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Multiple per-claim reinsurance based on maximizing the Lundberg exponent

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ABSTRACT

In this paper, we consider the optimal per-claim reinsurance problem for an insurer who designs a reinsurance contract with multiple reinsurance participants. In contrast to using the value-at-risk as a short-term risk measure, we take the Lundberg exponent in risk theory as a risk measure for the insurer over a long-term horizon because the Lundberg upper bound performs better in measuring the infinite-time ruin probability. To reflect various risk preferences of the reinsurance participants, we adopt a type of combined premium principle in which the expected premium principle, variance premium principle, and exponential premium principle are all special cases. Based on maximization of the insurer's Lundberg exponent, the optimal reinsurance is formulated within a static setting, and we derive optimal multiple reinsurance strategies within a general admissible policies set. In general, these optimal strategies are shown to have non-piecewise linear structures, differing from conventional reinsurance strategies such as quota-share, excess-of-loss, or linear layer reinsurance arrangements. In some special cases, the optimal reinsurance strategies reduce to classical results.

1. Introduction

Optimal risk control is a classical problem in economics and insurance. An early contribution to this problem can be traced back to the pioneering work by Borch in the 1960s (see Borch, 1962). In recent years, this problem has received increasing interest in the field of risk theory and insurance mathematics. For individuals and institutes, there are a few methods for performing risk management in the financial market in practice. Insurance or reinsurance is one of the most common and effective approaches. For example, individuals buy various insurance contracts to cede part or all of those risks expected to occur in their lifetime, and an insurer designs reinsurance contracts to cede business risks to one or several reinsurance companies. Because individuals and institutes may have different risk preferences (different risk measures), the optimal risk control (insurance or reinsurance) problem has been studied in many situations.

From the perspective of a single insurance company, risks are collected from policyholders. Because the risks are not perfectly homogeneous and the number of policyholders might not be sufficiently large, the insurer still has certain large aggregated risks in operations. To maintain a required solvency status for either a short-term or long-term horizon, the insurer usually takes reinsurance contracts to cede part of their business risk to reinsurance companies. The form of reinsurance usually depends on the reinsurance premium principle and the insurer's risk metrics. For a short-term period (usually one year), the value-at-risk (VaR) is an important risk measure, and insurance regulators in many countries require this measure to calculate the solvency capital. Therefore, many researchers have studied optimal reinsurance arrangements with VaR objectives or VaR constraints. For example, Cai and Tan (2007) considered optimal retention in stop-loss reinsurance by minimizing the VaR. Cai et al. (2008) further derived the optimal ceded loss functions for a class of increasing convex ceded loss functions and found that the optimal risk-sharing strategy can be in the form of stop-loss, quota-share, or change-loss arrangements under different cases. From a long-term viewpoint, a one-period model using VaR cannot reflect the dynamic process of a company's surplus. Correspondingly, in dynamic models, the ruin probability is usually taken as an objective function for optimization. For example, see Schmidli (2002) and Liang and Young (2018). However, in contrast to VaR, it is difficult to obtain an explicit expression for the ruin probability, even for classical risk models such

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as the Cramér-Lundberg model. Fortunately, ruin theory tells us that for the Cramér-Lundberg risk model, the ruin probability has a very simple exponential upper bound, called the Lundberg upper bound. Therefore, the Lundberg exponent serves as an alternative risk measure of ruin probability to determine a company's solvency capability over a long-term horizon. For details regarding ruin theory and the Lundberg exponent, we refer the readers to Albrecher et al. (2020), Asmussen and Albrecher (2010), Asmussen and Rolski (1994), Boxma and Mandjes (2021), Gerber (1979), Meilijson (2009), Liang and Young (2018), Rolski et al. (1999), and Schmidli (2002).

The optimal reinsurance problem is not usually approached by directly minimizing the ruin probability itself, because the ruin probability does not have an explicit expression in most situations, even in the classical risk model. Therefore, some scholars choose to minimize the Lundberg upper bound of the ruin probability as an alternative value function. In such cases, the Lundberg exponent is maximized by a possible reinsurance arrangement. The optimal reinsurance problem has been studied in terms of Lundberg exponent maximization over the past few decades. For example, Centeno (1986) studied optimal proportional reinsurance, excess-of-loss reinsurance, and their combination. Hipp and Schmidli (2004) and Centeno (2002) investigated the optimal proportional reinsurance problem for an insurer under the Cramér-Lundberg model and a general renewal risk model, respectively. Hald and Schmidli (2004) studied optimal proportional reinsurance under both the expected value and variance premium principles. Schmidli (2004) considered optimal excess-of-loss reinsurance under the Cramér-Lundberg model. Liang and Guo (2008) investigated the proportional reinsurance and investment strategy under a jump-diffusion model. Thus far, most authors have focused on special types of reinsurance, such as quota-share, excess-of-loss, or layer reinsurance arrangements, i.e., reinsurance policies belonging to a finite-dimensional reinsurance space. The optimal form of reinsurance based on Lundberg exponent maximization within a general infinite-dimensional admissible set has not been determined. To the best of our knowledge, only Gerber (1979) and Liang et al. (2020) have studied optimal risk-sharing among the class of plausible reinsurance treaties. Gerber (1979) showed that the optimal policy has the excess-of-loss form if the reinsurer takes the expected value premium principle. Liang et al. (2020) obtained the optimal reinsurance form (see Theorem 4.1) for the case in which the reinsurance company takes the mean-variance premium principle.

Optimal reinsurance problems are primarily studied between two parties, for example, one insurer and one reinsurance company. However, to effectively disperse risk, it is common and necessary to adopt multiple participants to share the whole risk. For an optimal insurance problem among multiple agents, Gerber (1979) considered the minimum premium problem under the exponential premium principle and showed that proportional insurance is the optimal form. Ludkovski and Young (2009) investigated distortion risk measures and showed that stop-loss risk-sharing is optimal in some special cases. Embrechts et al. (2018) considered a two-parameter class of quantile-based risk measures, and Chen et al. (2021) studied a dynamic Pareto optimal risk-sharing problem for a group of insurers under the mean-variance criterion. Bernard et al. (2020) investigated the optimal insurance design for an insurer with multiple policyholders in terms of the utility indifference premium principle and the objective of maximizing the insurer's utility. However, thus far, few studies have considered multiple reinsurance companies involved in a single reinsurance contract; for example, see Chen and Yuen (2016), Meng et al. (2016a), and Meng et al. (2016b, 2017) for a continuous time model and Boonen and Ghossoub (2021), Boonen et al. (2021), and Cai et al. (2017) for a static model. In these papers, it is assumed that one insurer faces two or multiple reinsurance companies. Considering the different risk preferences of participants and the nonlinearity of risk measures, the optimal multiple reinsurance for a single contract remains an interesting but unsolved problem.

In this paper, we consider one insurer and any number of reinsurance companies in a reinsurance contract. Meanwhile, we adopt a general premium principle with a combinational form, which was first introduced by Meng et al. (2019). Some classic principles, such as the expected value, variance, and exponential premium principles, can be applied as a special case of the general principle. In this manner, we can allow different risk preferences for the insurer and each reinsurer. With the objective of maximizing the insurer's Lundberg exponent, we study the optimal forms of risk shared by each participant, which exhibit non-piecewise linear structures. For all of the special cases considered, the optimal risk control policies can reduce to the classical results.

The remainder of this paper is organized as follows. We formulate the risk model and specify the multiple risk control optimization problem in Section 2. Section 3 presents the optimal multiple reinsurance policy, which is shown to exhibit non-piecewise linear structures. In Section 4, we study the optimal reinsurance policy under a few classic premium principles. Finally, a numerical analysis and economic explanations are given in Section 5.

2. Model formulation

We start with a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $\mathbb{E}$ and $\mathbb{V}$ denote the expectation and variance operations, respectively, under the probability measure $\mathbb{P}$.

For insurers, reinsurance is one of the most effective ways to perform risk management. In practice, the insurer usually transfers the risks of a business line to one or multiple reinsurance companies. In this paper, we assume that there exist m reinsurance companies to take the business risks ceded from the insurer in the market. For a risk Z_i with a specific reinsurance arrangement $\mathbf{g} = \{g_0(Z_i), g_1(Z_i), \cdots, g_m(Z_i)\}$, the insurer retains $g_0(Z_i)$, and $g_j(Z_i)$ is taken by the reinsurer j such that $0 \le g_i(x) \le x$, $i = 0, 1, \cdots, m$ and

$$Z_i - g_0(Z_i) = \sum_{j=1}^{m} g_j(Z_i).$$

Now, we define a specific reinsurance strategy $\mathbf{g}$ as admissible for $z \in [0, \infty)$ if it satisfies the following:

(i) $g_j(z) \ge 0$ and $z = \sum_{j=0}^{m} g_j(z)$.
(ii) $g_j(z)$ is increasing in z for all $j = 0, 1, \cdots, m$.

We denote by $\mathcal{G}$ the set of all admissible multiple reinsurance strategies. Note that the condition (ii) excludes some possible moral hazards, for example, the motivation of misrepresentation claims. In addition, for $z_2 \ge z_1 \ge 0$, it follows that

$$0 \le g_j(z_2) - g_j(z_1) = z_2 - \sum_{i \ne j}^{m} g_i(z_2) - \left(z_1 - \sum_{i \ne j}^{m} g_i(z_1) \right)$$
$$\le z_2 - z_1,$$

which implies that each $g_j(z)$ is a continuous function with respect to z for $j = 0, 1, \cdots, m$.

To determine how much risk should be ceded to the reinsurer, the insurer must consider the risk measure taken by the reinsurer. In previous studies, some popular premium principles, such as the expected value principle, variance principle, and exponential principle, have been considered. Different premium principles generally lead to different optimal reinsurance structures. To investigate the impact of the premium principle on the reinsurance structure, we introduce a type of combined premium principle $\pi^{\theta,a,\gamma}(Y)$ that can cover several commonly used principles.[1] Given a positive risk Y, the premium principle $\pi^{\theta,a,\gamma}(Y)$ is defined as

$$\pi^{\theta,a,\gamma}(Y)=\frac{1+\theta}{a}\ln\mathbb{E}[e^{aY}]+\gamma\mathbb{V}[Y], \tag{2.1}$$

where $\theta\geq 0,\gamma\geq 0$, and $a\geq 0$[2] and $\theta^2+\gamma^2+a^2\neq 0$. Clearly, this definition includes the following special premium principles:

- Expected value principle: If $\gamma=0$ and $a\to 0+$, then $\pi^{\theta,0,0}(Y)=(1+\theta)\mathbb{E}[Y]$.
- Mean-variance principle: If $a\to 0+$, then $\pi^{\theta,0,\gamma}(Y)=(1+\theta)\mathbb{E}[Y]+\gamma\mathbb{V}[Y]$. Furthermore, if $\theta=0$, then $\pi^{0,0,\gamma}(Y)=\mathbb{E}[Y]+\gamma\mathbb{V}[Y]$, which leads to the variance principle.
- Exponential principle: If $\theta=\gamma=0$, then $\pi^{0,a,0}(Y)=\frac{1}{a}\ln\mathbb{E}[e^{aY}]$.

In risk theory, an insurer's original surplus process $\{X_t,t\geq 0\}$ can be described by the classic Cramér-Lundberg risk model, i.e.,

$$X_t=x+ct-\sum_{i=1}^{N_t}Z_i,$$

where $x\geq 0$ is the initial capital; $c\geq 0$ is the premium rate; the claims $\{Z_i,i=1,2,\cdots\}$ are independent and identically distributed positive random variables; and $\{N_t,t\geq 0\}$ is a homogeneous Poisson process with intensity rate λ, independent of $\{Z_i,i=1,2,\cdots\}$. Without a loss of generality, we assume that $\lambda=1$ and Z has the same distribution as the claims Z_i. Throughout this paper, it is assumed that the random variable Z has an unbounded support $[0,\infty)$.[3] In addition, we assume that the distribution of Z has a light tail, i.e., there exists a constant $\epsilon>0$ such that $\mathbb{E}[e^{\epsilon Z}]<\infty$. We denote $\varsigma=\sup\{y:\mathbb{E}[e^{yZ}]<\infty\}$.

With a specific reinsurance arrangement $\mathbf{g}=\{g_0(z),g_1(z),\cdots,g_m(z)\}$, the total risk up to time t ceded from the insurer to the jth reinsurer is $\sum_{i=1}^{N_t}g_j(Z_i)$. Thus, the reinsurance premium up to time t that the insurer must pay to the jth reinsurer is

$$\begin{aligned}\pi_t^{\theta_j,a_j,\gamma_j}&\triangleq\pi^{\theta_j,a_j,\gamma_j}\left(\sum_{i=1}^{N_t}g_j(Z_i)\right)\\&=\frac{1+\theta_j}{a_j}\left(\mathbb{E}[e^{a_jg_j(Z)}]-1\right)t+\gamma_j\mathbb{E}[g_j^2(Z)]t\end{aligned} \tag{2.2}$$

for $0\leq a_j<\varsigma$. For convenience of the following analysis but without loss of generality, we here rank m reinsurance companies such that

$$\theta_1\leq\theta_2\leq\cdots\leq\theta_m.$$

If $\theta_i=\theta_{i+1}$, we assume $a_i\leq a_{i+1}$. Furthermore, if $\theta_i=\theta_{i+1}$ and $a_i=a_{i+1}$, we take $\gamma_i\leq\gamma_{i+1}$.

