Insurance: Mathematics and Economics 111 (2023) 257–278

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

On the area in the red of Lévy risk processes and related quantities

Mohamed Amine Lkabous [a], Zijia Wang [b,*]

[a] School of Mathematical Sciences, University of Southampton, Highfield, Southampton, SO17 1BJ, UK
[b] Department of Finance, The Chinese University of Hong Kong, Hong Kong

ARTICLE INFO

Article history:
Received October 2022
Received in revised form May 2023
Accepted 15 May 2023
Available online 19 May 2023

JEL classification:
C02
C46
G22

Keywords:
Cost of recovery
Area in the red
Lévy risk processes

ABSTRACT

Under contemporary insurance regulatory frameworks, an insolvent insurer placed in receivership may have the option of rehabilitation, during which a plan is devised to resolve the insurer's difficulties. The regulator and receiver must analyze the company's financial condition and determine whether a rehabilitation is likely to be successful or if its problems are so severe that the appropriate action is to liquidate the insurer. Therefore, it is essential to evaluate the cost required to support the insurer during its insolvent states in the decision-making process. To this end, we study areas in the red (below level 0) up to the recovery time, Poissonian, and continuous first passage times in this paper. Furthermore, we extend the study to the areas associated with Parisian ruin to evaluate the total cost until possible liquidation. For spectrally negative Lévy processes (SNLPs), also known as Lévy risk models, we derive the expectations of these quantities in terms of the well-known scale functions. Our results improve the existing literature, in which only expected areas for the Brownian motion and the Cramér-Lundberg risk process with exponential jumps are known.

1. Introduction

One of the central topics in risk theory is to analyze ruin-related quantities such as the time to ruin and the deficit at ruin, due to their significant roles in assessing an insurer's solvency risk. In practice, when an insurer enters a period of financial difficulty, an appropriate course of action will be explored first to help the company regain its financial footing before liquidation. Therefore, it is of interest to continue monitoring the insurer's surplus process after the ruin time. In this paper, we study the recovery of the risk process from insolvency under practical considerations.

As discussed in Li et al. (2014), declaring bankruptcy and implementing liquidation should not be treated as the same event and they are often highly regulated. Under Chapter 7 liquidation (which is also known as a "straight bankruptcy") of the U.S. Bankruptcy Code, a distressed firm ceases operations and liquidates its assets. This is in contrast to Chapter 11 reorganization, which allows the firm to continue operating its business and restructuring its debts and obligations within a grace period granted by the federal court. Contemporary insurance regulatory frameworks have also included similar features. According to the handbook published by the National Association of Insurance Commissioners (NAIC) (2021), an insurer may be placed in receivership if the state insurance department believes that its solvency situation cannot be corrected. During a conservation procedure, the receiver will analyze the insurer's financial condition and decide whether a liquidation or rehabilitation plan should be implemented. If rehabilitation is warranted, a plan is devised to resolve the insurer's difficulties and return it to the marketplace. As pointed out by NAIC,[1]

"The regulator must determine whether a rehabilitation of the company is likely to be successful or if its problems are so severe that the rehabilitation would significantly increase the risk of loss to policyholders. If the latter is true, the appropriate course of action is to liquidate the insurer." For more discussions on this topic, we refer readers to Li et al. (2020) and the references therein.

From the above discussion, one can conclude that it is crucial to assess the insurer's ability to recover from financial distress during the decision-making process. There has been a wide range of risk indicators proposed in the ruin theory literature to address this. For a risk process $\{X_t\}_{t\geq 0}$ with ruin time $\tau_0^- = \inf\{t \geq 0 : X_t < 0\}$ (where we adopt the convention that $\inf\emptyset = \infty$), Picard (1994) proposed two

* Corresponding author.
E-mail addresses: m.a.lkabous@soton.ac.uk (M.A. Lkabous), zijiawang@cuhk.edu.hk (Z. Wang).

[1] See the document on receivership updated on July 06, 2022 published by NAIC at https://content.naic.org/cipr-topics/receivership.

https://doi.org/10.1016/j.insmatheco.2023.05.005
0167-6687/

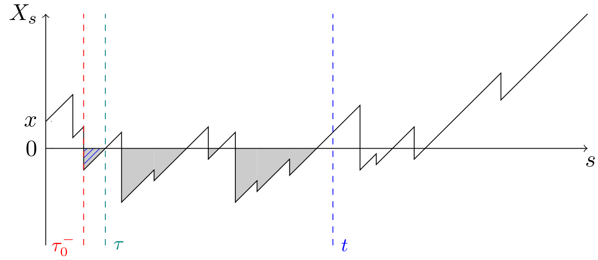

Fig. 1. A sample path of X, where the gray shaded region is the total area in the red $\mathcal{A}_t$ and the blue pattern shaded region is the cost of recovery $\mathcal{I}$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

indexes: (1) the *recovery time* $\tau - \tau_0^-$ where $\tau = \inf\{t > \tau_0^- : X_t = 0\}$, which represents the time required for the recovery of the insurer, and (2) the *maximum severity of ruin* $\sup_{\tau_0^- \leq t \leq \tau}\{|X_t|\}$ to describe the worst situation the company would experience before recovery. These quantities are further studied by Dickson and dos Reis (1997), dos Reis (2000), and Landriault et al. (2019) in the Cramér-Lundberg risk process, and Li (2008) in the Sparre Andersen risk model. Note that the aforementioned works relate to the first excursion of X below level 0. For risk processes perturbed by diffusion or business lines that may experience financial distress again after recovering from the first insolvent state, analyzing the recovery of the process in the long run may be more relevant. The *total time in the red* $\int_0^\infty \mathbf{1}_{\{X_s<0\}} ds$, first proposed and studied by dos Reis (1993) in the context of Cramér-Lundberg risk processes, was later extended to SNLPs in Landriault et al. (2011), Guérin and Renaud (2017), Landriault et al. (2020), among others.

It is worth noting that three key aspects should be addressed when evaluating an insurer's financial condition and the feasibility of a rehabilitation/recovery plan: duration, severity, and frequency of distress. If the financial distress lasts too long, it may be too expensive to keep the business ongoing, especially when taking the time value of money into account. If the company stays in the red too deeply, the decision may be negative as it will take a significant cost to save the business with little hope of recovery. Furthermore, if the company experiences many distress periods over time, even if the durations and severities are low, the cost may still be high due to court fees and arrears. Therefore, it is of practical interest to consider risk indicators that incorporate these factors.

In light of the above, we propose to study two main risk indicators that take both the time spent in the red and the severity of distress into account: the so-called *cost of recovery*, which is defined as the aggregate severity of ruin until recovery:

$$\mathcal{I} = \int_{\tau_0^-}^{\tau} |X_t|\, dt, \tag{1}$$

and the total *area in the red* up to a fixed time $t > 0$, defined by

$$\mathcal{A}_t = \int_0^t |X_s| \mathbf{1}_{\{X_s<0\}} ds. \tag{2}$$

In the case where $\tau_0^- = \infty$, we adopt the convention that $\mathcal{I} = 0$. See Fig. 1 for a sample path illustration. We note that the quantity $\mathcal{I}$ was first proposed by Picard (1994), and the author derived its expectation using a martingale approach for the Cramér-Lundberg risk process. For processes with bounded variation, the quantity $\mathcal{I}$ can be viewed as the area in the red of the first negative excursion of X. The second quantity $\mathcal{A}_t$ may be interpreted as the total costs required for keeping the business line alive during its stress periods until time t.

In the literature, there are a few results concerning the area in the red for the Brownian motion. For example, the distribution of $\mathcal{A}_1$ has been studied by Perman and Wellner (1996) using the results of Shepp (1982). For an infinite horizon time, the expectation of $\mathcal{A}_\infty$ has been calculated by Gerber et al. (2012). An application of the aforementioned results to structural credit risk models has been introduced by Yildirim (2006) and a new default time defined as $\xi = \inf\{t > 0 : \mathcal{A}_t > b\}$, that is the first time the cumulative area of the asset value of the firm below the threshold level 0 exceeds a fixed level $b > 0$, was studied. For general risk processes, Loisel (2005) discusses the relations between the expected area in the red and the expected time in the red through differentiation formulas, and a closed-form formula for $\mathbb{E}[\mathcal{A}_\infty]$ is derived for the Cramér-Lundberg risk process with exponential jumps. In Loisel and Trufin (2014), the authors propose a new relevant risk indicator defined as the minimum initial capital needed to ensure that the expected area in red is less than a predetermined value. More recently, Callant et al. (2022) studied a generalized version of the expected area in the red up to a fixed time making use of Schwartz's theory of distributions in Schwartz (1945). In Bayraktar and Young (2010), a minimization problem of the expected area in the red has been briefly studied. Undoubtedly, solving the distribution functions of $\mathcal{I}$ and $\mathcal{A}_t$ is more desirable. However, it is a highly non-trivial problem and, to the best of our knowledge, the distribution of $\mathcal{A}_t$ has only been found for the Brownian motion.

Our main contribution lies in deriving explicit expressions (in terms of scale functions) for the expectations of the aforementioned area-related quantities in a more general context of SNLPs. As the main results of our paper, we first provide analytical expressions for the expected cost of recovery and other related quantities of interest, such as the covariance between the recovery time and area. We further examine the expected area in the red up to a Poissonian first passage time[2] for general SNLPs. Consequently, the expected areas up to a continuous first passage time and up to an infinite horizon time are obtained as limiting cases, and the expected areas obtained in Gerber

[2] There is an extensive literature on Lévy risk processes under Poissonian observations in insurance mathematics. See, e.g., Albrecher et al. (2016) and Landriault et al. (2018).

et al. (2012) and Loisel (2005) are recovered as special cases. Moreover, we extend our results by investigating the areas associated with Parisian ruin, which occurs at the first time the risk process stays in the red for longer than a certain period (see, e.g., Loeffen et al. (2013) and Landriault et al. (2014) for more details). By evaluating the expected (total) cost required to recover from financial distress, our results provide insights into comprehensively assessing the feasibility of a recovery plan.

The rest of the paper is organized as follows. In Section 2, we first present the necessary background material on spectrally negative Lévy processes and scale functions. The main results of this paper are presented in Section 3, where analytical expressions for expectations of area-related quantities are derived. In Section 4, we study the cases of the Brownian risk process, the Cramér-Lundberg process with exponential jumps and the jump-diffusion risk process with phase-type claims and conduct a numerical study to illustrate the main results. All technical proofs are postponed to the Appendix.

2. Preliminaries on spectrally negative Lévy processes

First, we present the necessary background material on spectrally negative Lévy processes. A Lévy insurance risk process X is a process with stationary and independent increments and no positive jumps. To avoid trivialities, we exclude the case where X has monotone paths. As the Lévy process X has no positive jumps, its Laplace transform exists: for all $\lambda, t \geq 0$,

$$\mathbb{E}\left[e^{\lambda X_t}\right] = e^{t\psi(\lambda)},$$

where

$$\psi(\lambda) = \gamma\lambda + \frac{1}{2}\sigma^2\lambda^2 + \int_{(-\infty,0)} \left(e^{\lambda z} - 1 - \lambda z \mathbf{1}_{\{z>-1\}}\right) \Pi(dz),$$

for $\gamma \in \mathbb{R}$ and $\sigma \geq 0$, and Π is a σ-finite measure on $(0, \infty)$ called the Lévy measure of X such that

$$\int_{(-\infty,0)} (1 \wedge z^2)\Pi(dz) < \infty.$$

Recall that the function ψ is infinitely differentiable and strictly convex on $(0, \infty)$. For a SNLP, there exists a function $\Phi\colon [0, \infty) \to [0, \infty)$ defined by $\Phi_q = \sup\{\lambda \geq 0 : \psi(\lambda) = q\}$ (the right inverse of ψ) such that

$$\psi(\Phi_q) = q, \quad q \geq 0,$$

and thus

$$\Phi_q' = \frac{1}{\psi'(\Phi_q)}, \quad q \geq 0. \tag{3}$$

Throughout this paper, we will suppose that X satisfies the security loading condition, namely $\mathbb{E}[X_1] > 0$, and in this case we have

$$\lim_{q\to 0} \frac{q}{\Phi_q} = \psi'(0+) > 0.$$

Moreover, we will use the standard Markovian notation: the law of X when starting from $X_0 = x$ is denoted by $\mathbb{P}_x$ and the corresponding expectation by $\mathbb{E}_x$. We write $\mathbb{P}$ and $\mathbb{E}$ when $x = 0$.

2.1. Scale functions

We now present the definitions of the scale functions W_q and Z_q of X. For $q \geq 0$, the q-scale function of the process X is defined as the continuous function on $[0, \infty)$ with Laplace transform

$$\int_0^\infty e^{-\lambda y} W_q(y)dy = \frac{1}{\psi_q(\lambda)}, \quad \text{for } \lambda > \Phi_q,$$

where $\psi_q(\lambda) = \psi(\lambda) - q$. This function is unique, positive and strictly increasing for $x \geq 0$. We extend W_q to the whole real line by setting $W_q(x) = 0$ for $x < 0$. We write $W = W_0$ when $q = 0$. The initial values of W_q and W_q' are given by

$$W_q(0+) = \begin{cases} 1/c & \text{when } \sigma = 0 \text{ and } \int_{(-1,0)} z\Pi(dz) < \infty, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$W_q'(0+) = \begin{cases} 2/\sigma^2 & \text{when } \sigma > 0, \\ (\Pi(-\infty, 0) + q)/c^2 & \text{when } \sigma = 0 \text{ and } \Pi(-\infty, 0) < \infty, \\ \infty & \text{when } \sigma = 0 \text{ and } \Pi(-\infty, 0) = \infty. \end{cases} \tag{4}$$

where $c := \gamma + \int_{(-1,0)} z\Pi(dz) > 0$. On the other hand, when $\psi'(0+) > 0$, the limit of W is given by

$$\lim_{x\to\infty} W(x) = \frac{1}{\psi'(0+)} = \frac{1}{\mathbb{E}[X_1]}.$$

We also define another scale function $Z_q(x,\lambda)$ by

$$Z_q(x,\lambda) = \mathrm{e}^{\lambda x}\left(1 - \psi_q(\lambda)\int_0^x \mathrm{e}^{-\lambda y} W_q(y)\mathrm{d}y\right), \quad x \ge 0,$$

and $Z_q(x,\lambda) = \mathrm{e}^{\lambda x}$ for $x < 0$. We write $Z = Z_0$ when $q = 0$. For $\lambda = 0$,

$$Z_q(x,0) = 1 + q\int_0^x W_q(y)\mathrm{d}y, \quad x \in \mathbb{R}.$$

By straightforward calculations, one deduces that

$$Z'(x,0) = x - \psi'(0+)\int_0^x W(y)\mathrm{d}y, \tag{5}$$

$$Z''(x,0) = x^2 - \left(2x\psi'(0+) + \psi''(0+)\right)\int_0^x W(y)\mathrm{d}y + 2\psi'(0+)\int_0^x yW(y)\mathrm{d}y, \tag{6}$$

and

$$\begin{aligned} Z'''(x,0) =& x^3 - \left(3x^2\psi'(0+) + x\psi''(0+) + \psi'''(0+)\right)\int_0^x W(y)\mathrm{d}y \\ &+ \left(6x\psi'(0+) + 3\psi''(0+)\right)\int_0^x yW(y)\mathrm{d}y - 3\psi'(0+)\int_0^x y^2 W(y)\mathrm{d}y, \end{aligned} \tag{7}$$

where Z', Z'' and Z''' are the first, second and third order derivatives of Z with respect to the second argument, respectively.

