_2 \in (\varphi-\delta,\varphi)$, and then

$$V(\zeta,\varphi) - V(\zeta,\varphi-\delta) \ge -L(\zeta,\varphi)\delta, \tag{3.11}$$

where the constant L is given by (3.9). On the other hand, by choosing the optimal stopping time $\tau_\delta^+ := \tau^*(\zeta,\varphi+\delta)$ for $V(\zeta,\varphi+\delta)$, we have

$$\begin{aligned} &V(\zeta,\varphi+\delta) - V(\zeta,\varphi) \\ &\ge \mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\tau_\delta^+} e^{-\tilde{r}s}\left(H(\zeta+s,\Phi_s^{\varphi+\delta}) - H(\zeta+s,\Phi_s^{\varphi})\right)ds\Big] \end{aligned}$$

$$= \mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\tau_\delta^+} e^{-\tilde{r}s} H_\varphi(\zeta+s, \xi_3 Z_s)(\Phi_s^{\varphi+\delta} - \Phi_s^\varphi)ds\Big]$$

$$= \delta\mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\tau_\delta^+} e^{-\tilde{r}s}\left(e^{\epsilon(\zeta+s)}\beta(\xi_3 Z_s)^{\beta-1} + e^{\epsilon(\zeta+s)}(\beta+1)(\xi_3 Z_s)^\beta - K\right) Z_s ds\Big],$$

and

$$V(\zeta, \varphi+\delta) - V(\zeta, \varphi) \geq -L(\zeta, \varphi)\delta, \tag{3.12}$$

for some $\xi_3 \in (\varphi, \varphi+\delta)$, where the constant L is given by (3.9).

Using a similar argument and choosing $\tau_\delta^- := \tau^*(\zeta, \varphi-\delta)$, we have

$$V(\zeta, \varphi) - V(\zeta, \varphi-\delta) \leq \delta\mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\tau_\delta^-} e^{-\tilde{r}s}(e^{\epsilon(\zeta+s)}\beta(\xi_4 Z_s)^{\beta-1} + e^{\epsilon(\zeta+s)}(\beta+1)(\xi_4 Z_s)^\beta - K) Z_s ds\Big],$$

for some $\xi_4 \in (\varphi-\delta, \varphi)$, and

$$V(\zeta, \varphi) - V(\zeta, \varphi-\delta) \leq L(\zeta, \varphi)\delta, \tag{3.13}$$

where L is given by (3.9). Combining (3.8), (3.11), (3.12), and (3.13), gives that

$$|V(\zeta, \varphi\pm\delta) - V(\zeta, \varphi)| \leq L\delta,$$

for some constant L depending on ζ and φ, which proves $V(\zeta, \cdot)$ is locally Lipschitz continuous.

Similarly we show $V(\cdot, \varphi)$ is locally Lipschitz continuous. Direct computation shows that

$$\begin{aligned}
&V(\zeta+\delta, \varphi) - V(\zeta, \varphi)\\
&\leq \mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\tau^*} e^{-\tilde{r}s} H(\zeta+\delta+s, \Phi_s)ds\Big] - \mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\tau^*} e^{-\tilde{r}s} H(\zeta+s, \Phi_s)ds\Big]\\
&= (e^{\epsilon\delta}-1)\mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\tau^*} e^{-(\tilde{r}-\epsilon)s+\epsilon\zeta}(1+\Phi_s)(\Phi_s)^\beta ds\Big].
\end{aligned} \tag{3.14}$$

Letting $\nu_\delta^+ := \tau^*(\zeta+\delta, \varphi)$ to be optimal for $V(\zeta+\delta, \varphi)$, we have

$$\begin{aligned}
&V(\zeta+\delta, \varphi) - V(\zeta, \varphi)\\
&\geq \mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\nu_\delta^+} e^{-\tilde{r}s} H(\zeta+\delta+s, \Phi_s)ds\Big] - \mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\nu_\delta^+} e^{-\tilde{r}s} H(\zeta+s, \Phi_s)ds\Big]\\
&= (e^{\epsilon\delta}-1)\mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\nu_\delta^+} e^{-(\tilde{r}-\epsilon)s+\epsilon\zeta}(1+\Phi_s)(\Phi_s)^\beta ds\Big].
\end{aligned} \tag{3.15}$$

By symmetric arguments, we have

$$V(\zeta, \varphi) - V(\zeta-\delta, \varphi) \geq (1-e^{-\epsilon\delta})\mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\tau^*} e^{-(\tilde{r}-\epsilon)s+\epsilon\zeta}(1+\Phi_s)(\Phi_s)^\beta ds\Big], \tag{3.16}$$

and

$$V(\zeta, \varphi) - V(\zeta-\delta, \varphi) \leq (1-e^{-\epsilon\delta})\mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\nu_\delta^-} e^{-(\tilde{r}-\epsilon)s+\epsilon\zeta}(1+\Phi_s)(\Phi_s)^\beta ds\Big], \tag{3.17}$$

with $\nu_\delta^- := \tau^*(\zeta-\delta, \varphi)$ being the optimal stopping time for $V(\zeta-\delta, \varphi)$. Since

$$(\epsilon-1)\delta < e^{\epsilon\delta} - 1 < (1+\epsilon)\delta,\ (\epsilon-1)\delta < 1 - e^{-\epsilon\delta} < (1+\epsilon)\delta,$$

for sufficiently small $\delta > 0$, using (3.7), (3.14), (3.15), (3.16), and (3.17), we conclude that $V(\cdot, \varphi)$ is Lipschitz continuous.

Furthermore, we conclude that V is differentiable a.e. in $\mathbb{R}\times\mathbb{R}_+$. Suppose that (ζ,φ) is any differentiable point of V. Dividing (3.6), (3.10), (3.14) and (3.16) by δ and letting $\delta\to 0$ yield (3.4) and (3.5).

To prove (ii) we use standard argument of PDE analysis. Indeed, for any point $(\zeta,\varphi)\in\mathcal{C}$, we consider the following problem

$$\begin{aligned}&f_\zeta+\frac{1}{2}\omega^2\varphi^2 f_{\varphi\varphi}-\tilde{r}f=-H(\zeta,\varphi)\quad \text{in } \mathcal{D},\\ &f=V\quad \text{on } \partial\mathcal{D},\end{aligned}$$

where $\mathcal{D}:=\zeta_1,\zeta_2)\times(\varphi_1,\varphi_2)\subset\mathcal{C}$ and $\partial\mathcal{D}$ denotes the parabolic boundary of $\mathcal{D}$. Since we have showed that V is continuous, the classical theory for parabolic equations (see e.g., (Friedman, 1964, Chapter 3)) guarantees the existence of a unique solution f with f_{xx} and f_t being continuous in $\mathcal{D}$. On the other hand, a standard verification argument (see e.g., (Karatzas and Shreve, 1998, Theorem 2.7.7)) gives that $f=V$ in $\mathcal{D}$. We conclude $V\in C^{1,2}$ by the arbitrariness of (ζ,φ). □

Next we further characterize the stopping region and the continuation region by the free boundary and analyze the properties of the free boundary.

Proposition 3.2. *Assume $\epsilon\neq 0$ and define the free boundary as*

$$b(\zeta):=\sup\{\varphi: V(\zeta,\varphi)<0\} \tag{3.18}$$

for $\zeta\in\mathbb{R}$. Then

$$\mathcal{S}=\{(\zeta,\varphi): \varphi\geq b(\zeta)\}, \tag{3.19}$$

$$\mathcal{C}=\{(\zeta,\varphi): \varphi< b(\zeta)\}, \tag{3.20}$$

and $b(\zeta)<+\infty$ for any finite number $\zeta\in\mathbb{R}$.

Proof. From the variational inequality (3.1), it is known that the stopping region $\mathcal{S}$ satisfies that

$$\mathcal{S}\subset\{(\zeta,\varphi): H(\zeta,\varphi)\geq 0\},$$

where H is defined in (2.19). Letting $H(\zeta,\varphi)=0$ gives that

$$\varphi=K^{\frac{1}{\beta}}e^{-\frac{\epsilon}{\beta}\zeta}=:\Gamma(\zeta). \tag{3.21}$$

Since $H(\zeta,\varphi)\geq 0$ is equivalent to $\varphi\geq\Gamma(\zeta)$, we have

$$\mathcal{S}\subset\{(\zeta,\varphi): \varphi\geq\Gamma(\zeta)\}. \tag{3.22}$$

We now proceed the analysis in the following two steps.

Step 1. Assume that $\epsilon>0$. Since $H(\zeta,\varphi)$ is increasing in ζ, we see that $V(\zeta,\varphi)$ is increasing in ζ. Hence we deduce that if $(\zeta_0,\varphi_0)\in\mathcal{S}$ and $\zeta'>\zeta_0$, then $V(\zeta',\varphi_0)\geq V(\zeta_0,\varphi_0)=0$. Moreover the variational inequality (3.1) implies that $V(\zeta',\varphi_0)\leq 0$. Therefore we have

$$V(\zeta',\varphi_0)=0,\quad \text{for } \zeta'>\zeta_0. \tag{3.23}$$

On the other hand, we assume $(\zeta_0,\varphi_0)\in\mathcal{S}$ and $\varphi'>\varphi_0>0$. Denoting $\mathcal{R}:=\{\zeta\geq\zeta_0,\ \varphi\geq\varphi_0\}$, letting $\tau^*:=\tau^*(\zeta_0,\varphi')$ to be the optimal stopping time for $V(\zeta_0,\varphi')$ and $\tau_{\mathcal{R}}$ to be the first exit time from $\mathcal{R}$ for the process $(\zeta_0+s,\Phi_s^{\varphi'})$, then the definition of τ^* and (3.23) means that $\tau^*\leq\tau_{\mathcal{R}}$. That is

$$\Phi_s^{\varphi'}\geq\varphi_0\geq\Gamma(\zeta_0)\geq\Gamma(s+\zeta_0),$$

for $s\in[0,\tau^*]$, where the second inequality follows from (3.22) and the third inequality follows from the fact that $\Gamma(\zeta)$ is decreasing. Therefore we have

$$H(s+\zeta_0,\Phi_s^{\varphi'})\geq 0,\quad \text{for } s\in[0,\tau^*],$$

and consequently

$$V(\zeta_0,\varphi')=\mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\tau^*}e^{-\tilde{r}s}H(s+\zeta_0,\Phi_s^{\varphi'})ds\Big]\geq 0.$$

This together with $V(\zeta_0,\varphi')\leq 0$ that is implied from the variational inequality (3.1), yields

$$V(\zeta_0,\varphi')=0,\quad \text{for } \varphi'>\varphi_0. \tag{3.24}$$

Now if we define a function $b(t)$ as in (3.18), then the upward connectedness of $\mathcal{S}$ means that (3.19) and (3.20) hold.