Correspondingly, the insurer's surplus process with the reinsurance strategy $\mathbf{g}$ can be rewritten as

$$\begin{aligned}X_t^{\mathbf{g}}&=x+ct-\sum_{j=1}^{m}\pi_t^{\theta_j,a_j,\gamma_j}-\sum_{i=1}^{N_t}g_0(Z_i)\\&=x+\left\{c-\sum_{j=1}^{m}\left[\frac{1+\theta_j}{a_j}\left(\mathbb{E}[e^{a_jg_j(Z)}]-1\right)+\gamma_j\mathbb{E}[g_j^2(Z)]\right]\right\}t\\&\quad-\sum_{i=1}^{N_t}g_0(Z_i).\end{aligned} \tag{2.3}$$

In risk theory, the ruin probability over an infinite-time horizon plays an important role in measuring the long-term solvency. Mathematically, we respectively define the ruin time and ruin probability as follows:

$$\tau^{\mathbf{g}}=\inf\{t>0:X_t^{\mathbf{g}}<0\} \tag{2.4}$$

and

$$V^{\mathbf{g}}(x)=\mathbb{P}\{\tau^{\mathbf{g}}<\infty|X_0^{\mathbf{g}}=x\},\quad x\geq 0. \tag{2.5}$$

It is known that the ruin probability for the classic Cramér-Lundberg risk model rarely has explicit expressions, except for a class of phase-type claims, see also Asmussen and Albrecher (2010). To ensure that the ruin probability is tractable for all claims with light tails, we employ the results of the Cramér-Lundberg approximation:

$$\lim_{x\to\infty}V^{\mathbf{g}}(x)e^{R_{\mathbf{g}}x}=C_{\mathbf{g}},\quad x\geq 0 \tag{2.6}$$

and the exponential upper bound:

$$V^{\mathbf{g}}(x)\leq e^{-R_{\mathbf{g}}x},\quad x\geq 0, \tag{2.7}$$

where $C_{\mathbf{g}}$ is a constant and $R_{\mathbf{g}}$ is the so-called adjustment coefficient or Lundberg exponent. Therefore, we can control the upper bound of the ruin probability in terms of the Lundberg exponent. In general, a larger $R_{\mathbf{g}}$ corresponds to a smaller ruin probability. Now, we can define the optimization problem[4]:

$$V(x)=\inf_{\mathbf{g}\in\mathcal{G}}e^{-R_{\mathbf{g}}x},\quad x\geq 0.$$

Our aim is to find an optimal multiple reinsurance strategy $\mathbf{g}^*\in\mathcal{G}$ such that

$$R^*:=R_{\mathbf{g}^*}=\sup_{\mathbf{g}\in\mathcal{G}}R_{\mathbf{g}}. \tag{2.8}$$

It is worth mentioning that claims with light-tailed distributions guarantee the existence of the Lundberg exponent.

To avoid trivial cases, we include a no-arbitrage assumption for the premium rate c:

$$\mathbb{E}[Z]<c<A_m(Z), \tag{2.9}$$

where $A_m(Z):=\min_{\sum_{j=1}^m g_j(Z)=Z}\sum_{j=1}^m\left[\frac{1+\theta_j}{a_j}\left(\mathbb{E}[e^{a_jg_j(Z)}]-1\right)+\gamma_j\mathbb{E}[g_j^2(Z)]\right]$ denotes the minimum reinsurance premium rate paid by the insurers when all risks are ceded to m reinsurance companies.

On one hand, if the left side of (2.9) does not follow, i.e., $c\leq\mathbb{E}[Z]$, then we have

[1] The advantage of this setting is that it allows us to investigate the optimal reinsurance under a unified framework for reinsurers with different premium principles.

[2] Here, $a=0$ means $\lim_{a\to 0+}$.

[3] The results obtained in this paper are also applicable for the case of Z with a bounded support after some minor changes.

[4] The Lundberg upper bound of the ruin probability $e^{-R_{\mathbf{g}}x}$ resembles an exponential utility function, but the optimization problem of Lundberg exponent maximization $(\sup R_{\mathbf{g}})$ is essentially different from the expected utility maximization of terminal wealth $\left(\sup\mathbb{E}\left[e^{-X_T^{\mathbf{g}}}\right]\right)$.

$$c-\sum_{j=1}^{m}\left[\frac{1+\theta_j}{a_j}\left(\mathbb{E}[e^{a_j g_j(Z)}]-1\right)+\gamma_j\mathbb{E}[g_j^2(Z)]\right] \le \mathbb{E}[Z]-\sum_{j=1}^{m}\mathbb{E}g_j(Z)=\mathbb{E}(g_0(Z)) \tag{2.10}$$

because $e^x \ge 1+x$. Thus, for any $\mathbf{g}\in\mathcal{G}$, the surplus process (2.3) has a non-positive drift, and consequently, $V^{\mathbf{g}}(x)=1$ for any $x\ge 0$. On the other hand, if the right side of (2.9) does not hold, then $c\ge A_m(z)$, which means that there exists a reinsurance strategy $\mathbf{g}=(0,g_1,g_2,\cdots,g_m)$ with $\sum_{j=1}^{m} g_j(Z)=Z$, such that

$$c\ge\sum_{j=1}^{m}\left[\frac{1+\theta_j}{a_j}\left(\mathbb{E}[e^{a_j g_j(Z)}]-1\right)+\gamma_j\mathbb{E}[g_j(Z)]^2\right].$$

With this special reinsurance strategy, the insurer cedes all of the risks, but the total reinsurance premium rate is less than the premium rate c. Thus, we have $V^{\mathbf{g}}(x)=0$ for any $x\ge 0$.

3. Optimal multiple reinsurance strategy

In this section, we obtain the optimal reinsurance strategy for the optimization problem (2.8). Initially, we transfer the problem to a tractable form similar to an Hamilton-Jacobi-Bellman(HJB) equation for a dynamic control problem. We first define the following function:

$$K_{\boldsymbol{\theta},\mathbf{a},\boldsymbol{\gamma}}(\mathbf{g},r)=\sum_{j=1}^{m}\left[\frac{1+\theta_j}{a_j}\left(e^{a_j g_j}-1\right)+\gamma_j g_j^2\right]+\frac{1}{r}\left[e^{rg_0}-1\right], \tag{3.1}$$

where $r>0$ and vectors $\boldsymbol{\theta}=(\theta_1,\cdots,\theta_m)$, $\mathbf{a}=(a_1,\cdots,a_m)$, $\boldsymbol{\gamma}=(\gamma_1,\cdots,\gamma_m)$.

Lemma 3.1. *For a given admissible strategy $g_0(Z)$ such that $\mathbb{P}\{g_0(Z)>0\}>0$, the function $\frac{1}{r}\mathbb{E}\left[e^{rg_0(Z)}-1\right]$ is strictly increasing with respect to $r>0$.*

Proof. Firstly, we can easily know that the function $\frac{e^{r\xi}-1}{r}$ is strictly increasing with respect to $r>0$ for any fixed constant $\xi>0$.

Note that the function $g_0(\cdot)$ is continuously increasing. Then, there exists a constant $z_0\ge 0$ such that

$$A=:\{w:g_0(Z(w))>0\}=\{w:Z(w)>z_0\}$$

and $\mathbb{P}\{A\}>0$. Furthermore, there exists a large enough $k>0$ such that $g_0(z_0+\frac{1}{k})>0$ and $\mathbb{P}\{A_k\}>0$, where

$$A_k=\left\{w:Z(w)>z_0+\frac{1}{k}\right\}.$$

Therefore, for any $r_1>r_2$, we have

$$\begin{aligned}
&\frac{1}{r_1}\mathbb{E}\left[e^{r_1 g_0(Z)}-1\right]-\frac{1}{r_2}\mathbb{E}\left[e^{r_2 g_0(Z)}-1\right]\\
&=\int_A\left(\frac{1}{r_1}\left[e^{r_1 g_0(Z(w))}-1\right]-\frac{1}{r_2}\left[e^{r_2 g_0(Z(w))}-1\right]\right)\mathbb{P}(dw)\\
&\ge\int_{A_k}\left(\frac{1}{r_1}\left[e^{r_1 g_0(Z(w))}-1\right]-\frac{1}{r_2}\left[e^{r_2 g_0(Z(w))}-1\right]\right)\mathbb{P}(dw)\\
&\ge\left(\frac{1}{r_1}\left[e^{r_1 g_0(z_0+\frac{1}{k})}-1\right]-\frac{1}{r_2}\left[e^{r_2 g_0(z_0+\frac{1}{k})}-1\right]\right)\mathbb{P}\{A_k\}\\
&>0
\end{aligned}$$

where the second inequality follows because $\frac{1}{r_1}(e^{r_1 y}-1)-\frac{1}{r_2}(e^{r_2 y}-1)$ is strictly increasing with respect to $y>0$ for any fixed constants $r_1>r_2>0$. Thus, it implies that $\frac{1}{r}\mathbb{E}\left[e^{rg_0(Z)}-1\right]$ is strictly increasing with respect to $r>0$. □

Lemma 3.2. *If there exists $\mathbf{g}^*\in\mathcal{G}$ such that $R^*:=R_{\mathbf{g}^*}=\sup_{\mathbf{g}\in\mathcal{G}}R_{\mathbf{g}}$, then $R^*>0$ satisfies the following equation:*

$$\mathbb{E}(K_{\boldsymbol{\theta},\mathbf{a},\boldsymbol{\gamma}}(\mathbf{g}^*(Z),R^*))=\inf_{\mathbf{g}\in\mathcal{G}}\left\{\mathbb{E}(K_{\boldsymbol{\theta},\mathbf{a},\boldsymbol{\gamma}}(\mathbf{g}(Z),R^*))\right\}=c. \tag{3.2}$$

Conversely, if $\mathbf{g}^*$ and $R^*>0$ satisfy (3.2), then $R^*=R_{\mathbf{g}^*}\ge R_{\mathbf{g}}$ for any admissible strategy $\mathbf{g}$.

Proof. For any given admissible strategy $\mathbf{g}\in\mathcal{G}$, if

$$c\le\sum_{j=1}^{m}\left[\frac{1+\theta_j}{a_j}\left(\mathbb{E}[e^{a_j g_j(Z)}]-1\right)+\gamma_j\mathbb{E}[g_j^2(Z)]\right]+\mathbb{E}[g_0(Z)],$$

then we know that the surplus process (2.3) has a non-positive drift, which implies that $V^{\mathbf{g}}(x)=1$ for any $x\ge 0$. Therefore, $R_{\mathbf{g}}=0$. Otherwise, we have

$$c>\sum_{j=1}^{m}\left[\frac{1+\theta_j}{a_j}\left(\mathbb{E}[e^{a_j g_j(Z)}]-1\right)+\gamma_j\mathbb{E}[g_j^2(Z)]\right]+\mathbb{E}[g_0(Z)],$$

which implies $\mathbb{P}\{g_0(Z)>0\}>0$[5] from (2.9) and the associated Lundberg exponent $R_{\mathbf{g}}$ is the unique positive root of the following equation of r (see page 7 of Grandell, 1991 for the solution of Lundberg exponent):

$$c-\mathbb{E}[K_{\boldsymbol{\theta},\mathbf{a},\boldsymbol{\gamma}}(\mathbf{g},r)]=0. \tag{3.3}$$

Because $R^*\ge R_{\mathbf{g}}$, it follows from Lemma 3.1 that

$$c-\mathbb{E}[K_{\boldsymbol{\theta},\mathbf{a},\boldsymbol{\gamma}}(\mathbf{g},R^*)]\le 0 \tag{3.4}$$

for any given admissible strategy $\mathbf{g}\in\mathcal{G}$. Thus, taking the supremum on both sides of the above inequality, we obtain that R^* satisfies

$$\sup_{\mathbf{g}\in\mathcal{G}}\left[c-\mathbb{E}[K_{\boldsymbol{\theta},\mathbf{a},\boldsymbol{\gamma}}(\mathbf{g},R^*)]\right]\le 0.$$

Further, noting that $c-\mathbb{E}[K_{\boldsymbol{\theta},\mathbf{a},\boldsymbol{\gamma}}(\mathbf{g}^*,R^*)]=0$, we can conclude that (3.2) holds true.

Conversely, if $\mathbf{g}^*$ and $R^*>0$ satisfy (3.2), then we have

$$\begin{aligned}
&\mathbb{E}[K_{\boldsymbol{\theta},\mathbf{a},\boldsymbol{\gamma}}(\mathbf{g}^*(Z)),R^*]=\inf_{\mathbf{g}\in\mathcal{G}}\mathbb{E}[K_{\boldsymbol{\theta},\mathbf{a},\boldsymbol{\gamma}}(\mathbf{g}(Z),R^*)]=c \quad\text{and}\\
&\mathbb{E}[K_{\boldsymbol{\theta},\mathbf{a},\boldsymbol{\gamma}}(\mathbf{g}(Z),R_{\mathbf{g}})]=c
\end{aligned} \tag{3.5}$$

for any admissible strategy $\mathbf{g}\in\mathcal{G}$. Then, we can conclude that

$$\mathbb{E}[K_{\boldsymbol{\theta},\mathbf{a},\boldsymbol{\gamma}}(\mathbf{g}(Z),R^*)]\ge\mathbb{E}[K_{\boldsymbol{\theta},\mathbf{a},\boldsymbol{\gamma}}(\mathbf{g}(Z),R_{\mathbf{g}})]. \tag{3.6}$$

If $\mathbb{P}\{g_0(Z)=0\}=1$, then we know from (2.9) that the surplus process (2.3) has a non-positive drift, which implies that $V^{\mathbf{g}}(x)=1$ for any $x\ge 0$. Therefore, $R_{\mathbf{g}}=0$ and then $R^*\ge R_{\mathbf{g}}$. If $\mathbb{P}\{g_0(Z)>0\}>0$, then from (3.6) we have

$$\frac{1}{R^*}\mathbb{E}\left[e^{R^* g_0(Z)}-1\right]\ge\frac{1}{R_{\mathbf{g}}}\mathbb{E}\left[e^{R_{\mathbf{g}} g_0(Z)}-1\right],$$

which also implies that $R^*\ge R_{\mathbf{g}}$ according to Lemma 3.1. Thus, we can conclude that $R^*=R_{\mathbf{g}^*}=\sup_{\mathbf{g}\in\mathcal{G}}R_{\mathbf{g}}$. □

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[5] In fact, if $\mathbb{P}\{g_0(Z)>0\}=0$, we have $g_0(Z)=0$ a.s. and $\mathbb{E}[g_0(Z)]=0$. Noting $\mathbf{g}=\{g_0(z),g_1(z),\cdots,g_m(z)\}\in\mathcal{G}$, we have $Z=\sum_{j=0}^{m}g_j(Z)=\sum_{j=1}^{m}g_j(Z)$, that is $c>\sum_{j=1}^{m}\left[\frac{1+\theta_j}{a_j}\left(\mathbb{E}[e^{a_j g_j(Z)}]-1\right)+\gamma_j\mathbb{E}[g_j^2(Z)]\right]$ with $Z=\sum_{j=1}^{m}g_j(Z)$, which is in contradiction with (2.9).