2.2. Fluctuation identities

For $b \in \mathbb{R}$, we define the standard first passage times by

$$\tau_b^{+(-)} = \inf\{t > 0 \colon X_t > (<) b\}.$$

The Laplace transform of the deficit at ruin is given by

$$\mathbb{E}_x\left[\mathrm{e}^{\theta X_{\tau_0^-}} \mathbf{1}_{\{\tau_0^- < \infty\}}\right] = Z(x,\theta) - \frac{\psi(\theta)}{\theta - \Phi_0} W(x). \tag{8}$$

We also recall that the q-potential measure of X killed on exiting $(-\infty, 0]$ is given by

$$\mathbb{P}_x\left(X_{\mathrm{e}_q} \in \mathrm{d}z, \mathrm{e}_q < \tau_0^+\right) = q\left(\mathrm{e}^{\Phi_q x} W_q(-z) - W_q(x-z)\right)\mathrm{d}z,$$

for $x, z \le 0$ and e_q denotes an exponential random variable with rate $q > 0$ that is independent of X. Consequently,

$$\mathbb{E}_x\left[\mathrm{e}^{\theta X_{\mathrm{e}_q}} \mathbf{1}_{\{\mathrm{e}_q < \tau_0^+\}}\right] = \frac{q\left(\mathrm{e}^{\Phi_q x} - \mathrm{e}^{\theta x}\right)}{\psi_q(\theta)}, \tag{9}$$

for any $q, \theta \ge 0$ and $x \le 0$.

An extensive body of literature has recently emerged on the so-called Poissonian exit times, which are defined as

$$T_b^{+(-)} := \inf\{T_i : X_{T_i} > (<) b, i \in \mathbb{N}\}, \quad b \in \mathbb{R},$$

where $\{T_i\}_{i\in\mathbb{N}}$ are the arrival times of an independent Poisson process with intensity rate $\lambda > 0$. For $b, \lambda, \theta \ge 0$ and $x \le b$, we have the following useful identity taken from Albrecher et al. (2016),

$$\mathbb{E}_x\left[\mathrm{e}^{\theta X_{\tau_0^-}} \mathbf{1}_{\{\tau_0^- < T_b^+\}}\right] = Z(x,\theta) - \frac{W(x)}{\theta - \Phi_\lambda}\left(\psi(\theta) - \lambda\frac{Z(b,\theta)}{Z(b,\Phi_\lambda)}\right). \tag{10}$$

We refer the reader to Kyprianou (2014) and Albrecher et al. (2016) for more details on spectrally negative Lévy processes and fluctuation identities.

3. Main results

In this section, we first extend the results of Picard (1994) by deriving the expected cost of recovery $\mathbb{E}_x[\mathcal{I}]$ and other relevant quantities for SNLPs. Then, an analytical expression of $\mathbb{E}_x[\mathcal{A}_{T_b^+}]$ is obtained, which respectively leads to the expectations of $\mathcal{A}_{\tau_b^+}$ and $\mathcal{A}_\infty$ by taking appropriate limits. Extensions to the Parisian ruin time are also considered at the end of this section.

3.1. Expected cost of recovery

Theorem 1. *For $x \in \mathbb{R}$, the expected cost of recovery is given by*

$$\mathbb{E}_x[\mathcal{I}] = \left(\frac{\psi''(0+)^2}{4\psi'(0+)^2} - \frac{\psi'''(0+)}{6\psi'(0+)}\right) W(x) - \mathcal{R}(x), \tag{11}$$

where

$$\mathcal{R}(x) = \frac{\psi''(0+)}{2\psi'(0+)^2}x - \frac{x^2}{2\psi'(0+)} + x\int_0^x W(y)\mathrm{d}y - \int_0^x yW(y)\mathrm{d}y. \tag{12}$$

In particular, for X of unbounded variation, we have $\mathbb{E}[\mathcal{I}] = 0$.

The following proposition provides an expression for the covariance between $\tau - \tau_0^-$ and $\int_{\tau_0^-}^{\tau} |X_s|\mathrm{d}s$, which measures the relationship between the duration and cost of recovery.

Proposition 2. *For $x \in \mathbb{R}$,*

$$\begin{aligned}\mathrm{Cov}_x\left[\tau - \tau_0^-, \int_{\tau_0^-}^{\tau} |X_s|\ \mathrm{d}s\right] &= \left(\frac{\psi''(0+)}{\psi'(0+)^3} + \frac{\mathbb{E}_x\left[X_{\tau_0^-}\mathbf{1}_{\{\tau_0^-<\infty\}}\right]}{2\psi'(0+)^2}\right)\mathbb{E}_x\left[X^2_{\tau_0^-}\mathbf{1}_{\{\tau_0^-<\infty\}}\right] - \frac{\mathbb{E}_x\left[X^3_{\tau_0^-}\mathbf{1}_{\{\tau_0^-<\infty\}}\right]}{2\psi'(0+)^2} \\ &\quad - \left(\frac{\psi''(0+)^2}{\psi'(0+)^4} - \frac{\psi'''(0+)}{2\psi'(0+)^3} + \frac{\psi''(0+)\mathbb{E}_x\left[X_{\tau_0^-}\mathbf{1}_{\{\tau_0^-<\infty\}}\right]}{2\psi'(0+)^3}\right)\mathbb{E}_x\left[X_{\tau_0^-}\mathbf{1}_{\{\tau_0^-<\infty\}}\right],\end{aligned} \tag{13}$$

where expressions for the first and second moments of deficit $X_{\tau_0^-}$ are given in Eqs. (28) and (29), and

$$\mathbb{E}_x\left[X^3_{\tau_0^-}\mathbf{1}_{\{\tau_0^-<\infty\}}\right] = Z'''(x,0) - \frac{\psi''''(0+)}{4}W(x).$$

In particular, for $x \le 0$,

$$\mathrm{Cov}_x\left[\tau_0^+, \int_0^{\tau_0^+} |X_s|\,\mathrm{d}s\right] = \frac{\psi''(0+)x^2}{2\psi'(0+)^3} - \left(\frac{\psi''(0+)^2}{\psi'(0+)^4} - \frac{\psi'''(0+)}{2\psi'(0+)^3}\right)x. \tag{14}$$

Remark 3. Using similar arguments as in the proof of Theorem 1, one can derive an expression for a more general quantity, namely

$$\mathbb{E}_x\left[\int_{\tau_0^-}^{\tau} X_s^n \mathrm{d}s\right], \quad n \in \mathbb{N}^+.$$

For $x \le 0$ and $n \in \mathbb{N}^+$, applying Eq. (9) and Leibniz's rule yields

$$\begin{aligned}\mathbb{E}_x\left[\int_0^{\tau_0^+} X_s^n \mathrm{d}s\right] &= \lim_{q\to 0}\frac{1}{q}\mathbb{E}_x\left[X^n_{\mathrm{e}_q}\mathbf{1}_{\{\mathrm{e}_q<\tau_0^+\}}\right] \\ &= \lim_{q\to 0}\frac{1}{q}\frac{\mathrm{d}^n}{\mathrm{d}\lambda^n}\mathbb{E}_x\left[\mathrm{e}^{\lambda X_{\mathrm{e}_q}}\mathbf{1}_{\{\mathrm{e}_q<\tau_0^+\}}\right]\Big|_{\lambda=0} \\ &= \lim_{q\to 0}\left\{\mathrm{e}^{\Phi_q x}\hat{\psi}_q^{(n)}(0) - \sum_{k=0}^{n}\binom{n}{k}x^k\hat{\psi}_q^{(n-k)}(0)\right\},\end{aligned}$$

where $\hat{\psi}_q^{(n)}(\lambda)$ is the n-th derivative of $\left(\psi_q(\lambda)\right)^{-1}$ with respect to λ and it can be evaluated recursively using di Bruno's formula (see, e.g., Riordan (2012)). It follows that

$$\mathbb{E}_x\left[\int_{\tau_0^-}^{\tau} X_s^n \, \mathrm{d}s\right] = \mathbb{E}_x\left[\mathbb{E}_{X_{\tau_0^-}}\left[\int_0^{\tau_0^+} X_s^n \, \mathrm{d}s\right]\mathbf{1}_{\{\tau_0^-<\infty\}}\right]$$
$$= \mathbb{E}_x\left[\lim_{q\to 0}\left\{\mathrm{e}^{\Phi_q X_{\tau_0^-}}\hat{\psi}_q^{(n)}(0) - \sum_{k=0}^{n}\binom{n}{k}X_{\tau_0^-}^k \hat{\psi}_q^{(n-k)}(0)\right\}\mathbf{1}_{\{\tau_0^-<\infty\}}\right],$$

for $x\in\mathbb{R}$. For example, for the case where $n=2$, one can obtain from the above that

$$\mathbb{E}_x\left[\int_{\tau_0^-}^{\tau} X_s^2 \, \mathrm{d}s\right] = \mathbb{E}_x\left[\lim_{q\to 0}\left\{\mathrm{e}^{\Phi_q X_{\tau_0^-}}\hat{\psi}_q^{(2)}(0) - \sum_{k=0}^{2}\binom{2}{k}X_{\tau_0^-}^k \hat{\psi}_q^{(2-k)}(0)\right\}\mathbf{1}_{\{\tau_0^-<\infty\}}\right]$$
$$= -\frac{\mathbb{E}_x\left[X_{\tau_0^-}^3\mathbf{1}_{\{\tau_0^-<\infty\}}\right]}{3\psi'(0+)} + \frac{\psi''(0+)\mathbb{E}_x\left[X_{\tau_0^-}^2\mathbf{1}_{\{\tau_0^-<\infty\}}\right]}{2\psi'(0+)^2}$$
$$+\left(\frac{\psi'''(0+)}{3\psi'(0+)^2} - \frac{\psi''(0+)^2}{2\psi'(0+)^3}\right)\mathbb{E}_x\left[X_{\tau_0^-}\mathbf{1}_{\{\tau_0^-<\infty\}}\right]. \tag{15}$$

In the case where X is the Cramér-Lundberg risk process, Eqs. (14) and (15) recover the results in Corollary of Picard (1994).

Remark 4. One can also study the discounted cost of recovery defined as

$$\mathcal{I}^{(q)} = \int_{\tau_0^-}^{\tau} \mathrm{e}^{-qt}\,|X_t|\,\mathrm{d}t, \quad q>0.$$

Following the same step as in the proof of (11), one obtains

$$\mathbb{E}_x\left[\mathcal{I}^{(q)}\right] = -\mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^-}\mathbb{E}_{X_{\tau_0^-}}\left[\int_0^{\tau_0^+}\mathrm{e}^{-qs}X_s\,\mathrm{d}s\right]\mathbf{1}_{\{\tau_0^-<\infty\}}\right]$$
$$= W_q(x)\left(\frac{1}{\Phi_q^2} - \frac{\psi'(0+)}{q^2\Phi_q'}\right) - \frac{\psi'(0+)}{q^2}\left(Z_q(x) - \mathrm{e}^{\Phi_q x}\right) - \frac{Z_q'(x,0)}{q}.$$

The expression of $\mathbb{E}_x[\mathcal{I}]$ can be recovered by taking the limit as $q\to 0$. By L'Hôpital's rule, we have

$$\mathbb{E}_x[\mathcal{I}] = \lim_{q\to 0}\mathbb{E}_x\left[\mathcal{I}^{(q)}\right]$$
$$= \lim_{q\to 0} W_q(x)\left(\frac{1}{\Phi_q^2} - \frac{\psi'(0+)}{q^2\Phi_q'}\right) - \lim_{q\to 0}\left(\frac{\psi'(0+)}{q^2}\left(Z_q(x) - \mathrm{e}^{\Phi_q x}\right) - \frac{Z_q'(x,0)}{q}\right)$$
$$= W(x)\lim_{q\to 0}\left(\frac{2q\Phi_q' + q^2\Phi_q'' - 2\Phi_q'\Phi_q'\psi'(0+)}{2\Phi_q'\Phi_q}\right) + \frac{1}{2\psi'(0+)}Z''(x,0) - \frac{\psi''(0+)}{2\psi'(0+)^2}Z'(x,0)$$
$$= W(x)\left(\frac{\psi''(0+)^2}{4\psi'(0+)^2} - \frac{\psi'''(0+)}{6\psi'(0+)}\right) - \mathcal{R}(x).$$

For the sake of compactness and brevity of our proofs and results, we have omitted the discount factor in the analysis of other quantities.

3.2. Expected area in the red

The next result provides an expression for the expectation of the total area in the red up to a Poissonian passage time T_b^+, which can be interpreted as the expected total costs required to keep the business alive until the insurer's surplus reaches threshold $b\geq 0$ under Poissonian observations.

Theorem 5. *For $\lambda, b\geq 0$ and $x\leq b$,*

$$\mathbb{E}_x\left[\mathcal{A}_{T_b^+}\right] = \mathcal{R}(b) - \mathcal{R}(x) - \frac{1}{\Phi_\lambda}\left(\frac{Z'(b,0)}{\psi'(0+)} + \frac{1}{\psi'(0+)\Phi_\lambda} - \frac{\psi''(0+)}{2\psi'(0+)^2} - \frac{Z(b,\Phi_\lambda)}{\lambda}\right). \tag{16}$$

The total area in the red up the continuous first passage time τ_b^+, that is $\mathbb{E}_x\left[\mathcal{A}_{\tau_b^+}\right]$, can be obtained from the expression in Eq. (16) by letting the Poisson arrival rate go to infinity.

Corollary 6. *For $b \geq 0$ and $x \leq b$, we have*

$$\mathbb{E}_x\left[\mathcal{A}_{\tau_b^+}\right] = \mathcal{R}(b) - \mathcal{R}(x), \tag{17}$$

In particular, for $x = 0$,

$$\mathbb{E}\left[\mathcal{A}_{\tau_b^+}\right] = \mathcal{R}(b).$$

The total area in the red is then given by

$$\mathbb{E}_x\left[\mathcal{A}_\infty\right] = \frac{\psi''(0+)^2}{4\psi'(0+)^3} - \frac{\psi'''(0+)}{6\psi'(0+)^2} - \mathcal{R}(x), \tag{18}$$

for $x \in \mathbb{R}$.

Remark 7. Finally, note that the expressions for expectations of $\mathcal{A}_{T_b^+}$, $\mathcal{A}_{\tau_b^+}$ and $\mathcal{A}_\infty$ in Eqs. (16), (17) and (18) only rely on the scale functions and the Laplace exponent ψ. Hence, one can obtain closed-form expressions for the expected areas as long as $W(x)$ and ψ are explicit.

