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Inf-convolution and optimal allocations for mixed-VaRs

Zichao Xia, Zhenfeng Zou, Taizhong Hu *

International Institute of Finance, School of Management, University of Science and Technology of China, Hefei, Anhui 230026, China

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A mixed Value-at-Risk (VaR) is a two-parameter quantile-based risk measure, which is a convex combination of left-VaR and right-VaR. In this paper, we investigate optimal allocations in a risk sharing problem where the objectives of agents are mixed-VaRs. Explicit formulas of the inf-convolution and Pareto optimal allocations are obtained. The worst-case mixed VaR under model uncertainty is also presented.

1. Introduction

Let $(\Omega, \mathscr{F}, \mathrm{P})$ be an atomless probability space, and $\mathcal{X}$ be a convex cone of random variables defined on $(\Omega, \mathscr{F}, \mathrm{P})$. A risk measure is a functional $\rho : \mathcal{X} \to [-\infty, +\infty)$ (Artzner et al., 1999; Föllmer and Schied, 2016). In a risk sharing problem, there are n agents equipped with respective risk measures $\rho_1, \ldots, \rho_n$. Let $X \in \mathcal{X}$ denote the total risk, which is shared by n agents. X is splitted into an allocation $(X_1, \ldots, X_n) \in \mathbb{A}_n(X)$ among n agents, where $\mathbb{A}_n(X)$ is the set of all possible allocation of X, defined as

$$\mathbb{A}_n(X) = \left\{ (X_1, \ldots, X_n) \in \mathcal{X}^n : \sum_{i=1}^n X_i = X \right\}.$$

We refer to $\sum_{i=1}^n \rho_i(X_i)$ as the aggregate risk value of n agents under the allocation $(X_1, \ldots, X_n)$ of X. The inf-convolution of risk measures $\rho_1, \ldots, \rho_n$ is the mapping $\square_{i=1}^n \rho_i : \mathcal{X} \to [-\infty, +\infty)$, defined as

$$\square_{i=1}^n \rho_i(X) = \inf \left\{ \sum_{i=1}^n \rho_i(X_i) : (X_1, \ldots, X_n) \in \mathbb{A}_n(X) \right\}, \quad X \in \mathcal{X}. \tag{1.1}$$

An n-tuple $(X_1, \ldots, X_n) \in \mathbb{A}_n(X)$ is called an optimal allocation of X for $(\rho_1, \ldots, \rho_n)$ if $\square_{i=1}^n \rho_i(X) = \sum_{i=1}^n \rho_i(X_i)$. For more on inf-convolution, see Barrieu and El Karoui (2005), Acciaio (2007), Jouini et al. (2008), Filipović and Svindland (2008) and Tsanakas (2009) for the case of convex risk measures, Embrechts et al. (2018), Wang and Wei (2020) and Liu et al. (2022) for the non-convex case, and Liu et al. (2020) for conditions under which law invariance of inf-convolution holds.

Value-at-Risk (VaR) is the most common risk measure used in banking and finance. VaR is a non-convex risk measure and has two versions, the left- and right-quantiles. Let L^0 be the set of all random variables on $(\Omega, \mathscr{F}, \mathrm{P})$, and let L^r denote the set of all random

* Corresponding author.
E-mail addresses: zichaox@mail.ustc.edu.cn (Z. Xia), newzzf@mail.ustc.edu.cn (Z. Zou), thu@ustc.edu.cn (T. Hu).

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variables with finite rth moment, where $r > 0$. For any $X \in L^0$, a positive (negative) value of X represents a financial loss (profit), and F_X represents the distribution function of X. The left-VaR of X at confidence level $\alpha \in (0,1)$ is defined as

$$\mathrm{VaR}^L_\alpha(X) := F_X^{-1}(1-\alpha) = \inf\{x \in \mathbb{R} : F_X(x) \geq 1-\alpha\}, \quad X \in L^0,$$

and the right-VaR of X at confidence level $\alpha \in (0,1)$ is defined as

$$\mathrm{VaR}^R_\alpha(X) = \inf\{x \in \mathbb{R} : F_X(x) > 1-\alpha\}, \quad X \in L^0.$$

In addition, let $\mathrm{VaR}^L_0(X) = \mathrm{VaR}^R_0(X) = \text{ess-sup}(X) = \sup\{x \in \mathbb{R} : F_X(x) < 1\}$ and $\mathrm{VaR}^L_1(X) = \mathrm{VaR}^R_1(X) = \text{ess-inf}(X) = \inf\{x \in \mathbb{R} : F_X(x) > 0\}$. For the role of left-quantile (VaR^L) and right quantile (VaR^R) as risk measures, see the discussion in Acerbi and Tasche (2002) and Liu et al. (2022, Remark 5).