Now we prove $b(\zeta)<+\infty$ for any finite number $\zeta\in\mathbb{R}$ by methods of contradiction. To this end, we define the ζ-section of $\mathcal{S}$ and $\mathcal{C}$ respectively as

$$\mathcal{S}_\zeta := \{\varphi : \varphi \geq b(\zeta)\}, \quad \mathcal{C}_\zeta := \{\varphi : \varphi < b(\zeta)\}.$$

Suppose that $b(\zeta_0) = +\infty$ for some ζ_0, that is, $\mathcal{C}_{\zeta_0} = (0, +\infty)$, then $\mathcal{C}_\zeta = (0, +\infty)$ for $\zeta < \zeta_0$ by the monotone increasing property of $V(\cdot, \varphi)$. Hence, this implies

$$\mathcal{S} \subset \{(\zeta, \varphi) : \zeta > \zeta_0, \varphi \geq \Gamma(\zeta)\},$$

which further guarantees the optimal stopping time $\tau^*(\tilde{\zeta}, \varphi) \geq \zeta_0 - \tilde{\zeta} > 0$ for any $\tilde{\zeta} < \zeta_0$ and $\varphi > 0$, since the process $\{(\tilde{\zeta} + s, \Phi_s^\varphi)\}_{s \geq 0}$ starting at $(\tilde{\zeta}, \varphi)$ will take at least $\zeta_0 - \tilde{\zeta}$ to arrive at the stopping region $\mathcal{S}$. By dominated convergence theorem, it follows that

$$\begin{aligned}\lim_{\varphi \to +\infty} V(\tilde{\zeta}, \varphi) &= \lim_{\varphi \to +\infty} \mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\tau^*} e^{-\tilde{r}s} H(s + \tilde{\zeta}, \Phi_s^\varphi) ds\Big] \\ &= \mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\tau^*} \lim_{\varphi \to +\infty} e^{-\tilde{r}s} H(s + \tilde{\zeta}, \Phi_s^\varphi) ds\Big] \\ &= +\infty.\end{aligned}$$

This contradicts with $V(\zeta, \varphi) \leq 0$ for any $(\zeta, \varphi) \in \mathbb{R} \times \mathbb{R}_+$. Hence, $b(\zeta) < +\infty$ for any $\zeta \in \mathbb{R}$.

Step 2. Assume that $\epsilon < 0$. Let $\zeta = -\vartheta$ and $\mathcal{V}(\vartheta, \varphi) = V(-\vartheta, \varphi)$. Then we find that

$$\mathcal{V}(\vartheta, \varphi) = \inf_\tau \mathbb{E}^{\mathbb{Q}}\Big[\int_0^\tau e^{-\tilde{r}s} H(-\vartheta + s, \Phi_s) ds\Big]. \tag{3.25}$$

Since for $\epsilon < 0$, $H(\cdot, \varphi)$ defined in (2.19) is decreasing and function Γ defined in (3.21) is increasing, from (3.25) we see that $\mathcal{V}(\cdot, \varphi)$ is increasing and $\tilde{\Gamma}(\vartheta) := \Gamma(-\vartheta)$ is decreasing. Using a similar argument as Step 1, it follows that there exists a unique free boundary $\tilde{b}(\vartheta)$ such that

$$\mathcal{S} = \{(\vartheta, \varphi) : \varphi \geq \tilde{b}(\vartheta)\}, \quad \mathcal{C} = \{(\vartheta, \varphi) : \varphi < \tilde{b}(\vartheta)\}.$$

Then the desired results follow by letting $b(\zeta) := \tilde{b}(-\zeta)$. □

Now we establish the smooth pasting condition for the value function across the free boundary (see Proposition 3.3). To prove the results, we first introduce the following lemma.

Lemma 3.1. *Assume that $\epsilon \neq 0$. Let τ_ν be an optimal stopping time for $V(\zeta, b(\zeta) - \nu)$ given by*

$$\tau_\nu := \tau^*(\zeta, b(\zeta) - \nu) = \inf\{s \geq 0 : \Phi_s^{b(\zeta)-\nu} \geq b(s + \zeta)\}.$$

Then $\lim_{\nu \to 0} \tau_\nu = 0$.

Proof. The proof mainly follows Cox and Peskir (2015). Firstly, assume that $\nu > 0$. Direct computation shows that

$$\Phi_s^{b(\zeta)-\nu} = (b(\zeta) - \nu) Z_s,$$

where $Z_s = \exp\Big(-\frac{1}{2}\omega^2 s + \omega W_s\Big)$. Then we see that τ_ν is increasing in ν. Hence, we can denote $\lim_{\nu \to 0} \tau_\nu := \tau_+$. We introduce the truncated version of τ_ν by setting

$$\tau_\nu^\delta := \inf\Big\{s > \delta : Z_s \geq \frac{b(s + \zeta)}{b(\zeta) - \nu}\Big\}$$

with fixed $\delta > 0$ and sufficiently small ν such that $b(\zeta) - \nu > 0$. Note that τ_ν^δ is also increasing in ν. So we can also denote $\lim_{\nu \to 0} \tau_\nu^\delta := \tau_+^\delta$. Since $\tau_\nu^\delta \geq \tau_0^\delta$ for any $\nu > 0$, it follows that

$$\tau_+^\delta \geq \tau_0^\delta. \tag{3.26}$$

We claim that

$$\tau_+^\delta = \tau_0^\delta, \quad \mathbb{Q} - \text{a.s.} \tag{3.27}$$

Hence we derive that

$$\begin{aligned}\lim_{\nu \to 0} \tau_\nu &= \lim_{\nu \to 0} \lim_{\delta \to 0} \tau_\nu^\delta = \lim_{\delta \to 0} \lim_{\nu \to 0} \tau_\nu^\delta = \lim_{\delta \to 0} \tau_+^\delta \\ &= \lim_{\delta \to 0} \tau_0^\delta = \inf\{s > 0 : b(\zeta) Z_s \geq b(s + \zeta)\} \\ &= 0,\end{aligned}$$

where the two limits commute since τ_ν^δ is increasing in δ and ν. Thus the proof of this lemma is complete if we prove claim (3.27) is true.

To prove (3.27), we choose $\nu < \frac{b(\zeta)}{2}$ and by mean value theorem and the definition of τ_ν there exists some $\tilde{\xi} \in (b(\zeta)-\nu, b(\zeta))$ such that

$$\begin{aligned}
\mathbb{Q}(\tau_\nu^\delta > \xi) &= \mathbb{Q}\Big(W_s < \frac{1}{\omega}\log\Big(\frac{b(s+\zeta)}{b(\zeta)}\Big) + \frac{1}{\omega}\log\Big(\frac{b(\zeta)}{b(\zeta)-\nu}\Big) + \frac{1}{2}\omega s,\ s\in(\delta,\xi]\Big) \\
&= \mathbb{Q}\Big(W_s < \frac{1}{\omega}\log\Big(\frac{b(s+\zeta)}{b(\zeta)}\Big) + \frac{1}{\omega\tilde{\xi}}\nu + \frac{1}{2}\omega s,\ s\in(\delta,\xi]\Big) \\
&\le \mathbb{Q}\Big(W_s - \int_0^s H_r^\nu dr < \frac{1}{\omega}\log\Big(\frac{b(s+\zeta)}{b(\zeta)}\Big) + \frac{1}{2}\omega s,\ s\in(\delta,\xi]\Big),
\end{aligned} \tag{3.28}$$

where $H_r^\nu = \frac{2}{\omega b(\zeta)}\frac{\nu}{\delta} I(0\le r\le\delta)$ and $I(\cdot)$ denotes the indicator function. If we let

$$\frac{d\tilde{\mathbb{Q}}}{d\mathbb{Q}} = \exp\Big(\int_0^T H_r^\nu dW_r - \frac{1}{2}\int_0^T (H_r^\nu)^2 dr\Big) = \exp\Big(\frac{2\nu^2}{\omega^2 b(\zeta)^2\delta} + \frac{2\nu}{\omega b(\zeta)\delta}B_\delta\Big),$$

and

$$B_s = W_s - \int_0^s H_r^\nu dr,$$

then B_s is a standard Brownian motion under $\tilde{\mathbb{Q}}$ by Girsanov's theorem. Using the fact that τ_ν^δ is increasing in ν and (3.28), the dominated convergence theorem gives that

$$\begin{aligned}
\mathbb{Q}(\tau_+^\delta > \xi) &\le \lim_{\nu\to 0}\mathbb{Q}(\tau_\nu^\delta > \xi) \\
&\le \lim_{\nu\to 0}\mathbb{E}^{\tilde{\mathbb{Q}}}\Big[e^{-\frac{2\nu^2}{\omega^2 b(\zeta)^2\delta} - \frac{2\nu}{\omega b(\zeta)\delta}B_\delta} I\Big(B_s < \frac{1}{\omega}\log\Big(\frac{b(s+\zeta)}{b(\zeta)}\Big) + \frac{1}{2}\omega s,\ s\in(\delta,\xi]\Big)\Big] \\
&= \tilde{\mathbb{Q}}\Big(B_s < \frac{1}{\omega}\log\Big(\frac{b(s+\zeta)}{b(\zeta)}\Big) + \frac{1}{2}\omega s,\ s\in(\delta,\xi]\Big) \\
&= \mathbb{Q}\Big(b(\zeta)Z_s < b(s+\zeta),\ s\in(\delta,\xi]\Big) \\
&= \mathbb{Q}(\tau_0^\delta > \xi).
\end{aligned}$$

So the Fubini's theorem gives that

$$\mathbb{E}^{\mathbb{Q}}[\tau_+^\delta] = \mathbb{E}^{\mathbb{Q}}\Big[\int_0^\infty I(\xi < \tau_+^\delta)d\xi\Big] = \int_0^\infty \mathbb{Q}(\tau_+^\delta > \xi)d\xi \le \int_0^\infty \mathbb{Q}(\tau_0^\delta > \xi)d\xi = \mathbb{E}^{\mathbb{Q}}[\tau_0^\delta].$$

Now claim (3.27) follows from (3.26). If $\nu < 0$, the proof follows by the fact that $(\zeta, b(\zeta)-\nu) \in \mathcal{S}$. Thus the proof of this lemma is complete. □

Proposition 3.3. *Assume that $\epsilon \neq 0$. The smooth pasting condition holds true*

$$\lim_{\varphi\to b(\zeta)} V_\varphi(\zeta,\varphi) = V_\varphi(\zeta, b(\zeta)) = 0, \tag{3.29}$$

for any $\zeta \in \mathbb{R}$.

