Insurance: Mathematics and Economics 109 (2023) 52–68

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Managing reputational risk in the decumulation phase of a pension fund

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ARTICLE INFO

Article history:
Received August 2022
Received in revised form December 2022
Accepted 17 December 2022
Available online 30 December 2022

JEL classification:
C61
G22
G52
J26

Keywords:
Pensions
Collective Investment
Risk
Retirement phase
Reputational risk
Stochastic control

ABSTRACT

In this paper, we suggest strategies for a pension provider to avoid a loss of reputation due to possible pension reductions in the decumulation phase. In different settings, we determine optimal actions to keep the pension plan solvent, i.e. value of the assets always above the net present value of the pension liabilities. With this in mind, we solve suitable singular control problems. We show that, in expectation, the pension provider can cover the costs of the optimal action via sharing bonus payments with the policyholders.

1. Introduction

Pension funds are among the most important players in the financial markets of the OECD countries, managing more than 37.7 trillion of assets at the end of 2021 which represents over 70 percent of the OECD's area GDP. According to the European Insurance and Occupational Pensions Authority, more than half of European life insurers were then guaranteeing a return for investors that had been higher than the local 10-year government bond, thus creating an undesirable negative investment spread and worsening the solvency of the pension plan.

Ultra low-interest rates as we have seen then[1] together with the increasing requirements on solvency have put a greater strain on the solvency of pension funds, whose liabilities consist of a fixed investment return or promises of payment in the case of defined benefit plans – normally calculated by reference to individual's past salaries (Boado-Penas et al. (2020)).

Traditional with-profit insurance products with guarantees needed to be reinvented to meet consumer's needs in terms of stability after retirement and at the same time guarantee the solvency of the plan (Boado-Penas et al. (2022)). Guillén et al. (2006) analyse new pension saving schemes and discuss about the application of smoothing mechanisms. Bohnert et al. (2014) study dynamic hybrid life products and show that higher guarantees do not necessarily imply a higher willingness to pay. For workplace pension plans, under collective defined

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[1] Since December 2021 interest rates are rising. However, pensions will not be positively impacted in the short run as, in line with their liabilities, pension providers tend to enter long-term investment positions.

https://doi.org/10.1016/j.insmatheco.2022.12.005
0167-6687/

contribution (CDC) schemes contributions are pooled and managed on a collective basis while the plan has a target pension amount rather than a contractual guarantee (see, for example, Bams et al. (2016) and Chen et al. (2021)).

Pension providers need the trust of people as setting aside money from contributors for the long term requires trust and commitment. It can take a lifetime to build trust, but it can go in a flash.

A few studies deal with reputational risk in the financial sector. Fiordelisi et al. (2014) show that considerable reputational losses occur following the announcement of operational losses, with fraud being the event type that generates the greatest reputational impact. Eckert and Gatzert (2017) prove that reputational losses can undoubtedly exceed operational losses. The reputational losses are modelled using the abnormal returns around the date of the event of the operational loss. In the insurance sector, Gatzert et al. (2016) analyse reputational risk insurance solutions due to increased relevance of reputational risk – mainly as a result of the growing influence of social media. They show that most insurers do focus on loss control rather than covering actuarial financial losses of reputation risk. For example, Zurich's Brand Assurance offers protection for brand restoration expenses up to a limit of 100 million dollars while Munich Re focuses on lost profits as a result of a decline in revenues following a crisis event. Insuring reputational risk has major challenges such as the loss valuation together with the pricing and the lack of data.

Besley and Prat (2005) state that one of the main challenges for pensions (public and private) is their inability to develop a credible institutional framework in the sense that promises of payment may be reasonably respected. Holzmann (2007) claims that constant tinkering with parametric reforms (i.e. changes in retirement age and/or increases in contribution rates amongst others) to restore long-term solvency lowers the credibility of pension systems. This credibility problem is associated with reputational risk which can be the result of actions that give the pension system a bad image and give contributors a reason to not contribute or change to other systems, should this option be available to them (Vidal-Meliá et al. (2009)).

In our work, we focus on the decumulation phase of the pension plan and examine different strategies so that the pension benefits do not need to be decreased to keep the plan solvent. Solvency is measured through the degree of capital cover defined as the ratio of the value of the pension fund and the present value of future pension payments for all current retirees. Our main argument why a pension provider should avoid pension cuts in the decumulation phase is the corresponding loss of reputation. For instance, in non-life insurance a considerable fraction of individuals refuses any default risk. In general, consumers demand more than 20% reduction in premium to offset a 1% default risk (see, for example, Wakker et al. (1997) and Zimmer et al. (2009)). Although it is clear that a decrease in pensions could occur if there is no contractual guarantee for the height of future pension payments, the reputation of a pension fund provider will seriously suffer if pensions have to be decreased without an objective reason (such as deflation). Therefore, it is essential to look for strategies that at least reduce the risk of a pension cut or the costs of necessary capital injections by the fund provider.

Instead of explicitly looking at the strategy how the pension fund is managed (and thus do not include option- or CPPI-like positions), we will concentrate on possible strategies to stabilise the pension payments over time and the corresponding costs of the necessary actions.

The paper is organised as follows: in Section 2 we consider possible cost-minimising actions to avoid that pensions have to be decreased. The use of a simplified setting (with conservative estimates of the mortality of the pensioners) will allow us to obtain explicit solutions that enable us to illustrate our findings by some numbers. The more realistic setting of letting the customers participate in the good performance of the fund via bonus payments and thus also keeping the ratio between (total) fund value and value of potential pension payments in a given target interval is the subject of Section 3. There, we will also explore a possible participation in the bonus payments for the provider.

Pension reductions can lead to a loss of reputation potentially causing severe financial losses in the future. We show that despite avoiding pension reductions seems costly for the pension providers, their participation in the bonus payments will, in expectation, cover the costs and keep the reputational risk at arm's length. The interests of both parties are taken into account – in a way, one is running with the hare and hunting with the hounds.

2. Decumulation Phase - Basic Models And Minimal Costs

We consider the decumulation phase of a pension scheme without capital guarantees. In contrast to classical pensions, we assume that the accumulated capital of the pensioners is invested in a fund and will be paid out over time. Motivated by the Solvency II ratio, we consider DCC (the degree of capital cover),

$$\mathrm{DCC} = \frac{\text{the current fund value}}{\text{the present value of pensions to be paid}}$$

as the main index of solvency of the pension scheme.

For this quotient, we assume that we can approximate the value of the pensions to be paid for M members with a constant rate of mortality λ and the same pension rate P for everyone by

$$\frac{MP}{\lambda}\,.$$

Here, M can be regarded as the average number of members in the scheme while λ plays the role of an approximation of the mortality intensity that is chosen in a conservative way to care for longevity increasing over the next years.

The fund value per share is assumed to follow a geometric Brownian motion

$$S_t = S_0\exp\{\mu t + \sigma W_t\} = S_0\exp\{\Big(\nu - \tfrac{1}{2}\sigma^2\Big)t + \sigma W_t\}$$

with $\mu, \sigma > 0$ (i.e. $\nu > \frac{1}{2}\sigma^2$). Then, the index DCC is given by

$$C_t := \lambda\frac{S_0\exp\{\mu t + \sigma W_t\}}{MP} = \frac{S_0\lambda}{MP}\exp\{\Big(\nu - \tfrac{1}{2}\sigma^2\Big)t + \sigma W_t\}$$

where S_0 denotes the initial capital at the start of our considerations. We work on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{R}_+}$ generated by the Brownian motion W.

Remark 2.1 *(Modelling aspects and performance issues).*
1. **Going concern or run off**: In the definition of DCC we divide by the present value of the pensions of M members that have just started their pension phase. There is no decrease of the denominator over time. Thus, we are actually modelling a situation where the average age of the pensioners stays (approximately) constant over time. I.e. there is a kind of inflow-equals-outflow relation among the pensioners with regard to the age. We thus implicitly consider the situation of a non-ageing representative pensioner. This is a kind of *going concern assumption* and does not correspond to a decumulation of the total fund value. Even more, we implicitly assume that $S(0)$ is the standardised value of the accumulated contributions of the members when entering the retirement phase. If we accept the going concern framework then we can w.l.o.g. assume $M = 1$.

If, however, we really look at the closed group of M pensioners that for simplicity all entered the decumulation phase at the same time, then the situation gets more complicated. In this run-off-type framework, we also have to consider that money is actually paid out and that there are no new members that compensate for the payments by bringing in their accumulated money from the savings phase. As a consequence, we have to condition on survival times of the individuals, on the amount of money already paid out and further details. For reasons of tractability, we will thus from now on focus on the above going concern framework. A possible justification for this assumption can also be the stability/reliability of future pension payments that we want to ensure. The so obtained good reputation of the pension provider will be the main reason for the constant inflow of new members.
2. **Generalising the going concern framework**: The above going concern assumption can also be weakened by always looking at the quotient of the value of total funds and total future payments. However, this typically requires a detailed assignment of gains/bonuses/funds and pensions to different customers. Given such an assignment our optimisation problems below can be formulated in a similar – but notationally more involved – way.
3. **High pensions or low number of adjustments**: Looking at the representation of DCC above, the first idea is that it would make a very good impression to have a high value of DCC. However, this can have various reasons such as

- the initially promised pension rate of P is much smaller than one would assume it to be fair,
- the reserve generated by a very good past performance of the fund.

In both cases, an increase of the pension rate seems to be justified. However, a (temporarily) low value of DCC can make both the pensioner and the pension provider nervous, the first one fears for her pension, the latter might lose money by necessary capital injections to keep DCC above 1 to avoid a bad reputation. Depending on the modelling framework, we will below provide optimal solutions.

Notation 2.2. For convenience, we introduce some notation for frequently occurring expressions:

i) $x = \ln(\lambda S_0/(MP))$ and $\mathbb{E}_x[\cdot] = \mathbb{E}[\cdot\,|\ln(C_0) = x]$,
ii) $b := \ln(1.25)$,
iii) $\theta := \frac{-\mu+\sqrt{\mu^2+2\delta\sigma^2}}{\sigma^2} > 0$ and $\zeta := \frac{-\mu-\sqrt{\mu^2+2\delta\sigma^2}}{\sigma^2} < 0$,
iv) $\mathrm{sh}(x) = (e^x - e^{-x})/2$,
v) $1\!\!1_A = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$.