Embrechts et al. (2018) addressed the problem of risk sharing among n agents with a two-parameter class of quantile-based risk measure, the so-called Range-Value-at-Risk (RVaR). As a special consequence of the main result on RVaR, Corollary 2 in Embrechts et al. (2018) gives the VaR inf-convolution formula

$$\mathrm{VaR}^L_{\alpha_1} \square \mathrm{VaR}^L_{\alpha_2} = \mathrm{VaR}^L_{\alpha_1+\alpha_2}, \tag{1.2}$$

where $\alpha_1, \alpha_2 > 0$ such that $\alpha_1 + \alpha_2 < 1$. Liu et al. (2022) extended (1.2) to the inf-convolution for the case of a mixed collection of VaR^L and VaR^R, and the case of VaR^L or VaR^R and another tail risk measure, and obtain explicit forms of the inf-convolution as well as the corresponding optimal allocations. In particular, they obtain that

$$\mathrm{VaR}^L_{\alpha_1} \square \mathrm{VaR}^R_{\alpha_2} = \mathrm{VaR}^R_{\alpha_1} \square \mathrm{VaR}^L_{\alpha_2} = \mathrm{VaR}^R_{\alpha_1} \square \mathrm{VaR}^R_{\alpha_2} = \mathrm{VaR}^R_{\alpha_1+\alpha_2} \tag{1.3}$$

with $\alpha_1, \alpha_2 > 0$ such that $\alpha_1 + \alpha_2 < 1$, and their optimal allocations are presented in (3.1) and (3.2).

The purpose of this paper is to investigate optimal allocations in a risk sharing problem (1.1) where the objectives of agents are mixed-VaRs. A mixed-VaR is also a two-parameter quantile-based risk measure, which is defined as

$$\mathrm{VaR}^\lambda_\alpha(X) = (1-\lambda)\mathrm{VaR}^L_\alpha(X) + \lambda \mathrm{VaR}^R_\alpha(X), \quad X \in L^0,$$

where $\lambda \in [0,1]$. In particular, when $\lambda = 0, 1$, $\mathrm{VaR}^0_\alpha = \mathrm{VaR}^L_\alpha$ and $\mathrm{VaR}^1_\alpha = \mathrm{VaR}^R_\alpha$. For more on λ-quantile (mixed VaR) and its applications, see Dhaene et al. (2002) and Dhaene et al. (2012). For mixed-VaR, by monotonicity of inf-convolution and (1.3), we have

$$\mathrm{VaR}^\lambda_{\alpha_1} \square \mathrm{VaR}^R_{\alpha_2} = \mathrm{VaR}^R_{\alpha_1} \square \mathrm{VaR}^\lambda_{\alpha_2} = \mathrm{VaR}^R_{\alpha_1+\alpha_2}, \quad \lambda \in [0,1], \tag{1.4}$$

with $\alpha_1, \alpha_2 > 0$ such that $\alpha_1 + \alpha_2 < 1$.

The rest of this paper is organized as follows. In Section 2, we obtain the following explicit form of the inf-convolution for mixed-VaRs $(\mathrm{VaR}^{\lambda_1}_{\alpha_1}, \ldots, \mathrm{VaR}^{\lambda_n}_{\alpha_n})$,

$$\mathrm{VaR}^{\lambda_1}_{\alpha_1} \square \mathrm{VaR}^{\lambda_2}_{\alpha_2} \square \cdots \square \mathrm{VaR}^{\lambda_n}_{\alpha_n}(X) = \mathrm{VaR}^\lambda_\alpha(X), \quad X \in L^0, \tag{1.5}$$

where $\lambda_1, \ldots, \lambda_n \in [0,1]$, $\alpha_1 > 0, \ldots, \alpha_n > 0$ such that $\alpha := \sum_{i=1}^n \alpha_i < 1$, and $\lambda = \min\left\{\sum_{i=1}^n \lambda_i, 1\right\}$. The (Pareto) optimal allocations of (1.5) are constructed in Section 3. The worst-case mixed VaR under model uncertainty is presented in Section 4.