3.3. Extensions to Parisian ruin

The quantities $\mathbb{E}_x[\mathcal{I}]$ and $\mathbb{E}_x\left[\mathcal{A}_{T_b^+}\right]$ discussed in previous sections measure the cost required until the insurer recovers from the classical ruin and until it is financially stable, respectively. It is also interesting to consider the cost of recovery associated with the Parisian ruin time. More specifically, suppose that each time the insurer enters an insolvent state, it is granted a grace period during which a rehabilitation plan is implemented. The company will be liquidated at the end of the grace period unless its surplus recovers to level 0 within the period. In this context, it is of practical interest to consider the following questions:

- What is the total cost of supporting the insurer during its stressful periods until liquidation?
- If the Parisian ruin occurs, how much additional cost will be required for the insurer to recover?

This section aims to provide insights for answering the questions.

We first recall that the Parisian ruin time with a fixed grace period $r > 0$ is defined as

$$\kappa_r = \inf\{t > 0 : t - g_t > r\},$$

where $g_t := \sup\{0 \leq s \leq t : X_s \geq 0\}$. We refer readers to Loeffen et al. (2013) and Loeffen et al. (2018) for additional references on this topic. In this context, one may consider the cost of recovery from the Parisian ruin time:

$$\mathcal{I}_r = \int_{\kappa_r}^{\tau_r} |X_t|\,\mathrm{d}t,$$

where $\tau_r = \inf\{t > \kappa_r : X_t = 0\}$, and the total area in the red until recovery $\mathcal{A}_{\tau_r}$.

Proposition 8. *For $r > 0$ and $x \in \mathbb{R}$,*

$$\begin{aligned}\mathbb{E}_x\left[\mathcal{I}_r\right] = &\frac{\Lambda(x,r)}{\int_0^\infty \frac{z}{r}\mathbb{P}\left(X_r \in \mathrm{d}z\right)}\left(\frac{\psi''(0+)^2}{4\psi'(0+)^2} - \frac{\psi'''(0+)}{6\psi'(0+)} - \frac{\left(r\psi'(0+)\right)^2 + r\psi''(0+)}{2} - \psi'(0+)\int_0^r\int_0^\infty (r-s)\frac{z}{s}\mathbb{P}\left(X_s \in \mathrm{d}z\right)\mathrm{d}s\right) \\ &+ rZ'(x,0) + \frac{r^2\psi'(0+)}{2} - \psi'(0+)\int_0^r (r-s)\Lambda(x,s)\mathrm{d}s - \mathcal{R}(x),\end{aligned} \tag{19}$$

and

$$\mathbb{E}_x\left[\mathcal{A}_{\tau_r}\right] = \frac{\Lambda(x,r)}{\int_0^\infty \frac{z}{r}\mathbb{P}\left(X_r \in \mathrm{d}z\right)}\left(\frac{\psi''(0+)^2}{4\psi'(0+)^2} - \frac{\psi'''(0+)}{6\psi'(0+)}\right) - \mathcal{R}(x). \tag{20}$$

where $\Lambda(x,r) = \int_0^\infty W(x+z)\frac{z}{r}\mathbb{P}\left(X_r \in \mathrm{d}z\right)$.

If the length of the grace period is modeled by an exponential random variable e_β (independent of X) with mean $1/\beta > 0$, this leads to the exponential Parisian ruin time:

$$\kappa^\beta = \inf\left\{t > 0 : t - g_t > \mathrm{e}_\beta^{g_t}\right\},$$

where $\mathrm{e}_\beta^{g_t}$ denotes an independent copy of e_β associated with the negative excursion that began at time g_t. We refer the reader to Landriault et al. (2014), Baurdoux et al. (2016), and references therein for more discussions on this stopping time. The cost of recovery from the exponential Parisian ruin time is then

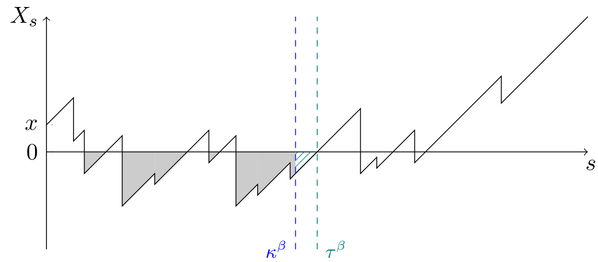

Fig. 2. A sample path of X, where the gray shaded region is the total area in the red until liquidation $\mathcal{A}_{\kappa^\beta}$ and the green pattern shaded region is the cost of recovery from the Parisian ruin time $\mathcal{I}^\beta$.

$$\mathcal{I}^\beta = \int_{\kappa^\beta}^{\tau^\beta} |X_t| \, \mathrm{d}t,$$

where $\tau^\beta = \inf\{t > \kappa^\beta : X_t = 0\}$. See Fig. 2 for a sample path illustration.

The next proposition is a counterpart to Proposition 8 under the assumption of exponential grace periods.

Proposition 9. *For $\beta > 0$ and $x \in \mathbb{R}$,*

$$\mathbb{E}_x\left[\mathcal{I}^\beta\right] = Z\left(x, \Phi_\beta\right) \frac{\Phi_\beta}{\beta} \left(\frac{\psi''(0+)^2}{4\psi'(0+)^2} - \frac{\psi''(0+)}{2\beta} - \frac{\psi'(0+)^2}{\beta^2} - \frac{\psi'''(0+)}{6\psi'(0+)} \right) + \frac{\psi'(0+)}{\beta^2} + \frac{Z'(x,0)}{\beta} - \mathcal{R}(x), \tag{21}$$

$$\mathbb{E}_x\left[\mathcal{A}_{\tau^\beta}\right] = \frac{\Phi_\beta}{\beta} Z(x, \Phi_\beta) \left(\frac{\psi''(0+)^2}{4\psi'(0+)^2} - \frac{\psi'''(0+)}{6\psi'(0+)} \right) - \mathcal{R}(x), \tag{22}$$

and

$$\mathbb{E}_x\left[\mathcal{A}_{\kappa^\beta} \mathbf{1}_{\{\kappa^\beta < \infty\}}\right] = \left(\frac{\psi''(0+)}{2\beta} + \frac{\psi'(0+)^2}{\beta^2} - \frac{\psi'(0+)}{\beta\Phi_\beta} + \frac{\psi'(0+)\Phi'_\beta}{\Phi_\beta^2} - \frac{\Phi''_\beta \psi'(0+)}{2\Phi'_\beta \Phi_\beta} \right) \frac{\Phi_\beta Z\left(x, \Phi_\beta\right)}{\beta} - \frac{\Phi_\beta \psi'(0+)}{2\beta} \left(\Phi'_\beta Z''\left(x, \Phi_\beta\right) + \frac{\Phi''_\beta}{\Phi'_\beta} Z'\left(x, \Phi_\beta\right) \right) - \frac{\psi'(0+)}{\beta^2} - \frac{Z'(x,0)}{\beta}. \tag{23}$$

Finally, it is worth mentioning that the expressions for $\mathbb{E}_x[\mathcal{I}]$ and $\mathbb{E}_x[\mathcal{A}_\infty]$ can be obtained from the above propositions as limiting cases by taking appropriate limits.

4. Examples

This section is devoted to provide some examples of the spectrally negative Lévy process X for the main results in Section 3. For cases of the Brownian risk process, the Cramér-Lundberg process with exponential jumps, and the jump-diffusion risk process with phase-type claims, we will provide explicit expressions for the areas in the red.

4.1. Brownian motion

Let X be a drifted Brownian motion, i.e.,

$$X_t = x + \mu t + \sigma B_t,$$

for $t \geq 0$, where $\mu, \sigma > 0$ and $\{B_t\}_{t\geq 0}$ is a standard Brownian motion. The Laplace exponent is then given by $\psi(\theta) = \mu\theta + \frac{\sigma^2}{2}\theta^2$ for $\theta \geq 0$, and consequently we have $\psi'(0+) = \mu$ and $\psi''(0+) = \sigma^2$. In this case, for $x \geq 0$ and $\lambda > 0$, it is well known that the scale functions of X are given by

$$W(x) = \frac{1}{\mu}\left(1 - \mathrm{e}^{-2\mu x/\sigma^2}\right),$$

and

$$Z(x, \Phi_\lambda) = \frac{\lambda}{\mu}\left(\frac{1}{\Phi_\lambda} - \frac{\mathrm{e}^{-2\mu x/\sigma^2}}{\Phi_\lambda + 2\mu/\sigma^2} \right),$$

where $\Phi_\lambda = \left(\sqrt{\mu^2 + 2\sigma^2\lambda} - \mu\right)\sigma^{-2}$. It follows from Eq. (12) that

$$\begin{aligned}\mathcal{R}(x) =& \frac{\sigma^2}{2\mu^2}x - \frac{x^2}{2\mu} + \frac{x}{\mu}\int_0^x 1 - \mathrm{e}^{-2\mu y/\sigma^2}\mathrm{d}y - \frac{1}{\mu}\int_0^x y\left(1-\mathrm{e}^{-2\mu y/\sigma^2}\right)\mathrm{d}y \\ =& \frac{\sigma^4}{4\mu^3}\left(1-\mathrm{e}^{-2\mu x/\sigma^2}\right).\end{aligned}$$

It is worth noting that $\mathbb{E}_x[\mathcal{I}] = 0$ for any $x \geq 0$, which is consistent with the fact that X creeps downward to 0 and 0 is regular for $(0, \infty)$. Using Theorem 5 and Corollary 6, we obtain

$$\mathbb{E}_x\left[\mathcal{A}_{T_b^+}\right] = \frac{\sigma^4}{4\mu^3}\mathrm{e}^{-2\mu x/\sigma^2} - \mathrm{e}^{-2\mu b/\sigma^2}\left(\frac{\sigma^4}{4\mu^3} + \frac{1}{\Phi_\lambda}\left(\frac{\sigma^2}{2\Phi_\lambda\mu^2} + \frac{1}{\mu\left(\Phi_\lambda + 2\mu\sigma^2\right)}\right)\right),$$

the total area in the red up to τ_b^+ is given by

$$\mathbb{E}_x\left[\mathcal{A}_{\tau_b^+}\right] = \frac{\sigma^4}{4\mu^3}\left(\mathrm{e}^{-2\mu x/\sigma^2} - \mathrm{e}^{-2\mu b/\sigma^2}\right),$$

and the total area in the red is

$$\mathbb{E}_x[\mathcal{A}_\infty] = \frac{\sigma^4}{4\mu^3}\mathrm{e}^{-2\mu x/\sigma^2},$$

which corresponds to the expression Eq. (59) obtained by Gerber et al. (2012).

4.2. Cramér-Lundberg process

Let X be a Cramér-Lundberg risk processes with exponentially distributed claims, i.e.,

$$X_t = x + ct - \sum_{i=1}^{N_t} C_i,$$

where $N = \{N_t\}_{t\geq 0}$ is a Poisson process with intensity $\eta > 0$ and $\{C_i\}_{i\in\mathbb{N}^+}$ is an iid sequence of exponential random variables with mean $1/\alpha$, independent of N. In what follows, we also assume that $c > \eta/\alpha$ so that the ruin probability is not trivially 1. In this case, the Laplace exponent of X is given by

$$\psi(\theta) = c\theta - \eta + \frac{\alpha\eta}{\theta+\alpha}, \quad \theta \geq 0,$$

and its first three derivatives evaluated at $\theta = 0$ are

$$\begin{aligned}\psi'(\theta) =& c - \frac{\alpha\eta}{(\theta+\alpha)^2} \Rightarrow \psi'(0+) = c - \frac{\eta}{\alpha} > 0, \\ \psi''(\theta) =& \frac{2\alpha\eta}{(\theta+\alpha)^3} \Rightarrow \psi''(0+) = \frac{2\eta}{\alpha^2}, \\ \psi'''(\theta) =& -\frac{6\alpha\eta}{(\theta+\alpha)^4} \Rightarrow \psi'''(0+) = -\frac{6\eta}{\alpha^3},\end{aligned}$$

and its right inverse $\Phi_\lambda = (\sqrt{(\eta+\lambda-c\alpha)^2 + 4c\lambda\alpha} + \eta + \lambda - c\alpha)/(2c)$. Moreover, for $x \geq 0$, the scale functions of X are given by

$$W(x) = \frac{1}{c-\eta/\alpha}\left(1 - \frac{\eta}{c\alpha}\mathrm{e}^{(\frac{\eta}{c}-\alpha)x}\right),$$

and

$$Z(x,\Phi_\lambda) = \frac{\lambda}{c-\eta/\alpha}\left(\frac{1}{\Phi_\lambda} - \frac{\eta}{c\alpha}\frac{\mathrm{e}^{(\frac{\eta}{c}-\alpha)x}}{\Phi_\lambda + \alpha - \eta/c}\right), \quad \lambda \geq 0.$$

Using Eq. (12), we obtain

$$\begin{aligned}\mathcal{R}(x) =& -\frac{x^2}{2\left(c-\eta/\alpha\right)} + \frac{\eta x}{(\alpha c-\eta)^2} + \frac{x}{c-\eta/\alpha}\left(x - \frac{\eta\left(\mathrm{e}^{(\frac{\eta}{c}-\alpha)x} - 1\right)}{c\alpha(\eta/c-\alpha)}\right) \\ & - \frac{1}{c-\eta/\alpha}\left(\frac{x^2}{2} - \frac{\eta}{c\alpha}\left(\frac{x\mathrm{e}^{(\frac{\eta}{c}-\alpha)x}}{\eta/c-\alpha} - \frac{\mathrm{e}^{(\frac{\eta}{c}-\alpha)x}-1}{(\eta/c-\alpha)^2}\right)\right) \\ =& \frac{c\eta}{(\eta-c\alpha)^3}\left(\mathrm{e}^{(\eta/c-\alpha)x} - 1\right).\end{aligned}$$

Table 1
Impact of λ on $\mathbb{E}_x\left[\mathcal{A}_{T_b^+}\right]$ for $b=35$ and $c=5$.

x	$\lambda=1$	$\lambda=10$	$\lambda=100$	$\lambda=500$	$\lambda=\infty$
1	71.0063	70.9426	70.9245	70.9223	70.9217
5	47.1418	47.0781	47.0600	47.0578	47.0572
10	28.0497	27.9860	27.9679	27.9657	27.9651
20	9.4462	9.3824	9.3644	9.3622	9.3616
30	2.6023	2.5386	2.5205	2.5183	2.5177

Table 2
Impact of b on $\mathbb{E}_x\left[\mathcal{A}_{T_b^+}\right]$ for $c=5$ and $\lambda=10$.

x	$b=25$	$b=50$	$b=100$	$b=\infty$
1	65.9137	71.8556	72.3834	72.3870
5	42.0492	47.9911	48.5189	48.5225
10	22.9571	28.8990	29.4268	29.4304
15	11.3771	17.3191	17.8468	17.8504
20	4.3535	10.2955	10.8232	10.8268