Next, we will solve (3.2) to determine the optimal multiple reinsurance in terms of a point-wise optimization approach. Let us define a Lagrange function:

$$H^{r,\ell}_{\boldsymbol{\theta},\boldsymbol{a},\boldsymbol{\gamma}}(z,\mathbf{g}) = K_{\boldsymbol{\theta},\boldsymbol{a},\boldsymbol{\gamma}}(\mathbf{g},r) - \ell\left(\sum_{j=0}^{m} g_j - z\right) = \sum_{j=1}^{m}\left[\frac{1+\theta_j}{a_j}\left(e^{a_j g_j}-1\right)+\gamma_j g_j^2\right]+\frac{1}{r}\left[e^{rg_0}-1\right] - \ell\left(\sum_{j=0}^{m} g_j - z\right), \quad (3.7)$$

where $\ell > 0$ is a Lagrange multiplier. Note that the function $H^{r,\ell}_{\boldsymbol{\theta},\boldsymbol{a},\boldsymbol{\gamma}}(z,\mathbf{g})$ is concave with respect to g_j for $j = 0, 1, 2, \cdots, m$. Thus, we can adopt the first optimality condition for the derivation of the solution. Taking the derivative of the function $H^{r,\ell}_{\boldsymbol{\theta},\boldsymbol{a},\boldsymbol{\gamma}}(z,\mathbf{g})$ with respect to g_j, we derive the first-order conditions:

$$e^{rg_0} - \ell = 0, \quad (3.8)$$

$$(1+\theta_j)e^{a_j g_j} + 2\gamma_j g_j - \ell = 0, \quad (3.9)$$

for $j = 1, 2, \cdots, m$.

Below, we analyze the solutions to the first-order condition system. Firstly, define functions $h^{r,\ell}(g_0) := e^{rg_0} - \ell$ and $h^{\ell}_{\theta_j,a_j,\gamma_j}(g_j) := (1+\theta_j)e^{a_j g_j} + 2\gamma_j g_j - \ell$ for $j = 1, 2, \cdots, m$, then we have the following lemma:

Lemma 3.3. *For $j = 1, 2, \cdots, m$, if $a_j \neq 0$ or $\gamma_j \neq 0$ and ℓ satisfies*

$$(1+\theta_j) < \ell < (1+\theta_j)e^{a_j z} + 2\gamma_j z, \quad (3.10)$$

then there always exists a unique positive root $g^{\ell}_{\theta_j,a_j,\gamma_j} \in (0, z)$ such that

$$h^{\ell}_{\theta_j,a_j,\gamma_j}(g^{\ell}_{\theta_j,a_j,\gamma_j}) = 0. \quad (3.11)$$

Furthermore, the function $g^{\ell}_{\theta_j,a_j,\gamma_j}$ is continuous and strictly increasing with respect to ℓ with the following boundary conditions:

$$g^{\ell}_{\theta_j,a_j,\gamma_j}\Big|_{\ell=1+\theta_j} = 0 \ \ and \ \ g^{\ell}_{\theta_j,a_j,\gamma_j}\Big|_{\ell=(1+\theta_j)e^{a_j z}+2\gamma_j z} = z.$$

Proof. From the definition of (3.9), we can easily find that the function $h^{\ell}_{\theta_j,a_j,\gamma_j}(g_j)$ is strictly increasing with respect to $g_j \in [0, z]$. Furthermore, we have

$$h^{\ell}_{\theta_j,a_j,\gamma_j}(0) = (1+\theta_j) - \ell < 0$$
$$h^{\ell}_{\theta_j,a_j,\gamma_j}(z) = (1+\theta_j)e^{a_j z} + 2\gamma_j z - \ell > 0$$

according to condition (3.10). Thus, the equation $h^{\ell}_{\theta_j,a_j,\gamma_j}(g_j) = 0$ uniquely determines a positive root in $(0, z)$, which is denoted by $g^{\ell}_{\theta_j,a_j,\gamma_j} \in (0, z)$.

In addition, (3.11) implies

$$\ell = (1+\theta_j)e^{a_j g^{\ell}_{\theta_j,a_j,\gamma_j}} + 2\gamma_j g^{\ell}_{\theta_j,a_j,\gamma_j}.$$

Define $\Phi_j(g) := (1+\theta_j)e^{a_j g} + 2\gamma_j g$. The function $\Phi_j(g)$ is continuous and strictly increasing with respect to g, so is the inverse function Φ_j^{-1}. Thus $g^{\ell}_{\theta_j,a_j,\gamma_j} = \Phi_j^{-1}(\ell)$ is continuous and strictly increasing with respect to ℓ. Note that $\Phi_j(0) = 1+\theta_j$ and $\Phi_j(z) = (1+\theta_j)e^{a_j z} + 2\gamma_j z$, then the two boundary conditions follow immediately. □

Now, we define the following function:

$$\hat{g}^{r,\ell}_j(z) := \arg\inf_{0\le g_j\le z} H^{r,\ell}_{\boldsymbol{\theta},\boldsymbol{a},\boldsymbol{\gamma}}(z,\mathbf{g})$$

for $j = 0, 1, 2, \cdots, m$. Then, the minimum point of the Lagrange function (3.7) is given as follows:

$$\hat{g}^{r,\ell}_0(z) = \begin{cases} 0, & 0 < \ell \le 1, \\ \min\{z, \frac{1}{r}\ln(\ell)\}, & \ell > 1. \end{cases} \quad (3.12)$$

For $j = 1, 2, \cdots, m$, if $a_j \neq 0$ or $\gamma_j \neq 0$, then

$$\hat{g}^{r,\ell}_j(z) = \begin{cases} 0, & 0 < \ell \le 1+\theta_j, \\ g^{\ell}_{\theta_j,a_j,\gamma_j}, & 1+\theta_j < \ell < (1+\theta_j)e^{a_j z} + 2\gamma_j z, \\ z, & \ell \ge (1+\theta_j)e^{a_j z} + 2\gamma_j z, \end{cases} \quad (3.13)$$

where $g^{\ell}_{\theta_j,a_j,\gamma_j}$ is obtained in Lemma 3.3, by which the function $\hat{g}^{r,\ell}_j(z)$ is continuous with respect to ℓ. If $a_j = \gamma_j = 0$, then

$$\hat{g}^{r,\ell}_j(z) = \begin{cases} 0, & 0 < \ell < 1+\theta_j, \\ \text{any value in } [0, z], & \ell = 1+\theta_j, \\ z, & \ell > 1+\theta_j. \end{cases} \quad (3.14)$$

Next, we will determine the optimal Lagrange multiplier.

Proposition 3.1. *For any fixed $r > 0$ and $z \in [0, \infty)$, there uniquely exists a bounded root, denoted by $\ell^r(z)$, such that*

$$\sum_{j=0}^{m} \hat{g}^{r,\ell^r(z)}_j(z) = z.$$

More specifically,

- **(I)** *$\ell^r(z) \equiv 1+\theta_1$ for $z \in [0, \frac{1}{r}\ln(1+\theta_1))$;*
- **(II)** *For $z \in [\frac{1}{r}\ln(1+\theta_1), \infty)$, there are following two cases.*
  If $a_j \neq 0$ or $\gamma_j \neq 0$ for all $j = 1, 2, \cdots, m$, we have $\ell^r(z) = \tilde{\ell}^r(z)$ such that

$$1+\theta_1 \le \ell^r(z) < (1+\theta_j)e^{a_j z} + 2\gamma_j z, \ \ for\ any \ \ j = 1, 2, \cdots, m, \quad (3.15)$$

  where $\tilde{\ell}^r(z)$ is determined by $\sum_{j=0}^{m} \hat{g}^{r,\tilde{\ell}^r(z)}_j(z) = z$.
  If $a_j = 0$ and $\gamma_j = 0$ for some $j = i_1, i_2, \cdots, i_n (1 \le i_1 < \cdots < i_n \le m)$, we have

$$\ell^r(z) = \begin{cases} \hat{\ell}^r(z), & z \in \left[\frac{1}{r}\ln(1+\theta_1), \dot{z}_{i_1}\right), \\ 1+\theta_{i_1}, & z \in [\dot{z}_{i_1}, \infty), \end{cases} \quad (3.16)$$

  such that $1+\theta_1 \le \ell^r(z) \le 1+\theta_{i_1}$, where $\hat{\ell}^r(z)$ and $\dot{z}_{i_1}$ are uniquely determined by $\sum_{j=0,}^{i_1-1} \hat{g}^{r,\hat{\ell}^r(z)}_j(z) = z$ and $\hat{\ell}^r(\dot{z}_{i_1}) = 1+\theta_{i_1}$ respectively.

Proof. (I) For any fixed $r > 0$, $z \in [0, \frac{1}{r}\ln(1+\theta_1))$ implies $e^{rz} < 1+\theta_1$. Then we can take $\ell^r(z) \equiv 1+\theta_1$. From (3.12)-(3.14), we have $\hat{g}^{r,\ell^r(z)}_0(z) = z$ and $\hat{g}^{r,\ell^r(z)}_j(z) = 0$ for $j = 1, 2, \cdots, m$, so that $\sum_{j=0}^{m} \hat{g}^{r,\ell^r(z)}_j(z) = z$ is satisfied.

(II) For any fixed $0 < r < \infty$, $z \in [\frac{1}{r}\ln(1+\theta_1), \infty)$ implies $e^{rz} \ge 1+\theta_1$. From (3.12) we know that $\hat{g}^{r,\ell}_0(z) = \frac{1}{r}\ln(\ell)$ for $\ell \in [1+\theta_1, e^{rz}]$, which is continuous and strictly increasing with respect to ℓ. In addition, for $j = 1, 2, \cdots, m$, we consider $\hat{g}^{r,\ell}_j(z)$ in two situations.

(II-1) Firstly, let us consider the case in which $a_j \neq 0$ or $\gamma_j \neq 0$ for all $j = 1, 2, \cdots, m$. According to Lemma 3.3, we know that

$\hat{g}_j^{r,\ell}(z)$ defined by (3.13) is also continuously increasing with respect to ℓ, where $j=1,2,\cdots,m$. Thus, $\sum_{j=0}^m \hat{g}_j^{r,\ell}(z)$ is continuous and strictly increasing with respect to $\ell\in[1+\theta_1,e^{rz}]$. In addition, it follows that

$$\lim_{\ell\to 1+\theta_1}\sum_{j=0}^m \hat{g}_j^{r,\ell}(z)=\sum_{j=0}^m \hat{g}_j^{r,1+\theta_1}(z)=\frac{1}{r}\ln(1+\theta_1)\le z, \tag{3.17}$$

$$\lim_{\ell\to e^{rz}}\sum_{j=0}^m \hat{g}_j^{r,\ell}(z)=z+\lim_{\ell\to e^{rz}}\sum_{j=1}^m \hat{g}_j^{r,\ell}(z)\ge z, \tag{3.18}$$

which implies that there exists a unique root, denoted by $\tilde{\ell}^r(z)\in[1+\theta_1,e^{rz}]$, such that

$$\sum_{j=0}^m \hat{g}_j^{r,\tilde{\ell}^r(z)}(z)=z. \tag{3.19}$$

In addition, we show that the bound (3.15) holds true. If $\tilde{\ell}^r(z)>(1+\theta_i)e^{a_iz}+2\gamma_iz$ for some i, then, from (3.12) and (3.13), it follows that

$$\hat{g}_i^{r,\tilde{\ell}^r(z)}(z)=z \text{ and } \hat{g}_0^{r,\tilde{\ell}^r(z)}(z)=\frac{1}{r}\ln(\tilde{\ell}^r(z))>0,$$

which lead to a contradiction with $\sum_{j=0}^m \hat{g}_j^{r,\tilde{\ell}^r(z)}(z)=z$.

(II-2) Secondly, we consider the case in which there exist $n(1\le n\le m)$ reinsurers such that $a_j=0$ and $\gamma_j=0$ for $j=i_1,i_2,\cdots,i_n$, which implies that the n reinsurers take the expected value premium principle with safety loading $\theta_k,\ k=i_1,i_2,\cdots,i_n$, respectively. Note that

$$\theta_{i_1}\le\theta_{i_2}\le\cdots\le\theta_{i_n}. \tag{3.20}$$

For fixed r and $z\in[\frac{1}{r}\ln(1+\theta_1),\infty)$, define a critical level

$$A_z:=\min\{e^{rz},(1+\theta_j)e^{a_jz}+2\gamma_jz,\ j=1,2,\cdots,i_1-1\}.$$

From (3.12) and (3.13), we know that, for $\ell\in[1+\theta_1,A_z]$, the function $\sum_{j=0}^{i_1-1}\hat{g}_j^{r,\ell}(z)$ can be written as

$$\sum_{j=0}^{i_1-1}\hat{g}_j^{r,\ell}(z)=\frac{1}{r}\ln(\ell)+\sum_{j=1}^{i_1-1}g^{\ell}_{\theta_j,a_j,\gamma_j}I_{\{\ell>1+\theta_j\}}=:\Psi_r(\ell)$$

where $I_{\{.\}}$ denotes the indicator function. Note that the function $\frac{1}{r}\ln(\ell)$ is continuous and strictly increasing with respect to ℓ. And, from Lemma 3.3, we know that each $g^{\ell}_{\theta_j,a_j,\gamma_j}I_{\{\ell>1+\theta_j\}}$ is continuous and increasing with respect to ℓ for $j=1,2,\cdots,i_1-1$. Thus, the function $\Psi_r(\ell)$ given above is continuous and strictly increasing with respect to ℓ. In addition, it follows that

$$\begin{aligned}\lim_{\ell\to 1+\theta_1}\Psi_r(\ell)&=\lim_{\ell\to 1+\theta_1}\sum_{j=0}^{i_1-1}\hat{g}_j^{r,\ell}(z)=\sum_{j=0}^{i_1-1}\hat{g}_j^{r,1+\theta_1}(z)\\&=\frac{1}{r}\ln(1+\theta_1)\le z\end{aligned} \tag{3.21}$$

$$\begin{aligned}\lim_{\ell\to A_z}\Psi_r(\ell)&=\lim_{\ell\to A_z}\sum_{j=0}^{i_1-1}\hat{g}_j^{r,\ell}(z)\\&=\lim_{\ell\to A_z}\frac{1}{r}\ln(\ell)+\lim_{\ell\to A_z}\sum_{j=1}^{i_1-1}\hat{g}_j^{r,\ell}(z)\ge z.\end{aligned} \tag{3.22}$$

Thus, we can conclude that there exists a unique root, denoted by $\hat{\ell}^r(z)\in[1+\theta_1,A_z]$, such that

$$\Psi_r(\hat{\ell}^r(z))=\sum_{j=0}^{i_1-1}\hat{g}_j^{r,\hat{\ell}^r(z)}(z)=z, \tag{3.23}$$

and $\hat{\ell}^r(z)=\Psi_r^{-1}(z)$ is also continuous and strictly increasing with respect to z.