The motivation of such a study can be seen from Remark 5 in Liu et al. (2022). For $X \in L^0$, the distinction of $\mathrm{VaR}^L_\alpha(X)$ and $\mathrm{VaR}^R_\alpha(X)$ arises only when the distribution function F_X of X is strictly flat on the interval $\{x : F_X(x) = \alpha\}$. Such a situation often occurs when F_X is the distribution function of a discrete random variable or F_X is the empirical distribution of some random sample. In this case, VaR^L_α can be regarded as the left end-point of the confidence interval, whereas VaR^R_α represents the right end-point. A strict risk regulator will choose VaR^R_α, a lenient risk regulator will impose VaR^L_α, whereas a moderate regulator might accept the mixed-VaR, $\mathrm{VaR}^\lambda_\alpha$ for some $\lambda \in (0,1)$. Here $1-\lambda$ can be regarded as a moderate risk index. Eq. (1.5) can be interpreted as that, if all agents in a risk sharing problem are under a moderate regime, then the corresponding inf-convolution is also a mixed-VaR. On the other hand, although our results are of more theoretical than practical interest, they do have some impact on risk sharing and can give us some insight on the understanding of some known results in the recent literature.

2. Inf-convolution of mixed-VaRs

We first give the explicit form of the inf-convolution for two agents with respective risk measures $\mathrm{VaR}^{\lambda_1}_{\alpha_1}$ and $\mathrm{VaR}^{\lambda_2}_{\alpha_2}$.

Lemma 2.1. *For $\lambda_1, \lambda_2 \in [0,1]$, and $\alpha_1 > 0$, $\alpha_2 > 0$ such that $\alpha_1 + \alpha_2 < 1$, we have*

$$\mathrm{VaR}^{\lambda_1}_{\alpha_1} \square \mathrm{VaR}^{\lambda_2}_{\alpha_2}(X) = \mathrm{VaR}^\lambda_{\alpha_1+\alpha_2}(X), \quad X \in L^0, \tag{2.1}$$

where $\lambda = \min\{\lambda_1 + \lambda_2, 1\}$.

Proof. In view of (1.2) and (1.4), we only consider the case $\lambda_1, \lambda_2 \in [0,1)$ and $\lambda > 0$. Denote $\alpha = \alpha_1 + \alpha_2$. For any $X \in L^0$, (2.1) holds trivially if $\mathrm{VaR}^R_\alpha(X) = \mathrm{VaR}^L_\alpha(X)$. Thus, we assume that

$$\mathrm{VaR}^L_\alpha(X) < \mathrm{VaR}^R_\alpha(X), \tag{2.2}$$

which implies that $\mathsf{P}(A)=\alpha$, where $A:=\{X\geq \mathrm{VaR}_\alpha^R(X)\}$. Let $(X_1,X_2)\in\mathbb{A}_2(X)$. By translation invariance of VaR-type risk measures, we further assume without loss of generality that $\mathrm{VaR}_\alpha^L(X)=0$, $\mathrm{VaR}_{\alpha_2}^R(X_2)=\mathrm{VaR}_\alpha^R(X)$. Denote $\mathrm{VaR}_{\alpha_2}^L(X_2)=t$, where t will be regarded as a parameter in the following discussion. Then $t\leq \mathrm{VaR}_\alpha^R(X)$. Therefore, we have

$$\mathrm{VaR}_{\alpha_1}^{\lambda_1}\Box\mathrm{VaR}_{\alpha_2}^{\lambda_2}(X)=\inf_{t\leq \mathrm{VaR}_\alpha^R(X)}\left\{\mathrm{VaR}_{\alpha_1}^{\lambda_1}(X_1)+\mathrm{VaR}_{\alpha_2}^{\lambda_2}(X_2)\;\middle|\;\begin{array}{l}(X_1,X_2)\in\mathbb{A}_2(X),\\ \mathrm{VaR}_{\alpha_2}^R(X_2)=\mathrm{VaR}_\alpha^R(X),\\ \mathrm{VaR}_{\alpha_2}^L(X_2)=t\end{array}\right\}. \tag{2.3}$$

Under the aforementioned assumptions, from the definition of VaR_α^R, it follows that $\mathsf{P}(X_2\geq \mathrm{VaR}_\alpha^R(X)+\epsilon)<\alpha_2$ for any $\epsilon>0$. Note that

$$A\subset\{X_2\geq \mathrm{VaR}_\alpha^R(X)+\epsilon\}\cup\{X_1\geq-\epsilon\}.$$

Then,

$$\mathsf{P}(A)\leq \mathsf{P}\left(X_2\geq \mathrm{VaR}_\alpha^R(X)+\epsilon\right)+\mathsf{P}(X_1\geq-\epsilon)<\alpha_2+\mathsf{P}(X_1\geq-\epsilon).$$

Thus, $\mathsf{P}(X_1\geq-\epsilon)>\alpha-\alpha_2=\alpha_1$, which implies that

$$\mathrm{VaR}_{\alpha_1}^L(X_1)\geq 0 \tag{2.4}$$

since ϵ is arbitrary.