Table 3
Impact of c on $\mathbb{E}_x\left[\mathcal{A}_{T_b^+}\right]$ for $b=25$ and $x=1$.

c	$\lambda=1$	$\lambda=10$	$\lambda=100$	$\lambda=\infty$
5	66.1994	65.9137	65.9137	65.8202
7	3.3303	3.3290	3.3290	3.3285
10	0.5484	0.5484	0.5484	0.5483
12	0.2686	0.2686	0.2686	0.2686

By Theorem 1, we obtain the following expression for the expected cost of recovery,

$$\begin{aligned}\mathbb{E}_x[\mathcal{I}] &= \left(\frac{4\eta^2}{4(c-\eta/\alpha)^3\alpha^4}+\frac{6\eta}{6\alpha^3(c-\eta/\alpha)^2}\right)\left(1-\frac{\eta}{c\alpha}\mathrm{e}^{(\eta/c-\alpha)x}\right)-\frac{c\eta\left(\mathrm{e}^{(\eta/c-\alpha)x}-1\right)}{(\eta-c\alpha)^3}\\ &=\frac{\eta\mathrm{e}^{(\eta/c-\alpha)x}}{\alpha(\eta-c\alpha)^2}.\end{aligned}$$

For the total area in the red up to T_b^+ and τ_b^+, using Theorem 5 and Corollary 6, we get

$$\mathbb{E}_x\left[\mathcal{A}_{T_b^+}\right]=\frac{c\eta}{(\eta-c\alpha)^3}\left(\mathrm{e}^{(\eta/c-\alpha)b}-\mathrm{e}^{(\eta/c-\alpha)x}\right)-\frac{\eta\mathrm{e}^{(\eta/c-\alpha)b}}{\Phi_\lambda(c\alpha-\eta)}\left(\frac{1}{\alpha(\eta-c\alpha)}+\frac{1}{\Phi_\lambda+\alpha-\eta/c}\right)$$

and

$$\mathbb{E}_x\left[\mathcal{A}_{\tau_b^+}\right]=\mathcal{R}(b)-\mathcal{R}(x)=\frac{c\eta\left(\mathrm{e}^{(\eta/c-\alpha)b}-\mathrm{e}^{(\eta/c-\alpha)x}\right)}{(\eta-c\alpha)^3}.$$

Consequently, the total area in the red can be obtained by taking limit as follows,

$$\mathbb{E}_x[\mathcal{A}_\infty]=\lim_{b\to\infty}\mathbb{E}_x\left[\mathcal{A}_{\tau_b^+}\right]=\frac{c\eta\mathrm{e}^{(\eta/c-\alpha)x}}{(c\alpha-\eta)^3},$$

which recovers Theorem 7 of Loisel (2005).

To examine the impact of the model parameters on the expected total area, we provide the values of $\mathbb{E}_x\left[\mathcal{A}_{T_b^+}\right]$ for X under the parameter setting $\eta=1/\alpha=2$ in Tables 1, 2 and 3. It can be observed from Table 1 and the left panel of Fig. 3 that $\mathbb{E}_x\left[\mathcal{A}_{T_b^+}\right]$ decreases and converges to $\mathbb{E}_x\left[\mathcal{A}_{\tau_b^+}\right]$ as λ increases, which is expected as $\tau_b^+<T_b^+$ almost surely for $\lambda>0$ and $T_b^+\to\tau_b^+$ as $\lambda\to\infty$. Also, note that expected total area decreases as the initial capital x increases. In Table 2, we notice that the cost for supporting the insurer during its insolvent states (i.e., $\mathbb{E}_x\left[\mathcal{A}_{T_b^+}\right]$) increases as the threshold level b increases. See the right panel of Fig. 3 for the convergence of $\mathbb{E}_x\left[\mathcal{A}_{T_b^+}\right]$ to $\mathbb{E}_x[\mathcal{A}_\infty]$. Table 3 contains values of $\mathbb{E}_x\left[\mathcal{A}_{T_b^+}\right]$ with different premium rates c and we note that the expected total area decreases significantly as the premium rate increases.

Remark 10. For general claim size distributions, one can use Theorem 1 and Proposition 2 to recover identities related to the cost of recovery (see p. 115 of Picard (1994)).

$$\mathbb{E}_x\left[\mathbb{E}_{X_{\tau_0^-}}\left[\int_0^{\tau_0^+}X_s\mathrm{d}s\right]\mathbf{1}_{\{\tau_0^-<\infty\}}\right]=\frac{\mathbb{E}_x\left[X_{\tau_0^-}^2\mathbf{1}_{\{\tau_0^-<\infty\}}\right]}{2(c-\eta\mathbb{E}[C_1])}-\frac{\eta\mathbb{E}[C_1^2]\mathbb{E}_x\left[X_{\tau_0^-}\mathbf{1}_{\{\tau_0^-<\infty\}}\right]}{2(c-\eta\mathbb{E}[C_1])^2},$$

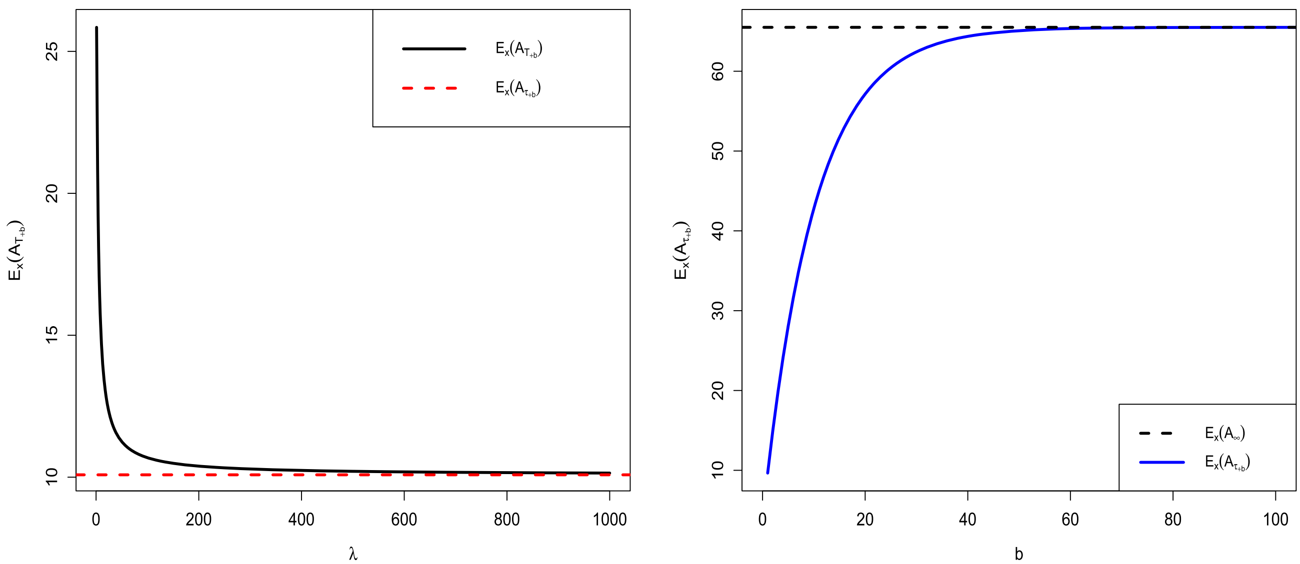

Fig. 3. Convergence of $\mathbb{E}_x\left[\mathcal{A}_{T_b^+}\right]$ to $\mathbb{E}_x\left[\mathcal{A}_{\tau_b^+}\right]$ and $\mathbb{E}_x\left[\mathcal{A}_\infty\right]$ as $\lambda\to\infty$ and $b\to\infty$ respectively.

and

$$\mathbb{E}_x\left[\mathbb{E}_{X_{\tau_0^-}}\left[\int_0^{\tau_0^+} X_s^2 \mathrm{d}s\right]\mathbf{1}_{\{\tau_0^-<\infty\}}\right] = -\frac{\mathbb{E}_x\left[X_{\tau_0^-}^3\mathbf{1}_{\{\tau_0^-<\infty\}}\right]}{3(c-\eta\mathbb{E}[C_1])} + \frac{\eta\mathbb{E}[C_1^2]\mathbb{E}_x\left[X_{\tau_0^-}^2\mathbf{1}_{\{\tau_0^-<\infty\}}\right]}{2(c-\eta\mathbb{E}[C_1])^2} - \left(\frac{\eta^2\mathbb{E}[C_1^2]^2}{2(c-\eta\mathbb{E}[C_1])^3} + \frac{\eta\mathbb{E}[C_1^3]\mathbb{E}_x\left[X_{\tau_0^-}\mathbf{1}_{\{\tau_0^-<\infty\}}\right]}{3(c-\eta\mathbb{E}[C_1])^2}\right),$$

and the covariance

$$\mathrm{Cov}_{X_{\tau_0^-}}\left[\tau_0^+, \int_0^{\tau_0^+} X_s \mathrm{d}s\right] = -\frac{\eta\mathbb{E}[C_1^2]X_{\tau_0^-}^2}{2(c-\eta\mathbb{E}[C_1])^3} - \left(\frac{\lambda\mathbb{E}[C_1^2]^2}{c-\eta\mathbb{E}[C_1]} + \frac{\mathbb{E}[C_1^3]}{2}\right)\frac{\eta X_{\tau_0^-}}{(c-\eta\mathbb{E}[C_1])^3}.$$

4.3. Jump diffusion risk process with phase-type claims

As a generalization of the previous two examples, we consider a jump diffusion risk process with phase-type claims, that is,

$$X_t = ct + \sigma B_t - \sum_{i=1}^{N_t} C_i,$$

where $\sigma \geq 0$, $\{B_t\}_{t\geq 0}$ is a standard Brownian motion, $\{N_t\}_{t\geq 0}$ is a Poisson process with intensity $\eta > 0$, and $\{C_1, C_2, \ldots\}$ are independent random variables with common phase-type distribution with the minimal representation $(m, \mathbf{T}, \boldsymbol{\alpha})$, i.e., its cumulative distribution function is given by $F(x) = 1 - \boldsymbol{\alpha}\mathrm{e}^{\mathbf{T}x}\mathbf{1}$, where $\mathbf{T}$ is an $m\times m$ matrix of a continuous-time killed Markov chain, its initial distribution is given by a simplex $\boldsymbol{\alpha} = [\alpha_1, \ldots, \alpha_m]$, and $\mathbf{1}$ denotes a column vector of ones. All of the aforementioned objects are mutually independent, see Egami and Yamazaki (2014) for more details.

The Laplace exponent of X is known to be of the form

$$\psi(\lambda) = c\lambda + \frac{\sigma^2\lambda^2}{2} + \eta\left(\boldsymbol{\alpha}(\lambda\mathbf{I}-\mathbf{T})^{-1}\mathbf{t} - 1\right),$$

where $\mathbf{t} = -\mathbf{T1}$. Let ρ_j be the roots with negative real parts of the equation $\theta \mapsto \psi(\theta) = 0$. Since we assume the net profit condition $\mathbb{E}[X_1] > 0$, from Proposition 5.4 in Kuznetsov et al. (2012), we have that the ρ_j's are distinct roots. Then, from Proposition 2.1 in Egami and Yamazaki (2014), we have

$$W(x) = \frac{1}{\psi'(0+)} + \sum_{j=1}^{n} A_j \mathrm{e}^{\rho_j x}, \tag{24}$$

where $A_j = \frac{1}{\psi'(\rho_j)}$ and $n = |I_\rho|$ where I_ρ is the set of indices corresponding to the ρ_j's. From Eqs. (42) and (43) in Strietzel and Behme (2022), we have

$$\frac{1}{\psi'(0+)}=\frac{1}{c}\mathbf{1}_{\{\sigma^2=0\}}-\sum_{i=1}^{n}\frac{1}{\psi'(\rho_i)},$$

and

$$\frac{\psi''(0+)}{\psi'(0+)^3}=-\sum_{i=1}^{n}\frac{\psi''(\rho_i)}{\psi'(\rho_i)^3}.$$

Using Eq. (24), we obtain

$$x\int_0^x W(y)\mathrm{d}y=\frac{x^2}{\psi'(0+)}+x\sum_{i=1}^{n}\frac{\mathrm{e}^{\rho_i x}-1}{A_i\rho_i},$$

and

$$\int_0^x yW(y)\mathrm{d}y=\frac{x^2}{2\psi'(0+)}+\sum_{i=1}^{n}\frac{1}{A_i\rho_i}\left(x\mathrm{e}^{\rho_i x}-\frac{\mathrm{e}^{\rho_i x}-1}{\rho_i}\right).$$

Thus,

$$x\int_0^x W(y)\mathrm{d}y-\int_0^x yW(y)\mathrm{d}y=\frac{x^2}{2\psi'(0+)}-x\sum_{i=1}^{n}\frac{1}{A_i\rho_i}+\sum_{i=1}^{n}\frac{1}{A_i\rho_i}\left(\frac{\mathrm{e}^{\rho_i x}-1}{\rho_i}\right).$$

Putting all the pieces together, one can obtain the expression for $\mathcal{R}(x)$.

5. Conclusion

Under contemporary insurance regulatory frameworks, it is essential to evaluate the cost required to recover from an insurer's insolvency when determining whether rehabilitation should be granted. To this end, we study several area-related quantities in the context of Lévy risk processes. We obtain explicit and compact expressions, expressed in terms of the scale functions and the Laplace exponent, for the expected areas up to the recovery times, the Poissonian first passage time, the continuous first passage time, and the infinite horizon time.

Declaration of competing interest

There is no competing interest.

Data availability

No data was used for the research described in the article.

Acknowledgements

We sincerely thank the Handling Editor and two anonymous referees for constructive comments which improve the quality of this paper. Mohamed Amine Lkabous acknowledges the support from a start-up grant from the University of Southampton. Zijia Wang acknowledges the support from a start-up grant from the Chinese University of Hong Kong Business School.