Infinite time horizon - Decumulation with minimal costs

As above, we consider the case of a no-guarantee product. If the value of C_t hits 1 from above, then to avoid a bad reputation by having to decrease promised (but not guaranteed!) pensions, the pension provider injects an amount of $(k-1)MP/\lambda > 0$ into the fund position via buying additional shares and assigning them in a proportional (to the positions) way to the pensioners to raise DCC to the value $k > 1$. Note also that under the assumption of $\mu > 0$, the process C_t converges to infinity almost surely for $t \to \infty$. We now look at the evolution of DCC in time when the provider follows the above type of capital injection strategy.

We define

$$\tau_{1;k} := \inf\{t > 0 : \ C_t = 1, \ \ln(C_0) = x\}$$

and in general the next injection time for $n \geq 2$ as

$$\tau_{n;k}|\tau_{n-1;k} < \infty := \inf\{t > \tau_{n-1;k} : \ C_t = 1, \ C_{\tau_{n-1:k}} = k\}$$

with the convention $\tau_{0;k} = 0$. Note that the inter-arrival times $\tau_{n;k} - \tau_{n-1;k}$, $n \geq 2$ are independent and identically distributed with

$$\mathbb{P}[\tau_{n;k} - \tau_{n-1;k} \leq z] = \int_0^z \frac{\ln(k)}{\sqrt{2\pi}\sigma t^{3/2}} e^{-\frac{(\ln(k)+\mu t)^2}{2\sigma^2 t}} \,\mathrm{d}t \,,$$

$$\mathbb{P}[\tau_{n;k} - \tau_{n-1;k} = \infty] = 1 - e^{-2\frac{\mu \ln(k)}{\sigma^2}} \,.$$

Further, we set

$$L := \sup\{n \in \mathbb{N} : \tau_{n;k} < \infty\}$$

L, the number of capital injections is geometrically distributed and thus almost surely finite. The discounted amount of injections is given by

$$(k-1)\frac{MP}{\lambda}\sum_{i=1}^{\infty} e^{-\delta\tau_{i;k}} . \tag{1}$$

As the sequence $\tau_{n;k}$ is monotonically increasing in k, the corresponding sequence $e^{-\delta\tau_{i;k}}$ is monotonically decreasing. We are now looking for the cheapest k in the sense that the value function is given via the following optimisation problem

$$v(x) = \inf_{k>1}\left\{(k-1)\frac{MP}{\lambda}\mathbb{E}_x\Big[\sum_{i=1}^{\infty} e^{-\delta\tau_{i;k}}\Big]\right\} , \tag{2}$$

i.e. the minimum of the expectation should be attained for the choice of k.

To make the following considerations rigorous, let us assume that we look at a slightly modified problem where k is assumed to be bounded away from 1, i.e. we temporarily assume that we have

$$k \ge \tilde{k} > 1.$$

In general, the choice of $\tilde{k}$ may depend on the preferences and the degree of risk aversion of the insurance company. For our considerations, we assume that $\tilde{k} > 1$ is arbitrary but fixed.

Let further $\tilde{\tau}$ be a generic random variable with the same distribution as $\tau_{2;k} - \tau_{1;k}$. Note that for $x = \ln(S_0\lambda/(MP))$ we have:

$$\begin{aligned}
\mathbb{E}[e^{-\delta\tau_{i;k}}\mathbb{1}_{[\tau_{i;k}<\infty]}] &= \mathbb{E}_x[e^{-\delta\tau_{1;k}}e^{-\delta(\tau_{2;k}-\tau_{1;k})}\cdots e^{-\delta(\tau_{i;k}-\tau_{i-1;k})}\mathbb{1}_{[\sum_{j=1}^{i}(\tau_{j;k}-\tau_{j-1,k})<\infty]}] \\
&= \mathbb{E}_x[e^{-\delta\tau_{1;k}}e^{-\delta(\tau_{2;k}-\tau_{1;k})}\cdots e^{-\delta(\tau_{i;k}-\tau_{i-1;k})}\prod_{j=1}^{i}\mathbb{1}_{[(\tau_{j;k}-\tau_{j-1,k})<\infty]}] \\
&= \mathbb{E}_x[e^{-\delta\tau_{1;k}}\mathbb{1}_{[\tau_{1;k}<\infty]}]\cdots\mathbb{E}_{\ln(k)}[e^{-\delta(\tau_{i;k}-\tau_{i-1,k})}\mathbb{1}_{[(\tau_{i;k}-\tau_{i-1,k})<\infty]}] \\
&= e^{\zeta x}\cdot e^{\zeta(i-1)\ln(k)} .
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
\sum_{i=1}^{\infty}\mathbb{E}\Big[e^{-\delta\tau_{i;k}}\mathbb{1}_{[\tau_{i;k}<\infty]}\Big] &= e^{\zeta x}\sum_{i=1}^{\infty}e^{(i-1)\zeta\ln(k)} \\
&= e^{\zeta x}\frac{1}{1-e^{\zeta\ln(k)}} .
\end{aligned}$$

The function

$$\gamma(k) := (k-1)e^{\zeta x}\frac{1}{1-e^{\zeta\ln(k)}}$$

is increasing for all values of x. Hence, the minimal value of expected discounted injections will be attained at the minimal value of k which obviously is not attained in the interval $(1,\infty)$. This can be seen as a heuristic motivation for a local time type capital injection strategy, i.e. the pension provider injects only the necessary amount of capital into the fund such that C_t never goes below 1. We will detail this in the following remark.

Remark 2.3. If we allow $k = 1$, the problem becomes similar to a dividend problem considered in Asmussen and Taksar (1997) or the capital injection problem in Eisenberg and Schmidli (2009). However, differently than in our setting the underlying process in the above references is an arithmetic Brownian motion. The process describing the reflection from below for a geometric Brownian motion as given in Boado-Penas et al. (2021) is $Y_t = -\min\big(\inf\{\ln(S_0\lambda/(MP)) + \mu t + \sigma W_t\},\ 0\big)$, i.e.

$$\mathbb{E}\Big[\int_0^{\infty} e^{-\delta s}\,\mathrm{d}Y_s\Big] = -\frac{e^{\zeta x}}{\zeta} ,$$

and $x > 0$. Note that we get

$$\lim_{k\to 1}\frac{k-1}{1-e^{\zeta\ln(k)}} = -\frac{1}{\zeta} ,$$

i.e. the result for $k = 1$ can be obtained as a limiting case of the result for $k > 1$.

Remark 2.4 *(Costs and pension).*
1. While it is clear that (a suitably discretised version of) the above local-time type strategy delivers the cheapest adjustment in the case of a positive value of μ, the limit cost is a function of the fund parameters, the discounting factor and the pension rate,

$$h(a,P)=\frac{M}{\lambda}P\frac{e^{-ax}}{a}, \qquad a:=-\zeta>0\,.$$

Clearly, $h(a,P)$ increases in $P>0$ and decreases in a. By assuming a finite value of a, we have implicitly assumed that a riskless investment (i.e. one with $\sigma^2=0$) is not sufficient for paying the promised pension rate. However, what we see is that in our setting, choosing the fund with the highest available a allows to promise the highest possible P while keeping the injection costs constant. Thus, in this sense, the customer already benefits from a well performing fund although we have not included a possible increase of the pension rate P in this simple setting.
2. We next look at the multiplying factor of

$$g(a)=\frac{e^{-ax}}{a}$$

in $h(a,P)$. If we interpret $h(a,P)$ as today's value of all future costs, then we have a rough upper bound due to the infinite time horizon. However, this can already give us an impression about the design of a possible *cost reserve*. Note that:

a) δ can be interpreted as the allowable factor to discount future liabilities,
b) $\mu+\frac{1}{2}\sigma^2-\delta$ can be interpreted as a risk premium for fund investment over risk-free investment for a rate of δ.

Then, we can see that for a risk premium of 5% (which can be generated by e.g. the choice of $\mu=0.04$, $\delta=0.01$, $\sigma=0.2$), the discounted future costs are below 5% for all $x\geq 1$. Thus, shifting 5% of the part of the accumulated money into a reserve fund that is invested at the riskless rate is already enough to cover all costs of keeping DCC above the critical value of 1. As, however, the situation with just one boundary is oversimplifying, there will be a second possibility to finance the costs of keeping DCC above 1 (and thus ensuring a good reputation for not reducing pensions), paying out a kind of bonus if an upper bound of DCC will be reached and keeping parts of the bonus with the company to build up reserves for capital injections in case of reaching $DCC=1$ in the future again. We will get back to this remark below, but will before have a look at another natural variant of capital injection to avoid a loss of reputation.

3. Two Boundaries and Infinite Time Horizon – Possible Pension Increases via Bonus Payments

As we have already noted that DCC will converge to infinity almost surely if we have $\mu>0$, a very good performance of the fund calls for a participation of its virtual owners, the pensioners. This can e.g. be obtained by increasing the pension rate P if DCC has reached a given level.

To illustrate this idea, we assume that C_t cannot leave the band $[1,1.25]$. The upper boundary is motivated by the current development of occupational plans in Germany, see, for instance, Chen and Rach (2021). But, in fact, the upper boundary can be an arbitrary real number bigger than 1. We further require that if C hits the upper boundary 1.25 then the pension rate P will be increased so that DCC is at the same level $k\in[1,1.25]$ as after the adjustments from below. Our target here is to find the optimal k such that the expected amount of payments is minimised. Note that by the assumption of the adjustment to the same level k no matter whether DCC exits through 1 or through 1.25, a local time type strategy cannot be optimal any longer.

Choosing a low value of k bears the danger that the DCC hits 1 soon (i.e. we increase the probability for the next claim) and we then have to inject money into the system to avoid a loss of reputation. Further, if we reach k from above, then we have to increase the pension rate. However, there remains to decide about the way how we are indeed doing this:

- Either, we increase the pension rate P by the full possible factor of $1.25/k$ when we have $\text{DCC}=1.25$. Then, the pension has been increased and we will face higher costs of $(k-1)\cdot\frac{MP\cdot 1.25}{k\cdot\lambda}$ at future times.
- Or, we increase the pension by a factor of - say - $1.2/k$ and assign the remaining factor of $0.05/k$ to the pension provider to cover future capital injections. As the virtual guarantee to keep the DCC above 1 is given by the pension provider without any contractual obligation, it seems to be fair to also let the provider benefit from the good fund performance.