Below, we will discuss the lower bound of the right hand side of (2.3) according to three cases: $t=\mathrm{VaR}_\alpha^R(X)$, $t\in[0,\mathrm{VaR}_\alpha^R(X))$ and $t<0$.

(i) Suppose $t=\mathrm{VaR}_\alpha^R(X)$. In this case, whenever λ_1 and λ_2 take any values, we have

$$\mathrm{VaR}_{\alpha_1}^{\lambda_1}(X_1)+\mathrm{VaR}_{\alpha_2}^{\lambda_2}(X_2)\geq 0+\mathrm{VaR}_\alpha^R(X)=\mathrm{VaR}_\alpha^R(X)\geq\lambda\,\mathrm{VaR}_\alpha^R(X). \tag{2.5}$$

(ii) Suppose $t\in[0,\mathrm{VaR}_\alpha^R(X))$. From (2.2) and the definition of VaR^L, we get that $\mathsf{P}(X_2\leq t)=1-\alpha_2$. Then,

$$\mathsf{P}\left(X_1\geq \mathrm{VaR}_\alpha^R(X)-t\right)\geq \mathsf{P}(A\cap\{X_2\leq t\})\geq \mathsf{P}(A)+\mathsf{P}(X_2\leq t)-1=\alpha_1, \tag{2.6}$$

implying that $\mathrm{VaR}_{\alpha_1}^R(X_1)\geq \mathrm{VaR}_\alpha^R(X)-t$. Thus, using (2.4),

$$\begin{aligned}&\inf_{t\in[0,\mathrm{VaR}_\alpha^R(X))}\left\{\mathrm{VaR}_{\alpha_1}^{\lambda_1}(X_1)+\mathrm{VaR}_{\alpha_2}^{\lambda_2}(X_2)\right\}\\ &\geq\inf_{t\in[0,\mathrm{VaR}_\alpha^R(X))}\left\{\lambda_1\left(\mathrm{VaR}_\alpha^R(X)-t\right)+(1-\lambda_2)t+\lambda_2\mathrm{VaR}_\alpha^R(X)\right\}\\ &=\inf_{t\in[0,\mathrm{VaR}_\alpha^R(X))}\left\{(\lambda_1+\lambda_2)\mathrm{VaR}_\alpha^R(X)+t(1-\lambda_1-\lambda_2)\right\}\\ &=\lambda\,\mathrm{VaR}_\alpha^R(X),\end{aligned} \tag{2.7}$$

where $\lambda=\min\{\lambda_1+\lambda_2,1\}$.

We now turn to prove that the strict equality holds in (2.7). For any $\epsilon>0$, we will construct two random variables $X_{1,\epsilon}, X_{2,\epsilon}$, which satisfy that

$$\mathrm{VaR}_{\alpha_2}^L(X_{2,\epsilon})=t,\qquad \mathrm{VaR}_{\alpha_2}^R(X_{2,\epsilon})=\mathrm{VaR}_\alpha^R(X), \tag{2.8}$$

and

$$\mathrm{VaR}_{\alpha_1}^R(X_{1,\epsilon})<\mathrm{VaR}_\alpha^R(X)-t+\epsilon,\qquad \mathrm{VaR}_{\alpha_1}^L(X_{1,\epsilon})=0. \tag{2.9}$$