Proof. Using the techniques of Gapeev (2021), we prove this proposition. Since for any $\varphi > b(\zeta)$, $V(\zeta,\varphi) = 0$, the right derivative in φ at $b(\zeta)$ is given by

$$V_\varphi^+(\zeta, b(\zeta)) = 0. \tag{3.30}$$

Moreover the left derivative in φ at $b(\zeta)$ satisfies

$$\liminf_{\nu\to 0+}\frac{V(\zeta,b(\zeta)) - V(\zeta,b(\zeta)-\nu)}{\nu} = \liminf_{\nu\to 0+}\frac{-V(\zeta,b(\zeta)-\nu)}{\nu} \ge 0. \tag{3.31}$$

On the other hand, for $\nu > 0$, by mean value theorem,

$$\begin{aligned}
&\frac{V(\zeta,b(\zeta))-V(\zeta,b(\zeta)-\nu)}{\nu}\\
&\le \frac{\mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\tau_\nu} e^{-\tilde{r}s}(H(\zeta+s,\Phi_s^{b(\zeta)})-H(\zeta+s,\Phi_s^{b(\zeta)-\nu}))ds\Big]}{\nu}\\
&=\frac{\mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\tau_\nu} e^{-\tilde{r}s}H_\varphi(\zeta+s,\eta Z_s)\nu Z_s ds\Big]}{\nu}\\
&=\mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\tau_\nu} e^{-\tilde{r}s}H_\varphi(\zeta+s,\eta Z_s)Z_s ds\Big],
\end{aligned}$$

for some $\eta\in\big(b(\zeta)-\nu,b(\zeta)\big)$, where τ_ν is an optimal stopping time for $V(\zeta,b(\zeta)-\nu)$. Applying Lemma 3.1 and dominated convergence theorem, we have

$$\limsup_{\nu\to 0+}\frac{V(\zeta,b(\zeta))-V(\zeta,b(\zeta)-\nu)}{\nu}\le 0. \tag{3.32}$$

Therefore combining (3.31) with (3.32) gives that the left derivative $V_\varphi^-(\zeta,b(\zeta))=0$. Further recalling (3.30) gives the second equality in (3.29) holds. The first equality in (3.29) follows from (3.4) and Lemma 3.1. Thus, the proof is completed. □

In the following proposition, we present the continuity and monotonicity of the free boundary.

Proposition 3.4. *Assume that $\epsilon\neq 0$. The free boundary $b(\zeta)$ defined by (3.18) is continuous. If $\epsilon<0$, then $b(\zeta)$ is strictly increasing; if $\epsilon>0$, then $b(\zeta)$ is strictly decreasing.*

Proof. Since $V(\zeta,\varphi)$ is increasing in ζ for $\epsilon>0$ and decreasing for $\epsilon<0$, from the definition of the free boundary (3.18), it is seen that $b(\zeta)$ is decreasing for $\epsilon>0$ and increasing for $\epsilon<0$. Next, we prove $b(\zeta)$ is strictly increasing for $\epsilon<0$. If it is not the case, then there exists $\zeta_1<\zeta_2$, $\varphi_0>0$, such that $b(\zeta)=\varphi_0$ for $\zeta\in[\zeta_1,\zeta_2]$. The region $[\zeta_1,\zeta_2]\times(0,\varphi_0]$ is a subset of continuation region (see (3.20)). Therefore from (3.1) and (3.3), in the region it holds that

$$V_\zeta+\frac{1}{2}\omega^2\varphi^2V_{\varphi\varphi}-\tilde{r}V+H(\zeta,\varphi)=0.$$

So V_ζ satisfies

$$\begin{aligned}
&V_{\zeta\zeta}+\frac{1}{2}\omega^2\varphi^2V_{\zeta\varphi\varphi}-\tilde{r}V_\zeta=-H_\zeta(\zeta,\varphi)\ge 0,\\
&V_\zeta(\zeta,\varphi_0)=0,\quad \text{for } \zeta\in(\zeta_1,\zeta_2),
\end{aligned}$$

where the second equation follows from the fact that $V(\zeta,\varphi_0)=0$ for $\zeta\in(\zeta_1,\zeta_2)$. On the other hand, $V_\zeta(\zeta,\varphi)\le 0$ for any $(\zeta,\varphi)\in[\zeta_1,\zeta_2]\times(0,\varphi_0]$. Now by Hopf lemma (see Lieberman (1996)), $V_{\zeta\varphi}(\zeta,\varphi_0)>0$ for any $\zeta\in(\zeta_1,\zeta_2)$. However, as $V_\varphi(\zeta,\varphi_0)=V_\varphi(\zeta,b(\zeta))=0$, we deduce that $V_{\zeta\varphi}(\zeta,\varphi_0)=0$ for $\zeta\in(\zeta_1,\zeta_2)$, which is contradiction. Thus $b(\zeta)$ is strictly increasing for $\epsilon<0$. Similarly it can be proved that $b(\zeta)$ is strictly decreasing for $\epsilon>0$.

The proof of the continuity of $b(\zeta)$ follows Peskir and Shiryaev (2006). We first consider the case $\epsilon>0$. Let $\zeta_n\downarrow\zeta_0$. It follows from $V(\zeta_n,b(\zeta_n))=0$ and the continuity of V that

$$V(\zeta_0,b(\zeta_0+))=0.$$

Then the definition of b implies that $b(\zeta_0+)\ge b(\zeta_0)$. Furthermore, the decreasing monotonicity of b gives that $b(\zeta_0+)\le b(\zeta_0)$. We conclude that $b(\zeta)$ is right continuous at ζ_0. Suppose that $b(\zeta_0+)\neq b(\zeta_0-)$. Then it must have $b(\zeta_0+)<b(\zeta_0-)$. The definition of b yields $V(\zeta_0,\varphi)=0$ for any $\varphi\in(b(\zeta_0+),b(\zeta_0-))$. Hence, $V_\varphi(\zeta_0,\varphi)=0$ for any $\varphi\in(b(\zeta_0+),b(\zeta_0-))$ from smooth pasting condition in Proposition 3.3. Consider the region $R=\{\zeta'\le s\le\zeta_0,\ \varphi'\le\varphi\le b(s)\}$ and $R'=\{\zeta'\le s\le\zeta_0,\ \varphi'\le\varphi\le b(\zeta')\}$ for some $\zeta'<\zeta_0$ and $\varphi'\in(b(\zeta_0+),b(\zeta_0-))$. As b is a decreasing function, we deduce that $R\subset R'$. Since in the continuation region $\mathcal{C}$,

$$\frac{1}{2}\omega^2\varphi^2V_{\varphi\varphi}=-V_\zeta-H+\tilde{r}V,$$

$V_\zeta\ge 0$ and $V\le 0$, we derive that for any $(s,\varphi)\in R$,

$$V_{\varphi\varphi}(s,\varphi)\le-\frac{2H(s,\varphi)}{\omega^2\varphi^2}\le-\min_{(s,\varphi)\in R}\frac{2H(s,\varphi)}{\omega^2\varphi^2}\le-\min_{(s,\varphi)\in R'}\frac{2H(s,\varphi)}{\omega^2\varphi^2}\le -C,$$

where C is a positive constant. By the continuity of V and the smooth pasting condition in Proposition 3.3, it follows that for any $\zeta'<s<\zeta_0$,

$$V(s,\varphi')=\int_{\varphi'}^{b(s)}\int_u^{b(s)}V_{\varphi\varphi}(s,v)dvdu\le -C\int_{\varphi'}^{b(s)}\int_u^{b(s)}dvdu=-\frac{C}{2}(b(s)-\varphi')^2.$$

Now the continuity of V implies that

$$\lim_{s\to\zeta_0-} V(s,\varphi') = V(\zeta_0,\varphi') \le -\frac{C}{2}(b(\zeta_0-)-\varphi')^2 < 0,$$

which contradicts the fact that $(\zeta_0,\varphi')\in\mathcal{S}$. Therefore it has $b(\zeta_0+)=b(\zeta_0-)$. The proof for the case $\epsilon>0$ is thus complete using that $b(\zeta)$ is right continuous at ζ_0.

For the case $\epsilon<0$, a similar argument also shows that $b(\zeta)$ is continuous. Consequently, the whole proof is complete. □

Now we derive the integral equation for the free boundary as follows.

Proposition 3.5. *Assume that $\epsilon\neq 0$. The free boundary $b(\zeta)$ defined by (3.18) satisfies the following integral equation:*

$$\begin{aligned}0=\int_0^{+\infty}\Big[&-Ke^{-\tilde r s}N\Big(d\big(s,b(\zeta+s),b(\zeta)\big)\Big)\\&-Ke^{-\tilde r s}b(\zeta)N\Big(d\big(s,b(\zeta+s),b(\zeta)\big)-\omega\sqrt{s}\Big)\\&+b(\zeta)^{\beta}e^{\epsilon\zeta+(\epsilon-\frac12\omega^2\beta+\frac12\omega^2\beta^2-\tilde r)s}N\Big(d\big(s,b(\zeta+s),b(\zeta)\big)-\omega\beta\sqrt{s}\Big)\\&+b(\zeta)^{\beta+1}e^{\epsilon\zeta+(\epsilon-\frac12\omega^2(\beta+1)+\frac12\omega^2(\beta+1)^2-\tilde r)s}N\Big(d\big(s,b(\zeta+s),b(\zeta)\big)-\omega(\beta+1)\sqrt{s}\Big)\Big]ds,\end{aligned}\tag{3.33}$$

where $d(s,b(\zeta+s),\varphi)=\frac{1}{\omega\sqrt{s}}\log\left(\frac{b(\zeta+s)}{\varphi}\right)+\frac12\omega\sqrt{s}$ and N is the CDF of standard normal distribution.