The value function then depends on both the value of the fund and on the current pension level P. To avoid this dependence on P, we will below rely on formally not changing the pensions, but assigning bonus payments to the customers in case of a good performance. These bonus payments can be used to buy additional shares of the fund which can be seen as formally increasing the number of customers.

3.1. Constant Pension – Profit Participation via Bonus Payments

In this section, we assume that the pension rate will never change. The pensioners benefit from a good performance of the fund, i.e. if C_t reaches 1.25, in form of lump sum payments reducing the current value of C_t. This way, the insurance company rewards the pensioners and avoids long-term liabilities connected to an increase of the pension rate.

We again assume that all processes live on the probability space $(\Omega,\mathcal{F},\mathbb{P})$ and $\mathbb{F}=(\mathcal{F}_t)_{t\in\mathbb{R}_+}$ is a filtration generated by the Brownian motion W. To shorten notations, we write $X_t=x+\mu t+\sigma W_t$ throughout this section.

Definition 3.1 *(Admissible strategies).*
• Let $\kappa=(k_n)_{n\in\mathbb{N}}$ with $k_n\in(1,1.25)$. Further, set $T_0:=0, k_0:=e^x$.
For

$$T_1^\kappa := \inf\{t > 0 : C_t \notin (1, 1.25)\}\,,$$
$$T_n^\kappa := \inf\{t > T_{n-1}^\kappa : k_{n-1}e^{\mu(t-T_{n-1}^\kappa)+\sigma(W_t - W_{T_{n-1}^\kappa})} \notin (1, 1.25)\}\,, \quad n \ge 2\,, \tag{3}$$

the DCC process under κ is given as

$$C_t^\kappa = e^{x+\mu t+\sigma W_t}\mathbb{1}_{[t\in[0,T_1^\kappa)]} + \sum_{n=2}^{\infty} k_{n-1}e^{\mu(t-T_{n-1}^\kappa)+\sigma(W_t-W_{T_{n-1}^\kappa})}\mathbb{1}_{[t\in[T_{n-1}^\kappa, T_n^\kappa)]}\,.$$

If k_n is $\mathcal{F}_{T_n^\kappa}$-measurable, we call κ an admissible adjustment strategy. The set of admissible strategies will be denoted by $\mathcal{A}$.
• Additionally, we allow all strategies from the closure $\bar{\mathcal{A}}$ of $\mathcal{A}$. The closure is meant in the Skorokhod topology (see e.g. (Protter, 2005, p. 225)).

Remark 3.2.
1. The sequence $\kappa = (k_n)_{n\in\mathbb{N}}$ denotes the points to which DCC is shifted at the n-th exit time from $(1, 1.25)$. To indicate the dependence of the corresponding post-adjustment process on the strategy, we denote it by X^κ.
2. Note that the constant "continuous time" strategies $\kappa \equiv 1$ (DCC is reflected at 1 and jumps down to 1 after hitting 1.25) and $\kappa \equiv 1.25$ (DCC is reflected at 1.25 and jumps up to 1.25 after hitting 1) are in the closure $\bar{\mathcal{A}}$. To see this, construct a sequence of constant discrete strategies converging to 1 or to 1.25 respectively.

Thus, a strategy from $\mathcal{A}$ to be applied at the first exit time of the DCC process from the interval $(1, 1.25)$ is decided at time 0, which is of crucial importance in our model. If x is sufficiently closer to 0 than to $\ln(1.25)$ then the probability to reach 0 first is higher. Choosing a higher k would mean a higher penalty, as we will pay $(k-1)MP/\lambda$ in case of reaching zero before $\ln(1.25)$. The crucial task is to determine the exact values when x is sufficiently closer to 0 or to $\ln(1.25)$ in the sense that will be relevant for our optimisation problem.

The return function V^κ, corresponding to a discrete admissible strategy $\kappa = (k_n)_{n\in\mathbb{N}}$ is defined as

$$V^\kappa(x) = \frac{MP}{\lambda}\mathbb{E}_x\Big[\sum_{n=1}^{\infty} e^{-\delta T_n^\kappa}\big(k_{n-1}-1\big)\mathbb{1}_{[C^\kappa_{T_n^\kappa-}=1]}\Big]\,,$$

where T_n^κ and C_t^κ are given in Definition 3.1.

We are seeking to minimise $V^\kappa(x)$ over all admissible strategies κ and let

$$V(x) = \inf_{\kappa\in\bar{\mathcal{A}}} V^\kappa(x)\,.$$

We will in the following look at special classes/examples of strategies as this will lead to the finally optimal strategy.

Example 3.3 *(Constant discrete strategies).*
To illustrate some important features of our optimisation problem, let us first consider a constant strategy $\kappa \equiv k \in (1, 1.25)$, i.e. under k the post-adjustment process X_t^k will be set equal to $\ln(k)$ every time when exiting the interval $(0, b)$ with $b = \ln(1.25)$.

For the special case of constant strategies, we can derive explicit forms of their return functions. Note that starting at x, T_1^k depends only on x but not on k and will be the same for any admissible strategy. For this reason, we write T_1 instead of T_1^k. To derive the corresponding return function V^k corresponding to $\kappa \equiv k$, note that one restarts the process after every exit from $(0, b)$, and the expression in every following time interval is independent of the past. With this we get

$$\begin{aligned}
V^k(x) &= \frac{MP}{\lambda}(k-1)\mathbb{E}_x\big[e^{-\delta T_1}\mathbb{1}_{[X_{T_1-}=0]}\big] \\
&\quad + \mathbb{E}_x\big[e^{-\delta T_1}\big]\cdot\mathbb{E}_{\ln(k)}\Big[e^{-\delta T_1-}\mathbb{1}_{[X_{T_1-}=0]}\Big]\cdot\frac{MP}{\lambda}(k-1)\sum_{n=1}^{\infty}\mathbb{E}_{\ln(k)}\big[e^{-\delta T_1}\big]^{n-1} \\
&= \frac{MP}{\lambda}(k-1)\Big\{\mathbb{E}_x\big[e^{-\delta T_1}\mathbb{1}_{[X_{T_1-}=0]}\big] + \frac{\mathbb{E}_x\big[e^{-\delta T_1}\big]\cdot\mathbb{E}_{\ln(k)}\big[e^{-\delta T_1}\mathbb{1}_{[X_{T_1-}=0]}\big]}{1-\mathbb{E}_{\ln(k)}\big[e^{-\delta T_1}\big]}\Big\}\,.
\end{aligned}$$

Here, we have also used that the independence of the increments of a Brownian motion (for $n \ge 2$) implies:

$$\mathbb{E}_x\Big[e^{-\delta T_n^k}\mathbb{1}_{[X^k_{T_n^k}=0]}\Big] = \mathbb{E}_x\big[e^{-\delta T_1}\big]\cdot\mathbb{E}_{\ln(k)}\big[e^{-\delta T_1}\big]^{n-2}\cdot\mathbb{E}_{\ln(k)}\Big[e^{-\delta T_1}\mathbb{1}_{[X^k_{T_1}=0]}\Big]\,.$$

The remaining expressions in the return functions are given by (see (Borodin and Salminen, 2002, p. 309)):

$$\begin{aligned}
\mathbb{E}_x\big[e^{-\delta T_1^k}\big] &= \frac{e^{-\frac{\mu}{\sigma^2}x}\mathrm{sh}\big(\frac{(b-x)\sqrt{\mu^2+2\delta\sigma^2}}{\sigma^2}\big) + e^{\frac{\mu}{\sigma^2}(b-x)}\mathrm{sh}\big(\frac{x\sqrt{\mu^2+2\delta\sigma^2}}{\sigma^2}\big)}{\mathrm{sh}\big(\frac{b\sqrt{\mu^2+2\delta\sigma^2}}{\sigma^2}\big)}\,, \\
\mathbb{E}_x\big[e^{-\delta T_1^k}\mathbb{1}_{[X_{T_1^k}=0]}\big] &= \frac{e^{-\frac{\mu}{\sigma^2}x}\mathrm{sh}\big(\frac{(b-x)\sqrt{\mu^2+2\delta\sigma^2}}{\sigma^2}\big)}{\mathrm{sh}\big(\frac{b\sqrt{\mu^2+2\delta\sigma^2}}{\sigma^2}\big)}\,.
\end{aligned} \tag{4}$$

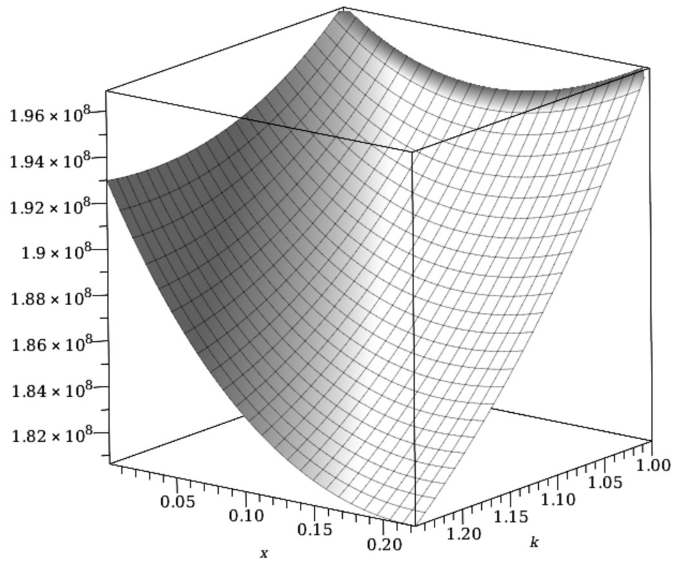


Fig. 1. The return function $(k,x)\mapsto V^k(x)$ for constant discrete strategies.

In particular, we get

$$V^k(\ln(k))=\frac{MP}{\lambda}(k-1)\frac{\mathbb{E}_{\ln(k)}\big[e^{-\delta T_1}1\!\!1_{[X_{T_1-}=0]}\big]}{1-\mathbb{E}_{\ln(k)}\big[e^{-\delta T_1}\big]}\,.$$

Formulas (4) determine V^k to be of the form

$$\alpha_1(k)e^{\frac{-\mu+\sqrt{\mu^2+2\sigma^2\delta}}{\sigma^2}x}+\alpha_2(k)e^{\frac{-\mu-\sqrt{\mu^2+2\sigma^2\delta}}{\sigma^2}x}=\alpha_1(k)e^{\theta x}+\alpha_2(k)e^{\zeta x}$$

for some $\alpha_1(k),\ \alpha_2(k)\in\mathbb{R}$.