To this end, define $B_\epsilon=\left\{\mathrm{VaR}_\alpha^R(X)\leq X<\mathrm{VaR}_\alpha^R(X)+\epsilon\right\}$. Then $\mathsf{P}(B_\epsilon)>0$. Since the underlying probability space is atomless, we can choose a measurable set $B_\epsilon^*\subset B_\epsilon$ such that

$$\mathsf{P}\left(B_\epsilon^*\right)=\min\left\{\frac{1}{2}\mathsf{P}(B_\epsilon),\ \alpha_1\right\}$$

and $\inf\{X(w):w\in B_\epsilon^*\}\geq\sup\{X(w):w\in B_\epsilon\setminus B_\epsilon^*\}$. Additionally, define $C_\epsilon=\left\{X\geq \mathrm{VaR}_\alpha^R(X)+\epsilon\right\}$, and choose another measurable set $C_\epsilon^*\subset C_\epsilon$ such that $\mathsf{P}(C_\epsilon^*)=\alpha_1-\mathsf{P}\left(B_\epsilon^*\right)$. Now construct two random variables

$$X_{1,\epsilon}=(X-t)(I_{C_\epsilon^*}+I_{B_\epsilon^*}),\qquad X_{2,\epsilon}=X-X_{1,\epsilon}.$$

It is easy to check that

$$\begin{aligned}&\mathsf{P}\left(X_{2,\epsilon}\geq \mathrm{VaR}_\alpha^R(X)\right)=\mathsf{P}(A\setminus(B_\epsilon^*\cup C_\epsilon^*))=\mathsf{P}(A)-\mathsf{P}(B_\epsilon^*\cup C_\epsilon^*)=\alpha_2,\\ &\mathrm{VaR}_{\alpha_2}^R(X_{2,\epsilon})=\mathrm{VaR}_\alpha^R(X),\qquad \mathrm{VaR}_{\alpha_2}^L(X_{2,\epsilon})=t.\end{aligned}$$

From the construction of $X_{1,\epsilon}$, we know that $\mathrm{VaR}^L_{\alpha_1}(X_{1,\epsilon}) = 0$,

$$\mathrm{P}\left(X_{1,\epsilon} \ge \mathrm{VaR}^R_\alpha(X) - t + \epsilon\right) = \mathrm{P}(C^*_\epsilon) < \alpha_1.$$

Then,

$$\mathrm{VaR}^R_{\alpha_1}(X_{1,\epsilon}) < \mathrm{VaR}^R_\alpha(X) - t + \epsilon.$$

Therefore, $X_{1,\epsilon}$ and $X_{2,\epsilon}$ constructed above satisfy (2.8) and (2.9). The infimum lower bound in the left hand side of (2.7) is also bounded from above as follows:

$$\begin{aligned}
&\inf_{t\in[0,\mathrm{VaR}^R_\alpha(X))} \left\{\mathrm{VaR}^{\lambda_1}_{\alpha_1}(X_1) + \mathrm{VaR}^{\lambda_2}_{\alpha_2}(X_2)\right\} \\
&\le \inf_{t\in[0,\mathrm{VaR}^R_\alpha(X))} \left\{\lambda_1\left(\mathrm{VaR}^R_\alpha(X) - t + \epsilon\right) + (1-\lambda_2)t + \lambda_2\mathrm{VaR}^R_\alpha(X)\right\} \\
&= \inf_{t\in[0,\mathrm{VaR}^R_\alpha(X))} \left\{(\lambda_1+\lambda_2)\mathrm{VaR}^R_\alpha(X) + t(1-\lambda_1-\lambda_2) + \epsilon\lambda_1\right\} \\
&= \lambda\,\mathrm{VaR}^R_\alpha(X) + \epsilon\lambda_1.
\end{aligned}$$

Setting $\epsilon \downarrow 0$ yields that

$$\inf_{t\in[0,\mathrm{VaR}^R_\alpha(X))} \left\{\mathrm{VaR}^{\lambda_1}_{\alpha_1}(X_1) + \mathrm{VaR}^{\lambda_2}_{\alpha_2}(X_2)\right\} = \lambda\,\mathrm{VaR}^R_\alpha(X). \tag{2.10}$$

(iii) Suppose $t < 0$. For this case, $\mathrm{P}(X_2 \le t) = 1 - \alpha_2$ and

$$\mathrm{P}\left(X_1 \ge \mathrm{VaR}^R_\alpha(X) - t\right) \ge \mathrm{P}\left(A \cap \{X_2 \le t\}\right) \ge \mathrm{P}(A) + \mathrm{P}(X_2 \le t) - 1 = \alpha_1.$$