Appendix A

A.1. Proof of Theorem 1

Proof. First, using Fubini's Theorem, we have

$$\begin{aligned}\mathbb{E}_x\left[\int_0^{\tau_0^+}X_s\mathrm{d}s\right]&=\mathbb{E}_x\left[\int_0^{\infty}X_s\mathbf{1}_{\{s<\tau_0^+\}}\mathrm{d}s\right]\\&=\lim_{q\to0}\int_0^{\infty}\mathrm{e}^{-qs}\mathbb{E}_x\left[X_s\mathbf{1}_{\{s<\tau_0^+\}}\right]\mathrm{d}s\\&=\lim_{q\to0}\frac{1}{q}\mathbb{E}_x\left[X_{\mathrm{e}_q}\mathbf{1}_{\{\mathrm{e}_q<\tau_0^+\}}\right],\end{aligned}\tag{25}$$

for $x\le0$. By Eq. (9), one obtains

$$\begin{aligned}\mathbb{E}_x\left[X_{e_q}\mathbf{1}_{\{e_q<\tau_0^+\}}\right] &= \frac{\mathrm{d}}{\mathrm{d}\lambda}\mathbb{E}_x\left[e^{\lambda X_{e_q}}\mathbf{1}_{\{e_q<\tau_0^+\}}\right]\Big|_{\lambda=0} \\ &= \frac{qxe^{\lambda x}(q-\psi(\lambda))+q\left(e^{\lambda x}-e^{\Phi_q x}\right)\psi'(\lambda)}{\psi_q(\lambda)^2}\Big|_{\lambda=0} \\ &= \frac{\left(1-e^{\Phi_q x}\right)\psi'(0+)+qx}{q},\end{aligned} \tag{26}$$

for $x \le 0$. Substituting Eq. (26) into Eq. (25) and noting that $\Phi_0'' = -\frac{\psi''(0+)}{\psi'(0+)^3}$, it follows that

$$\begin{aligned}\mathbb{E}_x\left[\int_0^{\tau_0^+} X_s \mathrm{d}s\right] &= \lim_{q\to 0}\frac{\left(1-e^{\Phi_q x}\right)\psi'(0+)+qx}{q^2} \\ &= -\frac{x}{2}\left(\frac{x}{\psi'(0+)}-\frac{\psi''(0+)}{\psi'(0+)^2}\right),\end{aligned} \tag{27}$$

for $x \le 0$.

Also, by taking the right-hand derivative of Eq. (8) with respect to θ at zero, we get an expression for the expected discounted deficit at ruin,

$$\begin{aligned}\mathbb{E}_x\left[X_{\tau_0^-}\mathbf{1}_{\{\tau_0^-<\infty\}}\right] &= \frac{\mathrm{d}}{\mathrm{d}\theta}\mathbb{E}_x\left[e^{\theta X_{\tau_0^-}}\mathbf{1}_{\{\tau_0^-<\infty\}}\right]\Big|_{\theta=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}\theta}\left\{Z(x,\theta)-\frac{\psi(\theta)}{\theta}W(x)\right\}\Big|_{\theta=0} \\ &= Z'(x,0)-\frac{\psi''(0+)}{2}W(x),\end{aligned} \tag{28}$$

and similarly,

$$\begin{aligned}\mathbb{E}_x\left[X^2_{\tau_0^-}\mathbf{1}_{\{\tau_0^-<\infty\}}\right] &= \frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\mathbb{E}_x\left[e^{\theta X_{\tau_0^-}}\mathbf{1}_{\{\tau_0^-<\infty\}}\right]\Big|_{\theta=0} \\ &= \left\{Z''(x,\theta)-\frac{\psi''(\theta)\theta^2-2\psi'(\theta)\theta+2\psi(\theta)}{\theta^3}W(x)\right\}\Big|_{\theta=0} \\ &= Z''(x,0)-\frac{\psi'''(0+)W(x)}{3}.\end{aligned} \tag{29}$$

Combining Eqs. (27) $\sim$ (29) and applying the strong Markov property of X, we obtain

$$\begin{aligned}\mathbb{E}_x[\mathcal{I}] &= -\mathbb{E}_x\left[\mathbb{E}_{X_{\tau_0^-}}\left[\int_0^{\tau_0^+} X_s\,\mathrm{d}s\right]\mathbf{1}_{\{\tau_0^-<\infty\}}\right] \\ &= \frac{1}{2\psi'(0+)}\mathbb{E}_x\left[X^2_{\tau_0^-}\mathbf{1}_{\{\tau_0^-<\infty\}}\right]-\frac{\psi''(0+)}{2\psi'(0+)^2}\mathbb{E}_x\left[X_{\tau_0^-}\mathbf{1}_{\{\tau_0^-<\infty\}}\right] \\ &= \frac{1}{2\psi'(0+)}\left(Z''(x,0)-\frac{\psi'''(0+)W(x)}{3}\right)-\frac{\psi''(0+)}{2\psi'(0+)^2}\left(Z'(x,0)-\frac{\psi''(0+)}{2}W(x)\right),\end{aligned} \tag{30}$$

for $x \in \mathbb{R}$. Applying Eqs. (5) and (6) and noting that

$$\mathcal{R}(x) = \frac{\psi''(0+)Z'(x,0)}{2\psi'(0+)^2}-\frac{Z''(x,0)}{2\psi'(0+)}, \tag{31}$$

Eq. (30) reduces to Eq. (11). ■

A.2. Proof of Proposition 2

Proof. For $x \in \mathbb{R}$, using the strong Markov property of X, we have

$$\mathrm{Cov}_x\left[\tau-\tau_0^-, \int_{\tau_0^-}^{\tau} X_s\,\mathrm{d}s\right] = \mathbb{E}_x\left[\mathbb{E}_{X_{\tau_0^-}}\left[\tau_0^+\int_0^{\tau_0^+} X_s\,\mathrm{d}s\right]\right]$$

$$
\begin{aligned}
&\quad - \mathbb{E}_x\left[\mathbb{E}_{X_{\tau_0^-}}\left[\tau_0^+\right]\right]\mathbb{E}_x\left[\mathbb{E}_{X_{\tau_0^-}}\left[\int_0^{\tau_0^+} X_s\,\mathrm{d}s\right]\right]\\
&= \mathbb{E}_x\left[\mathbb{E}_{X_{\tau_0^-}}\left[\tau_0^+\int_0^{\tau_0^+} X_s\,\mathrm{d}s\right]\right]\\
&\quad - \mathbb{E}_x\left[\frac{X_{\tau_0^-}}{\psi'(0+)}\right]\mathbb{E}_x\left[\frac{X_{\tau_0^-}}{2}\left(\frac{X_{\tau_0^-}}{\psi'(0+)}-\frac{\psi''(0+)}{\psi'(0+)^2}\right)\right],
\end{aligned}
\tag{32}
$$

where the last equation follows from Eq. (27) and the fact that,

$$
\begin{aligned}
\mathbb{E}_x\left[\tau_0^+\right] &= -\frac{\mathrm{d}}{\mathrm{d}q}\,\mathbb{E}_x\left[\mathrm{e}^{-q\tau_0^+}\right]\Big|_{q=0}\\
&= -\frac{\mathrm{d}}{\mathrm{d}q}\,\mathrm{e}^{\Phi_q x}\Big|_{q=0} = -\frac{x}{\psi'(0+)},
\end{aligned}
$$

for $x \le 0$.

To calculate the first term of Eq. (32), we note that, for $x \le 0$,

$$
\begin{aligned}
\mathbb{E}_x\left[\tau_0^+\int_0^{\tau_0^+} X_s\,\mathrm{d}s\right] &= \lim_{q\to 0}\frac{1}{q}\mathbb{E}_x\left[\tau_0^+ X_{\mathrm{e}_q}\mathbf{1}_{\{\mathrm{e}_q<\tau_0^+\}}\right]\\
&= -\lim_{q\to 0}\frac{1}{q}\frac{\mathrm{d}^2}{\mathrm{d}\lambda\mathrm{d}p}\mathbb{E}_x\left[\mathrm{e}^{-p\tau_0^+ +\lambda X_{\mathrm{e}_q}}\mathbf{1}_{\{\mathrm{e}_q<\tau_0^+\}}\right]\Big|_{\lambda=p=0},
\end{aligned}
\tag{33}
$$

where

$$
\begin{aligned}
\mathbb{E}_x\left[\mathrm{e}^{-p\tau_0^+ +\lambda X_{\mathrm{e}_q}}\mathbf{1}_{\{\mathrm{e}_q<\tau_0^+\}}\right] &= \mathbb{E}_x\left[\mathrm{e}^{\lambda X_{\mathrm{e}_q}-p\mathrm{e}_q}\mathbf{1}_{\{\mathrm{e}_q<\tau_0^+\}}\mathbb{E}_{X_{\mathrm{e}_q}}\left[\mathrm{e}^{-p\tau_0^+}\right]\right]\\
&= \frac{q}{p+q}\mathbb{E}_x\left[\mathrm{e}^{(\lambda+\Phi_p)X_{\mathrm{e}_{p+q}}}\mathbf{1}_{\{\mathrm{e}_{p+q}<\tau_0^+\}}\right]\\
&= \frac{q\left(\mathrm{e}^{(\lambda+\Phi_p)x}-\mathrm{e}^{\Phi_{p+q}x}\right)}{p+q-\psi\left(\lambda+\Phi_p\right)}.
\end{aligned}
\tag{34}
$$

Substituting Eq. (34) into Eq. (33) yields

$$
\begin{aligned}
\mathbb{E}_x\left[\tau_0^+\int_0^{\tau_0^+} X_s\mathrm{d}s\right] &= -\lim_{q\to 0}\frac{\mathrm{d}^2}{\mathrm{d}\lambda\mathrm{d}p}\frac{\mathrm{e}^{(\lambda+\Phi_p)x}-\mathrm{e}^{\Phi_{p+q}x}}{p+q-\psi\left(\lambda+\Phi_p\right)}\Bigg|_{\lambda=p=0}\\
&= -\lim_{q\to 0}\frac{xqx\Phi_0' + \left(x\Phi_0' - x\Phi_q'\mathrm{e}^{\Phi_q x}\right)\psi'(0+) + \left(1-\mathrm{e}^{\Phi_q x}\right)\psi''(0+)\,\Phi_0'}{q^2}\\
&= \left(\frac{\psi''(0+)^2}{\psi'(0+)^4}-\frac{\psi'''(0+)}{2\psi'(0+)^3}\right)x - \frac{\psi''(0+)}{\psi'(0+)^3}x^2 + \frac{1}{2\psi'(0+)^2}x^3,
\end{aligned}
\tag{35}
$$

where in the last equality, we applied L'Hôpital's rule and used $\Phi_0''' = \frac{3\psi''(0+)^2}{\psi'(0+)^5} - \frac{\psi'''(0+)}{\psi'(0+)^4}$. Combining Eq. (35) and Eq. (32) and using the linearity of covariance completes the proof of Eq. (13). ■

A.3. Proof of Theorem 5

Proof. We first consider the case where the paths of X are of bounded variation. Applying the strong Markov property of X, it follows that

$$
\begin{aligned}
\mathbb{E}_x\left[\mathcal{A}_{T_b^+}\right] &= -\mathbb{E}_x\left[\mathbb{E}_{X_{\tau_0^-}}\left[\int_0^{\tau_0^+} X_s\mathrm{d}s\right]\mathbf{1}_{\{\tau_0^-<T_b^+\}}\right] + \mathbb{P}_x\left(\tau_0^- < T_b^+\right)\mathbb{E}\left[\mathcal{A}_{T_b^+}\right]\\
&= -\mathbb{E}_x\left[\mathbb{E}_{X_{\tau_0^-}}\left[\int_0^{\tau_0^+} X_s\mathrm{d}s\right]\mathbf{1}_{\{\tau_0^-<T_b^+\}}\right] + \left(1-\frac{\lambda W(x)}{\Phi_\lambda Z(b,\Phi_\lambda)}\right)\mathbb{E}\left[\mathcal{A}_{T_b^+}\right],
\end{aligned}
\tag{36}
$$

where in the last equality we used Eq. (10) for $\theta = 0$.

Using a similar technique as in the proof of Theorem 1, one deduces that

$$\begin{aligned}\mathbb{E}_x\left[X_{\tau_0^-}\mathbf{1}_{\{\tau_0^-<T_b^+\}}\right] &= \frac{\mathrm{d}}{\mathrm{d}\theta}\mathbb{E}_x\left[\mathrm{e}^{\theta X_{\tau_0^-}}\mathbf{1}_{\{\tau_0^-<T_b^+\}}\right]\Bigg|_{\theta=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}\theta}\left\{Z(x,\theta) - \frac{W(x)}{\theta-\Phi_\lambda}\left(\psi(\theta) - \lambda\frac{Z(b,\theta)}{Z(b,\Phi_\lambda)}\right)\right\}\Bigg|_{\theta=0} \\ &= Z'(x,0) + W(x)\left(\frac{\psi'(0+)}{\Phi_\lambda} - \frac{\lambda\left(\Phi_\lambda Z'(b,0)+1\right)}{Z(b,\Phi_\lambda)\Phi_\lambda^2}\right),\end{aligned} \tag{37}$$

and

$$\begin{aligned}\mathbb{E}_x\left[X_{\tau_0^-}^2\mathbf{1}_{\{\tau_0^-<T_b^+\}}\right] &= \frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\mathbb{E}_x\left[\mathrm{e}^{\theta X_{\tau_0^-}}\mathbf{1}_{\{\tau_0^-<T_b^+\}}\right]\Bigg|_{\theta=0} \\ &= Z''(x,0) + W(x)\left(\frac{\psi''(0+)\Phi_\lambda + 2\psi'(0+)}{\Phi_\lambda^2}\right) \\ &\quad - \frac{\lambda W(x)}{Z(b,\Phi_\lambda)}\frac{Z''(b,0)}{\Phi_\lambda} - \frac{2\lambda W(x)}{Z(b,\Phi_\lambda)}\left(\frac{\Phi_\lambda Z'(b,0)+1}{\Phi_\lambda^3}\right).\end{aligned} \tag{38}$$

Combining Eqs. (27), (37) and (38), we obtain

$$\begin{aligned}\mathbb{E}_x\left[\mathbb{E}_{X_{\tau_0^-}}\left[\int_0^{\tau_0^+} X_s\,\mathrm{d}s\right]\mathbf{1}_{\{\tau_0^-<T_b^+\}}\right] &= \frac{\psi''(0+)}{2\psi'(0+)^2}\mathbb{E}_x\left[X_{\tau_0^-}\mathbf{1}_{\{\tau_0^-<T_b^+\}}\right] - \frac{1}{2\psi'(0+)}\mathbb{E}_x\left[X_{\tau_0^-}^2\mathbf{1}_{\{\tau_0^-<T_b^+\}}\right] \\ &= \frac{\psi''(0+)}{2\psi'(0+)^2}\left(Z'(x,0) + W(x)\left(\frac{\psi'(0+)}{\Phi_\lambda} - \frac{\lambda\left(\Phi_\lambda Z'(b,0)+1\right)}{Z(b,\Phi_\lambda)\Phi_\lambda^2}\right)\right) \\ &\quad - \frac{1}{2\psi'(0+)}\left(Z''(x,0) + W(x)\left(\frac{\psi''(0+)\Phi_\lambda + 2\psi'(0+)}{\Phi_\lambda^2}\right)\right) \\ &\quad + \frac{1}{2\psi'(0+)}\frac{\lambda W(x)}{Z(b,\Phi_\lambda)}\left(\frac{Z''(b,0)}{\Phi_\lambda} + 2\left(\frac{\Phi_\lambda Z'(b,0)+1}{\Phi_\lambda^3}\right)\right) \\ &= \mathcal{R}(x) - \frac{\lambda W(x)}{Z(b,\Phi_\lambda)\Phi_\lambda}\mathcal{R}(b) - \frac{W(x)}{\Phi_\lambda^2} \\ &\quad + \frac{\lambda W(x)}{2\psi'(0+)Z(b,\Phi_\lambda)\Phi_\lambda^2}\left(\frac{2\Phi_\lambda\left(b-\psi'(0+)\int_0^b W(y)\mathrm{d}y\right)+2}{\Phi_\lambda} - \frac{\psi''(0+)}{\psi'(0+)}\right),\end{aligned} \tag{39}$$

for $x \le b$ and $b \ge 0$. Specifically, letting $x = 0$ in Eq. (39), one obtains

$$\begin{aligned}&\mathbb{E}\left[\mathbb{E}_{X_{\tau_0^-}}\left[\int_0^{\tau_0^+} X_s\,\mathrm{d}s\right]\mathbf{1}_{\{\tau_0^-<T_b^+\}}\right] \\ &= \frac{W(0)}{\Phi_\lambda^2 Z(b,\Phi_\lambda)}\left(\frac{\lambda\Phi_\lambda\left(b-\psi'(0+)\int_0^b W(y)\mathrm{d}y\right)+\lambda}{\psi'(0+)\Phi_\lambda} - \frac{\lambda\psi''(0+)}{2\psi'(0+)^2} - Z(b,\Phi_\lambda) - \lambda\Phi_\lambda\mathcal{R}(b)\right).\end{aligned} \tag{40}$$