Therefore, it is easy to see that V^k solves the differential equation $\frac{\sigma^2}{2}f_{xx}+\mu f_x-\delta f=0$ with the boundary conditions $f(0)=\frac{MP}{\lambda}(k-1)+f(\ln(k))$ and $f(\ln(1.25))=f(\ln(k))$. Now, we can easily determine $\alpha_1(k)$ and $\alpha_2(k)$:

$$\alpha_1(k)=\frac{MP}{\lambda}(k-1)\frac{e^{\zeta b}-e^{\zeta\ln(k)}}{\big(e^{\theta\ln(k)}-1\big)\big(e^{\zeta b}-1\big)+\big(e^{\zeta\ln(k)}-1\big)\big(1-e^{\theta b}\big)}\,,$$
$$\alpha_2(k)=\alpha_1(k)\frac{e^{\theta b}-e^{\theta\ln(k)}}{e^{\zeta\ln(k)}-e^{\zeta b}}\,.$$

Because $\zeta<0$, $\theta>0$, $0<\ln(k)<b$ it holds that $\frac{e^{\theta b}-e^{\theta\ln(k)}}{e^{\zeta\ln(k)}-e^{\zeta b}}>0$, meaning that $\alpha_1(k),\alpha_2(k)>0$. Thus, we can conclude that $V^k_{xx}>0$, i.e. the function $V^k(x)$ is convex in x for any $k\in(1,1.25)$.

However,

$$V^k_x(0)=\frac{\alpha_1(k)}{e^{\zeta\ln(k)}-e^{\zeta b}}\big\{\theta(e^{\zeta\ln(k)}-e^{\zeta b})+\zeta(e^{\theta b}-e^{\theta\ln(k)})\big\}<0\,,$$
$$V^k_x(b)=\frac{\alpha_1(k)}{e^{\zeta\ln(k)}-e^{\zeta b}}\big\{\theta(e^{\zeta\ln(k)}-e^{\zeta b})e^{\theta b}+\zeta(e^{\theta b}-e^{\theta\ln(k)})e^{\zeta b}\big\}>0$$

for all $k\in(1,1.25)$. This can be easily seen by looking at auxiliary functions (representing the curly brackets above) $h_1(y):=\theta(e^{\zeta y}-e^{\zeta b})+\zeta(e^{\theta b}-e^{\theta y})$ and $h_2(y):=\theta(e^{\zeta y}-e^{\zeta b})e^{\theta b}+\zeta(e^{\theta b}-e^{\theta y})e^{\zeta b}$. One gets for $y\in(0,b)$ that $h_1'(y)>0$ and $h_2'(y)<0$ with $h_1(b)=0=h_2(b)$.

We conclude that every $V^k(x)$, $k\in(1,1.25)$, is non-monotone in x and has a unique global k-dependent minimum. It means that coming closer to 1.25 may be as bad as being close to 0, because by hitting 1.25 one is forced to jump down to the chosen level k. Fig. 1 shows the function $(k,x)\mapsto V^k(x)$ for $(k,x)\in(1,1.25)\times(0,b)$ for the parameter set $\mu=0.02$, $\delta=0.01$, $\sigma=0.1$, $M=10000$, $P=100$, $\lambda=0.02$. ■

The constant strategies for $k\in(1,1.25)$ are exercised discretely in time. But what happens if one chooses $k=1$ or $k=1.25$? Intuitively it is clear that for $k=1$ the DCC process has to be reflected at 0 and jumps back to 0 after hitting 1.25. And vice versa, for $k=1.25$ DCC is reflected at 1.25 and is adjusted to 1.25 after hitting 1.

Remark 3.4 *($\kappa\equiv1.25$ and $\kappa\equiv1$).*
We now consider the two extreme cases of constant strategies:

1. If $\kappa \equiv 1.25$, the process C is reflected at 1.25 and jumps back to 1.25 after hitting 1. The term describing the accumulated withdrawals from the geometric Brownian motion up to time t is

$$Z_t := 1.25\big(\sup_{0<s\le t}\{X_s\} - b\big)^+ ,$$

see Boado-Penas et al. (2021). Let now

$$\varrho_1 := \inf\{s > 0 : X_t - \big(\sup_{0<s\le t}\{X_s\} - b\big)^+ = 0\} .$$

Up to time ρ_1 the post-κ process C^κ is given by

$$C_t^\kappa = \exp\{X_t - \big(\sup_{0<s\le t}\{X_s\} - b\big)^+\} .$$

At ϱ_1, we restart the DCC process with initial value e^b. The post-κ process between the first and the second hitting time of 0 has the same distribution as C_t^κ for $t < \varrho_1$.

A process $\tilde{C}^\kappa$ with the same process law as the post-κ process C^κ is then a sum of independent copies of C_t^κ. The reflection term is the sum of reflection terms between 0-hitting times.

2. If $\kappa \equiv 1$, the process C is reflected at 1 and jumps back to 1 after hitting 1.25.

The process

$$Y_t = \Big(-\inf_{0\le s\le t}\{x + \mu s + \sigma W_s\}\Big)^+ ,$$

gives the accumulated amount of injections up to time t. Let now

$$\varrho_1 := \inf\{s > 0 : e^{X_t + Y_t} = 1.25\}$$

Up to time ϱ_1 the post-κ process C^κ is given by

$$C_t^\kappa = \exp\Big\{x + \mu t + \sigma W_t + Y_t\Big\} .$$

At time ϱ_1 the process is adjusted to 1. The process $\tilde{C}_t^\kappa$, $t \ge \varrho_1$ written as a sum of independent copies of C_t^κ, $t < \varrho_1$ has the same law like C_t^κ, $t \ge \varrho_1$.

Also in this case, the reflection term is the sum of reflection terms between 1.25-hitting times.

In the following lemma, we derive the return functions corresponding to the two extreme constant strategies.

Lemma 3.5.

1. *Consider the strategy $\kappa \equiv 1.25$. The corresponding return function $V^{1.25}(x)$ is*

$$V^{1.25}(x) = B_1 e^{\theta x} + B_2 e^{\zeta x} , \tag{5}$$

where

$$B_1 := \frac{MP}{\lambda} \cdot \frac{e^{\zeta b}\zeta(1.25-1)}{e^{\theta b}\theta\big(e^{\zeta b}-1\big) + e^{\zeta b}\zeta\big(1-e^{\theta b}\big)} ,$$
$$B_2 := -\frac{MP}{\lambda} \cdot \frac{e^{\theta b}\theta(1.25-1)}{e^{\theta b}\theta\big(e^{\zeta b}-1\big) + e^{\zeta b}\zeta\big(1-e^{\theta b}\big)} .$$

2. *The return function $V^1(x)$ corresponding to the constant strategy $\kappa \equiv 1$ is*

$$V^1(x) = C_1 e^{\theta x} + C_2 e^{\zeta x} , \tag{6}$$

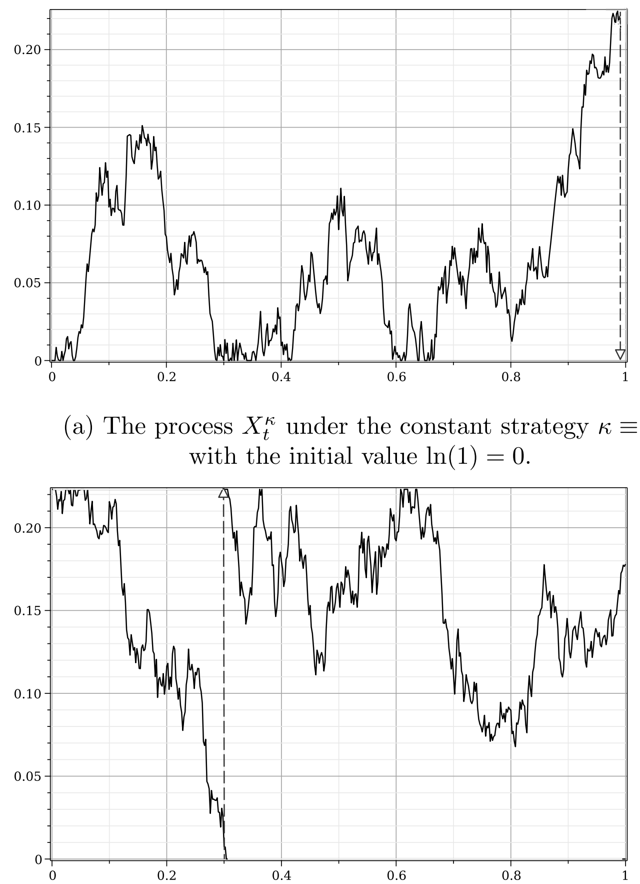
where

$$C_1 := \frac{MP}{\lambda} \cdot \frac{(e^{\zeta b}-1)}{\theta(1-e^{\zeta b}) - \zeta(1-e^{\theta b})} \quad \text{and} \quad C_2 := -\frac{MP}{\lambda} \cdot \frac{(e^{\theta b}-1)}{\theta(1-e^{\zeta b}) - \zeta(1-e^{\theta b})} .$$

Proof. See Appendix. □

Example 3.6. For $\mu = 0.02$, $\delta = 0.01$, $\sigma = 0.1$, $M = 10000$, $P = 100$, $\lambda = 0.02$, Fig. 2a shows a path of the process X_t^κ under the constant strategy $\kappa \equiv 1$. The process starts at 0, is reflected at 0 and is pushed down (dashed line) to 0 after hitting the level b.

Fig. 2b illustrates a possible evolution of the process X_t^κ with $\kappa \equiv 1.25$. X_t^κ is reflected at b, and is adjusted back to b after hitting the level 0.



(a) The process X_t^κ under the constant strategy $\kappa \equiv 1$ with the initial value $\ln(1) = 0$.

(b) The process X_t^κ under the constant strategy $\kappa \equiv 1.25$ with the initial value $\ln(1.25) = b$.

Fig. 2. Possible evolution of the logarithmic DCC process X under the fixed point strategies 1 and 1.25.

A well-known result, see for instance Eisenberg and Schmidli (2009), states that the minimal value of expected discounted (with a positive preference rate) capital injections, in a Brownian risk model, is achieved if one injects only if the surplus becomes negative and only as much as it is necessary to shift the process back to 0. In our case, Fig. 1 suggests that $\kappa \equiv 1$ is not optimal.