In addition, for $z\to\frac{1}{r}\ln(1+\theta_1)+$, it follows that $A_z\to 1+\theta_1$ and then $\hat{\ell}^r(z)\to 1+\theta_1$. On the other hand, for $z\to\infty$, it follows that $\hat{\ell}^r(z)\to\infty$. Then, there uniquely exists a point, $\dot{z}_{i_1}\in[\frac{1}{r}\ln(1+\theta_1),\infty)$, such that

$$\hat{\ell}^r(\dot{z}_{i_1})=1+\theta_{i_1}. \tag{3.24}$$

Now define $\ell^r(z)$ by (3.16), then it follows that $\ell^r(z)$ is uniquely determined such that $\ell^r(z)\le 1+\theta_{i_1}$.

Furthermore, from (3.13) and (3.14), we have[6]

$$\hat{g}_j^{r,\ell^r(z)}(z)\equiv 0,\ \text{ for }\ j=i_1+1,i_1+2,\cdots,m \tag{3.25}$$

and

$$\sum_{j=0,j\ne i_1}^m \hat{g}_j^{r,\ell^r(z)}(z)=\begin{cases}z, & 0\le z\le\dot{z}_{i_1},\\ \dot{z}_{i_1}, & z\ge\dot{z}_{i_1}.\end{cases}$$

For $j=i_1$, we know from (3.14) that $\hat{g}_{i_1}^{r,1+\theta_{i_1}}(z)$ can choose any value in $[0,z]$. However, in order to satisfy $\sum_{j=0}^m \hat{g}_j^{r,\ell^r(z)}(z)=z$ for all $z\ge 0$, we must have

$$\hat{g}_{i_1}^{r,\ell^r(z)}(z)=\begin{cases}0, & 0\le z\le\dot{z}_{i_1},\\ z-\dot{z}_{i_1}, & z\ge\dot{z}_{i_1}.\end{cases} \tag{3.26}$$

This completes the proof. □

Based on the above analysis, we now can construct a reinsurance arrangement with

$$\hat{\mathbf{g}}^{r,\ell^r(z)}(z)=\left(\hat{g}_0^{r,\ell^r(z)}(z),\cdots,\hat{g}_m^{r,\ell^r(z)}(z)\right). \tag{3.27}$$

Clearly, these expressions satisfy the first-order optimality condition system (3.8) and (3.9). In addition, with respect to variable z, we know that $\ell^r(z)$ is increasing function. And, for $j=0,1,\cdots,m$, $\hat{g}_j^{r,\ell}(z)$ is an increasing function with respect to both ℓ and z. Therefore, we can conclude that $\hat{g}_j^{r,\ell^r(z)}(z)$ is increasing with respect to variable z, so it belongs to the admissible set, that is, $\hat{\mathbf{g}}^{r,\ell^r(z)}(z)\in\mathcal{G}$.

Then, for any $\mathbf{g}=(\mathbf{g_0},\mathbf{g_1},\cdots,\mathbf{g_m})\in\mathcal{G}$, we must have

$$K_{\boldsymbol{\theta},\mathbf{a},\boldsymbol{\gamma}}\left(\hat{\mathbf{g}}^{r,\ell^r(z)}(z),r\right)\le K_{\boldsymbol{\theta},\mathbf{a},\boldsymbol{\gamma}}(\mathbf{g},r). \tag{3.28}$$

By replacing z with a random variable Z and taking the expectation, we have

$$\mathbb{E}[K_{\boldsymbol{\theta},\mathbf{a},\boldsymbol{\gamma}}(\hat{\mathbf{g}}^{r,\ell^r(Z)}(Z),r)]\le\mathbb{E}[K_{\boldsymbol{\theta},\mathbf{a},\boldsymbol{\gamma}}(\mathbf{g}(Z),r)]. \tag{3.29}$$

Therefore, we can conclude from (3.2) that, for any fixed $r>0$, the optimal multiple reinsurance strategy has the same form with $\hat{\mathbf{g}}^{r,\ell^r(z)}(z)$ given by (3.27).

Finally, we need to determine the optimal Lundberg exponent R^* via the following lemma.

[6] Note that $\ell^r(z)\le 1+\theta_{i_1}$. If $\theta_j=\theta_{i_1}$ for $j>i_1$ and $j\ne i_k,k=2,3,\cdots,n$, then we always have $\hat{g}_j^{r,\ell^r(z)}(z)=0$ by (3.13); If $\theta_{i_k}>\theta_{i_1}$ for some $2\le k\le n$, then we also have $\hat{g}_j^{r,\ell^r(z)}(i_k)=0$ by (3.14); In addition, if $\theta_{i_1}=\theta_{i_k}$ for some $2\le k\le n$, which means some reinsurers take the expected value premium principle with the same safety loading θ_{i_1}, then we can arbitrarily divide the given risk $Z-z_{i_1}$ between them without changing reinsurance premium. Specifically, we can set $\hat{g}_{i_k}^{r,\ell^r(z)}(z)=0$. Therefore, for the sake of formal unity, we here can take (3.25).

Lemma 3.4. *(Existence and uniqueness of R^*.) The optimal Lundberg exponent, given as R^*, is uniquely determined by the positive root of the equation with respect to r:*

$$\mathbb{E}[K_{\boldsymbol{\theta},\boldsymbol{a},\boldsymbol{\gamma}}(\hat{\mathbf{g}}^{r,\ell^r(Z)}(Z),r)]=c. \tag{3.30}$$

Proof. First, let us show that $\mathbb{E}\left[K_{\boldsymbol{\theta},\boldsymbol{a},\boldsymbol{\gamma}}(\hat{\mathbf{g}}^{r,\ell^r(Z)}(Z),r)\right]$ is a continuous and strictly increasing function with respect to r. Obviously, $\mathbb{E}\left[K_{\boldsymbol{\theta},\boldsymbol{a},\boldsymbol{\gamma}}(\hat{\mathbf{g}}^{r,\ell^r(Z)}(Z),r)\right]$ is continuous with respect to r. From the definition of $\hat{\mathbf{g}}^{r,\ell^r(z)}(z)$ and (3.28), we know that, for $\mathbf{g}=(\mathbf{g_0},\mathbf{g_1},\cdots,\mathbf{g_m})\in\mathcal{G}$,

$$K_{\boldsymbol{\theta},\boldsymbol{a},\boldsymbol{\gamma}}\left(\hat{\mathbf{g}}^{r,\ell^r(z)}(z),r\right)\le K_{\boldsymbol{\theta},\boldsymbol{a},\boldsymbol{\gamma}}(\mathbf{g},\mathbf{r}).$$

For any $0<r_1<r_2$, with $\mathbf{g}=\hat{\mathbf{g}}^{r_2,\ell^{r_2}(z)}(z)$, we have

$$\begin{aligned}
&\mathbb{E}\left[K_{\boldsymbol{\theta},\boldsymbol{a},\boldsymbol{\gamma}}(\hat{\mathbf{g}}^{r_1,\ell^{r_1}(Z)}(Z),r_1)\right]\le\mathbb{E}\left[K_{\boldsymbol{\theta},\boldsymbol{a},\boldsymbol{\gamma}}(\hat{\mathbf{g}}^{r_2,\ell^{r_2}(Z)}(Z),r_1)\right]\\
&=\sum_{j=1}^m\mathbb{E}\left[\frac{1+\theta_j}{a_j}\left(e^{a_j\hat{g}_j^{r_2,\ell^{r_2}(Z)}(Z)}-1\right)+\gamma_j\left(\hat{g}_j^{r_2,\ell^{r_2}(Z)}(Z)\right)^2\right]\\
&\quad+\frac{1}{r_1}\mathbb{E}\left[e^{r_1\hat{g}_0^{r_2,\ell^{r_2}(Z)}(Z)}-1\right]\\
&<\sum_{j=1}^m\mathbb{E}\left[\frac{1+\theta_j}{a_j}\left(e^{a_j\hat{g}_j^{r_2,\ell^{r_2}(Z)}(Z)}-1\right)+\gamma_j\left(\hat{g}_j^{r_2,\ell^{r_2}(Z)}(Z)\right)^2\right]\\
&\quad+\frac{1}{r_2}\mathbb{E}\left[e^{r_2\hat{g}_0^{r_2,\ell^{r_2}(Z)}(Z)}-1\right]\\
&=\mathbb{E}\left[K_{\boldsymbol{\theta},\boldsymbol{a},\boldsymbol{\gamma}}(\hat{\mathbf{g}}^{r_2,\ell^{r_2}(Z)}(Z),r_2)\right],
\end{aligned}$$

where the inequality above follows by Lemma 3.1. This result implies that $\mathbb{E}[K_{\boldsymbol{\theta},\boldsymbol{a},\boldsymbol{\gamma}}(\hat{\mathbf{g}}^{r,\ell^r(Z)}(Z),r)]$ is a continuous and strictly increasing function with respect to r.

Furthermore, from the proof of Proposition 3.1, we know that

$$\hat{g}_0^{r,\ell^r(z)}(z)=z,\quad \hat{g}_j^{r,\ell^r(z)}(z)=0,\quad z\in[0,\frac{1}{r}\ln(1+\theta_1)),$$

which results in

$$\lim_{r\to0+}\hat{g}_0^{r,\ell^r(z)}(z)=z,\quad \lim_{r\to0+}\hat{g}_j^{r,\ell^r(z)}(z)=0,\quad z\in[0,\infty)$$

for $j=1,2,\cdots,m$. Note that the function $\frac{e^{r\xi}-1}{r}$ is strictly increasing with respect to $r>0$ for any fixed constant $\xi>0$, and there exists a constant $\epsilon>0$ such that $\mathbb{E}[e^{\epsilon Z}]<\infty$. Therefore, we have

$$\begin{aligned}
\lim_{r\to0+}\mathbb{E}[K_{\boldsymbol{\theta},\boldsymbol{a},\boldsymbol{\gamma}}(\hat{\mathbf{g}}^{r,\ell^r(Z)}(Z),r)]&=\mathbb{E}\left[\lim_{r\to0+}\frac{1}{r}(e^{r\hat{g}_0^{r,\ell^r(Z)}(Z)}-1)\right]\\
&=\mathbb{E}[Z]<c,
\end{aligned}$$

in which we exchange the order of expectation and limit according to Lebesgue dominated convergence theorem and the last equality holds by $\hat{g}_0^{r,\ell^r(Z)}(Z)\le\frac{1}{r}(e^{r\hat{g}_0^{r,\ell^r(Z)}(Z)}-1)\le\frac{1}{r}(e^{rZ}-1)$ with $r>0$.[7]

On the other hand, from (3.12), we know that $r\hat{g}_0^{r,\ell^{r(z)}}(z)=\ln(\ell^r(z))$ is bounded when r is large enough, because $\ell^r(z)$ is bounded according to Proposition 3.1. Therefore, we have

$$\lim_{r\to\infty}\hat{g}_0^{r,\ell^r(z)}(z)=\lim_{r\to\infty}\frac{1}{r}\ln(\ell^r(z))=0\quad\text{and}$$
$$\lim_{r\to\infty}\frac{1}{r}\mathbb{E}\left[e^{r\hat{g}_0^{r,\ell^{r(z)}}(z)}-1\right]=0.$$

Denote $g_j^\infty(z):=\lim_{r\to\infty}\hat{g}_j^{r,\ell^r(z)}(z),\ \ j=1,2,\cdots,m$, then we have

$$\sum_{j=1}^m g_j^\infty(z)=\lim_{r\to\infty}\sum_{j=0}^m\hat{g}_j^{r,\ell^r(z)}(z)=z.$$

Thus, by taking $r\to\infty$, it follows

$$\begin{aligned}
&\lim_{r\to\infty}\mathbb{E}[K_{\boldsymbol{\theta},\boldsymbol{a},\boldsymbol{\gamma}}(\hat{\mathbf{g}}^{r,\ell^r(Z)}(Z),r)]\\
&=\lim_{r\to\infty}\left\{\sum_{j=1}^m\left[\frac{1+\theta_j}{a_j}\mathbb{E}\left[e^{a_j\hat{g}_j^{r,\ell^r(Z)}(Z)}-1\right]\right.\right.\\
&\quad\left.\left.+\gamma_j\mathbb{E}\left[\left(\hat{g}_j^{r,\ell^r(Z)}(Z)\right)^2\right]\right]\right\}\\
&=\sum_{j=1}^m\left[\frac{1+\theta_j}{a_j}\mathbb{E}\left[e^{a_jg_j^\infty(Z)}-1\right]+\gamma_j\mathbb{E}\left[\left(g_j^\infty(Z)\right)^2\right]\right]\\
&\ge\min_{\sum_{j=1}^m g_j(Z)=Z}\sum_{j=1}^m\left[\frac{1+\theta_j}{a_j}\left(\mathbb{E}\left[e^{a_jg_j(Z)}\right]-1\right)+\gamma_j\mathbb{E}[g_j^2(Z)]\right]\\
&>c,
\end{aligned}$$

where we exchange the order of expectation and limit according to Lebesgue dominated convergence theorem again and the last step follows from the no-arbitrage assumption (2.9). So far, we can conclude that equation (3.30) determines a unique positive root, denoted by R^*. □

In conclusion, based on the above analysis, we can obtain the main theorem as follows:

Theorem 3.1. *For the optimization problem (3.2), the optimal reinsurance strategies are*

$$g_j^*(z)=\hat{g}_j^{R^*,\ell^{R^*}(z)}(z),\ \ j=0,1,\cdots,m,$$

where $\hat{g}_0^{r,\ell^r(z)}(z)$ is given by (3.12), R^* is determined by (3.30), and the following are satisfied:

1. *If $a_j\neq0$ or $\gamma_j\neq0$ follows for all $j=1,2,\cdots,m$, then $\hat{g}_j^{r,\ell^r(z)}(z)$ and $\ell^r(z)$ are given by (3.13) and Proposition 3.1, respectively.*
2. *If there exist $n(1\le n\le m)$ reinsurance companies such that $a_j=0$ and $\gamma_j=0$ for $j=i_1,i_2,\cdots,i_n$, then*

   $$\hat{g}_k^{r,\ell^r(z)}(z)\equiv0,\quad k=i_1+1,i_1+2,\cdots,m.$$

   For $k=1,2,\cdots,i_1-1$, $\hat{g}_k^{r,\ell^r(z)}(z)$, $\hat{g}_{i_1}^{r,\ell^r(z)}(z)$, and $\ell^r(z)$ are given by (3.13), (3.26), and Proposition 3.1, respectively.