Proof. Applying the local time-space formula (see Peskir and Shiryaev (2006) and Johnson and Peskir (2017a)) and the smooth pasting condition, we deduce that

$$\begin{aligned}d(e^{-\tilde r s}V(\zeta+s,\Phi_s)) &= e^{-\tilde r s}\Big[V_\zeta+\frac12\omega^2\varphi^2V_{\varphi\varphi}-\tilde r V\Big](\zeta+s,\Phi_s)I(\Phi_s\neq b(\zeta+s))ds\\&\quad+e^{-\tilde r s}\omega\Phi_sV_\varphi(\zeta+s,\Phi_s)I(\Phi_s\neq b(\zeta+s))dW_s\\&\quad+\frac12e^{-\tilde r s}(V_\varphi(\zeta+s,\Phi_s^+)-V_\varphi(\zeta+s,\Phi_s^-))I(\Phi_s=b(\zeta+s))d\ell_s^b(\Phi)\\&=-e^{-\tilde r s}H(\zeta+s,\Phi_s)I(\Phi_s<b(\zeta+s))ds\\&\quad+e^{-\tilde r s}\omega\Phi_sV_\varphi(\zeta+s,\Phi_s)I(\Phi_s\neq b(\zeta+s))dW_s,\end{aligned}$$

where $\ell_s^b(\Phi)$ is the local time of Φ at the curve b given by

$$\ell_s^b(\Phi)=\mathbb{Q}\text{-}\lim_{\delta\to0}\frac{1}{2\delta}\int_0^s I\big(b(r)-\delta<\Phi_r<b(r)+\delta\big)d[\Phi,\Phi]_r.$$

Integrating from 0 to T and taking expectation under $\mathbb{Q}$, we have

$$\mathbb{E}^{\mathbb{Q}}[e^{-\tilde r T}V(\zeta+T,\Phi_T)]-V(\zeta,\varphi)=-\mathbb{E}^{\mathbb{Q}}\Big[\int_0^T e^{-\tilde r s}H(\zeta+s,\Phi_s)I(\Phi_s<b(\zeta+s))ds\Big].\tag{3.34}$$

We choose τ^* as the optimal stopping time for $V(\zeta,\varphi)$. By assumption (2.23), the Fubini's theorem gives that

$$\begin{aligned}|V(\zeta,\varphi)|&=\Big|\mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\tau^*}e^{-\tilde r s}H(\zeta+s,\Phi_s)ds\Big]\Big|\\&\le\mathbb{E}^{\mathbb{Q}}\Big[\int_0^{+\infty}e^{-\tilde r s}\big|H(\zeta+s,\Phi_s)\big|ds\Big]\\&=\int_0^{+\infty}e^{-\tilde r s}\mathbb{E}^{\mathbb{Q}}\Big[\big|H(\zeta+s,\Phi_s)\big|\Big]ds\\&\le\frac{K}{\tilde r}(1+\varphi)+\frac{e^{\epsilon\zeta}\varphi^{\beta}}{\tilde r-\epsilon+\frac12\omega^2\beta(1-\beta)}+\frac{e^{\epsilon\zeta}\varphi^{\beta+1}}{\tilde r-\epsilon-\frac12\omega^2(\beta+1)\beta},\end{aligned}$$

where $Z_s=\exp\big(-\frac12\omega^2s+\omega W_s\big)$, which leads to

$$\begin{aligned}\mathbb{E}^{\mathbb{Q}}\Big[e^{-\tilde{r}T}\big|V(\zeta+T,\Phi_T)\big|\Big] &\le \frac{K}{\tilde{r}}\mathbb{E}^{\mathbb{Q}}\big[e^{-\tilde{r}T}(1+\Phi_T)\big]+\frac{e^{-(\tilde{r}-\epsilon)T+\epsilon\zeta}\mathbb{E}^{\mathbb{Q}}\big[(\Phi_T)^{\beta}\big]}{\tilde{r}-\epsilon+\frac{1}{2}\omega^2\beta(1-\beta)}\\&\quad+\frac{e^{-(\tilde{r}-\epsilon)T+\epsilon\zeta}\mathbb{E}^{\mathbb{Q}}[(\Phi_T)^{\beta+1}]}{\tilde{r}-\epsilon-\frac{1}{2}\omega^2(\beta+1)\beta}\\&=\frac{K}{\tilde{r}}e^{-\tilde{r}T}(1+\varphi)+\frac{e^{-(\tilde{r}-\epsilon+\frac{1}{2}\omega^2\beta(1-\beta))T+\epsilon\zeta}\varphi^{\beta}}{\tilde{r}-\epsilon+\frac{1}{2}\omega^2\beta(1-\beta)}\\&\quad+\frac{e^{-(\tilde{r}-\epsilon-\frac{1}{2}\omega^2(\beta+1)\beta)T+\epsilon\zeta}\varphi^{\beta+1}}{\tilde{r}-\epsilon-\frac{1}{2}\omega^2(\beta+1)\beta}.\end{aligned}\tag{3.35}$$

Due to the assumption (2.23), (3.34) and (3.35), letting $T\to+\infty$, the dominated convergence theorem gives that

$$V(\zeta,\varphi)=\mathbb{E}^{\mathbb{Q}}\Big[\int_0^{+\infty}e^{-\tilde{r}s}H(\zeta+s,\Phi_s)I\big(\Phi_s<b(\zeta+s)\big)ds\Big],$$

which is further calculated as

$$\begin{aligned}V(\zeta,\varphi)&=\mathbb{E}^{\mathbb{Q}}\Big[\int_0^{+\infty}e^{-\tilde{r}s}(1+\Phi_s)(e^{\epsilon(\zeta+s)}(\Phi_s)^{\beta}-K)I\big(\Phi_s<b(\zeta+s)\big)ds\Big]\\&=\int_0^{+\infty}e^{-\tilde{r}s}\big[\mathcal{G}_1(s)+\mathcal{G}_2(s)+\mathcal{G}_3(s)+\mathcal{G}_4(s)\big]ds,\end{aligned}\tag{3.36}$$

with

$$\begin{aligned}\mathcal{G}_1(s)&:=e^{\epsilon(\zeta+s)}\mathbb{E}^{\mathbb{Q}}\Big[(\Phi_s)^{\beta}I\big(\Phi_s<b(\zeta+s)\big)\Big]\\&=\varphi^{\beta}e^{\epsilon\zeta+(\epsilon-\frac{1}{2}\omega^2\beta+\frac{1}{2}\omega^2\beta^2)s}N\big(d(s,b(\zeta+s),\varphi)-\omega\beta\sqrt{s}\big),\\\mathcal{G}_2(s)&:=e^{\epsilon(\zeta+s)}\mathbb{E}^{\mathbb{Q}}\Big[(\Phi_s)^{\beta+1}I\big(\Phi_s<b(\zeta+s)\big)\Big]\\&=\varphi^{\beta+1}e^{\epsilon\zeta+(\epsilon-\frac{1}{2}\omega^2(\beta+1)+\frac{1}{2}\omega^2(\beta+1)^2)s}N\big(d(s,b(\zeta+s),\varphi)-\omega(\beta+1)\sqrt{s}\big),\\\mathcal{G}_3(s)&:=-K\mathbb{E}^{\mathbb{Q}}\Big[I\big(\Phi_s<b(\zeta+s)\big)\Big]=-KN\big(d(s,b(\zeta+s),\varphi)\big),\\\mathcal{G}_4(s)&:=-K\mathbb{E}^{\mathbb{Q}}\Big[\Phi_sI\big(\Phi_s<b(\zeta+s)\big)\Big]=-K\varphi N\big(d(s,b(\zeta+s),\varphi)-\omega\sqrt{s}\big).\end{aligned}$$

Since the left-hand side of (3.36) is 0 if taking $\varphi=b(\zeta)$, evaluating (3.36) by $\varphi=b(\zeta)$ gives (3.33). □

Now we give the asymptotic properties of the free boundary b in the following proposition.

Proposition 3.6. *The following relations hold:*

$$\lim_{\zeta\to+\infty}b(\zeta)=0,\quad \lim_{\zeta\to-\infty}b(\zeta)=+\infty,$$

for $\epsilon>0$, *and*

$$\lim_{\zeta\to+\infty}b(\zeta)=+\infty,\quad \lim_{\zeta\to-\infty}b(\zeta)=0,$$

for $\epsilon<0$.

Proof. For $\epsilon>0$, it has that $b(\zeta)\ge\Gamma(\zeta)\to0$ as $\zeta\to+\infty$ and $b(\zeta)\ge\Gamma(\zeta)\to+\infty$ as $\zeta\to-\infty$ where Γ is defined in (3.21). Therefore $\lim_{\zeta\to-\infty}b(\zeta)=+\infty$. Furthermore assume that $\lim_{\zeta\to+\infty}b(\zeta):=A\ge0$. Then multiplying (3.33) by $e^{-\epsilon\zeta}$, we derive that

$$\begin{aligned}&e^{-\epsilon\zeta}\int_0^{+\infty}\Big[Ke^{-\tilde{r}s}N\Big(d\big(s,b(\zeta+s),b(\zeta)\big)\Big)\\&+Ke^{-\tilde{r}s}b(\zeta)N\Big(d\big(s,b(\zeta+s),b(\zeta)\big)-\omega\sqrt{s}\Big)\Big]ds\end{aligned}$$

$$= \int_0^{+\infty} \Big[b(\zeta)^\beta e^{(\epsilon - \frac{1}{2}\omega^2\beta + \frac{1}{2}\omega^2\beta^2 - \tilde{r})s} N\Big(d\big(s, b(\zeta+s), b(\zeta)\big) - \omega\beta\sqrt{s}\Big)$$

$$+ b(\zeta)^{\beta+1} e^{(\epsilon - \frac{1}{2}\omega^2(\beta+1) + \frac{1}{2}\omega^2(\beta+1)^2 - \tilde{r})s} N\Big(d\big(s, b(\zeta+s), b(\zeta)\big) - \omega(\beta+1)\sqrt{s}\Big)\Big] ds.$$

Letting $\zeta \to +\infty$, and noting that $d\big(s, b(\zeta+s), b(\zeta)\big) \to \frac{1}{2}\omega\sqrt{s}$ as $\zeta \to +\infty$, the dominated convergence theorem gives that

$$0 = A^\beta \int_0^{+\infty} e^{(\epsilon - \frac{1}{2}\omega^2\beta + \frac{1}{2}\omega^2\beta^2 - \tilde{r})s} N\Big(\Big(\frac{1}{2} - \beta\Big)\omega\sqrt{s}\Big)$$

$$+ A e^{(\epsilon - \frac{1}{2}\omega^2(\beta+1) + \frac{1}{2}\omega^2(\beta+1)^2 - \tilde{r})s} N\Big(\Big(\frac{1}{2} - \beta - 1\Big)\omega\sqrt{s}\Big) ds.$$

Hence, we have $A = 0$.