A significant drawback of the strategy $\kappa \equiv 1$ is that one is forced to reduce DCC to the level 1 when hitting 1.25. The advantage of being far from 1 quickly becomes a disadvantage. The question arises which strategy can have the advantages of $\kappa \equiv 1$ and avoids its disadvantages. As DCC is modelled by a process with continuous paths, a natural choice would be an opportunistic strategy: keep DCC at the upper boundary 1.25 as long as the financial market allows and use $k = 1$ when hitting the level 1. In the following, we will denote this strategy by k^{dr} where the index dr indicates that the DCC process has to be doubly reflected: at 1 and at 1.25. First, we look at DCC under the strategy κ^{dr} and construct the corresponding return function. For the generator of a doubly reflected non-degenerate Markov processes we refer to Ethier and Kurtz (1986), chapter "Examples of generators". Denoting the reflection term at 0 by Y_t and the reflection term at b by Z_t, the post-κ^{dr} process is

$$C_t^{dr} = e^{X_t + Y_t - Z_t} .$$

Lemma 3.7. *The return function corresponding to the doubly reflected strategy κ^{dr} is given by*

$$V^{dr}(x) = A_1 e^{\theta x} + A_2 e^{\zeta x} = \frac{MPe^{\zeta b}}{\lambda\theta(e^{\theta b} - e^{\zeta b})} e^{\theta x} - \frac{MPe^{\theta b}}{\lambda\zeta(e^{\theta b} - e^{\zeta b})} e^{\theta x} . \tag{7}$$

Proof. See Appendix. □

Remark 3.8. A doubly reflected strategy κ^{dr} fulfils $\kappa^{dr} \notin \bar{\mathcal{A}}$.

This can be easily seen as the strategies in $\mathcal{A}$ are allowed either to minimize the jumps downwards or the jumps upwards. There is no sequence of strategies that can minimise the jumps in both directions simultaneously, which would lead to κ^{dr}.

Although we have $\kappa^{dr} \notin \bar{\mathcal{A}}$, it is worth considering it as in the next proposition, we show that κ^{dr} outperforms the strategies from $\bar{\mathcal{A}}$.

Proposition 3.9. *For $x \in [0, b]$, the strategy κ^{dr} is an optimal strategy, and its corresponding return function $V^{dr}(x)$ fulfils $V^{dr}(x) \le V^{\kappa}(x)$ for any admissible strategy $\kappa \in \bar{\mathcal{A}}$.*

Proof. See Appendix. □

Since it is clear that a pension provider would like to avoid continuous bonus payments or continuous DCC adjustments, a strategy derived below may be of a particular interest.

Assuming that the pension rate will never change and that the excess in the fund will be paid out in the course of gain sharing procedures, we come to the following result.

Consider the strategy: "choose $k \in (1, 1.25]$ at $t = 0$ and follow κ^{dr} after". The return function $\tilde{V}$ corresponding to this strategy is

$$\begin{aligned}
\tilde{V}(x) &= \mathbb{E}_x\Big[e^{-\delta\tau_1}\mathbb{1}_{[C_{\tau_1-}=1]}\frac{MP(k-1)}{\lambda} + e^{-\delta\tau_1}V^{dr}(\ln(k))\Big] \\
&= e^{-\frac{\mu x}{\sigma^2}}\frac{\mathrm{sh}\big(\frac{\sqrt{\mu^2+2\delta\sigma^2}}{\sigma^2}(b-x)\big)}{\mathrm{sh}\big(\frac{\sqrt{\mu^2+2\delta\sigma^2}}{\sigma^2}b\big)}\Big\{(k-1)\frac{MP}{\lambda} + V(\ln(k))\Big\} \\
&\quad + e^{\frac{\mu(b-x)}{\sigma^2}}\frac{\mathrm{sh}\big(\frac{\sqrt{\mu^2+2\delta\sigma^2}}{\sigma^2}x\big)}{\mathrm{sh}\big(\frac{\sqrt{\mu^2+2\delta\sigma^2}}{\sigma^2}b\big)}V^{dr}(\ln(k)) \\
&= \frac{1}{2}\cdot e^{\theta x}\Bigg\{\frac{V^{dr}(\ln(k))e^{\frac{\mu}{\sigma^2}b}}{\mathrm{sh}\big(\frac{\sqrt{\mu^2+2\delta\sigma^2}}{\sigma^2}b\big)} - \Big((k-1)\frac{MP}{\lambda} + V^{dr}(\ln(k))\Big)\frac{e^{-\frac{\sqrt{\mu^2+2\delta\sigma^2}}{\sigma^2}b}}{\mathrm{sh}\big(\frac{\sqrt{\mu^2+2\delta\sigma^2}}{\sigma^2}b\big)}\Bigg\} \\
&\quad + \frac{1}{2}\cdot e^{\zeta x}\Bigg\{\Big((k-1)\frac{MP}{\lambda} + V^{dr}(\ln(k))\Big)\frac{e^{\frac{\sqrt{\mu^2+2\delta\sigma^2}}{\sigma^2}b}}{\mathrm{sh}\big(\frac{\sqrt{\mu^2+2\delta\sigma^2}}{\sigma^2}b\big)} - \frac{V^{dr}(\ln(k))e^{\frac{\mu}{\sigma^2}b}}{\mathrm{sh}\big(\frac{\sqrt{\mu^2+2\delta\sigma^2}}{\sigma^2}b\big)}\Bigg\}.
\end{aligned} \tag{8}$$

The question arises, which k should be chosen for the first exit time when DCC exits $(1, 1.25)$. The strategy is a trade-off between paying bigger claims and being pushed down after hitting the upper boundary.

In the next steps, we will find the value of k that minimizes the expected discounted first adjustment.

Lemma 3.10. *Recalling that*

$$V^{dr}(x) = A_1e^{\theta x} + A_2e^{\zeta x} = \frac{MPe^{\zeta b}}{\lambda\theta(e^{\theta b}-e^{\zeta b})}e^{\theta x} - \frac{MPe^{\theta b}}{\lambda\zeta(e^{\theta b}-e^{\zeta b})}e^{\theta x},$$

we define

$$\begin{aligned}
\psi(k,x) := e^{\theta x}&\Bigg\{\frac{V^{dr}(\ln(k))e^{\frac{\mu}{\sigma^2}b}}{\mathrm{sh}\big(\frac{\sqrt{\mu^2+2\delta\sigma^2}}{\sigma^2}b\big)} - \Big((k-1)\frac{MP}{\lambda} + V^{dr}(\ln(k))\Big)\frac{e^{-\frac{\sqrt{\mu^2+2\delta\sigma^2}}{\sigma^2}b}}{\mathrm{sh}\big(\frac{\sqrt{\mu^2+2\delta\sigma^2}}{\sigma^2}b\big)}\Bigg\} \\
+ e^{\zeta x}&\Bigg\{\Big((k-1)\frac{MP}{\lambda} + V^{dr}(\ln(k))\Big)\frac{e^{\frac{\sqrt{\mu^2+2\delta\sigma^2}}{\sigma^2}b}}{\mathrm{sh}\big(\frac{\sqrt{\mu^2+2\delta\sigma^2}}{\sigma^2}b\big)} - \frac{V^{dr}(\ln(k))e^{\frac{\mu}{\sigma^2}b}}{\mathrm{sh}\big(\frac{\sqrt{\mu^2+2\delta\sigma^2}}{\sigma^2}b\big)}\Bigg\}.
\end{aligned}$$

The function $k^*(x)$ implicitly given by

$$\psi_k(k,x) = 0$$

is strictly increasing, continuously differentiable in x, $k^*(x) \in (1, 1.25)$ for $x \in (0, b)$ and fulfils $k^*(x) < e^x$ for $x \in (0, b)$, $k^*(0) = 1$, $k^*(b) = 1.25$.

Proof. See Appendix. □

The function $\ln(k^*(x))$ has 2 fixed points: $x = 0$ and $x = b$. Note that $x = 0$ is an attracting fixed point, whereas $x = b$ is repelling. Thus, starting at any $x \in (0, b)$ the function k^* would generate a sequence $k_n^* = k^*(\ln(k_{n-1}^*))$ converging to 1, i.e. $\ln(k_n^*)$ would converge to 0. Starting at $x = b$ produces a constant strategy $\kappa \equiv 1.25$, but starting at any $x < b$ will never lead back to 1.25. The following example illustrates the recursive strategy $k_n^* = k^*(\ln(k_{n-1}^*))$.

Example 3.11. For $\mu = 0.02$, $\delta = 0.01$, $\sigma = 0.1$, $M = 10000$, $P = 100$, $\lambda = 0.02$, the function $\ln(k^*(x))$ in dependence on x is illustrated in Fig. 3. The dashed line represents the identity function x. We see that the function $\ln(k^*(x))$ lies indeed below x, i.e. $k^*(x) < e^x$.

Also, we see that a recursive application of k^* on $x = 0.15$ generates a decreasing sequence (intersections of the grey line with the black solid line representing the function $k^*(x)$) of $k_n^* = k(\ln(k_{n-1}^*))$ converging to 1 as $n \to \infty$.

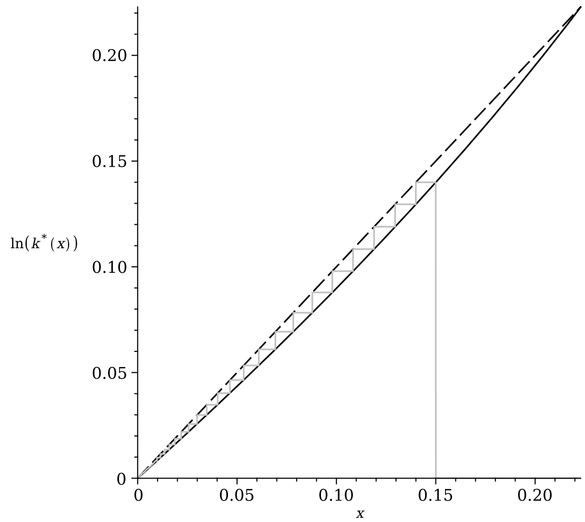


Fig. 3. The level $k^*(x)$ on the logarithmic scale (solid line) compared to the linear function x (dashed line). The strategy κ^* for the initial value $x=0.15$ (grey).