4. Some explicit cases

In the above section, we established the optimal form of multiple reinsurance strategy and obtained the maximized Lundberg exponent for a general setting. These results are exciting, but the optimal forms are very abstract. To clarify our results, we present the optimal reinsurance strategy for some special cases in this section. For example, we give the explicit optimal reinsurance policy for the case in which there exists only one reinsurance company with the general reinsurance premium principle (2.1), the optimal reinsurance policy for two reinsurance companies with general principles but different parameters, and the optimal reinsurance policy for a finite number of reinsurance companies with exponential premium principles. Although these results are for special cases of the general setting, this is the first report for some of these results.

[7] The left inequality follows due to $e^{rx}\ge1+rx$.

4.1. The case of $m=1$

In this subsection, we consider that there exists only one reinsurer, i.e., $m=1$. First, we have the following general proposition.

Proposition 4.1. *(General premium principle) If the reinsurer takes the general reinsurance premium principle given by (2.1) with $a_1 \neq 0$ or $\gamma_1 \neq 0$, then the optimal reinsurance strategy is*

$$g_0^*(Z) = \begin{cases} Z, & 0 \le Z \le \frac{\ln(1+\theta_1)}{R^*} \\ \tilde{g}_0^*(Z), & Z \ge \frac{\ln(1+\theta_1)}{R^*} \end{cases} \tag{4.1}$$

and $g_1^*(Z) = Z - g_0^*(Z)$, where $\tilde{g}_0^*(z)$ satisfies the following equation:

$$(1+\theta_1)e^{a_1(z-\tilde{g}_0^*(z))} + 2\gamma_1(z-\tilde{g}_0^*(z)) - e^{R^*\tilde{g}_0^*(z)} = 0, \tag{4.2}$$

and R^* is the solution to

$$\frac{1+\theta_1}{a_1}\mathbb{E}\left[e^{a_1(Z-\tilde{g}_0^*(Z))}-1\right] + \gamma_1\mathbb{E}\left[Z-\tilde{g}_0^*(Z)\right]^2 + \frac{1}{R^*}\mathbb{E}\left[e^{R^*\tilde{g}_0^*(Z)}-1\right] = c. \tag{4.3}$$

Proof. Recall (3.14) and (3.19). For $z \le \frac{\ln(1+\theta_1)}{R^*}$, the optimal multiplier $\ell^{R^*}(z) = 1+\theta_1$ and

$$g_0^*(z) = \hat{g}_0^{R^*,1+\theta_1}(z) = \min\left\{z, \frac{\ln(1+\theta_1)}{R^*}\right\} = z.$$

In contrast, when $z \ge \frac{\ln(1+\theta_1)}{R^*}$, we know from (3.8) and (3.9) that the functions $g_0^*(z)$ and $g_1^*(z)$ satisfy

$$e^{R^* g_0^*(z)} - \ell^{R^*}(z) = 0$$

$$(1+\theta_1)e^{a_1 g_1^*(z)} + 2\gamma_1 g_1^*(z) - \ell^{R^*}(z) = 0,$$

respectively. Note that $g_1^*(z) = z - g_0^*(z)$. Consequently, we find that $g_0^*(z)$ is the solution of the following equation:

$$(1+\theta_1)e^{a_1(z-x)} + 2\gamma_1(z-x) - e^{R^*x} = 0. \tag{4.4}$$

The left-hand side of the above equation is a strictly decreasing function with respect to x. The value is positive for $x=0$ and is negative for $x=z$ because $z \ge \frac{\ln(1+\theta_1)}{R^*}$. Therefore, equation (4.2) or (4.4) uniquely determines $g_0^*(z) \in (0,z)$. This ends the proof. □

The following corollary presents the optimal reinsurance policy for a reinsurance company with the mean-variance premium principle, including the expected value principle and variance principle as special cases.

Corollary 4.1. *(Mean-variance premium principle) If the reinsurer takes the mean-variance premium principle, i.e., $\pi^{\theta_1,0,\gamma_1}(Y) = (1+\theta_1)\mathbb{E}[Y] + \gamma_1\mathbb{V}[Y]$ ($a \to 0$, $\theta = \theta_1$ and $\gamma = \gamma_1$ in (2.1)), then the optimal reinsurance strategy is given by*

$$g_0^*(Z) = \min\{Z, f_{\theta_1}^{-1}(Z)\} = \begin{cases} Z, & 0 \le Z \le \frac{\ln(1+\theta_1)}{R^*} \\ f_{\theta_1}^{-1}(Z), & Z \ge \frac{\ln(1+\theta_1)}{R^*} \end{cases} \tag{4.5}$$

and $g_1^*(Z) = Z - g_0^*(Z) = \max\{0, Z - f_{\theta_1}^{-1}(Z)\}$, where

$$f_{\theta_1}(y) = \frac{1}{2\gamma_1}\left(e^{R^*y} - (1+\theta_1)\right) + y. \tag{4.6}$$

More specifically, if the expected value premium principle is taken (i.e., $\gamma_1 = 0$), then

$$g_0^*(Z) = \min\left\{Z, \frac{\ln(1+\theta_1)}{R^*}\right\},$$

which implies a stop-loss reinsurance. If the variance premium principle is taken (i.e., $\theta_1 = 0$), then the optimal reinsurance policy reduces to

$$g_0^*(Z) = f_0^{-1}(Z), \tag{4.7}$$

which has a non-piecewise linear structure.

Proof. Because these results can be directly obtained from Proposition 4.1, we omit the proof here. □

Corollary 4.2. *(Exponential premium principle) If the reinsurer takes the exponential premium principle (i.e., $\theta_1 = 0$ and $\gamma_1 = 0$), then the optimal reinsurance policy reduces to*

$$g_0^*(Z) = \frac{a_1}{a_1 + R^*}Z,$$

which implies a quota-share reinsurance.

Proof. This result can also be directly obtained from Proposition 4.1, and thus, we omit the proof here. □

Remark 4.1. From the above two corollaries, we observe the following. By maximizing the Lundberg exponent, we obtain an optimal reinsurance policy that corresponds to an excess-of-loss reinsurance under the expected value premium principle, which is consistent with the findings of Gerber (1979) and many current studies, even those with different value functions (see Asmussen et al. (2000), Meng and Zhang (2010), Hipp and Taksar (2010), Zhou and Cai (2014), and Liang and Young (2018)). In addition, the quota-share reinsurance is shown to be the optimal form under the exponential premium principle, which is a novel result. However, for the general combined premium principle $\pi^{\theta,a,\gamma}(Y)$ in (2.1) and even a simple variance premium principle, the optimal reinsurance policy has a non-piecewise linear structure (see (4.1), (4.5), and (4.7)), which strongly differs from some results reported in the literature. For example, Hipp and Taksar (2010), Meng et al. (2016b), and Zhang et al. (2016) showed that quota-share reinsurance is the optimal form under the variance premium principle.

4.2. The case of $m=2$

In this subsection, we assume that there exist two reinsurance companies to share the ceded risks from the insurer, i.e., $m=2$. Recall the assumption $\theta_1 \le \theta_2$. Here, we explore how the optimal reinsurance strategy changes with the different premium principles of the two reinsurance companies. Recall the definition of $\dot{z}_j \in [\frac{1}{r}\ln(1+\theta_1), \infty)$ in (3.24), that is, $\ell^{R^*}(\dot{z}_j) = 1+\theta_j$. First, we give the result for a general case.

Proposition 4.2. *(Two general premium principles) We assume that $a_j \neq 0$ or $\gamma_j \neq 0$ for $j = 1,2$. Then, the optimal reinsurance strategy is*

$$g_0^*(Z) = \begin{cases} Z, & 0 \le Z \le \frac{\ln(1+\theta_1)}{R^*}, \\ \tilde{g}_0^*(Z), & \frac{\ln(1+\theta_1)}{R^*} \le Z \le \dot{z}_2, \\ \bar{g}_0^*(Z), & Z \ge \dot{z}_2, \end{cases} \tag{4.8}$$

$$g_1^*(Z) = \begin{cases} 0, & 0 \le Z \le \frac{\ln(1+\theta_1)}{R^*}, \\ Z - \tilde{g}_0^*(Z), & \frac{\ln(1+\theta_1)}{R^*} \le Z \le \dot{z}_2, \\ \bar{g}_1^*(Z), & Z \ge \dot{z}_2, \end{cases} \tag{4.9}$$

and

$$g_2^*(Z) = \begin{cases} 0, & 0 \le Z \le \dot{z}_2, \\ Z - \bar{g}_0^*(Z) - \bar{g}_1^*(Z), & Z \ge \dot{z}_2, \end{cases} \tag{4.10}$$

where the function $\tilde{g}_0^*(z)$ satisfies the following equations:

$$(1+\theta_1)e^{a_1(z-\tilde{g}_0^*(z))}+2\gamma_1(z-\tilde{g}_0^*(z))-e^{R^*\tilde{g}_0^*(z)}=0 \tag{4.11}$$

and $\bar{g}_0^*(z)$ *and* $\bar{g}_1^*(z)$ *satisfy the following equations:*

$$(1+\theta_1)e^{a_1\bar{g}_1^*(z)}+2\gamma_1\bar{g}_1^*(z)-e^{R^*\bar{g}_0^*(z)}=0, \tag{4.12}$$

$$(1+\theta_2)e^{a_2(z-\bar{g}_0^*(z)-\bar{g}_1^*(z))}+2\gamma_2(z-\bar{g}_0^*(z)-\bar{g}_1^*(z))-e^{R^*\bar{g}_0^*(z)}=0. \tag{4.13}$$

Proof. Noting the assumption $a_j\neq 0$ or $\gamma_j\neq 0$ for $j=1,2$, the result corresponds to the first case in Theorem 3.1. Let us recall (3.14) and (3.19). For $z\le \frac{\ln(1+\theta_1)}{R^*}$, we know that $\ell^{R^*}(z)=1+\theta_1$, and then

$$g_0^*(z)=\hat{g}_0^{R^*,\ell^{R^*}(z)}(z)=\hat{g}_0^{R^*,1+\theta_1}(z)=\min\left\{z,\frac{1}{R^*}\ln(1+\theta_1)\right\}$$
$$=z.$$

For $z>\frac{\ln(1+\theta_1)}{R^*}$, we know, from (3.19), that $\ell^{R^*}(z)=\tilde{\ell}^{R^*}(z)>1+\theta_1$ and $g_0^*(z)=\hat{g}_0^{R^*,\ell^{R^*}(z)}(z)=\frac{1}{R^*}\ln(\ell^{R^*}(z))$ or, equivalently,

$$e^{R^*g_0^*(z)}=\ell^{R^*}(z).$$

From the definition of $\dot{z}_2$ and noting that $\tilde{\ell}^{R^*}(z)$ is continuous and strictly increasing with respect to z, we have $\dot{z}_2\ge\frac{\ln(1+\theta_1)}{R^*}$. Thus, for $\frac{\ln(1+\theta_1)}{R^*}\le z\le\dot{z}_2$, we have $1+\theta_1\le\ell^{R^*}(z)\le 1+\theta_2$, and then

$$g_2^*(z)=\hat{g}_2^{R^*,\ell^{R^*}(z)}(z)=0,$$
$$g_1^*(z)=\hat{g}_1^{R^*,\ell^{R^*}(z)}(z)=z-g_0^*(z).$$

At the same time, $g_1^*(z)$ satisfies

$$(1+\theta_1)e^{a_1(g_1^*(z))}+2\gamma_1(g_1^*(z))-\ell^{R^*}(z)=0,$$

and we then obtain (4.11).