For the case $\epsilon < 0$, by $\zeta = -\zeta'$ and $b(\zeta) = \tilde{b}(\zeta')$, a similar argument shows that $\lim_{\zeta\to+\infty} b(\zeta) = +\infty$ and $\lim_{\zeta\to-\infty} b(\zeta) = 0$. □

Now we consider the case $\epsilon = 0$. Following Ekström and Vannestål (2019), we study problem (2.21) by varying the initial point ψ for the process Φ,

$$\tilde{V}(\psi; \hat{K}) := \inf_{\tau\in\mathcal{T}} \mathbb{E}^{\mathbb{Q}}\Big[\int_0^\tau e^{-\tilde{r}s} \tilde{H}(\Phi_s^\psi) ds\Big], \tag{3.37}$$

where

$$\tilde{H}(\psi) = (1+\psi)(\psi^\beta - \hat{K}), \quad \hat{K} = \frac{\tilde{r} P \varphi^\beta}{\gamma \lambda_0 x}.$$

Obviously the value function defined by (2.21) satisfies that

$$V(\varphi) = \tilde{V}(\varphi; \hat{K}).$$

Using the dynamic programming principle to problem (3.37) yields

$$\min\Big\{\frac{1}{2}\omega^2\psi^2\tilde{V}'' - \tilde{r}\tilde{V} + \tilde{H}(\psi), -\tilde{V}\Big\} = 0.$$

Define the stopping region and the continuation region as

$$\mathscr{S} := \{\psi : \tilde{V}(\psi; \hat{K}) = 0\}, \quad \mathscr{C} := \{\psi : \tilde{V}(\psi; \hat{K}) < 0\}.$$

Using a similar argument as Proposition 3.2 and Proposition 3.3, we find that there exists a number $b_0(\hat{K}) > 0$ depending on φ and x such that

$$\mathscr{S} := [b_0, +\infty), \quad \mathscr{C} := (0, b_0),$$

and $\tilde{V}(b_0; \hat{K}) = 0$. From Peskir and Shiryaev (2006), we see that the optimal stopping time for problem (3.37) is

$$\tilde{\tau}^* = \inf\{s \ge 0 : \Phi_s^\psi \in \mathscr{S}\}. \tag{3.38}$$

Therefore the optimal stopping problem (3.37) is converted into the following free boundary problem

$$\frac{1}{2}\omega^2\psi^2\tilde{V}'' - \tilde{r}\tilde{V} = -(1+\psi)(\psi^\beta - \hat{K}), \quad \text{for } \psi < b_0, \tag{3.39}$$

$$\tilde{V}(\psi; \hat{K}) = 0, \quad \text{for } \psi \ge b_0, \tag{3.40}$$

$$\tilde{V}'(b_0; \hat{K}) = 0. \tag{3.41}$$

The above problem has a unique solution and can be solved explicitly in the following theorem.

Proposition 3.7. *Assume that $\epsilon = 0$ and denote*

$$\theta_+ = \frac{1}{2} + \frac{\sqrt{\omega^2 + 8\tilde{r}}}{2\omega}, \quad \Lambda(\psi) = C_1\psi^{\theta_+} + C_3 + C_3\psi + C_4\psi^\beta + C_5\psi^{\beta+1},$$

where

$$C_3 = -\frac{\hat{K}}{\tilde{r}}, \; C_4 = \frac{1}{\tilde{r} - \frac{1}{2}\omega^2\beta(\beta-1)}, \; C_5 = \frac{1}{\tilde{r} - \frac{1}{2}\omega^2\beta(\beta+1)}, \tag{3.42}$$

$$C_1 = -C_3 b_0^{-\theta_+} - C_3 b_0^{1-\theta_+} - C_4 b_0^{\beta-\theta_+} - C_5 b_0^{\beta-\theta_+ +1}.$$

Then the free boundary problem (3.39) - (3.41) *has the following closed-form solution*

$$\tilde{V}(\psi;\hat{K}) = \Lambda(\psi)I(\psi < b_0), \tag{3.43}$$

where $b_0(\hat{K})$ *depending on* x *and* φ *is the unique solution to the following equation in* b,

$$C_5(\beta+1-\theta_+)b^{\beta+1} + C_4(\beta-\theta_+)b^{\beta} + C_3(1-\theta_+)b - C_3\theta_+ = 0. \tag{3.44}$$

Proof. Firstly, we set $\theta_\pm = \frac{1}{2} \pm \frac{\sqrt{\omega^2+8\tilde{r}}}{2\omega}$. Solving ODE (3.39) gives that

$$\tilde{V}(\psi;\hat{K}) = \tilde{C}_1\psi^{\theta_+} + C_2\psi^{\theta_-} + C_3 + C_3\psi + C_4\psi^{\beta} + C_5\psi^{\beta+1},$$

where C_3, C_4, C_5 are defined by (3.42) and $\tilde{C}_1$ and C_2 are to be determined. Denote $\tilde{\tau}^*$ as the optimal stopping time for (3.37). Then by the dominated convergence theorem we derive that

$$\begin{aligned} 0 \geq \lim_{\psi\to 0}\tilde{V}(\psi;\hat{K}) &= \lim_{\psi\to 0}\mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\tilde{\tau}^*} e^{-\tilde{r}s}(1+\psi Z_s)(\psi^{\beta}(Z_s)^{\beta} - \hat{K})ds\Big] \\ &= \mathbb{E}^{\mathbb{Q}}\Big[\lim_{\psi\to 0}\int_0^{\tilde{\tau}^*} e^{-\tilde{r}s}(1+\psi Z_s)(\psi^{\beta}(Z_s)^{\beta} - \hat{K})ds\Big] \\ &= -\hat{K}\mathbb{E}^{\mathbb{Q}}\Big[\int_0^{\tilde{\tau}^*} e^{-\tilde{r}s}ds\Big] \geq -\frac{\hat{K}}{\tilde{r}}, \end{aligned}$$

where $Z_s = \exp\Big(-\frac{1}{2}\omega^2 s + \omega W_s\Big)$. This means that $\lim_{\psi\to 0}\tilde{V}(\psi;\hat{K})$ is bounded from below. Therefore it must have $C_2 = 0$. It then follows from the continuous pasting condition and smooth pasting condition that

$$\begin{aligned} &C_1 b^{\theta_+} + C_3 + C_3 b + C_4 b^{\beta} + C_5 b^{\beta+1} = 0, \\ &C_1\theta_+ b^{\theta_+-1} + C_3 + C_4\beta b^{\beta-1} + C_5(\beta+1)b^{\beta} = 0, \end{aligned}$$

which leads to equation (3.44). Now we verify that equation (3.44) has a unique solution. Denote the left-hand side of equation (3.44) by $G(b)$. Then differentiating G gives that

$$\begin{aligned} &G'(b) = C_5(\beta+1)(\beta+1-\theta_+)b^{\beta} + C_4\beta(\beta-\theta_+)b^{\beta-1} + C_3(1-\theta_+), \\ &G''(b) = C_5\beta(\beta+1)(\beta+1-\theta_+)b^{\beta-1} + C_4\beta(\beta-\theta_+)(\beta-1)b^{\beta-2}. \end{aligned}$$

From the definition of β in (2.16), $\beta > 1$ for $\epsilon = 0$. A direct computation shows that

$$\beta + 1 - \theta_+ = \frac{\sigma}{\omega} + \frac{1}{2} - \frac{\sqrt{\omega^2+8\tilde{r}}}{2\omega} = \frac{2\sigma + \omega - \sqrt{\omega^2+8\tilde{r}}}{2\omega}.$$

By assumption (2.23), namely $\tilde{r} > \frac{1}{2}\omega^2\beta(\beta+1)$, we deduce that

$$(2\sigma+\omega)^2 - \omega^2 - 8\tilde{r} = 4\omega^2(\beta^2+\beta) - 8\tilde{r} < 0.$$

Hence, we have $\beta - \theta_+ < \beta + 1 - \theta_+ < 0$. Assumption (2.23) also gives that $C_4, C_5 > 0$. So we have $G''(b) < 0$. Obviously, we see that $G'(0) = C_3(1-\theta_+) > 0$, $\lim_{b\to+\infty} G'(b) = -\infty$. Therefore, there exists a unique $b^* \in (0, +\infty)$ such that G is increasing in $(0, b^*)$ and decreasing in $(b^*, +\infty)$. Moreover

$$\begin{aligned} &G(0) = -C_3\theta_+ > 0, \\ &\lim_{b\to+\infty} G(b) = b^{\beta}(C_3 b^{1-\beta}(1-\theta_+) - C_3\theta_+ b^{-\beta} + C_4(\beta-\theta_+) + C_5(\beta+1-\theta_+)b) = -\infty. \end{aligned}$$

Therefore equation (3.44) has a unique solution $b_0 \in (b^*, +\infty)$. □

Using the previous results, the optimal stopping problem (2.9) is solved and the result is summarized in the following proposition.

Proposition 3.8. *If* $\epsilon \neq 0$, *then the value function of problem* (2.9) *is given by*

$$\hat{V}(x,\pi) = (1-\pi)V\Big(\frac{1}{\epsilon}\log\frac{\gamma\lambda_0 x(1-\pi)^{\beta}}{\pi^{\beta}}, \frac{\pi}{1-\pi}\Big), \tag{3.45}$$

for any $x > 0$, $0 < \pi < 1$, *where* $V(\zeta,\varphi)$ *is given by* (3.36). *The optimal stopping time in the problem* (2.9) *is given by*

$$\hat{\tau}^* = \inf\Big\{s \geq 0 : X_s \geq \hat{b}(\Pi_s)\Big\}, \tag{3.46}$$

where

$$\hat{b}(\pi) := \frac{\pi^{\beta}}{\gamma\lambda_0(1-\pi)^{\beta}} \exp\Big(\epsilon b^{-1}\big(\pi/(1-\pi)\big)\Big). \tag{3.47}$$

If $\epsilon = 0$, then the value function of problem (2.9) is given by

$$\hat{V}(x,\pi) = \frac{\gamma\lambda_0 x(1-\pi)^{\beta+1}}{\pi^{\beta}} \tilde{V}(\pi/(1-\pi); \hat{K}), \tag{3.48}$$

where $\tilde{V}$ is given by (3.43) and

$$\hat{K} = \frac{\tilde{r}P\varphi^{\beta}}{\gamma\lambda_0 x} = \frac{\tilde{r}P}{\gamma\lambda_0 x}\left(\frac{\pi}{1-\pi}\right)^{\beta}.$$

The optimal stopping time in the problem (2.9) has the same form as (3.46) with

$$\hat{b}(\pi) := \frac{(\theta_+ - \pi)P}{\gamma\lambda_0\big[C_4(\theta_+ - \beta)(1-\pi) + C_5(\theta_+ - \beta - 1)\pi\big]} \tag{3.49}$$

and C_4, C_5 are given by (3.42).

Proof. For $\epsilon \neq 0$, since $\hat{V}(x,\pi) = \frac{1}{1+\varphi}V(\zeta,\varphi)$ by (2.29) where $V(\zeta,\varphi)$ is given by (3.36), using the transformations $\zeta = \frac{1}{\epsilon}\log\frac{\gamma\lambda_0 x}{\varphi^{\beta}}$, $\varphi = \frac{\pi}{1-\pi}$ gives (3.45). Moreover since $b(\cdot)$ is strictly monotone (see Proposition 3.4), equation $\varphi = b(\zeta)$ with $b(\zeta)$ in (3.33) gives $x = \hat{b}(\pi)$ with $\hat{b}(\pi)$ in (3.47).