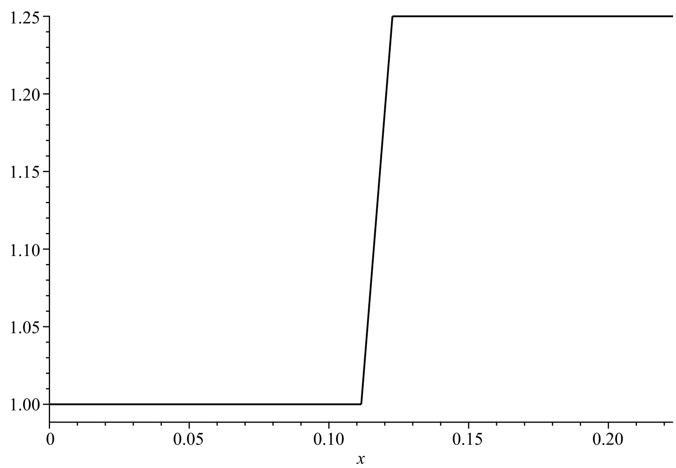


Fig. 4. The strategy after the second step in the strategy improvement recursion in Remark 3.12.

Remark 3.12. The optimal strategy can be found via strategy improvement approach. In the first step, one finds the minimum point k of the new function

$$\psi(k,x;1) := \frac{MP(k-1)}{\lambda}\mathbb{E}_x[e^{-\delta T_1}\mathbb{1}_{[X_{T_1-}=0]}] + \mathbb{E}_x[e^{-\delta T_1}]\psi\big(k^*(\ln(k)),\ln(k)\big)\,.$$

Note that for any $k\in(1,1.25)$, the function $\psi(k,x;1)$ is the return function corresponding to the strategy: "adjust the DCC to k after the first exit from $(1,1.25)$, adjust the DCC to $k^*(\ln(k))$ at the second exit from $(1,1.25)$ and follow the strategy κ^{dr} after". Using $\psi_k\big(k^*(\ln(k)),\ln(k)\big)=0$ one gets

$$\begin{aligned}\psi_k(k,x;1) &= \frac{MP}{\lambda}\mathbb{E}_x[e^{-\delta T_1}\mathbb{1}_{[X_{T_1-}=0]}]\\ &\quad+\frac{1}{k}\mathbb{E}_x[e^{-\delta T_1}]\Big\{\psi_k\big(k^*(\ln(k)),\ln(k)\big)\cdot(k^*)'(\ln(k))+\psi_x\big(k^*(\ln(k)),\ln(k)\big)\Big\}\\ &= \frac{MP}{\lambda}\mathbb{E}_x[e^{-\delta T_1}\mathbb{1}_{[X_{T_1-}=0]}]+\frac{1}{k}\mathbb{E}_x[e^{-\delta T_1}]\psi_x\big(k^*(\ln(k)),\ln(k)\big)\,.\end{aligned}$$

In Fig. 4 one sees the function $k^*(x;1)$ minimising $\psi_k(k,x;1)$ with respect to k. The function has 3 areas: $k\equiv 1$, $k\equiv 1.25$ and a curve in between.

Define recursively for $n\ge 2$

$$\psi(k,x;n) := \frac{MP(k-1)}{\lambda}\mathbb{E}_x[e^{-\delta T_1}\mathbb{1}_{[X_{T_1-}=0]}] + \mathbb{E}_x[e^{-\delta T_1}]\psi\big(k^*(\ln(k);n-1),\ln(k);n-1\big)\,,$$

where $k^*(x;n-1)$ is the minimum point k of the function $\psi(k,x;n-1)$. Proceeding in this way, calculating the minimum point k for every step n, would lead to an optimal strategy.

An optimal strategy $k^*(x;opt)$ obtained as a limit of $k^*(x;n)$ will be admissible, i.e. in $\bar{\mathcal{A}}$, by construction.

The optimal adjustment level will depend on the initial value of the DCC. However, such a strategy would produce some problems concerning its practical implementation. We omit the explicit derivation of the optimal strategy as it is only of technical interest as it is explained in the following remark.

Remark 3.13 *(Practical issues of strategies' implementation).*
1. We have already pointed out that the next action (i.e. capital injection or reduction of DCC) has to be determined either at the initial time $t = 0$ or right after the most recent claim occurrence time T_n. While the optimal strategies are then totally determined, their main drawback is that they depend on the initial time $t = 0$ and the initial value of x of DCC. If the initial value of DCC has been inside the interval $[1, 1.25]$ then the next action k_n^* depends on the index n and thus on the past. Hence, two different pensioners entering the decumulation phase at different times t_1 and t_n, will experience different next actions by the fund provider although they might have entered the decumulation phase when DCC had exactly the same value x.

To cope with this slightly irritating issue there are two natural solutions,

- either the pension provider has to continuously announce the next action value k_n^*, i.e. the height of the capital injection or the bonus payment,
- or – to simplify the situation – use only constant strategies and of course make them public. In the latter case, we still have to calculate an optimal constant strategy. Fig. 1 illustrating $(k, x) \mapsto V^k(x)$ suggests, for instance, that the optimal constant strategy is $V^{1.25}$ for the chosen set of parameters.

As all the values k_n^* are already computed before the decumulation phase of the first customers start, we go for the first suggested solution.
2. As a similar issue, we assume that the entrance of a new customer to the group of pensioners at time t does not change the DCC-value of C_t. This simply means that for gaining the standardised expected pension of P/λ, the pensioner has to bring in an accumulated wealth of $C_t \cdot P/\lambda$.

3.2. Bonus payments

After the value of the expected discounted claims – adjustments of DCC – to be paid by the insurance company has been found, one can look at the corresponding value of the accumulated discounted bonus payments to the pensioners. The insurance company – keeping DCC above 1 for reputational reasons – may want to participate in the gains of the fund if the DCC level hits 1.25. Therefore, it makes sense to compare the expected discounted adjustment payments and the expected discounted bonus payments with the target to let the insurance company participate in the gains and thus enable it to compensate the expected loss from the DCC-adjustments.

Denote the value of the bonus payments corresponding to an admissible strategy $\kappa = (k_n)_{n\in\mathbb{N}}$ with T_n^κ given in (3) by J^κ. Then,

$$J^\kappa(x) = \frac{MP}{\lambda}\mathbb{E}_x\Big[\sum_{n=1}^{\infty} e^{-\delta T_n^\kappa}(1.25 - k_n)\mathbb{1}_{[X^\kappa_{T_n-}=b]}\Big].$$

Especially, denote by J^k the return function for the constant strategy $\kappa \equiv k$. Then,

$$J^k(x) = (1.25 - k)\frac{MP}{\lambda}\sum_{n=1}^{\infty}\mathbb{E}_x[e^{-\delta T_n^\kappa}\mathbb{1}_{[X^\kappa_{T_n-}=b]}].$$

Like the function V^k, its counterpart function J^k solves Differential equation (9) with the boundary conditions $J^k(0) = J^k(\ln(k))$ and $J^k(b) = \frac{MP}{\lambda}(1.25 - k) + J^k(\ln(k))$. Now, we compare the two functions J^1 and V^1.

Theorem 3.14. *If $\theta \le 1$, then for $x \in [0, b]$ the expected discounted bonus payments $J^1(x)$ are strictly larger than the value of expected discounted costs $V^1(x)$.*

Proof. See Appendix. □

3.2.1. How to share gains and losses – Participation in a bonus scheme

Theorem 3.14 is a key result for our whole approach. It states that the expected sum of bonus payments strictly exceeds the expected costs of keeping the DCC above one for the case of $\theta < 1$. Thus, we can in this case apply classical insurance principles for sharing the gains.[2] Let us first point out that for reputational reasons, the insurer takes over all the costs to keep the DCC above zero. This situation can be interpreted as a classical non-life insurance contract – the insurer pays the clients' losses. For the premium, the insurer can apply the expected value principle and claim $(1 + \eta)V^1, \eta > 0$, as a share in the gains. The remaining part of $J^1 - (1 + \eta)V^1$ will then be assigned to the policy holder as a bonus payment. While in theory these payments are infinitesimal payments, the values in reality will be strictly positive for gains and strictly negative for losses due to the impossibility of continuous rebalancing. Thus, the positive gains are shared according to the rule that the pension provider gets a fraction of $(1 + \eta)V^1/J^1$ of every positive bonus. Of course, determining the exact height of η is the same problem as when it is determined in the usual case of an insurance premium.

Let us also point out that the assumption of $\theta \le 1$ is a natural one. The relation $\theta > 1$, is equivalent to $\mu + \sigma^2/2 < \delta$, i.e. the fund would perform worse in expectation than the riskless investment at the interest rate δ, an absolutely not acceptable performance. A risk-averse

[2] For readers interested in the valuation of with-profit contracts we refer, for instance, to Grosen and Jørgensen (2000), Jensen et al. (2001), Bacinello (2003), or Kling et al. (2007) and references therein.

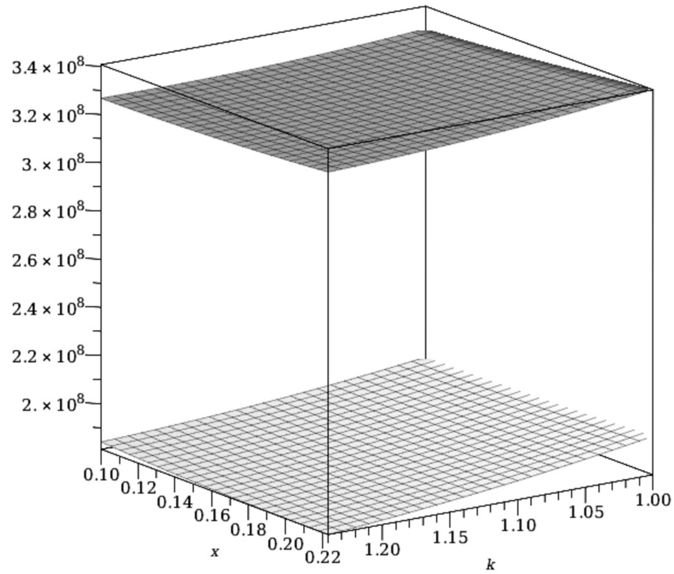


Fig. 5. The functions $J^k(x)$ (the upper function), $V^k(x)$ (the lower function).

insurer, in particular, a pension provider would only invest in a risky position if the relation $\theta < 1$ is valid. Thus, while managing their funds targeting $\theta < 1$ is mandatory for a pension provider.