Furthermore, if $z>\dot{z}_2$, we have $\ell^{R^*}(z)>1+\theta_2$, and then $g_2^*(z)=z-g_0^*(z)-g_1^*(z)$. At the same time, we know that $g_0^*(z)$, $g_1^*(z)$, and $g_2^*(z)$ satisfy

$$e^{R^*g_0^*(z)}-\ell^{R^*}(z)=0,$$
$$(1+\theta_1)e^{a_1g_1^*(z)}+2\gamma_1g_1^*(z)-\ell^{R^*}(z)=0,$$
$$(1+\theta_2)e^{a_2g_2^*(z)}+2\gamma_2g_2^*(z)-\ell^{R^*}(z)=0.$$

Thus, we obtain (4.12) and (4.13). □

Roughly speaking, Proposition 4.2 indicates that the insurer would like to retain all of the small part of the loss ($0\le Z\le\frac{\ln(1+\theta_1)}{R^*}$), share a large part of the loss ($Z\ge\dot{z}_2$) with the two reinsurance companies, and share a medium part of the loss ($\frac{\ln(1+\theta_1)}{R^*}\le Z\le\dot{z}_2$) with only the first reinsurance company, because the first reinsurance company has a lower loading on the expected value of loss than the second company ($\theta_1\le\theta_2$).

Now consider $\theta_1=\theta_2$. Note that

$$\tilde{g}_0^*\left(\frac{\ln(1+\theta_1)}{R^*}\right)=\frac{\ln(1+\theta_1)}{R^*}.$$

Thus, $\dot{z}_2=\frac{\ln(1+\theta_1)}{R^*}$ when $\theta_1=\theta_2$. From Proposition 4.2, we can see that the insurer keeps all of the small part of the loss ($Z\le\dot{z}_2$) and shares the remaining loss ($Z\ge\dot{z}_2$) with both reinsurance companies. Furthermore, if $a_1=a_2$ and $\gamma_1=\gamma_2$, then the corresponding parameters of equation (4.12) and equation (4.13) are consistent. Thus, $\bar{g}_1^*(z)=z-\bar{g}_0^*(z)-\bar{g}_1^*(z)$, which results in $g_1^*(z)=g_2^*(z)$ with $\dot{z}_2=\frac{\ln(1+\theta_1)}{R^*}$. Hence, we have

$$g_1^*(Z)=g_2^*(Z)=\frac{1}{2}(Z-g_0^*(Z)).$$

Now, let us consider the case in which one of the two reinsurers takes the expected value premium principle.

Proposition 4.3. *(One expected value principle) Here, we assume that one reinsurer takes the expected value premium principle while the other takes the general form. First, we assume that* $a_2=0$ *and* $\gamma_2=0$*, indicating that the second reinsurer takes the expected value principle with safety loading* $\theta_2(\theta_2>\theta_1)$*. Then, the optimal reinsurance strategy is*

$$g_0^*(Z)=\begin{cases} Z, & 0\le Z\le\frac{\ln(1+\theta_1)}{R^*},\\ \tilde{g}_0^*(Z), & \frac{\ln(1+\theta_1)}{R^*}\le Z\le\dot{z}_2,\\ \frac{\ln(1+\theta_2)}{R^*}, & Z\ge\dot{z}_2,\end{cases} \tag{4.14}$$

$$g_1^*(Z)=\begin{cases} 0, & 0\le Z\le\frac{\ln(1+\theta_1)}{R^*},\\ Z-\tilde{g}_0^*(Z), & \frac{\ln(1+\theta_1)}{R^*}\le Z\le\dot{z}_2,\\ \dot{z}_2-\frac{\ln(1+\theta_2)}{R^*}, & Z\ge\dot{z}_2,\end{cases} \tag{4.15}$$

and

$$g_2^*(Z)=\begin{cases} 0, & 0\le Z\le\dot{z}_2,\\ Z-\dot{z}_2, & Z\ge\dot{z}_2,\end{cases} \tag{4.16}$$

where the function $\tilde{g}_0^*(z)$ *is still given by (4.11). Second, if* $a_1=0$ *and* $\gamma_1=0$*, indicating that the first reinsurer takes the expected value principle with safety loading* $\theta_1>0$*, then the optimal reinsurance strategy is*

$$g_0^*(Z)=\min\left\{Z,\frac{\ln(1+\theta_1)}{R^*}\right\},\quad g_1^*(Z)=\left(Z-\frac{\ln(1+\theta_1)}{R^*}\right)_+$$

and $g_2^*(Z)=0$.

Proof. Noting that $a_j=0$ and $\gamma_j=0$ for $j=1,2$, this result corresponds to the second case of Theorem 3.1. If $a_2=0$ and $\gamma_2=0$, then the result implies that $i_1=2$. For $0<z<\dot{z}_2$, the proof is similar to that for Proposition (4.2); thus, we omit this part of the proof here. For $z\ge\dot{z}_2$, we have from (3.23) and (3.26) that $\ell^{R^*}(z)\equiv 1+\theta_2$ and (4.16) follows.

If $a_1=0$ and $\gamma_1=0$, then $i_1=1$ from (3.20). According to (3.23), we have $\ell^{R^*}(z)\equiv 1+\theta_1$ for all $z\ge 0$. Then, from (3.14) and (3.26), we can obtain

$$g_0^*(Z)=\min\left\{Z,\frac{\ln(1+\theta_1)}{R^*}\right\},\quad g_1^*(Z)=\left(Z-\frac{\ln(1+\theta_1)}{R^*}\right)_+$$

and $g_2^*(Z)=0$. □

Below, we consider some additional special cases.

Corollary 4.3. *(Exponential premium principle + Expected value principle) Consider the case in which the first reinsurance company takes the exponential premium principle* ($\theta_1=\gamma_1=0$) *and the second company takes the expected value principle* ($a_2=\gamma_2=0$)*. The optimal reinsurance strategy is*

$$g_0^*(Z)=\begin{cases}\frac{a_1}{a_1+R^*}Z, & 0\le Z\le\left(\frac{1}{a_1}+\frac{1}{R^*}\right)\ln(1+\theta_2),\\ \frac{\ln(1+\theta_2)}{R^*}, & Z\ge\left(\frac{1}{a_1}+\frac{1}{R^*}\right)\ln(1+\theta_2),\end{cases} \tag{4.17}$$

$$g_1^*(Z)=\begin{cases}\frac{R^*}{a_1+R^*}Z, & 0\le Z\le\left(\frac{1}{a_1}+\frac{1}{R^*}\right)\ln(1+\theta_2),\\ \frac{\ln(1+\theta_2)}{a_1}, & Z\ge\left(\frac{1}{a_1}+\frac{1}{R^*}\right)\ln(1+\theta_2),\end{cases} \tag{4.18}$$

and

$$g_2^*(Z)=\begin{cases}0, \quad 0\le Z\le\left(\frac{1}{a_1}+\frac{1}{R^*}\right)\ln(1+\theta_2),\\ Z-\left(\frac{1}{a_1}+\frac{1}{R^*}\right)\ln(1+\theta_2),\\ \qquad Z\ge\left(\frac{1}{a_1}+\frac{1}{R^*}\right)\ln(1+\theta_2),\end{cases} \tag{4.19}$$

which implies that the optimal reinsurance arrangement for the insurer is excess-of-loss after quota-share reinsurance.

Proof. The results can be directly obtained from Proposition 4.3; thus, we omit the proof here. □

Corollary 4.4. *(Variance premium principle + Variance premium principle) Consider the case in which both reinsurance companies take the variance premium principle ($a_j=0$ and $\theta_j=0$ for $j=1,2$). Then, the optimal reinsurance strategy is*

$$\begin{aligned}&g_0^*(Z)=h_0^{-1}(Z), \quad g_1^*(Z)=h_1^{-1}(Z),\\ &g_2^*(Z)=Z-h_0^{-1}(Z)-h_1^{-1}(Z),\end{aligned} \tag{4.20}$$

where

$$h_0(y)=\left(\frac{1}{2\gamma_1}+\frac{1}{2\gamma_2}\right)(e^{R^*y}-1)+y, \tag{4.21}$$

$$h_1(y)=(1+\gamma_1/\gamma_2)y+\frac{\ln(1+2\gamma_1 y)}{R^*}. \tag{4.22}$$

Specifically, if $\gamma_1=\gamma_2$, then

$$g_1^*(Z)=g_2^*(Z)=\frac{1}{2}(Z-h_0^{-1}(Z)).$$

Proof. Because $\theta_1=\theta_2=0$, we know that $\frac{\ln(1+\theta_1)}{R^*}=\dot z_2=0$. According to Proposition 4.2, we know that $\bar g_1^*(z)$ and $\bar g_2^*(z)$ satisfy the following equations:

$$1+2\gamma_1\bar g_1^*(z)-e^{R^*\bar g_0^*(z)}=0, \tag{4.23}$$

$$1+2\gamma_2(z-\bar g_0^*(z)-\bar g_1^*(z))-e^{R^*\bar g_0^*(z)}=0. \tag{4.24}$$

From direct calculations, we find that

$$h_0(\bar g_0^*(z))=z,\ h_1(\bar g_1^*(z))=z, \tag{4.25}$$

where the two functions $h_0(y)$ and $h_1(y)$ are given by (4.21) and (4.22), respectively. Noting the monotonic property of the two functions $h_0(y)$ and $h_1(y)$, we have

$$g_0^*(z)=\bar g_0^*(z)=h_0^{-1}(z) \text{ and } g_1^*(z)=\bar g_1^*(z)=h_1^{-1}(z). \tag{4.26}$$

Specifically, if $\gamma_1=\gamma_2$, we can easily determine from (4.23) and (4.24) that

$$\bar g_1^*(z)=z-\bar g_0^*(z)-\bar g_1^*(z)=\bar g_2^*(z).$$

Thus, the proof is completed. □

For the case in which both reinsurers take the exponential premium principle, we will give a more general result for any m reinsurers in the next subsection (see Proposition 4.4 below). Of course, there still remain other interesting premium combinations for two reinsurers, such as the combination of the exponential principle and variance principle. With similar arguments, the corresponding results can be directly obtained from Proposition 4.2 or Proposition 4.3; thus, we do not present a further analysis here.

4.3. The general case of m

In this subsection, we consider the general case in which there exist any number m of reinsurance companies. From Theorem 3.1, we know that the optimal reinsurance strategy has a complex structure when all reinsurance companies apply the general premium principle. Here, we consider some special cases in which the optimal reinsurance has a simple structure.

First, let us assume that all of the reinsurance companies take the exponential premium principle.

Proposition 4.4. *(Exponential premium principle) Consider the case in which all reinsurers take the exponential premium principle with different parameters ($\theta_j=0$ and $\gamma_j=0$ for all $j=1,2,\cdots,m$). Then, the optimal reinsurance strategy is*

$$\begin{aligned}&g_0^*(Z)=\frac{\frac{1}{R^*}}{\frac{1}{R^*}+\sum_{j=1}^m\frac{1}{a_j}}Z \quad \text{and}\\ &g_j^*(Z)=\frac{\frac{1}{a_j}}{\frac{1}{R^*}+\sum_{j=1}^m\frac{1}{a_j}}Z, \quad j=1,\cdots,m,\end{aligned} \tag{4.27}$$

which suggests that the risk is proportionally shared among the insurer and m reinsurance companies.

Proof. According to (3.9) and (3.8), we know that $g_0^*(z)$ and $g_j^*(z)$ ($j=1,2,\cdots,m$) satisfy

$$\begin{aligned}&e^{R^*g_0^*(z)}-\ell^{R^*}(z)=0,\\ &e^{a_jg_j^*(z)}-\ell^{R^*}(z)=0.\end{aligned}$$

With some direct calculations, we have

$$\begin{aligned}&g_0^*(z)=\frac{1}{R^*}\ln(\ell^{R^*}(z)) \quad \text{and}\\ &g_j^*(z)=\frac{1}{a_j}\ln(\ell^{R^*}(z)), \quad j=1,2,\cdots,m.\end{aligned}$$

Moreover, according to

$$\sum_{j=0}^m g_j^*(z)=\frac{1}{R^*}\ln(\ell^{R^*}(z))+\sum_{j=1}^m\frac{1}{a_j}\ln(\ell^{R^*}(z))=z,$$

it follows that

$$\ln(\ell^{R^*}(z))=\frac{1}{\frac{1}{R^*}+\sum_{j=1}^m\frac{1}{a_j}}z.$$

Then, (4.27) follows, and the proof is completed. □

In addition, we evaluate the case in which the first reinsurance company takes the expected value principle with the minimum loading factor, and m reinsurers take the same premium principle.

Corollary 4.5. *Consider that the first reinsurer takes the expected value premium principle ($a_1=0$ and $\gamma_1=0$). Then, the optimal reinsurance strategy is*

$$\begin{aligned}&g_0^*(Z)=\min\left\{Z,\frac{\ln(1+\theta_1)}{R^*}\right\},\ g_1^*(Z)=\left(Z-\frac{\ln(1+\theta_1)}{R^*}\right)_+,\\ &g_j^*(Z)=0,\ j=2,3,\cdots,m.\end{aligned} \tag{4.28}$$

In other words, the insurer will only cede risk to the first reinsurance company in terms of stop-loss reinsurance.