Substituting Eq. (40) into Eq. (36) leads to

$$\begin{aligned}\mathbb{E}\left[\mathcal{A}_{T_b^+}\right] &= -\frac{\Phi_\lambda Z(b,\Phi_\lambda)}{\lambda W(0)}\mathbb{E}\left[\mathbb{E}_{X_{\tau_0^-}}\left[\int_0^{\tau_0^+} X_s\mathrm{d}s\right]\mathbf{1}_{\{\tau_0^-<T_b^+\}}\right] \\ &= \frac{\psi''(0+)}{2\psi'(0+)^2\Phi_\lambda} + \frac{Z(b,\Phi_\lambda)}{\lambda\Phi_\lambda} + \mathcal{R}(b) - \frac{\Phi_\lambda\left(b-\psi'(0+)\int_0^b W(y)\mathrm{d}y\right)+1}{\psi'(0+)\Phi_\lambda^2}.\end{aligned} \tag{41}$$

Substituting Eqs. (41) and (39) into Eq. (36) completes the proof of Eq. (16) for the case where the paths of X are of bounded variation.

Now, if X has paths of unbounded variation, we consider an approximation approach similar to Yin and Yuen (2014) and Zhang and Wu (2002). More specifically, for $\epsilon > 0$, let

$$L_1(\epsilon)=\tau_{-\epsilon}^{-}=\inf\{t\geq 0: X_t<-\epsilon\},$$

and

$$R_1(\epsilon)=\inf\{t\geq L_1(\epsilon): X_t=0\}.$$

Recursively, we define two sequences of stopping times $\{L_k(\epsilon)\}_{k\geq 1}$ and $\{R_k(\epsilon)\}_{k\geq 1}$ as follows: for $k\geq 2$, let

$$L_k(\epsilon)=\inf\left\{t\geq R_{k-1}(\epsilon): X_t<-\epsilon\right\},$$

and

$$R_k(\epsilon)=\inf\{t\geq L_k(\epsilon): X_t=0\}.$$

Define

$$\mathcal{A}_{T_b^+}^{(\epsilon)}=\sum_{i=1}^{N_b(\epsilon)}\int\limits_{L_i(\epsilon)}^{R_i(\epsilon)}|X_s|\mathrm{d}s,$$

where

$$N_b(\epsilon)=\sup\left\{k: L_k(\epsilon)<T_b^+\right\},$$

with the convention that $\sup\emptyset=0$ and $\sum_{i=1}^{0}=0$. Note that $N_b(\epsilon)$ follows a geometric distribution:

$$\mathbb{P}_x\left(N_b(\epsilon)=0\right)=\mathbb{P}_x\left(T_b^+\leq\tau_{-\epsilon}^-\right),$$

and

$$\mathbb{P}_x\left(N_b(\epsilon)=n\right)=\mathbb{P}_x\left(\tau_{-\epsilon}^-<T_b^+\right)\left(\mathbb{P}\left(\tau_{-\epsilon}^-<T_b^+\right)\right)^{n-1}\mathbb{P}\left(\tau_{-\epsilon}^-\geq T_b^+\right),$$

for $n=1,2,\ldots$. Moreover, given that $N_b(\epsilon)=n$, one can show that $\left\{\int_{L_i(\epsilon)}^{R_i(\epsilon)}|X_s|\mathrm{d}s\right\}_{i\in\mathbb{N}^+}$ is a sequence of independent random variables and $\left\{\int_{L_i(\epsilon)}^{R_i(\epsilon)}|X_s|\mathrm{d}s\right\}_{i\geq 2}$ are identically distributed (see p. 108 of Bertoin (1996)). Then, applying the law of total expectation and the strong Markov property of X, we obtain

$$\begin{aligned}
&\mathbb{E}_x\left[\mathcal{A}_{T_b^+}^{(\epsilon)}\right]\\
=&\sum_{n=1}^{\infty}\left(\mathbb{E}_x\left[\int\limits_{L_1(\epsilon)}^{R_1(\epsilon)}|X_s|\,\mathrm{d}s\middle|N_b(\epsilon)=n\right]+\mathbb{E}_x\left[\sum_{i=2}^{n}\int\limits_{L_i(\epsilon)}^{R_i(\epsilon)}|X_s|\,\mathrm{d}s\middle|N_b(\epsilon)=n\right]\right)\mathbb{P}_x\left(N_b(\epsilon)=n\right)\\
=&\mathbb{E}_x\left[\int\limits_{L_1(\epsilon)}^{R_1(\epsilon)}|X_s|\,\mathrm{d}s\middle|N_b(\epsilon)\geq 1\right]\mathbb{P}_x\left(\tau_{-\epsilon}^-<T_b^+\right)\\
&+\mathbb{P}_x\left(\tau_{-\epsilon}^-<T_b^+\right)\sum_{n=2}^{\infty}\mathbb{E}\left[\sum_{i=1}^{n-1}\int\limits_{L_i(\epsilon)}^{R_i(\epsilon)}|X_s|\,\mathrm{d}s\middle|N_b(\epsilon)=n-1\right]\left(\mathbb{P}\left(\tau_{-\epsilon}^-<T_b^+\right)\right)^{n-1}\mathbb{P}\left(\tau_{-\epsilon}^->T_b^+\right)\\
=&-\mathbb{E}_x\left[\mathbb{E}_{X_{\tau_{-\epsilon}^-}}\left[\int\limits_0^{\tau_0^+}X_s\,\mathrm{d}s\right]\mathbf{1}_{\{\tau_{-\epsilon}^-<T_b^+\}}\right]+\mathbb{P}_x\left(\tau_{-\epsilon}^-<T_b^+\right)\mathbb{E}\left[\mathcal{A}_{T_b^+}^{(\epsilon)}\right].
\end{aligned}\tag{42}$$

Using a similar argument as in the bounded variation case and noting that

$$\mathbb{E}_x\left[X_{\tau_{-\epsilon}^-}^2\mathbf{1}_{\{\tau_{-\epsilon}^-<T_b^+\}}\right]=\mathbb{E}_{x+\epsilon}\left[X_{\tau_0^-}^2\mathbf{1}_{\left\{\tau_0^-<T_{b+\epsilon}^+\right\}}\right]-2\epsilon\mathbb{E}_x\left[X_{\tau_{-\epsilon}^-}\mathbf{1}_{\{\tau_{-\epsilon}^-<T_b^+\}}\right]-\epsilon^2\mathbb{P}_x\left(\tau_{-\epsilon}^-<T_b^+\right),$$

and

$$\mathbb{E}_x\left[X_{\tau_{-\epsilon}^-}\mathbf{1}_{\{\tau_{-\epsilon}^-<T_b^+\}}\right]=\mathbb{E}_{x+\epsilon}\left[X_{\tau_0^-}\mathbf{1}_{\left\{\tau_0^-<T_{b+\epsilon}^+\right\}}\right]-\epsilon\mathbb{P}_x\left(\tau_{-\epsilon}^-<T_b^+\right),$$

one can deduce that

$$\begin{aligned}
\mathbb{E}_x\left[\mathbb{E}_{X_{\tau^-_{-\epsilon}}}\left[\int_0^{\tau_0^+} X_s\,\mathrm{d}s\right]\mathbf{1}_{\{\tau^-_{-\epsilon}<T_b^+\}}\right] =& \left(\frac{\psi''(0+)}{2\psi'(0+)^2}+\frac{\epsilon}{\psi'(0+)}\right)\mathbb{E}_{x+\epsilon}\left[X_{\tau_0^-}\mathbf{1}_{\left\{\tau_0^-<T^+_{b+\epsilon}\right\}}\right]\\
&-\frac{1}{2\psi'(0+)}\mathbb{E}_{x+\epsilon}\left[X^2_{\tau_0^-}\mathbf{1}_{\left\{\tau_0^-<T^+_{b+\epsilon}\right\}}\right]\\
&-\left(\frac{\epsilon\psi''(0+)}{2\psi'(0+)^2}+\frac{\epsilon^2}{2\psi'(0+)}\right)\mathbb{P}_x\left(\tau^-_{-\epsilon}<T_b^+\right)\\
=&\mathcal{R}(x+\epsilon)-\frac{\lambda W(x+\epsilon)}{Z(b+\epsilon,\Phi_\lambda)\Phi_\lambda}\mathcal{R}(b+\epsilon)+\frac{\epsilon Z'(x+\epsilon,0)}{\psi'(0+)}\\
&-\left(\frac{\epsilon\psi''(0+)}{2\psi'(0+)^2}+\frac{\epsilon^2}{2\psi'(0+)}\right)\\
&+\frac{(\epsilon\Phi_\lambda-1)W(x+\epsilon)}{2\Phi_\lambda^2\psi'(0+)}\left(2\psi'(0+)-\frac{2\lambda Z'(b+\epsilon,0)}{Z(b+\epsilon,\Phi_\lambda)}\right)\\
&+\frac{(\epsilon\Phi_\lambda-1)W(x+\epsilon)}{2\Phi_\lambda^2\psi'(0+)}\left(\frac{\lambda\psi''(0+)}{\psi'(0+)Z(b+\epsilon,\Phi_\lambda)}-\frac{2\lambda}{Z(b+\epsilon,\Phi_\lambda)\Phi_\lambda}\right)\\
&+\frac{\lambda\epsilon^2W(x+\epsilon)}{2\psi'(0+)\Phi_\lambda Z(b+\epsilon,\Phi_\lambda)}.
\end{aligned}\tag{43}$$

Combining Eqs. (42) and (43) leads to

$$\begin{aligned}
\mathbb{E}\left[\mathcal{A}^{(\epsilon)}_{T_b^+}\right]=&-\frac{\Phi_\lambda Z(b+\epsilon,\Phi_\lambda)}{\lambda W(\epsilon)}\left\{\mathcal{R}(\epsilon)+\frac{\epsilon Z'(\epsilon,0)}{\psi'(0+)}-\frac{\epsilon\psi''(0+)}{2\psi'(0+)^2}-\frac{\epsilon^2}{2\psi'(0+)}\right\}+\mathcal{R}(b+\epsilon)\\
&-\frac{\epsilon\Phi_\lambda-1}{\Phi_\lambda\psi'(0+)}\left\{\frac{\psi''(0+)}{2\psi'(0+)}-Z'(b+\epsilon,0)-\frac{1}{\Phi_\lambda}-\frac{\psi'(0+)Z(b+\epsilon,\Phi_\lambda)}{\lambda}\right\}-\frac{\epsilon^2}{2\psi'(0+)}.
\end{aligned}\tag{44}$$

Also, it follows from Eq. (4) and L'Hôpital's rule that

$$\lim_{\epsilon\downarrow 0}\frac{\int_0^\epsilon \mathrm{e}^{-\theta y}W(y)\mathrm{d}y}{W(\epsilon)}=0,\tag{45}$$

for $\theta\geq 0$. Substituting Eqs. (43) and (44) into Eq. (42), Eq. (16) follows by taking the limit of $\mathbb{E}_x\left[\mathcal{A}^{(\epsilon)}_{T_b^+}\right]$ as $\epsilon\to 0$ and applying Eq. (45). ∎

A.4. Proof of Corollary 6

Proof. Taking limits as $\lambda\to\infty$ in Eq. (16), we see that Eq. (17) is proved by showing that

$$\lim_{\lambda\to\infty}\frac{1}{\Phi_\lambda}\left\{\frac{Z'(b,0)}{\psi'(0+)}+\frac{1}{\psi'(0+)\Phi_\lambda}-\frac{\psi''(0+)}{2\psi'(0+)^2}-\frac{Z(b,\Phi_\lambda)}{\lambda}\right\}=0.\tag{46}$$

Note that $Z(b,\Phi_\lambda)$ can be rewritten as

$$Z(b,\Phi_\lambda)=\lambda\int_0^\infty \mathrm{e}^{-\Phi_\lambda y}W(b+y)\mathrm{d}y,$$

for $\lambda>0$, and it follows from the dominated convergence theorem that

$$\lim_{\lambda\to\infty}\frac{Z(b,\Phi_\lambda)}{\lambda}=0.$$

Then Eq. (46) follows.