This implies that $\mathrm{VaR}^R_{\alpha_1}(X_1) \ge \mathrm{VaR}^R_\alpha(X) - t$. On the other hand, we have $\mathrm{P}(X \ge -\epsilon) > \alpha$ since $\mathrm{VaR}^L_\alpha(X) = 0$. Note that

$$\begin{aligned}
\mathrm{P}(X \ge -\epsilon) &\le \mathrm{P}(\{X_2 > t\} \cup \{X_1 \ge -t-\epsilon\}) \\
&\le \mathrm{P}(X_2 > t) + \mathrm{P}(X_1 \ge -t-\epsilon) = \alpha_2 + \mathrm{P}(X_1 \ge -t-\epsilon),
\end{aligned}$$

which implies that $\mathrm{P}(X_1 \ge -t-\epsilon) > \alpha_1$. Thus, $\mathrm{VaR}^L_{\alpha_1}(X_1) \ge -t$ since ϵ is arbitrary. Therefore,

$$\begin{aligned}
\mathrm{VaR}^{\lambda_1}_{\alpha_1}(X_1) + \mathrm{VaR}^{\lambda_2}_{\alpha_2}(X_2) &\ge -t(1-\lambda_1) + \lambda_1\left(\mathrm{VaR}^R_\alpha(X) - t\right) + (1-\lambda_2)t + \lambda_2\mathrm{VaR}^R_\alpha(X) \\
&= (\lambda_1+\lambda_2)\mathrm{VaR}^R_\alpha(X) - \lambda_2 t \\
&> (\lambda_1+\lambda_2)\mathrm{VaR}^R_\alpha(X) \\
&\ge \lambda\,\mathrm{VaR}^R_\alpha(X),
\end{aligned} \tag{2.11}$$

where the last inequality follows from the fact that $\mathrm{VaR}^L_\alpha(X) = 0$.

Based on the above discussions of three cases, it follows from (2.3), (2.5), (2.10) and (2.11) that

$$\mathrm{VaR}^{\lambda_1}_{\alpha_1}\Box\mathrm{VaR}^{\lambda_2}_{\alpha_2}(X) = \lambda\,\mathrm{VaR}^R_\alpha(X) = \mathrm{VaR}^\lambda_\alpha(X).$$

This proves the lemma. □

Remark 2.2. Following the notation in Embrechts et al. (2018), the RVaR at level $(\alpha,\beta) \in [0,1]^2$ with $\alpha + \beta \le 1$ is defined by

$$\mathrm{RVaR}_{\alpha,\beta}(X) = \frac{1}{\beta}\int_\alpha^{\alpha+\beta} \mathrm{VaR}^L_u(X)\,\mathrm{d}u, \quad X \in L^1,$$

for $\beta > 0$, and $\mathrm{RVaR}_{\alpha,\beta}(X) = \mathrm{VaR}^L_\alpha(X)$ for $\beta = 0$. Since $\mathrm{VaR}^L_\alpha(X)$ is right continuous in $\alpha \in [0,1)$ and, hence, we have

$$\lim_{\beta\to 0^+} \mathrm{RVaR}_{\alpha,\beta}(X) = \mathrm{VaR}^L_\alpha(X), \quad X \in L^1,$$

and

$$\lim_{\beta\to 0^+} \mathrm{RVaR}_{\alpha-\lambda\beta,\beta}(X) = \mathrm{VaR}^\lambda_\alpha(X), \quad X \in L^1, \tag{2.12}$$

where $0 \le \lambda \le 1$, $\alpha \ge \lambda\beta$ and $\alpha + (1-\lambda)\beta < 1$. By Theorem 2 in Embrechts et al. (2018), we have

$$\mathrm{RVaR}_{\alpha_1-\lambda_1\beta,\beta}\Box\,\mathrm{RVaR}_{\alpha_2-\lambda_2\beta,\beta}(X) = \mathrm{RVaR}_{\alpha_1+\alpha_2-(\lambda_1+\lambda_2)\beta,\beta}(X), \quad X \in L^1, \tag{2.13}$$

for $\lambda_i \in [0,1]$, $\alpha_i \geq \lambda_i \beta \geq 0$ $(i=1,2)$ and $\alpha_1+\alpha_2+(1-\lambda_1-\lambda_2)\beta<1$. One reviewer pointed out (2.12) to us and wondered whether (2.1) in Lemma 2.1 can be deduced from (2.13) by setting $\beta \to 0^+$ for $X \in L^1$ and $\lambda_1+\lambda_2 \leq 1$.