Proof. Recall our assumption that $\theta_1 \le \theta_2 \le \cdots \le \theta_m$. Hence, this result can be immediately obtained from the second case of Theorem 3.1 by noting that $i_1 = 1$ and $\dot{z}_1 = \frac{\ln(1+\theta_1)}{R^*}$. □

Corollary 4.6. *Let us assume that all m reinsurers take the same premium principle, i.e., $\theta_j = \theta$, $\gamma_j = \gamma$, and $a_j = a$ for $j = 1, 2, \cdots, m$, and $\gamma \neq 0$ or $a \neq 0$ follows. Then, the optimal reinsurance strategy is*

$$g_0^*(Z) = \begin{cases} Z, & 0 \le Z \le \frac{\ln(1+\theta)}{R^*} \\ \tilde{g}_0^*(Z), & Z \ge \frac{\ln(1+\theta)}{R^*} \end{cases} \tag{4.29}$$

and

$$g_j^*(Z) = \begin{cases} 0, & 0 \le Z \le \frac{\ln(1+\theta)}{R^*}, \\ \frac{1}{m}(Z - \tilde{g}_0^*(Z)), & Z \ge \frac{\ln(1+\theta)}{R^*}, \end{cases} \tag{4.30}$$

where $\tilde{g}_0^*(z)$ satisfies the following equation:

$$(1+\theta)e^{\frac{a}{m}(z-\tilde{g}_0^*(z))} + 2\frac{\gamma}{m}(z - \tilde{g}_0^*(z)) - e^{R^*\tilde{g}_0^*(z)} = 0. \tag{4.31}$$

Proof. These results can be directly obtained from the first case of Theorem 3.1, and thus, we omit the proof here. □

Finally, let us consider the case of $m \to \infty$ when all reinsurance companies take the same principle. Recall the no-arbitrage assumption (2.9). Noting that $A_m(Z)$ is a decreasing function, we define

$$A(Z) = \lim_{m\to\infty} A_m(Z). \tag{4.32}$$

We can show that it is possible for $c > A(Z)$ in some cases. For example, if all reinsurers take the same premium principle with $\theta_j = 0$, $\gamma_j = \gamma$, and $a_j = a$ for $j = 1, 2, \cdots, m$, we have, from (2.9),

$$\frac{1}{a}\left(\mathbb{E}\left[e^{aZ}\right] - 1\right) + \gamma\mathbb{E}[Z^2] > c. \tag{4.33}$$

By taking $g_j(Z) = \frac{1}{m}Z$, $j = 1, 2, \cdots, m$, it follows that

$$A_m(Z) \le m\left[\frac{1}{a}\left(\mathbb{E}\left[e^{a\frac{Z}{m}}\right] - 1\right) + \frac{1}{m^2}\gamma\mathbb{E}[Z^2]\right]. \tag{4.34}$$

As $m \to \infty$, we obtain $A(Z) \le \mathbb{E}[Z] < c$. According to this observation, we have the following corollary.

Corollary 4.7. *Assume that all reinsurers take the same premium principle and*

$$c > (1+\theta)\mathbb{E}[Z].$$

Then, there uniquely exists $\bar{m} > 1$ such that

$$(1+\theta)\frac{\bar{m}}{a}\left(\mathbb{E}\left[e^{a\frac{Z}{\bar{m}}}\right] - 1\right) + \frac{1}{\bar{m}}\gamma\mathbb{E}[Z^2] = c \tag{4.35}$$

and

$$(1+\theta)\frac{m}{a}\left(\mathbb{E}\left[e^{a\frac{Z}{m}}\right] - 1\right) + \frac{1}{m}\gamma\mathbb{E}[Z^2] < c, \quad m > \bar{m}. \tag{4.36}$$

In other words, if there exist $m (m \ge \ulcorner \bar{m} \urcorner)$ participants in the reinsurance market, then the insurer can equally cede all of the risk Z to the m reinsurers at a cost lower than the received premium rate c. Therefore, the optimal Lundberg exponent $R^* = \infty$ and $V(x) = 0$ for all $x \ge 0$.

Proof. Note that, for any fixed $z > 0$, the function $\frac{e^{az}-1}{a}$ is strictly increasing with respect to a for $a > 0$. Thus, the function $(1+\theta)\frac{m}{a}\left(\mathbb{E}\left[e^{a\frac{Z}{m}}\right] - 1\right) + \frac{1}{m}\gamma\mathbb{E}[Z^2]$ is strictly decreasing with respect to m such that

$$\lim_{m\to\infty}(1+\theta)\frac{m}{a}\left(\mathbb{E}\left[e^{a\frac{Z}{m}}\right] - 1\right) + \frac{1}{m}\gamma\mathbb{E}[Z^2] = (1+\theta)\mathbb{E}[Z] < c,$$

and at $m = 1$, the function equals $\frac{1+\theta}{a}\left(\mathbb{E}\left[e^{aZ}\right] - 1\right) + \gamma\mathbb{E}[Z^2] > c$. Therefore, we can uniquely determine $\bar{m}$ such that (4.35) and (4.36). □

Remark 4.1. Let us suppose that m reinsurance companies enter into a reinsurance treaty with an insurance company. Similar to Proposition 4.2 and Proposition 4.3 for $m = 2$, the optimal reinsurance strategies also maintain a multi-layer nonlinear structure. Specifically, the optimal reinsurance policies have at most $m + 1$ layers. The first layered point is $\frac{\ln(1+\theta_1)}{R^*}$, and the subsequent layered points are $\dot{z}_k$, where $\dot{z}_k$ can be determined by

$$\tilde{\ell}^{R^*}(\dot{z}_k) = 1 + \theta_k, \; k = 2, 3, \cdots, m, \quad \textit{or}$$
$$\hat{\ell}^{R^*}(\dot{z}_h) = 1 + \theta_h, \; h = 2, \cdots, i_1.$$

$\tilde{\ell}^{R^*}(z)$ and $\hat{\ell}^{R^*}(z)$ are defined by (3.19) and (3.24), respectively.

5. Numerical analysis

In the previous section, we derived the optimal multiple reinsurance strategy and optimal Lundberg exponent for a general case and presented results for some special cases. In this section, we perform a sensitivity analysis for the optimal reinsurance strategy and the optimal Lundberg exponent.

For this purpose, we first assume that the claim Z_i follows a *gamma*(α, β) distribution with the following probability density function:

$$f(x, \alpha, \beta) = \frac{\beta^\alpha x^{\alpha-1}e^{-\beta x}}{\Gamma(\alpha)}, \; x \ge 0,$$

where $\alpha > 0$ is the shape parameter, $\beta > 0$ is the inverse scale parameter, and Γ is the gamma function with the following formula:

$$\Gamma(\alpha) = \int_0^\infty t^{\alpha-1}e^{-t}dt.$$

Then, we determine the parameter setting for the base scenario: $\alpha = 3$ and $\beta = \frac{1}{2}$ in the density function, and the mean of each claim is $\frac{\alpha}{\beta} = 6$. The premium rate $c = 9$. We assume that there exist two reinsurance companies ($m = 2$) and that they take variance premium principles with different safety loading parameters $\gamma_1 = 0.5$, $\gamma_2 = 0.3$, and $\theta_1 = \theta_2 = a_1 = a_2 = 0$, that is,

$$\pi^{0,0,\gamma_i}(Y) = \mathbb{E}[Y] + \gamma_i\mathbb{V}[Y], \; i = 1, 2.$$

Now, we first conduct a sensitivity analysis for the premium rate c and safety loadings of the reinsurance premium γ_1 and γ_2 on the optimal Lundberg exponent. When we change one parameter, the remaining parameters are assumed to be unchanged in the base scenario. Recalling Theorem 3.1, we know that the optimal Lundberg exponent R^* is determined by (3.30). In this case, the Lundberg exponent satisfies

$$\sum_{j=1}^{2}\left(\mathbb{E}[g_j^*(Z)] + \gamma_j\mathbb{E}[(g_j^*(Z))^2]\right) + \frac{1}{R^*}[\mathbb{E}[e^{R^*g_0(Z)}] - 1] - c = 0, \tag{5.1}$$

Table 1
Impact of the premium rate c on the Lundberg exponent R^*.

c	6.5	7	7.5	8	8.5	9	9.5	10
R^*	0.0189	0.0386	0.0597	0.0816	0.1051	0.1315	0.1609	0.1948

Table 2
Impact of the safety loading γ_2 on the Lundberg exponent R^*.

γ_2	0.3	0.4	0.5	0.6	0.7	0.8
R^*	0.1315	0.1219	0.1168	0.1137	0.1114	0.1099

where $g_j^*(Z)$, $j = 0, 1, 2$ are given by Corollary 4.4. Note that each $g_j^*(Z)$ is also related to R^*; thus, it is very difficult to solve for R^* from (5.1) explicitly. To give a numerical solution to R^*, we employ *Monte Carlo simulations* to calculate the expectations in (5.1). And then R^* can be obtained in terms of one iterative algorithm, for example the *Bisection method*[8]:

Step 1 Simulate scenarios of the claim Z.
Step 2 Give initial values: $R_{down} = 0$ and a large enough R_{up} such that the left side of (5.1) is positive for $R^* = R_{up}$.
Step 3 Set $R^* = \frac{R_{down}+R_{up}}{2}$.
Step 4 Solve $g_0(Z)$, $g_1(Z)$, and $g_2(Z)$ according to Corollary 4.4.
Step 5 If (5.1) follows (within an acceptable error, for example 0.01%), then we obtain R^* and stop here.
Step 6 Otherwise, if the left side of equation (5.1) is positive, we set $R_{up} = R^*$ and then go back to Step 3; if the left side of equation (5.1) is negative, we set $R_{down} = R^*$ and go back to Step 3.

The impacts of the premium rate c and safety loading γ_2 on the optimal Lundberg exponent R^* are displayed in Tables 1 and 2, respectively. As shown in these tables, the optimal Lundberg exponent R^* increases with respect to the premium rate c, indicating that the upper bound of the ruin probability of the insurer decreases with increasing premium rate. In contrast, the optimal Lundberg exponent R^* decreases with respect to the safety loading of the reinsurance γ_2, implying that the upper bound of the ruin probability of the insurer increases with increasing safety loading. For a given risk, as the insurer charges a higher premium or as the reinsurance becomes cheaper, the ruin probability of the insurer decreases. These findings are fairly consistent with our intuition.

Once R^* is obtained, the optimal reinsurance arrangements $g_j^*(Z)$, $j = 0, 1, 2$ are then clear according to Corollary 4.4. Below, we proceed to analyze the impacts of the premium rate and safety loading of reinsurance on the optimal reinsurance strategy. Fig. 1 displays the optimal reinsurance arrangement (retained risk $g_0^*(z)$ and two ceded risks $g_1^*(z)$ and $g_2^*(z)$) for the base scenario. The results show that all three functions $g_0^*(z)$, $g_1^*(z)$, and $g_2^*(z)$ increase with respect to the claim amount z. As the total loss increases, the losses taken by the insurer and the reinsurers also increase. In addition, all of the optimal risk functions $g_i^*(z), i = 0, 1, 2$, are general curves, which differ from some common optimal reinsurance polices, such as quota-share, excess-of-loss, and piecewise linear optimal structures. Furthermore, because the second reinsurer charges less than the first reinsurer ($\gamma_1 = 0.5, \gamma_2 = 0.3$), the risk ceded to the second reinsurer is higher than that of the first reinsurer ($g_2^*(z) > g_1^*(z)$). All of these findings are also consistent with our intuition.

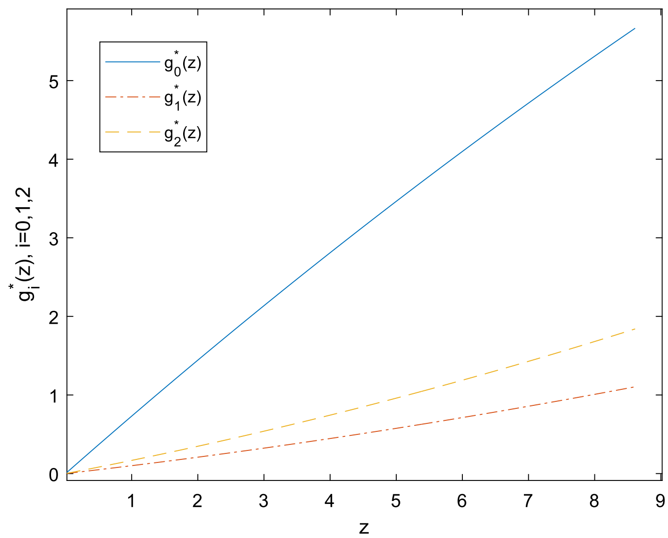

Fig. 1. Optimal risk-sharing policies $g_j(z)$.

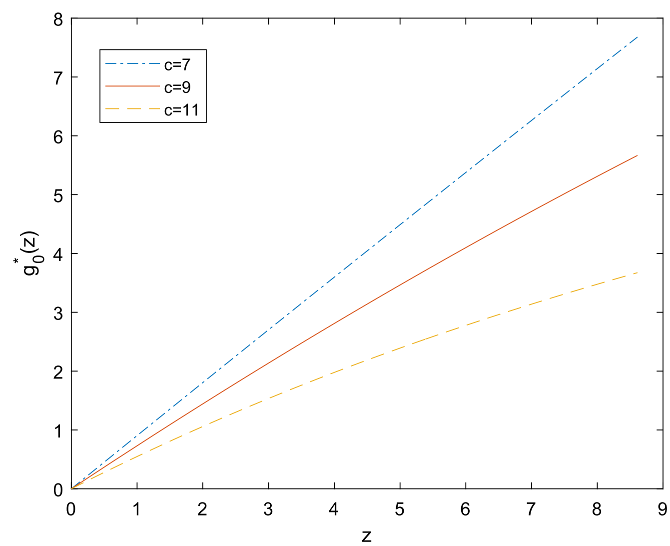

Fig. 2. Impact of the premium rate c on the retained risk $g_0(z)$.

Figs. 2–4 display optimal multiple reinsurance strategies for different premium rates ($c = 7, 9, 11$). The results show that the retained risk of the insurer decreases and both risks ceded to the two reinsurers increase as the premium rate increases. This trend is due to our optimization objective of maximizing the Lundberg exponent, which controls the ruin probability. For a given risk, if the insurer can charge a higher premium, then the insurer can afford to cede more risks in order to reduce the ruin probability. Correspondingly, the risks ceded to the reinsurers increase if the reinsurance premium principles are unchanged.

Figs. 5–9 present the impact of the safety loading of the reinsurance premium on the retained risk of the insurer and the

[8] It is worth to point that the numerical iterative algorithm for solving R^* given here has some limitations. In step 4, $g_j(Z)$, $j = 1, 2$ can be numerically obtained because they are uniquely determined by equations with analytical form. If the equations are not analytic or the number of equations m is very large, then it will be very challenging to obtain the numerical solutions. At that time, we will need to analyze specific issues on a case-by-case basis.