To prove Eq. (18), we see that we need to show

$$\lim_{b\to\infty}\mathcal{R}(b)=\frac{\psi''(0+)^2}{4\psi'(0+)^3}-\frac{\psi'''(0+)}{6\psi'(0+)^2},$$

which follows immediately from Eq. (31) if we can show that

$$\lim_{b\to\infty}Z'(b,0)=\frac{\psi''(0+)}{2\psi'(0+)},\tag{47}$$

and

$$\lim_{b\to\infty} Z''(b,0) = \frac{\psi'''(0+)}{3\psi'(0+)}. \tag{48}$$

Eq. (47) can be proved using Eq. (31) in Loeffen and Renaud (2010), that is

$$\lim_{b\to\infty} \mathbb{E}_b\left[X_{\tau_0^-}\mathbf{1}_{\{\tau_0^-<\infty\}}\right] = \lim_{b\to\infty}\left(Z'(b,0) - \frac{\psi''(0+)}{2}W(b)\right) = 0. \tag{49}$$

We now provide an alternative proof to the above limit. First, using Eq. (11) in Albrecher et al. (2016), one deduces that

$$\lim_{b\to\infty} Z(b,\Phi_\lambda) = \frac{\lambda}{\psi'(0+)\,\Phi_\lambda},$$

and consequently,

$$\begin{aligned}\lim_{b\to\infty}\lim_{\lambda\to 0}\frac{\partial}{\partial\lambda}Z(b,\Phi_\lambda) &= \lim_{\lambda\to 0}\frac{\partial}{\partial\lambda}\frac{\lambda}{\psi'(0+)\Phi_\lambda}\\ &= \frac{\psi''(0+)}{2\psi'(0+)^2}.\end{aligned} \tag{50}$$

Also, note that

$$\begin{aligned}\lim_{b\to\infty}\lim_{\lambda\to 0}\frac{\partial}{\partial\lambda}Z(b,\Phi_\lambda) &= \lim_{b\to\infty}\lim_{\lambda\to 0}\Phi'_\lambda Z'(b,\Phi_\lambda)\\ &= \lim_{b\to\infty}\frac{Z'(b,0)}{\psi'(0+)}.\end{aligned} \tag{51}$$

Combining Eqs. (51) and (50) proves Eq. (47). Using similar arguments, one can prove that

$$\lim_{b\to\infty} \mathbb{E}_b\left[X^2_{\tau_0^-}\mathbf{1}_{\{\tau_0^-<\infty\}}\right] = \lim_{b\to\infty}\left(Z''(b,0) - \frac{\psi'''(0+)W(b)}{3}\right) = 0,$$

and thus Eq. (48) follows. ■

A.5. Proof of Proposition 8

Proof. Using Eq. (27), one obtains

$$\begin{aligned}\mathbb{E}_x[\mathcal{I}_r] &= -\mathbb{E}_x\left[\mathbb{E}_{X_{\kappa_r}}\left[\int_0^{\tau_0^+} X_s\,\mathrm{d}s\right]\mathbf{1}_{\{\kappa_r<\infty\}}\right]\\ &= \frac{1}{2\psi'(0+)}\mathbb{E}_x\left[X^2_{\kappa_r}\mathbf{1}_{\{\kappa_r<\infty\}}\right] - \frac{\psi''(0+)}{2\psi'(0+)^2}\mathbb{E}_x\left[X_{\kappa_r}\mathbf{1}_{\{\kappa_r<\infty\}}\right].\end{aligned} \tag{52}$$

It is known from Corollary 3.5 in Loeffen et al. (2018) that

$$\mathbb{E}_x\left[\mathrm{e}^{\lambda X_{\kappa_r}}\mathbf{1}_{\{\kappa_r<\infty\}}\right] = \mathrm{e}^{\psi(\lambda)r}Z(x,\lambda) - \psi(\lambda)\int_0^r \mathrm{e}^{-\psi(\lambda)(s-r)}\Lambda(x,s)\,\mathrm{d}s - \frac{\mathcal{H}(r,\lambda)\,\Lambda(x,r)}{\int_0^\infty \frac{z}{r}\mathbb{P}(X_r\in\mathrm{d}z)},$$

for $x\in\mathbb{R}$, where

$$\mathcal{H}(r,\lambda) = \psi(\lambda)\,\mathrm{e}^{\psi(\lambda)r}\left(\frac{1}{\lambda} + \int_0^r \mathrm{e}^{-\psi(\lambda)s}\int_0^\infty \frac{z}{s}\mathbb{P}(X_s\in\mathrm{d}z)\,\mathrm{d}s\right).$$

A straightforward calculation shows that

$$\left.\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathcal{H}(r,\lambda)\right|_{\lambda=0} = r\psi'(0+)^2 + \frac{\psi''(0+)}{2} + \psi'(0+)\int_0^r\int_0^\infty \frac{z}{s}\mathbb{P}(X_s\in\mathrm{d}z)\,\mathrm{d}s,$$

and

$$\begin{aligned}\left.\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}\mathcal{H}(r,\lambda)\right|_{\lambda=0} =& \frac{\psi'''(0+)}{3} + r\psi'(0+)\left(r\psi'(0+)^2 + 2\psi''(0+)\right)\\ &+ \int_0^r\int_0^\infty \left(\psi''(0+) + 2\psi'(0+)^2(r-s)\right)\frac{z}{s}\mathbb{P}(X_s\in\mathrm{d}z)\,\mathrm{d}s.\end{aligned}$$

Consequently,

$$\begin{aligned}\mathbb{E}_x\left[X_{\kappa_r}\mathbf{1}_{\{\kappa_r<\infty\}}\right]=&\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathbb{E}_x\left[\mathrm{e}^{\lambda X_{\kappa_r}}\mathbf{1}_{\{\kappa_r<\infty\}}\right]\Big|_{\lambda=0}\\ =&\psi'(0+)r+Z'(x,0)-\psi'(0+)\int_0^r\Lambda(x,s)ds\\ &-\frac{\Lambda(x,r)}{\int_0^\infty \frac{z}{r}\mathbb{P}\left(X_r\in\mathrm{d}z\right)}\left(r\psi'(0+)^2+\frac{\psi''(0+)}{2}+\psi'(0+)\int_0^r\int_0^\infty\frac{z}{s}\mathbb{P}\left(X_s\in\mathrm{d}z\right)\mathrm{d}s\right),\end{aligned}\tag{53}$$

and

$$\begin{aligned}\mathbb{E}_x\left[X_{\kappa_r}^2\mathbf{1}_{\{\kappa_r<\infty\}}\right]=&\frac{\mathrm{d}^2}{\mathrm{d}\lambda^2}\mathbb{E}_x\left[\mathrm{e}^{\lambda X_{\kappa_r}}\mathbf{1}_{\{\kappa_r<\infty\}}\right]\Big|_{\lambda=0}\\ =&r\psi''(0+)+\left(r\psi'(0+)\right)^2+2r\psi'(0+)Z'(x,0)+Z''(x,0)\\ &-\int_0^r\left(\psi''(0+)+2\psi'(0+)^2(r-s)\right)\Lambda(x,s)ds\\ &-\frac{\Lambda(x,r)}{\int_0^\infty \frac{z}{r}\mathbb{P}\left(X_r\in\mathrm{d}z\right)}\left(\frac{\psi'''(0+)}{3}+r\psi'(0+)\left(r\psi'(0+)^2+2\psi''(0+)\right)\right.\\ &\left.+\int_0^r\int_0^\infty\left(\psi''(0+)+2\psi'(0+)^2(r-s)\right)\frac{z}{s}\mathbb{P}\left(X_s\in\mathrm{d}z\right)\mathrm{d}s\right).\end{aligned}\tag{54}$$

Substituting Eqs. (53) and (54) into Eq. (52) completes the proof of Eq. (19).

We only prove Eq. (20) for the case where X has paths of bounded variation. One can use similar arguments as in the proof of Theorem 5 for the unbounded variation case, and the details are thus omitted. For $x\in\mathbb{R}$, an application of the strong Markov property at τ_0^- and τ yields

$$\begin{aligned}\mathbb{E}_x\left[\mathcal{A}_{\tau_r}\right]=&-\mathbb{E}_x\left[\mathbb{E}_{X_{\tau_0^-}}\left[\int_0^{\tau_0^+}X_s\mathrm{d}s\right]\mathbf{1}_{\{\tau_0^-<\infty\}}\right]+\mathbb{E}_x\left[\mathbb{P}_{X_{\tau_0^-}}\left[\tau_0^+\le r\right]\mathbf{1}_{\{\tau_0^-<\infty\}}\right]\mathbb{E}\left[\mathcal{A}_{\tau_r}\right]\\ =&\left(\frac{\psi''(0+)^2}{4\psi'(0+)^2}-\frac{\psi'''(0+)}{6\psi'(0+)}\right)W(x)-\mathcal{R}(x)+\left(\int_0^\infty\left(W(x+z)-W(x)\right)\frac{z}{r}\mathbb{P}\left(X_r\in\mathrm{d}z\right)\right)\mathbb{E}\left[\mathcal{A}_{\tau_r}\right],\end{aligned}\tag{55}$$

where the last equation follows from Eq. (11) and Eq. (22) in Lkabous et al. (2017). Solving Eq. (55) for $x=0$ and then plugging the expression for $\mathbb{E}\left[\mathcal{A}_{\tau_r}\right]$ into Eq. (55) completes the proof of Eq. (20). ■

A.6. Proof of Proposition 9

Proof. By Eq. (27), we obtain

$$\begin{aligned}\mathbb{E}_x\left[\mathcal{I}^\beta\right]=&-\mathbb{E}_x\left[\mathbb{E}_{X_{\kappa^\beta}}\left[\int_0^{\tau_0^+}X_s\,\mathrm{d}s\right]\mathbf{1}_{\{\kappa^\beta<\infty\}}\right]\\ =&\frac{1}{2\psi'(0+)}\mathbb{E}_x\left[X_{\kappa^\beta}^2\mathbf{1}_{\{\tau_0^-<\infty\}}\right]-\frac{\psi''(0+)}{2\psi'(0+)^2}\mathbb{E}_x\left[X_{\kappa^\beta}\mathbf{1}_{\{\tau_0^-<\infty\}}\right],\end{aligned}\tag{56}$$

for $x\in\mathbb{R}$. From Eq. (14) in Albrecher et al. (2016), we have

$$\mathbb{E}_x\left[\mathrm{e}^{\theta X_{\kappa^\beta}}\mathbf{1}_{\{\kappa_r<\infty\}}\right]=\frac{\beta}{\beta-\psi(\theta)}\left(Z(x,\theta)-Z(x,\Phi_\beta)\frac{\Phi_\beta\psi(\theta)}{\beta\theta}\right).$$

Then, the first and second moments of X_{κ^β} are given by

$$\begin{aligned}\mathbb{E}_x\left[X_{\kappa^\beta}\mathbf{1}_{\{\kappa^\beta<\infty\}}\right]=&\frac{\mathrm{d}}{\mathrm{d}\theta}\mathbb{E}_x\left[\mathrm{e}^{\theta X_{\kappa^\beta}}\mathbf{1}_{\{\kappa_r<\infty\}}\right]\Big|_{\theta=0}\\ =&\frac{\beta\psi'(\theta)}{(\beta-\psi(\theta))^2}\left(Z(x,\theta)-Z(x,\Phi_\beta)\frac{\Phi_\beta\psi(\theta)}{\beta\theta}\right)\Big|_{\theta=0}\end{aligned}$$

$$
\begin{aligned}
&+\frac{\beta}{\beta-\psi(\theta)}\left(Z'(x,\theta)-Z(x,\Phi_\beta)\frac{\Phi_\beta}{\beta}\frac{\psi'(\theta)\theta-\psi(\theta)}{\theta^2}\right)\Bigg|_{\theta=0}\\
=&\frac{\psi'(0+)}{\beta}\left(1-Z(x,\Phi_\beta)\frac{\Phi_\beta\psi'(0+)}{\beta}\right)+Z'(x,0)-\frac{\psi''(0+)}{2}\frac{\Phi_\beta}{\beta}Z(x,\Phi_\beta),
\end{aligned}
\tag{57}
$$

and

$$
\begin{aligned}
\mathbb{E}_x\left[X_{\kappa^\beta}^2\mathbf{1}_{\{\kappa^\beta<\infty\}}\right]=&\frac{\mathrm{d}^2}{\mathrm{d}\theta^2}\mathbb{E}_x\left[\mathrm{e}^{\theta X_{\kappa^\beta}}\mathbf{1}_{\{\kappa^\beta<\infty\}}\right]\Bigg|_{\theta=0}\\
=&\frac{\beta\psi''(\theta)(\beta-\psi(\theta))^2+2\beta(\psi'(\theta))^2(\beta-\psi(\theta))}{(\beta-\psi(\theta))^4}\left(Z(x,\theta)-Z(x,\Phi_\beta)\frac{\Phi_\beta\psi(\theta)}{\beta\theta}\right)\Bigg|_{\theta=0}\\
&+\frac{2\beta\psi'(\theta)}{(\beta-\psi(\theta))^2}\left(Z'(x,\theta)-Z(x,\Phi_\beta)\frac{\Phi_\beta}{\beta}\frac{\psi'(\theta)\theta-\psi(\theta)}{\theta^2}\right)\Bigg|_{\theta=0}\\
&+\frac{\beta}{\beta-\psi(\theta)}\left(Z''(x,\theta)-Z(x,\Phi_\beta)\frac{\Phi_\beta}{\beta}\frac{\theta^2\psi''(\theta)-2(\psi'(\theta)\theta-\psi(\theta))}{\theta^3}\right)\Bigg|_{\theta=0}\\
=&\frac{\beta\psi''(0+)+2\psi'(0+)^2}{\beta^2}\left(1-Z(x,\Phi_\beta)\frac{\Phi_\beta\psi'(0+)}{\beta}\right)\\
&+\frac{2\psi'(0+)}{\beta}\left(Z'(x,0)-Z(x,\Phi_\beta)\frac{\Phi_\beta}{\beta}\frac{\psi''(0+)}{2}\right)+Z''(x,0)-Z(x,\Phi_\beta)\frac{\Phi_\beta}{\beta}\frac{\psi'''(0+)}{3}.
\end{aligned}
\tag{58}
$$

Substituting Eqs. (57) and (58) into Eq. (56) leads to Eq. (21).

Once again, we only provide the proof of Eqs. (22) and (23) for the bounded variation case. For $x\in\mathbb{R}$, by the strong Markov property and Eqs. (8) and (11),

$$
\begin{aligned}
\mathbb{E}_x\left[\mathcal{A}_{\tau^\beta}\right]=&-\mathbb{E}_x\left[\mathbb{E}_{X_{\tau_0^-}}\left[\int_0^{\tau_0^+}X_s\mathrm{d}s\right]\mathbf{1}_{\{\tau_0^-<\infty\}}\right]+\mathbb{E}_x\left[\mathbb{P}_{X_{\tau_0^-}}\left[\tau_0^+\le \mathrm{e}_\beta\right]\mathbf{1}_{\{\tau_0^-<\infty\}}\right]\mathbb{E}\left[\mathcal{A}_{\tau^\beta}\right]\\
=&\left(\frac{\psi''(0+)^2}{4\psi'(0+)^2}-\frac{\psi'''(0+)}{6\psi'(0+)}\right)W(x)-\mathcal{R}(x)+\left(Z(x,\Phi_\beta)-\frac{\beta}{\Phi_\beta}W(x)\right)\mathbb{E}\left[\mathcal{A}_{\tau^\beta}\right].
\end{aligned}
$$

Eq. (22) then follows immediately.