However, the above assertion is negative because the following equation does not hold in general:

$$\lim_{\beta\to 0^+} \mathrm{RVaR}_{k_1(\beta),k_2(\beta)} \,\square \lim_{\beta\to 0^+} \mathrm{RVaR}_{g_1(\beta),g_2(\beta)}(X) = \lim_{\beta\to 0^+} \mathrm{RVaR}_{k_1(\beta)+g_1(\beta),k_2(\beta)\vee g_2(\beta)}(X), \tag{2.14}$$

where $k_1(\beta), k_2(\beta), g_1(\beta), g_2(\beta)$ are continuous functions, taking values in $(0,1)$, such that $k_1(\beta)+g_1(\beta)+k_2(\beta)\vee g_2(\beta)<1$. To see it, we give the following counterexample. Choose $k_1(\beta)=\alpha_1$, $k_2(\beta)=\beta$, $g_1(\beta)=\alpha_2-c\beta$ and $g_2(\beta)=c\beta$ with $c\in(0,1)$. Then, by applying the property of RVaR, we have

$$\lim_{\beta\to 0^+} \mathrm{RVaR}_{k_1(\beta),k_2(\beta)} = \mathrm{VaR}^L_{\alpha_1}(X),$$

$$\lim_{\beta\to 0^+} \mathrm{RVaR}_{g_1(\beta),g_2(\beta)} = \lim_{\beta\to 0^+} \mathrm{RVaR}_{\alpha_2-c\beta,c\beta}(X) = \mathrm{VaR}^R_{\alpha_2}(X)$$

and

$$\lim_{\beta\to 0^+} \mathrm{RVaR}_{k_1(\beta)+g_1(\beta),k_2(\beta)\vee g_2(\beta)}(X) = \lim_{\beta\to 0^+} \mathrm{RVaR}_{\alpha_1+\alpha_2-c\beta,\beta}(X) = \mathrm{VaR}^c_{\alpha_1+\alpha_2}(X).$$

However,

$$\lim_{\beta\to 0^+} \mathrm{RVaR}_{k_1(\beta),k_2(\beta)} \,\square \lim_{\beta\to 0^+} \mathrm{RVaR}_{g_1(\beta),g_2(\beta)}(X) = \mathrm{VaR}^L_{\alpha_1} \square \mathrm{VaR}^R_{\alpha_2}(X) = \mathrm{VaR}^R_{\alpha_1+\alpha_2}(X).$$

This means that (2.14) does not hold. $\lhd$

Lemma 2 in Liu et al. (2020) states that for any risk measures $\rho_1,\ldots,\rho_n$, we have $\square_{i=1}^n \rho_i = \rho_1 \square \rho_2 \square \cdots \square \rho_n$. As a direct consequence of this fact, together with Lemma 2.1, we conclude the following main result in this paper.

Theorem 2.3. *For $\lambda_1,\ldots,\lambda_n \in [0,1]$, and $\alpha_1>0,\ldots,\alpha_n>0$ such that $\alpha := \sum_{i=1}^n \alpha_i < 1$, we have*

$$\mathop{\square}_{i=1}^{n} \mathrm{VaR}^{\lambda_i}_{\alpha_i}(X) = \mathrm{VaR}^{\lambda_1}_{\alpha_1} \square \mathrm{VaR}^{\lambda_2}_{\alpha_2} \square \cdots \square \mathrm{VaR}^{\lambda_n}_{\alpha_n}(X) = \mathrm{VaR}^{\lambda}_{\alpha}(X), \quad X \in L^0, \tag{2.15}$$

where $\lambda = \min\left\{\sum_{i=1}^n \lambda_i, 1\right\}$.

It should be mentioned that for any finite-valued monetary risk measures $\rho_1,\ldots,\rho_n$, an allocation is optimal if and only if it is also Pareto optimal (see, for example, Embrechts et al., 2018, Proposition 1). An n-tuple $(X_1,\ldots,X_n)\in\mathbb{A}_n(X)$ is called a Pareto optimal allocation if for any $(Y_1,\ldots,Y_n)\in\mathbb{A}_n(X)$ satisfying $\rho_i(Y_i)\leq\rho_i(X_i)$ for all $i=1,\ldots,n$, we have $\rho_i(X_i)=\rho_i(Y_i)$ for $i=1,\ldots,n$.