Example 3.15. Let again $\mu = 0.02$, $\delta = 0.01$, $\sigma = 0.1$, $M = 10000$, $P = 100$, $\lambda = 0.02$. Fig. 5 illustrates the functions V^k and J^k for different values of x. We clearly see that the insurance company can, in expectation, get back the money paid to avoid pension cuts, and even get compensated for the undertaken risk if the expected value principle is used for the bonus sharing.

Remark 3.16 *(Alternative Ways of Avoiding Pension Cuts).* The characteristic feature of our approach is that we assume a free fund that is not protected against downside risks. As we want to avoid a loss of reputation, we believe that the introduction of pension cuts is unacceptable. Also, the counterpart to the bonus payments, a kind of malus payments from the policy holders to the pension provider to stay at the same pension level, would lead to a substantial loss of reputation.

Thus, only the introduction of protections of the fund value remain as alternatives. Of course, buying a series of put options with a suitable strike seems to be a solution. However, why should an option on the fund be traded at all? Following a corresponding replication strategy to imitate the options' payments then bears the classical risk of limited liquidity in the market. A similar issue occurs with the gap risk of using a CPPI strategy, besides the fact that it will change the return characteristics of the fund.

We thus believe that our approach is the only simple way that avoids a loss in reputation.

4. Conclusion

Two essential goals in pension finance are guaranteeing the solvency of the plan and maintaining confidence in the scheme among participants in the sense of harmonising their expectations with the economic and financial realities of the plan. In the interests of long-term solvency, pension plan reforms involve unpopular measures for current workers and pensioners, i.e., cuts in pension benefits or increases in contributions, amongst others.

While a capital injection can be very costly, a pension cut might lead to a significant loss in reputation of the provider with even more serious consequences for the pension provider, i.e., a decrease in the amount of contributions. A capital injection made by the pension provider can be a solution if it can be financed.

In this paper, we have therefore formulated suitable control problems to determine cost minimizing strategies to avoid pension cuts. These strategies can be (semi-) explicitly calculated and are illustrated for particular examples. As a further conceptual ingredient, we have introduced the concept of bonus payments instead of an adaptation of the pensions. These bonus payments are due whenever DCC exits the given target interval on the upper side.

Our main result shows that under a reasonable fund performance indeed a participation of the provider in the bonus payments is possible. Moreover, we can show that – at least in expectation – the bonus payments are more than sufficient to finance the capital injections. Therefore, the proposed strategies would guarantee a certain level of pension to the policyholders and at the same time maintain the solvency of the plan.

Declaration of competing interest

The authors have no competing interest.

Data availability

No data was used for the research described in the article.

Acknowledgements

The research of Julia Eisenberg was funded by the Austrian Science Fund (FWF), Project number V 603-N35.
M. Carmen Boado-Penas acknowledges the Grant PID2020-114563GB-I00 funded by MCIN/AEI/10.13039/501100011033.
All authors would like to thank the unknown referees for their helpful comments and suggestions.

Appendix A

Proof of Lemma 3.5. **1.** Consider the constant strategy $\kappa \equiv 1.25$. The function $V^{1.25}$ as given in (5) is twice continuously differentiable in $(0,b)$, solves

$$\frac{\sigma^2}{2} f_{xx} + \mu f_x - \delta f = 0 \tag{9}$$

and fulfils $V_x^{1.25}(b) = 0$, $V^{1.25}(0) = \frac{MP \cdot 0.25}{\lambda} + V^{1.25}(b)$. Also, because $\theta > 0$, $\zeta < 0$ the coefficients B_1 and B_2 are strictly positive, implying $V^{1.25} > 0$ and consequently $V_{xx}^{1.25} > 0$.

Let Z_t denote the reflection term at b and let $(\tilde{T}_n)_{n\in\mathbb{N}}$ denote the times when the process, reflected at b, hits zero and $\tilde{\mathcal{N}}_t = \sup\{n : \tilde{T}_n \le t\}$. Since $\frac{\sigma^2}{2} V_{xx}^{1.25} + \mu V_x^{1.25} - \delta V^{1.25} = 0$, $V_x^{1.25}(b) = 0$ and $V^{1.25}(b) + \frac{MP \cdot 0.25}{\lambda} = V^{1.25}(0)$, Ito's formula (Protter (2005), p. 221, Theorem 71) yields

$$\begin{aligned} e^{-\delta t} V^{1.25}(X_t^\kappa) &= V^{1.25}(x) + \sum_{0<s\le t} e^{-\delta s}\{V^{1.25}(X_s^\kappa) - V^{1.25}(X_{s-}^\kappa)\} + \int_0^t e^{-\delta s} \sigma V_x^{1.25}\, \mathrm{d}W_s \\ &= V^{1.25}(x) - \sum_{n=1}^{\tilde{\mathcal{N}}_t} e^{-\delta \tilde{T}_n} \frac{MP \cdot 0.25}{\lambda} \mathbb{1}_{[X_{\tilde{T}_n -}=0]} + \int_0^t e^{-\delta s} \sigma V_x^{1.25}\, \mathrm{d}W_s\,. \end{aligned}$$

As $V^{1.25}$ and $V_x^{1.25}$ are bounded, taking expectations and letting $t \to \infty$ yields

$$V^{1.25}(x) = \frac{MP \cdot 0.25}{\lambda} \mathbb{E}_x\Big[\sum_{n=1}^{\infty} e^{-\delta \tilde{T}_n} \mathbb{1}_{[X_{\tilde{T}_n -}=0]}\Big]\,.$$

2. Consider the constant strategy $\kappa \equiv 1$. The function V^1 as given in (6) is continuously differentiable in $(0,b)$, solves (9) with $V^1(b) = V^1(0)$.

The coefficients C_1 and C_2 in (6) are strictly positive. This follows from the fact that the denominator $\theta(1 - e^{\zeta b}) - \zeta(1 - e^{\theta b})$ is strictly decreasing in b attaining 0 at $b = 0$. Therefore, we can conclude that $V_{xx}^1 > 0$, i.e. V^1 is strictly convex.

Similar to **1.** above, we apply Ito's formula (Protter (2005), p. 221, Theorem 71). Denoting by Y_t the reflection term of the log DCC process at 0, one gets:

$$\begin{aligned} e^{-\delta t} V^1(X_t^\kappa) &= V^1(0) + \sum_{0<s\le t} e^{-\delta s}\{V^1(X_s^\kappa) - V^1(X_{s-}^\kappa)\} + \int_0^t e^{-\delta s} \sigma V_x^1\, \mathrm{d}W_s + \int_0^t e^{-\delta s} V_x^1\, \mathrm{d}Y_s \\ &= V^1(x) + \int_0^t e^{-\delta s} \sigma V_x^1\, \mathrm{d}W_s + \int_0^t e^{-\delta s} V_x^1\, \mathrm{d}Y_s\,. \end{aligned}$$

Note that $\mathrm{d}Y_t \neq 0$ only if $V_x^1 = -\frac{MP}{\lambda}$, i.e. at $x = 0$. Taking expectations and letting $t \to \infty$, one gets

$$V^1(x) = \frac{MP}{\lambda} \mathbb{E}_x\Big[\int_0^\infty e^{-\delta s}\, \mathrm{d}Y_s\Big]\,. \quad \square$$

Proof of Lemma 3.7. The function in (7) solves Differential equation (9) with boundary conditions $f_x(0) = -\frac{MP}{\lambda}$, $f_x(b) = 0$. Denote by Y_t the reflection term at 0 and by Z_t the reflection term at b like described above. Then, using Ito's formula (Protter (2005), p. 221, Theorem 71)

$$V^{dr}(X_t^{dr}) = V^{dr}(x) + \int_0^t e^{-\delta s}\, \mathrm{d}W_s + \int_0^t V_x^{dr}\, \mathrm{d}Y_t - \int_0^t V_x^{dr}\, \mathrm{d}Z_t\,.$$

The stochastic integral is a martingale. The reflection term Y_t changes only if the process X^{dr} equals 0, the term Z_t changes if $X^{dr} = b$. Applying expectations and letting $t \to \infty$ yields the desired result. $\square$

Proof of Proposition 3.9. Every strategy in the closure $\bar{\mathcal{A}}$ can be approximated by a sequence $(\kappa_n)_{n\ge 1}$, $\kappa_n \in \mathcal{A}$. Therefore, it suffices to consider the strategies $\kappa \in \mathcal{A}$. Let $\kappa = (k_n)_{n\in\mathbb{N}}$ be an arbitrary admissible strategy from $\mathcal{A}$ and $X_t^\kappa = \ln(C_t^\kappa)$, i.e. the logarithm of the DCC process under κ. Like in Definition 3.1, we denote by T^κ the exit times of X^κ from the interval $(0, b)$ and by N_t^κ the number of exits up to time t. Then, by Ito's formula (Protter (2005), p. 221, Theorem 71), we get

$$e^{-\delta t}V^{dr}(X_t^\kappa) = V^{dr}(x) + \int_0^t e^{-\delta s}\Big\{\frac{\sigma^2}{2}V_{xx}^{dr} + \mu V_x^{dr} - \delta V^{dr}\Big\}\,\mathrm{d}s + \int_0^t e^{-\delta t}\sigma V_x^{dr}\,\mathrm{d}W_s$$
$$+ \sum_{0<s\le t} e^{-\delta s}\big\{V^{dr}(X_s^\kappa) - V^{dr}(X_{s-}^\kappa)\big\}$$