For $\epsilon = 0$, to prove (3.46), we first show that $b_0^{-1}(\cdot)$ exists in $(0,+\infty)$. To this end, we differentiate equation (3.44) in $\hat{K}$ and deduce that

$$-\frac{1}{\tilde{r}}b_0(1-\theta_+) + \frac{\theta_+}{\tilde{r}} = \Big[-C_3(1-\theta_+) - C_4(\beta-\theta_+)\beta b_0^{\beta-1} - C_5(\beta+1-\theta_+)(\beta+1)b_0^{\beta}\Big]\frac{\partial b_0}{\partial \hat{K}}.$$

Using (3.44) and the fact that $C_3 < 0$, $C_5 > 0$, $\beta > 1$, $\beta + 1 - \theta_+ < 0$ and $\theta_+ > 1$, we have

$$\begin{aligned}
&b_0\Big[-C_3(1-\theta_+) - C_4(\beta-\theta_+)\beta b_0^{\beta-1} - C_5(\beta+1-\theta_+)(\beta+1)b_0^{\beta}\Big]\\
&= -C_3(1-\theta_+)\beta b_0 - C_4(\beta-\theta_+)\beta b_0^{\beta} - C_5(\beta+1-\theta_+)\beta b_0^{\beta+1}\\
&\quad - C_5(\beta+1-\theta_+)b_0^{\beta+1} + C_3(1-\theta_+)b_0(\beta-1)\\
&= -C_3\beta\theta_+ + C_3(1-\theta_+)b_0(\beta-1) - C_5(\beta+1-\theta_+)b_0^{\beta} > 0,
\end{aligned}$$

and

$$-\frac{1}{\tilde{r}}b_0(1-\theta_+) + \frac{\theta_+}{\tilde{r}} = \frac{1}{\tilde{r}}\left[\theta_+(1+b_0) - b_0\right] > 0.$$

Hence, we have $\frac{\partial b_0}{\partial \hat{K}} > 0$. Taking $\hat{K} \to 0$ and $\hat{K} \to +\infty$ in (3.44), respectively we have $\lim_{\hat{K}\to 0} b_0(\hat{K}) = 0$ and $\lim_{\hat{K}\to+\infty} b_0(\hat{K}) = +\infty$. This implies that $b_0^{-1}(\cdot)$ exists in $(0,+\infty)$. Therefore (3.46) follows from (3.38), where the free boundary $\hat{b}(\pi)$ can be represented as

$$\hat{b}(\pi) := \frac{\tilde{r}P\pi^{\beta}}{\gamma\lambda_0(1-\pi)^{\beta}b_0^{-1}(\pi/(1-\pi))}.$$

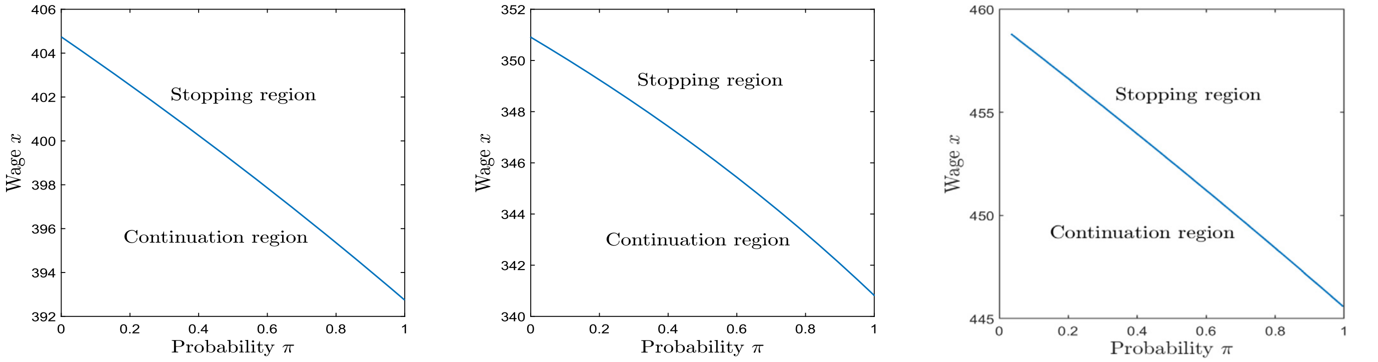
Thus, the free boundary of problem (2.9) is unique. Furthermore plugging $\varphi = b_0(\hat{K})$ and $\hat{K} = \frac{\tilde{r}P}{\gamma\lambda_0 x}\left(\frac{\pi}{1-\pi}\right)^{\beta}$ into (3.44) gives $x = \hat{b}(\pi)$ with $\hat{b}(\pi)$ in (3.49). Using $\hat{V}(x,\pi) = \frac{\gamma\lambda_0 x}{(1+\varphi)\varphi^{\beta}}V(\varphi) = \frac{\gamma\lambda_0 x}{(1+\varphi)\varphi^{\beta}}\tilde{V}(\varphi;\hat{K})$ by (2.30) with $\tilde{V}$ in (3.43) and

$$\varphi = \frac{\pi}{1-\pi}, \quad \hat{K} = \frac{\tilde{r}P}{\gamma\lambda_0 x}\left(\frac{\pi}{1-\pi}\right)^{\beta},$$

gives (3.48). □

Remark 3.1. From (2.10) and Proposition 3.8, it follows that the individual's value function (2.4) is given by

$$\begin{aligned}
\mathbb{V}(x,\pi) &= \mathbb{E}\Big[\int_0^{\infty} e^{-\tilde{r}t}\lambda_0\gamma X_t dt\Big] - \hat{V}(x,\pi) - P\\
&= \lambda_0\gamma x\int_0^{\infty} e^{-\tilde{r}t}\mathbb{E}[e^{\mu t}]dt - \hat{V}(x,\pi) - P\\
&= \lambda_0\gamma x\Big(\frac{\pi}{\tilde{r}-\mu_h} + \frac{1-\pi}{\tilde{r}-\mu_l}\Big) - \hat{V}(x,\pi) - P,
\end{aligned} \tag{3.50}$$



(a) $\epsilon = 0$. (b) $\epsilon > 0$. (c) $\epsilon < 0$.

Fig. 1. The free boundaries $x = \hat{b}(\pi)$.

where the second equality follows from that μ is independent of $\bar{W}_t$ and the third equality follows from that the assumption (2.20) is equivalent to $\tilde{r} > \mu_h$. The function $\hat{V}$ can be calculated by (3.45) and (3.48) for $\epsilon \neq 0$ and $\epsilon = 0$, respectively. Moreover, the optimal stopping time for problem (2.4) can also be represented as (3.46) with the free boundary given by (3.47) for $\epsilon \neq 0$ and (3.49) for $\epsilon = 0$, respectively.

4. Numerical examples

In this section, we take the benefit function $h(s)$ from Anquandah and Bogachev (2019),

$$h(s) = \begin{cases} h_0, & 0 \le s \le s_0, \\ h_0 e^{-\kappa(s-s_0)}, & s \ge s_0, \end{cases}$$

where $0 < h_0 \le 1$, $0 < s_0 < \infty$, and $\kappa > 0$. Thus the insurant receives a certain fraction of his final wage $h_0 X_{\tau_0}$ for a grace period s_0, after which the benefit is falling down exponentially with rate κ. This benefit schedule encourages the insurant to look for a new job. With this function h, constant γ defined in (2.3) is calculated as

$$\gamma = \frac{h_0(1 - e^{-(r+\lambda_1)s_0})}{r + \lambda_1} + \frac{h_0 e^{-(r+\lambda_1)s_0}}{r + \lambda_1 + \kappa}.$$

Example 4.1. This example studies the case $\epsilon = 0$ by implementing the theoretical formulas and simulating individual's wage path and performing sensitivity analysis. As Anquandah and Bogachev (2019), the parameters are set as follows. The inflation rate is taken as $r = 0.0004$, the unemployment rate $\lambda_0 = 0.01$, benefit exponentially falling down rate $\kappa = 0.0094$ (per week), certain fraction benefit $h_0 = 0.574$ and the grace period $s_0 = 34.7$ (weeks). The waiting rate λ_1 is chosen such that $\gamma = 30$. The initial wage is set $x = 346$ (euro per week) and the payment premium $P = 9000$ (euro). The volatility is set $\sigma = 0.04$, top value of wage drift $u_h = 0.0012$ and low value $u_l = 0.0004$. Thus, $\epsilon = \frac{1}{2}(\mu_h + \mu_l - \sigma^2) = 0$.

For the case that $\epsilon = 0$, the free boundary $\hat{b}$ has a closed-form (3.49), thus can be calculated directly. We draw the free boundary $x = \hat{b}(\pi)$, the stopping region and the continuation region in Fig. 1(a). Then we simulate the processes Π_t and X_t with different initial probability using the Euler methods and simulate the running process. By the definition of optimal stopping time (3.46), namely,

$$\hat{\tau}^* = \inf\left\{s \ge 0 : X_s \ge \hat{b}(\Pi_s)\right\},$$

we know that it is the optimal entry time when the path of X_t meets the path of $\hat{b}(\Pi_t)$ for the first time. Figs. 3(a), 4(a) and 5(a) shows the paths of Π_t, X_t and $\hat{b}(\Pi_t)$ and the optimal entry time $\hat{\tau}^*$ with initial probability $\pi = 0.75$, 0.5 and 0.25 respectively. After finding the free boundary $\hat{b}$, we can further calculate the value function $\mathbb{V}(x, \pi)$ of the initial individual's problem (2.4) using Proposition 3.8 and Remark 3.1. We depict the $x-$sections of the value functions in Fig. 6(a). Finally, in order to understand how varying values of exogenous parameters affect the free boundary $\hat{b}$, we conduct some comparative static analysis. To be specific, we change the value of P, λ_1, λ_0 and show the free boundary $\hat{b}$ in Fig. 2.

Fig. 1 shows that the optimal entry boundary is decreasing with respect to π. The economic explanation is clear as a higher π means that the individual is more optimistic about his/her wage, which adds incentive to an earlier entry of the UI contract and thus makes the threshold $\hat{b}$ lower.

From Figs. 3(a), 4(a) and 5(a), we know that at the origin time $t = 0$, an individual has a job with wage X_t which follows (2.7). It is the optimal to join the UI scheme by paying a premium P when the wage process X_t hits the free boundary $\hat{b}$. When the policyholder loses his/her job at time τ_0, he/she receives benefits from the insurance company which is dependent on his final wage X_{τ_0} and benefit function h, till to finding a new job after a period τ_1.

Fig. 6(a) reveals that the value of contract is increasing with respect to the initial belief π. Economically, as π increases, the individual becomes more and more optimistic about his/her career prospects, which means that the contract is more valuable to the individual.