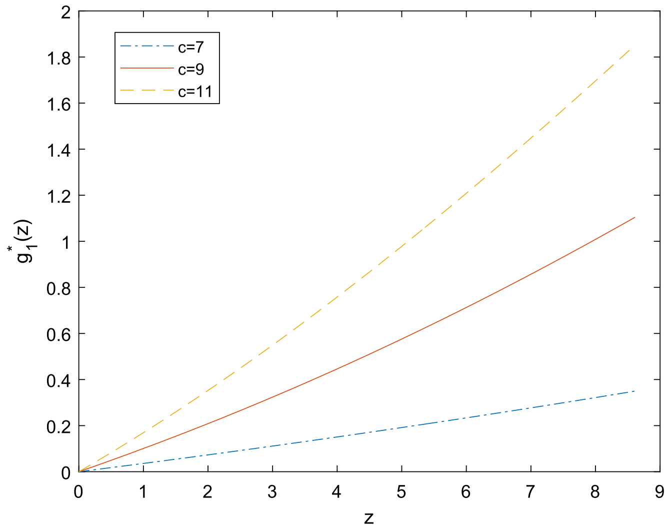

Fig. 3. Impact of the premium rate c on the risk-sharing policy $g_1(z)$.

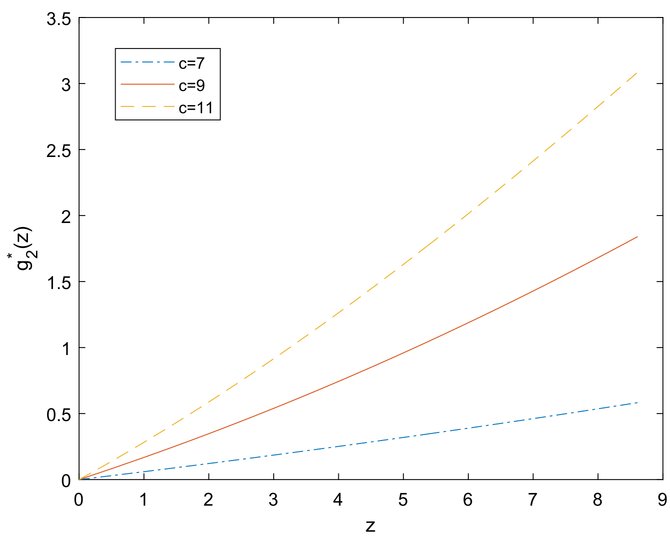

Fig. 4. Impact of the premium rate c on the risk-sharing policy $g_2(z)$.

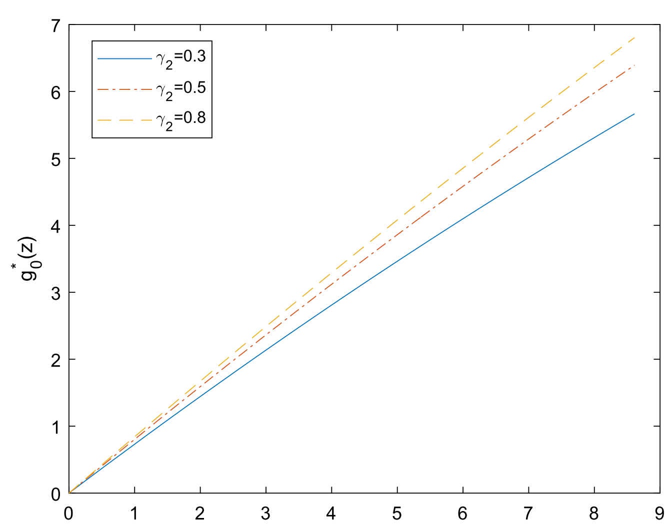

Fig. 5. Impact of the safety loading γ_2 on the retained risk $g_0(z)$.

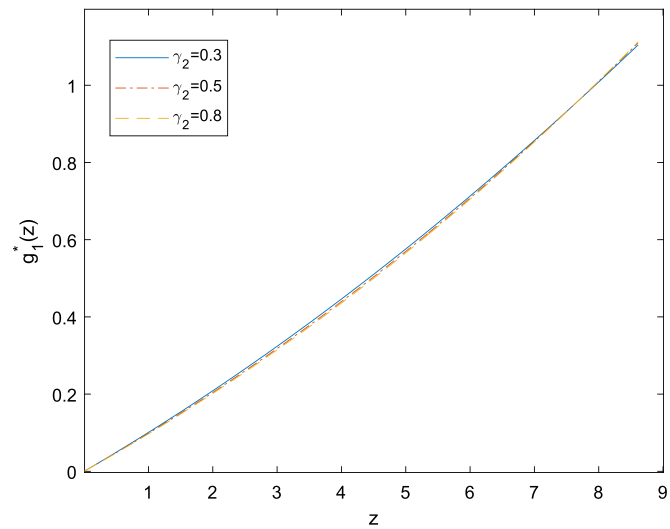

Fig. 6. Impact of the safety loading γ_2 on the risk-sharing policy $g_1(z)$.

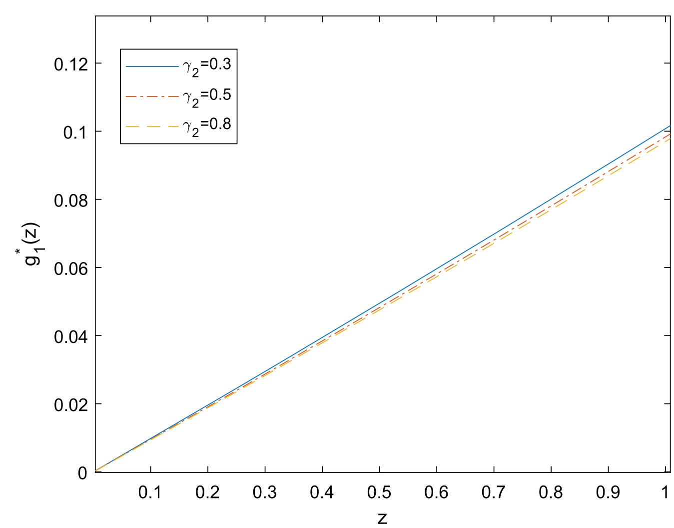

Fig. 7. Impact of the safety loading γ_2 on the risk-sharing policy $g_1(z)$: small claims.

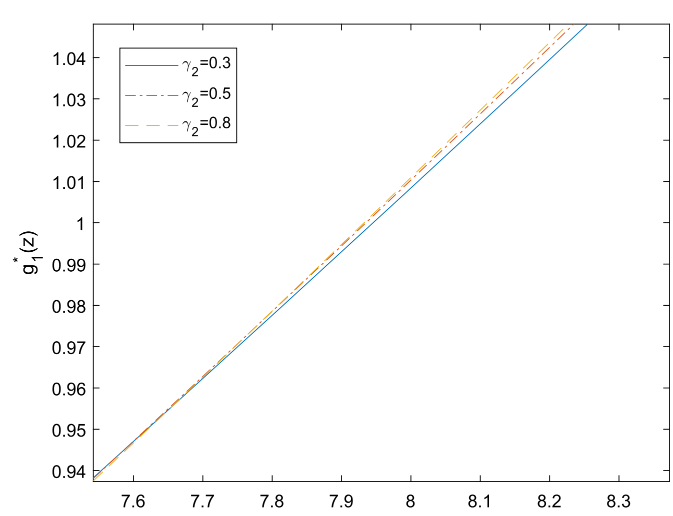

Fig. 8. Impact of the safety loading γ_2 on the risk-sharing policy $g_1(z)$: large claims.

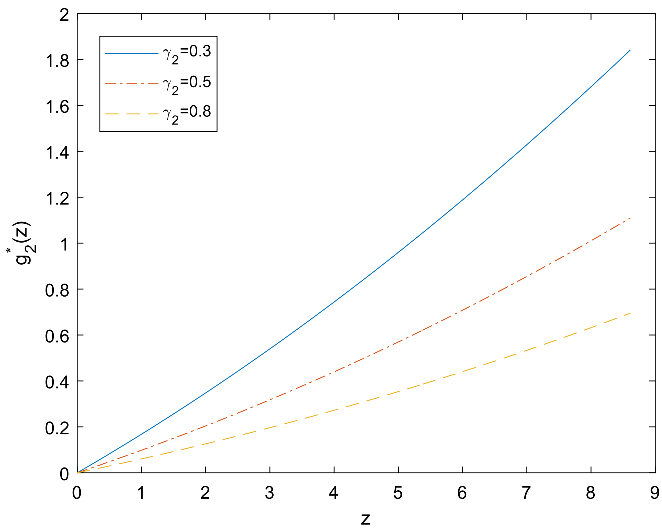

Fig. 9. Impact of the safety loading γ_2 on the risk-sharing policy $g_2(z)$.

risks ceded to the two reinsurers. The figures show the following phenomena. First, as the safety loading γ_2 increases, the premium charged by the second reinsurer for the same risk increases, whereas the premium charged by the first reinsurer does not change. Thus, the risk ceded to the second reinsurer decreases (see Fig. 9), and the risk retained by the insurer increases (see Fig. 5). However, the risk ceded to the first reinsurer does not show a significant change. We can conclude that if one of the reinsurers increases the price of its own undertaken risk, then the insurer will reduce the risk ceded to that reinsurer, but may not obviously increase the risk ceded to the other reinsurers. Basically, almost all of the reduced risk from the reinsurer will be retained by the insurer. To clearly show the impact on the risk ceded to the first reinsurer, we enlarged the results from Fig. 6 in Figs. 7 and 8 for small claims and large claims, respectively. Here, we can see that for small claims, the risk ceded to the first reinsurer is slightly reduced, whereas for large claims, the risk ceded to the first reinsurer increases slightly. From the perspective of the insurer, this result is easy to understand because the insurer would prefer small claims over large claims.

Finally, we further investigated the optimal reinsurance policy. In this paper, we obtained the optimal reinsurance policy by maximizing the Lundberg exponent or, equivalently, by minimizing the Lundberg bound of the ruin probability in terms of the optimal reinsurance policy. However, we note that the results may be different from the optimal reinsurance policy obtained by minimizing the ruin probability directly. The left graph of Fig. 10 displays the Lundberg bounds of the ruin probability without reinsurance and with the optimal reinsurance policy. We can see that the optimal reinsurance policy can significantly reduce the Lundberg upper bound. For the same optimal reinsurance policy, we also examine the impact on the ruin probability itself. It is almost impossible to give an explicit expression for the ruin probability because the retained risk $g_0^*(Z)$ may not have a simple distribution. Thus, we employ the *De Vylder approximation* (see Rolski et al., 1999) to calculate the ruin probability. The right graph in Fig. 10 shows that the obtained optimal reinsurance policy also greatly reduces the ruin probability, and the percentage reduction of the ruin probability is better than that for the Lundberg upper bound. This result suggests that the optimal reinsurance policy obtained by maximizing the Lundberg exponent also works well for controlling the ruin probability, to a certain extent.

6. Concluding remarks

Based on the objective of maximizing the Lundberg exponent, we have studied the optimal multiple reinsurance arrangement problem for an insurer with a general admissible policies set. Using a point-wise optimization approach for a type of combined premium principle, we showed that optimal multiple reinsurance strategies have non-piecewise linear structures, which differ from conventional reinsurance strategies such as quota-share, excess-of-loss, and layer reinsurance arrangements. In general, it seems that only claims Z_i with light-tailed distributions are formulated in the model setting in order to guarantee the existence of the Lundberg exponent and the risk measure of the exponential premium principle. In fact, our results can also be applied for claims Z_i with heavy-tailed distributions, as long as the risk retained by the insurer $g_0(Z_i)$ is light-tailed and the reinsurer undertaking the heavy-tailed part does not take the exponential premium principle, for example, when one reinsurance company takes the expected value principle.

Declaration of competing interest

There is no competing interest.

Data availability

No data was used for the research described in the article.

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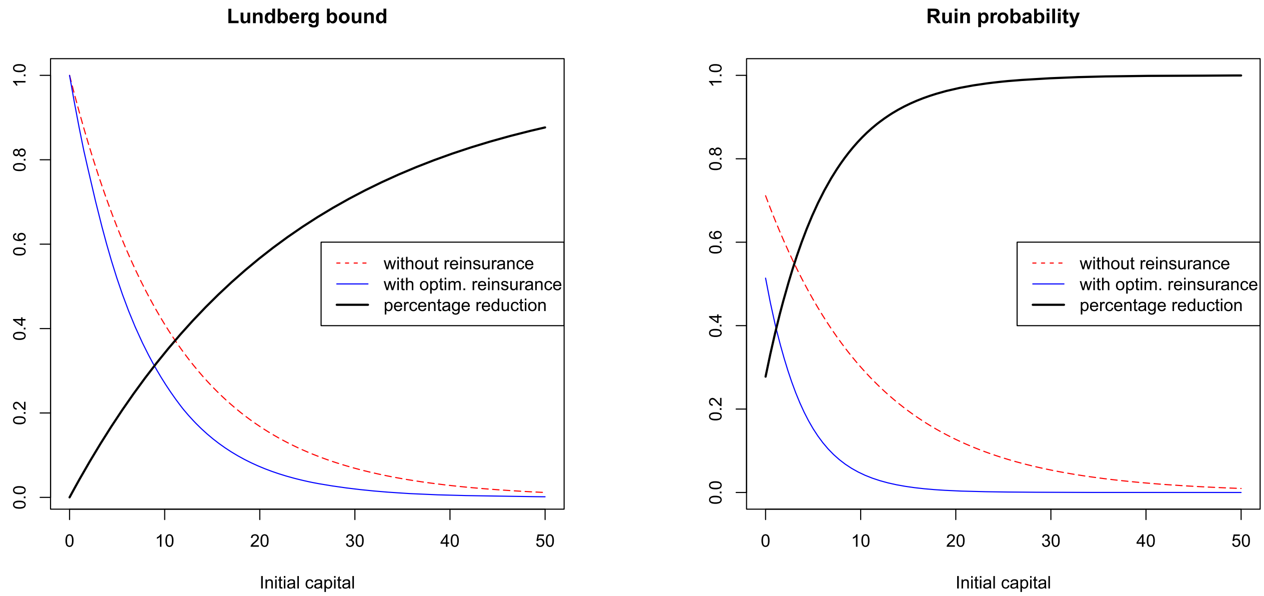

Fig. 10. Impact of the optimal reinsurance policy on the Lundberg bound and ruin probability.

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