Since V^{dr} solves Differential equation (9) and V_x^{dr} is bounded, the $\mathrm{d}s$ integral equals 0, and the stochastic integral is a martingale with 0-expectation. Consider now the last term. Note that the function V^{dr} is convex and decreasing (see Lemma 3.5), meaning that for all $x, y \in [0, b]$:

$$V^{dr}(y) - V^{dr}(x) \ge V_x^{dr}(x)(y-x)\,.$$

Letting for convenience $k_{-1} := e^x$ and noting that $-\ln(x)$ is a convex function on $[1, \infty)$, one gets

$$\begin{aligned}\sum_{0<s\le t} e^{-\delta s}\big\{V^{dr}(X_s^\kappa) - V^{dr}(X_{s-}^\kappa)\big\} &= \sum_{i=1}^{N_t^\kappa} e^{-\delta T_i^\kappa}\big\{V^{dr}(X_{T_i}^\kappa) - V^{dr}(X_{T_i-}^\kappa)\big\}\\ &\ge \sum_{i=1}^{N_t^\kappa} e^{-\delta T_i^\kappa} V_x^{dr}(X_{T_i-}^\kappa)\big(X_{T_i}^\kappa - X_{T_i-}^\kappa\big)\\ &= -\frac{MP}{\lambda}\sum_{i=1}^{N_t^\kappa} e^{-\delta T_i^\kappa}\ln(k_i)1\!\!1_{[X_{T_i-}^\kappa=0]}\\ &\ge -\frac{MP}{\lambda}\sum_{i=1}^{N_t^\kappa} e^{-\delta T_i^\kappa}\big(k_i-1\big)1\!\!1_{[X_{T_i-}^\kappa=0]}\,.\end{aligned}$$

In total,

$$\mathbb{E}_x\big[e^{-\delta t}V^{dr}(X_t^\kappa)\big] \ge V^{dr}(x) - \frac{MP}{\lambda}\mathbb{E}_x\Big[\sum_{i=1}^{N_t^\kappa} e^{-\delta T_i^\kappa}\big(k_i-1\big)1\!\!1_{[X_{T_i-}^\kappa=0]}\Big]\,.$$

Since $V^{dr}(x)$ is bounded, the lhs of the above inequality converges to 0 as $t\to\infty$, and one gets $V^\kappa(x) \ge V^{dr}(x)$. □

Proof of Lemma 3.10. • First, we derive the function $\psi(k, x)$ with respect to k. It holds that

$$\psi_k(k,x) = \frac{e^{\theta x}}{k}\Bigg\{\frac{V_x^{dr}(\ln(k))e^{\frac{\mu}{\sigma^2}b}}{\mathrm{sh}\big(\frac{\sqrt{\mu^2+2\delta\sigma^2}}{\sigma^2}b\big)} - \Big(k\frac{MP}{\lambda} + V_x^{dr}(\ln(k))\Big)\frac{e^{-\frac{\sqrt{\mu^2+2\delta\sigma^2}}{\sigma^2}b}}{\mathrm{sh}\big(\frac{\sqrt{\mu^2+2\delta\sigma^2}}{\sigma^2}b\big)}\Bigg\}$$
$$+ \frac{e^{\zeta x}}{k}\Bigg\{\Big(k\frac{MP}{\lambda} + V_x^{dr}(\ln(k))\Big)\frac{e^{\frac{\sqrt{\mu^2+2\delta\sigma^2}}{\sigma^2}b}}{\mathrm{sh}\big(\frac{\sqrt{\mu^2+2\delta\sigma^2}}{\sigma^2}b\big)} - \frac{V_x^{dr}(\ln(k))e^{\frac{\mu}{\sigma^2}b}}{\mathrm{sh}\big(\frac{\sqrt{\mu^2+2\delta\sigma^2}}{\sigma^2}b\big)}\Bigg\}\,.$$

• In order to simplify ψ_k, we define an auxiliary function

$$\begin{aligned}\xi(k,x) &:= \psi_k(k,x)\frac{\mathrm{sh}\big(\frac{\sqrt{\mu^2+2\delta\sigma^2}}{\sigma^2}b\big)}{e^{\frac{\mu}{\sigma^2}}}\\ &= e^{\theta x}\Big\{\frac{V_x^{dr}(\ln(k))}{k}\big(1-e^{\zeta b}\big) - e^{\zeta b}\frac{MP}{\lambda}\Big\} + e^{\zeta x}\Big\{\frac{V_x^{dr}(\ln(k))}{k}\big(e^{\theta b}-1\big) + e^{\theta b}\frac{MP}{\lambda}\Big\}\,.\end{aligned}$$

Note that for $k \in (1, 1.25)$ it holds that

$$\begin{aligned}&\frac{V_x^{dr}(\ln(k))}{k}\big(1-e^{\zeta b}\big) - e^{\zeta b}\frac{MP}{\lambda} \le -e^{\zeta b}\frac{MP}{\lambda} < 0\,,\\ &\frac{V_x^{dr}(\ln(k))}{k}\big(e^{\theta b}-1\big) + e^{\theta b}\frac{MP}{\lambda} \ge -\frac{MP}{k\lambda}\big(e^{\theta b}-1\big) + e^{\theta b}\frac{MP}{\lambda} > 0\,.\end{aligned}$$

Therefore, $\xi_x < 0$ and, because $\phi'' > 0$ and $\phi' \le 0$, it also holds that $\xi_k > 0$.

Further, using $\phi'(0) = -\frac{MP}{\lambda}$ and $\phi'(b) = 0$, we get for $x \in (0, b)$:

$$\xi(1, x) = e^{\theta x}\Big\{ -\frac{MP}{\lambda}\big(1 - e^{\zeta b}\big) - e^{\zeta b}\frac{MP}{\lambda}\Big\} + e^{\zeta x}\Big\{ -\frac{MP}{\lambda}\big(e^{\theta b} - 1\big) + e^{\theta b}\frac{MP}{\lambda}\Big\} < 0\,,$$
$$\xi(1.25, x) = e^{(\theta+\zeta)b}\frac{MP}{\lambda}\big\{-e^{\theta(x-b)} + e^{\zeta(x-b)}\big\} > 0\,.$$

That is, there is a unique function $k^*(x)$, continuously differentiable and strictly increasing in x with $\xi(k^*(x), x) = 0$. Also, we can conclude that

$$\psi_k(k, x) = \xi(k, x)\frac{e^{\frac{\mu}{\sigma^2}}}{\mathrm{sh}\big(\frac{\sqrt{\mu^2+2\delta\sigma^2}}{\sigma^2}b\big)}\begin{cases} < 0 & k < k^*(x), \\ > 0 & \text{otherwise}, \end{cases}$$

meaning that $k^*(x)$ is indeed the minimum of $\psi(k, x)$.

- At $x = 0$ and at $x = b$ it holds that $k^*(0) = 1$, $k^*(b) = 1.25$.
- It remains to prove $k^*(x) < e^x$ for $x \in (0, b)$. It suffices to show that $\xi(e^x, x) > 0$ for $x \in (0, b)$.
  It holds that

$$\xi(e^x, x) = e^{\theta x}\Big\{\phi'(x)e^{-x}\big(1 - e^{\zeta b}\big) - e^{\zeta b}\frac{MP}{\lambda}\Big\} + e^{\zeta x}\Big\{\phi'(x)e^{-x}\big(e^{\theta b} - 1\big) + e^{\theta b}\frac{MP}{\lambda}\Big\}$$
$$= \phi'(x)e^{-x}\Big\{e^{\theta x}\big(1 - e^{\zeta b}\big) + e^{\zeta x}\big(e^{\theta b} - 1\big)\Big\} - \Big\{e^{\theta x}\theta\frac{e^{\zeta b}\cdot MP}{\lambda\theta} - e^{\zeta x}\zeta\frac{e^{\theta b}\cdot MP}{\lambda\zeta}\Big\}$$
$$= \phi'(x)e^{-x}\Big\{e^{\theta x}\big(1 - e^{\zeta b}\big) + e^{\zeta x}\big(e^{\theta b} - 1\big)\Big\} - \phi'(x)(e^{\theta b} - e^{\zeta b})$$
$$= \phi'(x)\Big\{e^{\theta x - x}\big(1 - e^{\zeta b}\big) + e^{\zeta x - x}\big(e^{\theta b} - 1\big) - e^{\theta b} + e^{\zeta b}\Big\}\,.$$

Let

$$g(x) := e^{\theta x - x}\big(1 - e^{\zeta b}\big) + e^{\zeta x - x}\big(e^{\theta b} - 1\big) - e^{\theta b} + e^{\zeta b}\,.$$

One easily gets $g(0) = 0$, $g''(x) > 0$. At $x = b$ one has

$$g(b) = e^{\theta b - b}\big(1 - e^{\zeta b}\big) + e^{\zeta b - b}\big(e^{\theta b} - 1\big) - e^{\theta b} + e^{\zeta b} = e^{-b}(e^{\theta b} - e^{\zeta b}) - e^{\theta b} + e^{\zeta b} < 0\,.$$

Therefore, we can conclude that $g(x) < 0$ on $(0, b)$, leading to $\xi(e^x, x) > 0$ on $(0, b)$. This means $k^*(x) < e^x$ for $x \in (0, b)$. □

Proof of Lemma 3.14. The function V^1 is decreasing and the function J^1 is increasing. Therefore, it suffices to look at the values $J^1(0)$ and $V^1(0)$. It holds that

$$J^1(0) = -\frac{MP(e^b - 1)}{\lambda}\cdot\frac{\theta - \zeta}{\theta(1 - e^{\zeta b}) - \zeta(1 - e^{\theta b})}\,,$$
$$V^1(0) = \frac{MP}{\lambda}\cdot\frac{(e^{\zeta b} - e^{\theta b})}{\theta(1 - e^{\zeta b}) - \zeta(1 - e^{\theta b})} = \frac{e^{\theta b} - e^{\zeta b}}{(\theta - \zeta)(e^b - 1)}\cdot J^1(0)\,, \tag{10}$$

where we used $1.25 = e^b$.

- We define an auxiliary function

$$\xi(x) := \frac{e^{\theta x} - e^{\zeta x}}{(\theta - \zeta)(e^x - 1)}$$

representing the factor in front of $J^1(0)$ in (10).

Note first that the following holds true:

$$\lim_{x\to 0}\xi(x) = 1 \quad \text{and} \quad \lim_{x\to\infty}\xi(x) = \begin{cases} \infty & : \theta > 1, \\ \frac{1}{\theta-\zeta} & : \theta = 1, \\ 0 & : \theta < 1. \end{cases}$$

- We are interested in the behaviour of the function ξ on $(0, \infty)$. Note that if $\theta \le 1$ it holds for $x > 0$ that

$$\theta e^{\theta x} - \zeta e^{\zeta x} - (\theta - \zeta)e^x < \zeta(e^x - e^{\zeta x}) < 0\,,$$

meaning that $e^{\theta x} - e^{\zeta x} - (\theta - \zeta)(e^x - 1)$ is strictly decreasing and negative for $x > 0$. Therefore, we conclude $\xi(x) < 1$ for all $x > 0$, meaning that $V^1(0) < J^1(0)$. □

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