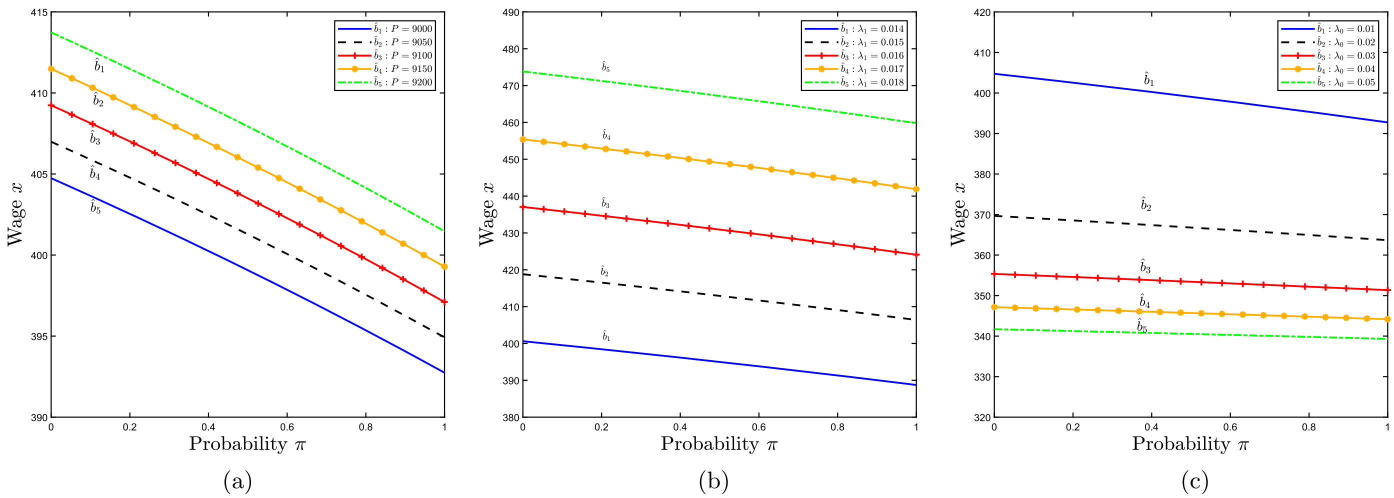


Fig. 2. Sensitivity of the free boundary $\hat{b}(\pi)$ to the premium P, the intensities λ_1 of unemployment duration and λ_0 of losing a job.

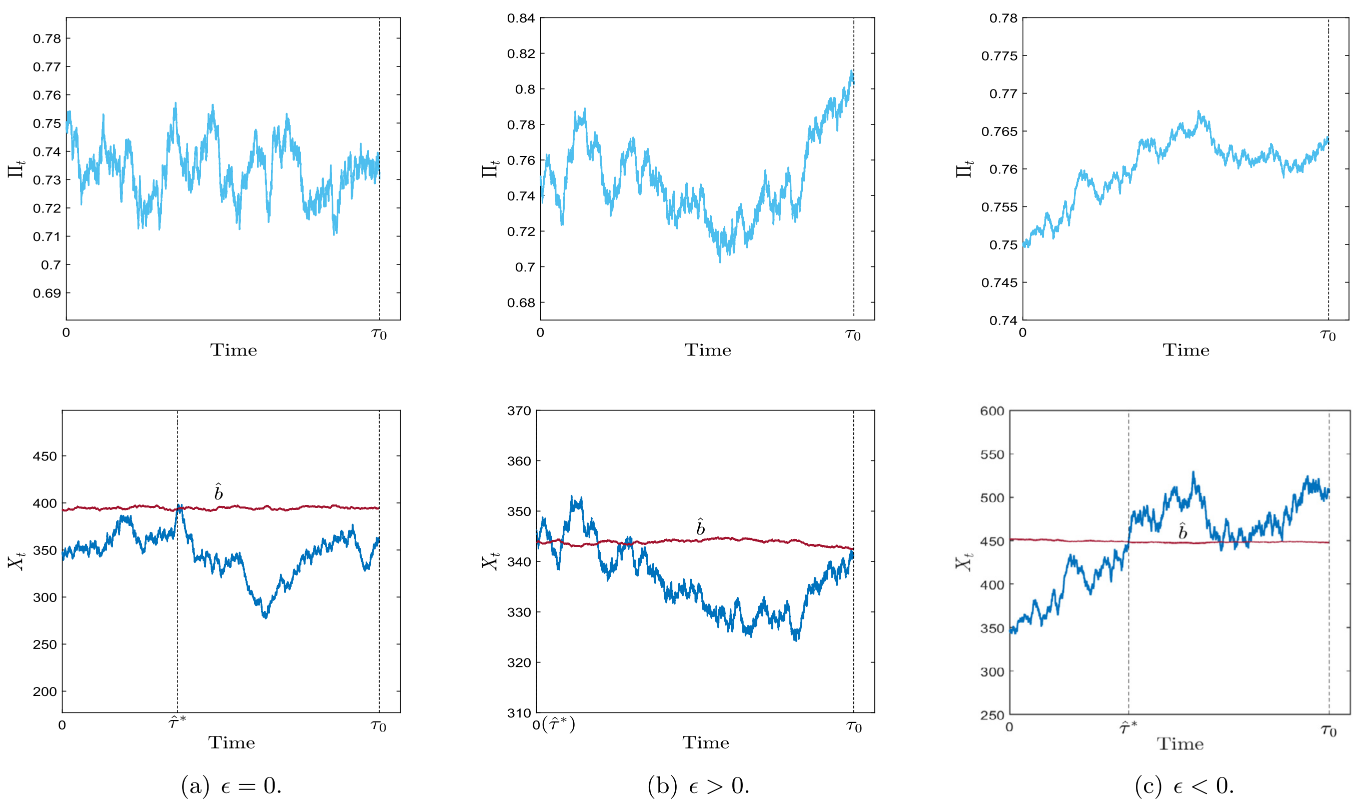


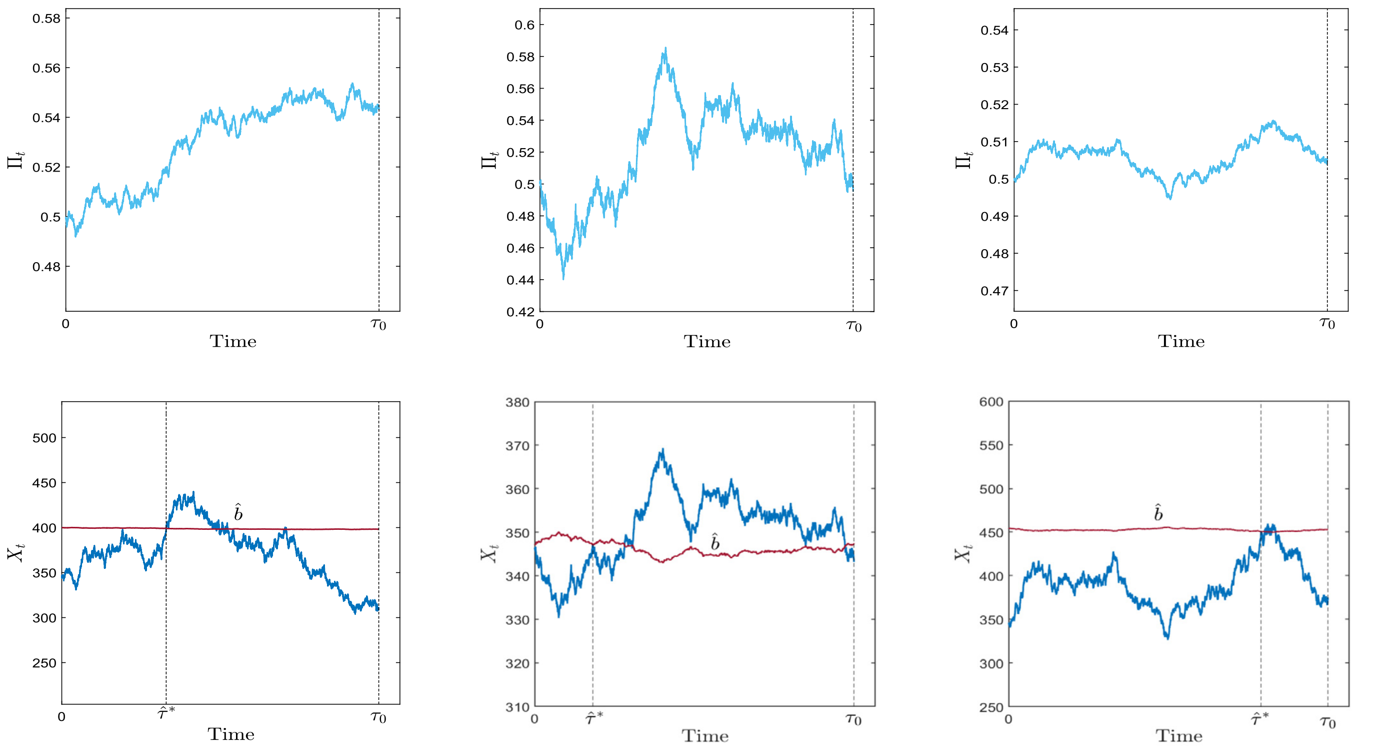
Fig. 3. Simulation of individual's wage path, the probability process and optimal stopping time for initial probability $\pi = 0.75$.

Fig. 2(a) shows that as the level of the premium increases, the corresponding threshold wage is increasing. Economically, increasing premium makes it more expensive to enter the UI contract, which further results in a higher optimal entry boundary.

It is also interesting to analyze how the parameter λ_1 affects the decision of the individual to enter the UI contract. From Fig. 2(b), it is evident that the optimal entry boundary moves up as λ_1 is increased. Economically, when λ_1 increases, the average period of unemployment will decease. It follows that the benefit γX_{τ_0} paid by the insurance company will also decrease and the UI contract appears to be less attractive to the individual, which will reduce the incentive to an early entry.