To prove Eq. (23), note that

$$
\mathbb{E}_x\left[\mathcal{A}_{\kappa^\beta}\mathbf{1}_{\{\kappa^\beta<\infty\}}\right]=\mathbb{E}_x\left[\mathcal{A}_{\tau^\beta}\mathbf{1}_{\{\kappa^\beta<\infty\}}\right]-\mathbb{E}_x\left[\mathcal{I}_\beta\right],
\tag{59}
$$

where

$$
\begin{aligned}
\mathbb{E}_x\left[\mathcal{A}_{\tau^\beta}\mathbf{1}_{\{\kappa^\beta<\infty\}}\right]=&-\mathbb{E}_x\left[\mathbb{E}_{X_{\tau_0^-}}\left[\int_0^{\tau_0^+}X_s\mathrm{d}s\mathbf{1}_{\{\kappa^\beta<\infty\}}\right]\mathbf{1}_{\{\tau_0^-<\infty\}}\right]\\
&+\left(Z(x,\Phi_\beta)-\frac{\beta}{\Phi_\beta}W(x)\right)\mathbb{E}\left[\mathcal{A}_{\tau^\beta}\mathbf{1}_{\{\kappa^\beta<\infty\}}\right],
\end{aligned}
\tag{60}
$$

for $x\in\mathbb{R}$. By Eq. (27) and Theorem 1 of Landriault et al. (2011), one derives that

$$
\begin{aligned}
\mathbb{E}_x\left[\int_0^{\tau_0^+}X_s\mathrm{d}s\mathbf{1}_{\{\kappa^\beta<\infty\}}\right]=&\mathbb{E}_x\left[\int_0^{\tau_0^+}X_s\mathrm{d}s\right]-\mathbb{E}_x\left[\int_0^{\tau_0^+}X_s\mathrm{d}s\mathbf{1}_{\{\kappa^\beta=\infty\}}\right]\\
=&\mathbb{E}_x\left[\int_0^{\tau_0^+}X_s\mathrm{d}s\right]-\mathbb{E}_x\left[\int_0^{\tau_0^+}X_s\mathrm{d}s\mathbf{1}_{\{\tau_0^+\le \mathrm{e}_\beta\}}\mathbb{P}\left[\kappa^\beta=\infty\right]\right]\\
=&-\frac{x}{2}\left(\frac{x}{\psi'(0+)}-\frac{\psi''(0+)}{\psi'(0+)^2}\right)-\psi'(0+)\frac{\Phi_\beta}{\beta}\mathbb{E}_x\left[\int_0^{\tau_0^+}X_s\mathrm{d}s\mathbf{1}_{\{\tau_0^+\le \mathrm{e}_\beta\}}\right],
\end{aligned}
\tag{61}
$$

for $x<0$. Using Eq. (34), it follows that

$$
\begin{aligned}
\mathbb{E}_x\left[\int_0^{\tau_0^+} X_s \mathrm{d}s \mathbf{1}_{\{\tau_0^+<\mathrm{e}_\beta\}}\right] &= \lim_{q\to 0}\frac{1}{q}\mathbb{E}_x\left[\mathrm{e}^{-\beta\tau_0^+}X_{\mathrm{e}_q}\mathbf{1}_{\{\mathrm{e}_q<\tau_0^+\}}\right] \\
&= \lim_{q\to 0}\frac{1}{q}\left\{\frac{\mathrm{d}}{\mathrm{d}\theta}\mathbb{E}_x\left[\mathrm{e}^{-\beta\tau_0^+}\mathrm{e}^{\theta X_{\mathrm{e}_q}}\mathbf{1}_{\{\mathrm{e}_q<\tau_0^+\}}\right]\Big|_{\theta=0}\right\} \\
&= \lim_{q\to 0}\frac{\mathrm{d}}{\mathrm{d}\theta}\frac{\left(\mathrm{e}^{(\theta+\Phi_\beta)x}-\mathrm{e}^{\Phi_{q+\beta}x}\right)}{q+\beta-\psi\left(\theta+\Phi_\beta\right)}\Big|_{\theta=0} \\
&= \lim_{q\to 0}\frac{x\mathrm{e}^{\Phi_\beta x}q+\left(\mathrm{e}^{\Phi_\beta x}-\mathrm{e}^{\Phi_{q+\beta}x}\right)\psi'(\Phi_\beta)}{q^2} \\
&= -x\mathrm{e}^{\Phi_\beta x}\left(\frac{\Phi_\beta''}{2\Phi_\beta'}+\frac{x\Phi_\beta'}{2}\right),
\end{aligned}
\tag{62}
$$

for $x<0$. Combining Eqs. (61) and (62) yields

$$
\begin{aligned}
\mathbb{E}_x\left[\mathbb{E}_{X_{\tau_0^-}}\left[\int_0^{\tau_0^+} X_s\mathrm{d}s\mathbf{1}_{\{\kappa^\beta<\infty\}}\right]\mathbf{1}_{\{\tau_0^-<\infty\}}\right] =& -\frac{1}{2\psi'(0+)}\mathbb{E}_x\left[X_{\tau_0^-}^2\mathbf{1}_{\{\tau_0^-<\infty\}}\right]+\frac{\psi''(0+)}{2\psi'(0+)^2}\mathbb{E}_x\left[X_{\tau_0^-}\mathbf{1}_{\{\tau_0^-<\infty\}}\right] \\
&+\frac{\Phi_\beta\Phi_\beta'\psi'(0+)}{2\beta}\mathbb{E}_x\left[X_{\tau_0^-}^2\mathrm{e}^{\Phi_\beta X_{\tau_0^-}}\mathbf{1}_{\{\tau_0^-<\infty\}}\right] \\
&+\frac{\Phi_\beta\Phi_\beta''\psi'(0+)}{2\beta\Phi_\beta'}\mathbb{E}_x\left[X_{\tau_0^-}\mathrm{e}^{\Phi_\beta X_{\tau_0^-}}\mathbf{1}_{\{\tau_0^-<\infty\}}\right].
\end{aligned}
\tag{63}
$$

By taking derivatives of Eq. (8), we obtain

$$
\mathbb{E}_x\left[X_{\tau_0^-}\mathrm{e}^{\Phi_\beta X_{\tau_0^-}}\mathbf{1}_{\{\tau_0^-<\infty\}}\right]=Z'(x,\Phi_\beta)-\frac{\Phi_\beta/\Phi_\beta'-\beta}{\Phi_\beta^2}W(x),
\tag{64}
$$

and

$$
\mathbb{E}_x\left[X_{\tau_0^-}^2\mathrm{e}^{\Phi_\beta X_{\tau_0^-}}\mathbf{1}_{\{\tau_0^-<\infty\}}\right]=Z''(x,\Phi_\beta)+\frac{\Phi_\beta''\Phi_\beta^2/\left(\Phi_\beta'\right)^3+2\Phi_\beta/\Phi_\beta'-2\beta}{\Phi_\beta^3}W(x).
\tag{65}
$$

Substituting Eqs. (28), (29), (64), and (65) into Eq. (63), it follows that

$$
\begin{aligned}
\mathbb{E}_x\left[\mathbb{E}_{X_{\tau_0^-}}\left[\int_0^{\tau_0^+} X_s\mathrm{d}s\mathbf{1}_{\{\kappa^\beta<\infty\}}\right]\mathbf{1}_{\{\tau_0^-<\infty\}}\right] =& \left(\frac{1}{\beta\Phi_\beta}-\frac{\Phi_\beta'}{\Phi_\beta^2}+\frac{\Phi_\beta''}{2\Phi_\beta'\Phi_\beta}+\frac{\psi'''(0+)}{6\psi'(0+)^2}-\frac{\psi''(0+)^2}{4\psi'(0+)^3}\right)\psi'(0+)W(x) \\
&+\frac{\Phi_\beta\psi'(0+)}{2\beta}\left(\Phi_\beta'Z''(x,\Phi_\beta)+\frac{\Phi_\beta''}{\Phi_\beta'}Z'(x,\Phi_\beta)\right)+\mathcal{R}(x).
\end{aligned}
\tag{66}
$$

Substituting Eq. (66) into Eq. (60) yields

$$
\begin{aligned}
\mathbb{E}_x\left[\mathcal{A}_{\tau^\beta}\mathbf{1}_{\{\kappa^\beta<\infty\}}\right] =& \left(\frac{\psi''(0+)^2}{4\psi'(0+)^3}-\frac{1}{\beta\Phi_\beta}+\frac{\Phi_\beta'}{\Phi_\beta^2}-\frac{\Phi_\beta''}{2\Phi_\beta'\Phi_\beta}-\frac{\psi'''(0+)}{6\psi'(0+)^2}\right)\frac{\psi'(0+)\Phi_\beta Z\left(x,\Phi_\beta\right)}{\beta} \\
&-\frac{\Phi_\beta\psi'(0+)}{2\beta}\left(\Phi_\beta'Z''(x,\Phi_\beta)+\frac{\Phi_\beta''}{\Phi_\beta'}Z'(x,\Phi_\beta)\right)-\mathcal{R}(x).
\end{aligned}
\tag{67}
$$

The proof of Eq. (23) is then completed by substituting Eqs. (67) and (21) into Eq. (59). Finally, we note that a straightforward calculation leads to

$$
\begin{aligned}
\Phi_\beta'Z''\left(x,\Phi_\beta\right)+\frac{\Phi_\beta''}{\Phi_\beta'}Z'\left(x,\Phi_\beta\right) =& \left(\Phi_\beta'x^2+\frac{\Phi_\beta''}{\Phi_\beta'}x\right)Z(x,\Phi_\beta)-2x\mathrm{e}^{\Phi_\beta x}\int_0^x\mathrm{e}^{-\Phi_\beta y}W(y)\mathrm{d}y \\
&+\mathrm{e}^{\Phi_\beta x}\int_0^x\mathrm{e}^{-\Phi_\beta y}\left(\beta\Phi_\beta'\left(2xy-y^2\right)+\left(2+\frac{\beta\Phi_\beta''}{\Phi_\beta'}\right)y\right)W(y)\mathrm{d}y. \quad \blacksquare
\end{aligned}
$$

References

Albrecher, H., Ivanovs, J., Zhou, X., 2016. Exit identities for Lévy processes observed at Poisson arrival times. Bernoulli 22 (3), 1364–1382.
Baurdoux, E.J., Pardo, J.C., Pérez, J.L., Renaud, J.-F., 2016. Gerber-Shiu distribution at Parisian ruin for Lévy insurance risk processes. Journal of Applied Probability.
Bayraktar, E., Young, V.R., 2010. Optimal investment strategy to minimize occupation time. Annals of Operations Research 176 (1), 389–408.
Bertoin, J., 1996. Lévy Processes. Cambridge University Press.
Callant, J., Trufin, J., Zuyderhoff, P., 2022. Some expressions of a generalized version of the expected time in the red and the expected area in red. Methodology and Computing in Applied Probability, 1–17.
Dickson, D.C.M., dos Reis, A.E., 1997. The effect of interest on negative surplus. Insurance. Mathematics & Economics 21 (1), 1–16.
dos Reis, A.E., 1993. How long is the surplus below zero? Insurance. Mathematics & Economics 12 (1), 23–38.
dos Reis, A.E., 2000. On the moments of ruin and recovery times. Insurance. Mathematics & Economics 27 (3), 331–343.
Egami, M., Yamazaki, K., 2014. Phase-type fitting of scale functions for spectrally negative Lévy processes. Journal of Computational and Applied Mathematics 264, 1–22.
Gerber, H.U., Shiu, E.S.W., Yang, H., 2012. The Omega model: from bankruptcy to occupation times in the red. European Actuarial Journal 2 (2), 259–272.
Guérin, H., Renaud, J.-F., 2017. On the distribution of cumulative Parisian ruin. Insurance. Mathematics & Economics 73, 116–123.
Kuznetsov, A., Kyprianou, A.E., Rivero, V., 2012. The theory of scale functions for spectrally negative Lévy processes. In: Lévy Matters. In: Springer Lecture Notes in Mathematics.
Kyprianou, A.E., 2014. Fluctuations of Lévy Processes with Applications - Introductory Lectures, second edition. Universitext. Springer, Heidelberg.
Landriault, D., Renaud, J.-F., Zhou, X., 2011. Occupation times of spectrally negative Lévy processes with applications. Stochastic Processes and Their Applications 121 (11), 2629–2641.
Landriault, D., Renaud, J.-F., Zhou, X., 2014. An insurance risk model with Parisian implementation delays. Methodology and Computing in Applied Probability 16 (3), 583–607.
Landriault, D., Li, B., Wong, J.T.Y., Xu, D., 2018. Poissonian potential measures for Lévy risk models. Insurance. Mathematics & Economics 82, 152–166.
Landriault, D., Li, B., Shi, T., Xu, D., 2019. On the distribution of classic and some exotic ruin times. Insurance. Mathematics & Economics 89, 38–45.
Landriault, D., Li, B., Lkabous, M.A., 2020. On occupation times in the red of Lévy risk models. Insurance. Mathematics & Economics 92, 17–26.
Li, B., Tang, Q., Wang, L., Zhou, X., 2014. Liquidation risk in the presence of Chapters 7 and 11 of the US bankruptcy code. The Journal of Financial Engineering 1 (03), 1450023.
Li, S., 2008. The time of recovery and the maximum severity of ruin in a Sparre Andersen model. North American Actuarial Journal 12 (4), 413–425.
Li, X., Liu, H., Tang, Q., Zhu, J., 2020. Liquidation risk in insurance under contemporary regulatory frameworks. Insurance. Mathematics & Economics 93, 36–49.
Lkabous, M.A., Czarna, I., Renaud, J.-F., 2017. Parisian ruin for a refracted Lévy process. Insurance. Mathematics & Economics 74, 153–163.
Loeffen, R.L., Renaud, J.-F., 2010. De Finetti's optimal dividends problem with an affine penalty function at ruin. Insurance. Mathematics & Economics 46 (1), 98–108.
Loeffen, R.L., Czarna, I., Palmowski, Z., 2013. Parisian ruin probability for spectrally negative Lévy processes. Bernoulli 19 (2), 599–609.
Loeffen, R.L., Palmowski, Z., Surya, B.A., 2018. Discounted penalty function at Parisian ruin for Lévy insurance risk process. Insurance. Mathematics & Economics 83, 190–197.
Loisel, S., 2005. Differentiation of some functionals of risk processes, and optimal reserve allocation. Journal of Applied Probability 42 (2), 379–392.
Loisel, S., Trufin, J., 2014. Properties of a risk measure derived from the expected area in red. Insurance. Mathematics & Economics 55, 191–199.
Strietzel, P. Lukas, Behme, A., 2022. Moments of the ruin time in a Lévy risk model, Methodology and Computing in Applied Probability 24, 3075–3099.
The National Association of Insurance Commissioners (NAIC), 2021. Receiver's Handbook for Insurance Company Insolvencies.
Perman, M., Wellner, J.A., 1996. On the distribution of Brownian areas. The Annals of Applied Probability 6 (4), 1091–1111.
Picard, P., 1994. On some measures of the severity of ruin in the classical Poisson model. Insurance. Mathematics & Economics 14 (2), 107–115.
Riordan, J., 2012. Introduction to Combinatorial Analysis. Courier Corporation.
Schwartz, L., 1945. Généralisation de la notion de fonction, de dérivation, de transformation de fourier et applications mathématiques et physiques. Annales de l'université de Grenoble. Nouvelle série. Section sciences mathématiques et physiques 21, 57–74.
Shepp, L.A., 1982. On the integral of the absolute value of the pinned Wiener process. The Annals of Probability 10 (1), 234–239.
Yildirim, Y., 2006. Modeling default risk: a new structural approach. Finance Research Letters 3 (3), 165–172.
Yin, C., Yuen, K., 2014. Exact joint laws associated with spectrally negative Lévy processes and applications to insurance risk theory. Frontiers of Mathematics in China 9, 1453–1471.
Zhang, C., Wu, R., 2002. Total duration of negative surplus for the compound Poisson process that is perturbed by diffusion. Journal of Applied Probability 39 (3), 517–532.