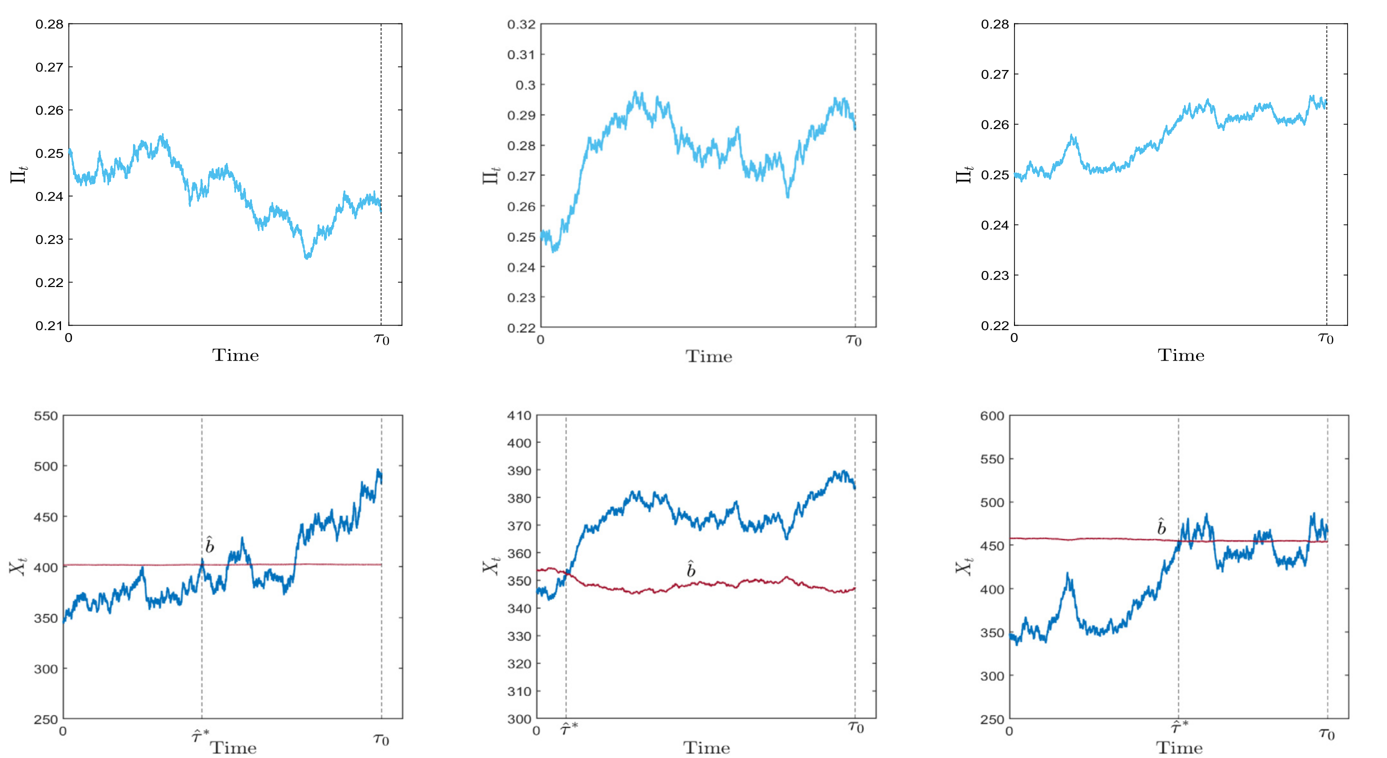
Finally, we study the impact of varying λ_0 on the optimal entry boundary. Increasing λ_0 leads to the lower entry boundaries as revealed in Fig. 2(c). Economically, the bigger λ_0 means a higher risk of losing the job, which motivates the individual to join the UI contract earlier and thus makes the threshold $\hat{b}$ lower.

Example 4.2. This example studies the case $\epsilon \neq 0$, by solving integral equation (3.33), simulating individual's wage path, the probability process and optimal stopping time and plotting $x-$sections of the solution to problem (2.4). The setting is as the same as Example 4.1 except $\sigma = 0.02$ and $\sigma = 0.06$ such that $\epsilon = \frac{1}{2}(\mu_h + \mu_l - \sigma^2) = 0.0006$ and $\epsilon = \frac{1}{2}(\mu_h + \mu_l - \sigma^2) = -0.001$ respectively.



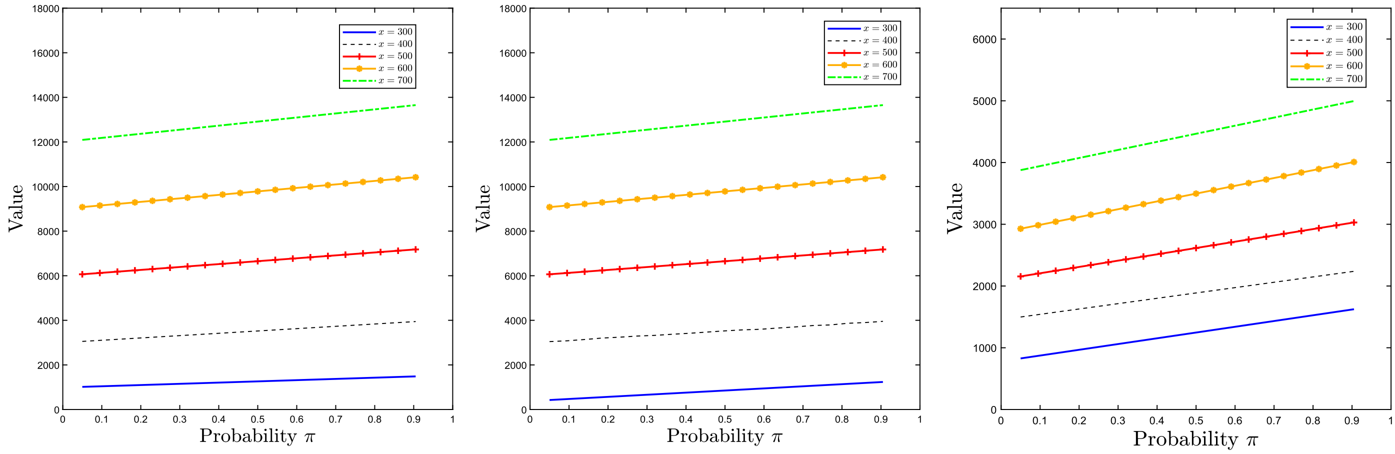
(a) $\epsilon = 0$. (b) $\epsilon > 0$. (c) $\epsilon < 0$.

Fig. 4. Simulation of individual's wage path, the probability process and optimal stopping time for initial probability $\pi = 0.5$.



(a) $\epsilon = 0$. (b) $\epsilon > 0$. (c) $\epsilon < 0$.

Fig. 5. Simulation of individual's wage path, the probability process and optimal stopping time for initial probability $\pi = 0.25$.



(a) $\epsilon = 0$. (b) $\epsilon > 0$. (c) $\epsilon < 0$.

Fig. 6. The values of individual's problem (2.4).

For the case that $\epsilon \neq 0$, the free boundary $\hat{b}$ is given by (3.47), which is dependent on b^{-1}. Since b is the solution of the integral equation (3.33), there is no closed-form formula for $\hat{b}$. We now develop the numerical method based on the ideas put forward by Johnson and Peskir (2017a). Replacing T with $T-\zeta$ in (3.34) gives that

$$\begin{aligned} V(\zeta,\varphi) &= \mathbb{E}^{\mathbb{Q}}\Big[\int_0^{T-\zeta} e^{-\tilde{r}s} H(\zeta+s,\Phi_s) I(\Phi_s < b(\zeta+s)) ds\Big] + \mathbb{E}^{\mathbb{Q}}\big[e^{-\tilde{r}(T-\zeta)} V(T,\Phi_{T-\zeta})\big] \\ &\approx \mathbb{E}^{\mathbb{Q}}\Big[\int_0^{T-\zeta} e^{-\tilde{r}s} H(\zeta+s,\Phi_s) I(\Phi_s < b(\zeta+s)) ds\Big] \\ &:= \int_0^{T-\zeta} J(s,\zeta,b(\zeta+s),\varphi) ds, \end{aligned}$$

where the second term in the first equality tends to zero by (3.35). Thus, instead of solving (3.33) immediately, using the continuous pasting condition, we truncate the upper limit of integration by $T-\zeta$ (i.e., replace $+\infty$ by $T-\zeta$) and set $\zeta \in [-T, T]$. That is, we shall solve

$$\int_0^{T-\zeta} J(s,\zeta,b(\zeta+s),b(\zeta)) ds = 0. \tag{4.1}$$

We compute the boundary b by backward recursion. Set the discretization mesh $\zeta_i = -T + i\Delta$ for $i = 0,\dots,n$ with $\Delta = 2T/n$. Then we use rectangular rule to discretize the integral

$$\begin{aligned} \int_0^{T-\zeta_i} J\big(s,\zeta_i,b(\zeta_i+s),b(\zeta_i)\big) ds &= \sum_{j=1}^{n-i} \int_{(j-1)\Delta}^{j\Delta} J\big(s,\zeta_i,b(\zeta_i+s),b(\zeta_i)\big) ds \\ &\approx \sum_{j=1}^{n-i} J\big(j\Delta,\zeta_i,b(\zeta_i+j\Delta),b(\zeta_i)\big)\Delta. \end{aligned}$$

Let $b_i \approx b(\zeta_i)$ such that

$$0 = \sum_{j=1}^{n-i} J(j\Delta,\zeta_i,b_{i+j},b_i)\Delta. \tag{4.2}$$

According to Proposition 3.6, we set

$$b_n = 0, \quad b_0 = T,$$

for $\epsilon > 0$, and

$$b_n = T, \quad b_0 = 0,$$

for $\epsilon < 0$. If b_{i+j} is known for all $j \geq 1$, then (4.2) is a nonlinear equation for b_i. Thus, in each state the free boundary approximation b_i can be solved by Newton iteration methods. And all b_i can be obtained by iterating from $i = n-1$ to 0. After getting b_i for $i = 0, 1, \dots, n$, we can find a ζ_i such that $b_i = \frac{\pi}{1-\pi}$ for a given π, and then use this relationship $\zeta = \frac{1}{\epsilon} \log \frac{\gamma\lambda_0 x}{\varphi^\beta}$ to find the corresponding x. Consequently the free boundary is solved.

We draw the free boundary $x = \hat{b}(\pi)$, the stopping region and the continuation region in Fig. 1(b) for $\epsilon > 0$ and Fig. 1(c) for $\epsilon < 0$ respectively. Then we simulate the processes Π_t and X_t with different initial probability using the Euler methods and simulate the running process for $\epsilon > 0$ and $\epsilon < 0$ respectively. The results are shown in Figs. 3, 4 and 5. After solving the free boundary, we can further calculate the prime values by Monte-Carlo simulations. To be more specific, we simulate the path of X_t and record the time when a path of the wage first touches the free boundary. Then we use the numerical integration formula to calculate $\hat{V}(x,\pi)$. And the value $\mathbb{V}(x,\pi)$ can be calculated by (3.50) using the results of $\hat{V}(x,\pi)$. The $x-$sections of the value function $\mathbb{V}(x,\pi)$ are shown in Figs. 6(b) and 6(c).

We notice that the properties of the optimal entry boundary and the value function in this example are similar to the case $\epsilon = 0$ in Example 4.1.

5. Conclusions

This paper studies the optimal time for the individual to join an UI scheme under uncertainty drift of the wage process. This problem is a kind of degenerate two-dimensional optimal stopping problem. The degeneration makes the analysis and solution extremely hard. This paper solves it by converting the problem into an equivalent auxiliary non-degenerate optimal stopping problem and studying the auxiliary free boundary problem instead. The monotonicity and continuity of the auxiliary free boundary are rigorously proved and then the results for the original problem are established by the inverse transformation. The running process of the UI scheme is simulated and the dynamic optimal entry decision is clearly shown in the simulation.

Declaration of competing interest

There is no competing interest.

Data availability

No data was used for the research described in the article.

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