3. Optimal allocations for mixed-VaRs

For different parameters $\{\lambda_i\}$ and $\{\alpha_i\}$, Theorem 2.3 in the above section gives the explicit form of the inf-convolution of X for risk measures $(\mathrm{VaR}^{\lambda_1}_{\alpha_1},\ldots,\mathrm{VaR}^{\lambda_n}_{\alpha_n})$. A corresponding optimal allocation will be constructed explicitly in this section. Optimal allocations for some special cases of the parameters are available in the literature. To state these, denote $\lambda=\max\{\sum_{i=1}^n \lambda_i, 1\}$ and $\alpha=\sum_{i=1}^n \alpha_i$.

- For $\lambda=0$ (that is, $\lambda_i=0$ for all i), an optimal allocation of X for $(\mathrm{VaR}^L_{\alpha_1},\ldots,\mathrm{VaR}^L_{\alpha_n})$ has the form (see Embrechts et al., 2018, Corollary 2):

$$\begin{aligned} X_k &= (X-m) I_{\{1-\sum_{i=1}^k \alpha_i < U_X \leq 1-\sum_{i=1}^{k-1}\alpha_i\}}, \quad k=1,\ldots,n-1, \\ X_n &= (X-m) I_{\{U_X \leq 1-\sum_{i=1}^{n-1}\alpha_i\}} + m, \end{aligned} \tag{3.1}$$

  where $m\in\left(-\infty, \mathrm{VaR}^L_\alpha(X)\right]$ is a constant, and U_X is a uniform random variable on $(0,1)$ such that $F_X^{-1}(U_X)=X$, a.s..
- For $\lambda=1$ and $\lambda_i\in\{0,1\}$ for each i, an optimal allocation of X for $(\mathrm{VaR}^{\lambda_1}_{\alpha_1},\ldots,\mathrm{VaR}^{\lambda_n}_{\alpha_n})$ has the form (see Liu et al., 2022, Theorem 1):

$$X_k = \left(X-\mathrm{VaR}^R_\alpha(X)\right) I_{A_k} + \frac{1}{n}\mathrm{VaR}^R_\alpha(X), \quad k=1,\ldots,n, \tag{3.2}$$

  where $\{A_1,\ldots,A_n\}$ is a partition of Ω with $\mathrm{P}(A_k)=\alpha_k/\alpha$ for each k. To see it, from the construction given in the proof of Theorem 1 in Liu et al. (2022), we have

$$\sum_{k=1}^n \mathrm{VaR}^{\lambda_k}_{\alpha_k}(X_k) \leq \sum_{k=1}^n \mathrm{VaR}^R_{\alpha_k}(X_k) \leq \mathrm{VaR}^R_\alpha(X). \tag{3.3}$$

  On the other hand, by Theorem 2.3,

$$\sum_{k=1}^n \mathrm{VaR}^{\lambda_k}_{\alpha_k}(X_k) \geq \mathrm{VaR}^{\lambda_1}_{\alpha_1} \square \mathrm{VaR}^{\lambda_2}_{\alpha_2} \square \cdots \square \mathrm{VaR}^{\lambda_n}_{\alpha_n}(X) = \mathrm{VaR}^R_\alpha(X).$$

  Therefore, $\sum_{k=1}^n \mathrm{VaR}^{\lambda_k}_{\alpha_k}(X_k) = \mathrm{VaR}^R_\alpha(X)$.

- When $\mathrm{VaR}^L_\alpha(X) = \mathrm{VaR}^R_\alpha(X)$, $(X_1, \ldots, X_n)$ define by (3.2) is also an optimal allocation of X for $(\mathrm{VaR}^{\lambda_1}_{\alpha_1}, \ldots, \mathrm{VaR}^{\lambda_n}_{\alpha_n})$, where parameters $\{\lambda_i\}$ and $\{\alpha_i\}$ are arbitrary. To see it, applying Theorem 2.3 yields that

$$\sum_{k=1}^{n} \mathrm{VaR}^{\lambda_k}_{\alpha_k}(X_k) \geq \mathrm{VaR}^{\lambda}_{\alpha}(X) = \mathrm{VaR}^R_\alpha(X). \tag{3.4}$$

Therefore, by (3.3) and (3.4), we have $\sum_{k=1}^{n} \mathrm{VaR}^{\lambda_k}_{\alpha_k}(X_k) = \mathrm{VaR}^R_\alpha(X)$.

Theorem 3.1. *Suppose that* $\lambda := \sum_{i=1}^{n} \lambda_i < 1$, $\alpha := \sum_{i=1}^{n} \alpha_i < 1$, *and* $\mathrm{VaR}^L_\alpha(X) < \mathrm{VaR}^R_\alpha(X)$. *Then there exists an optimal allocation of* X *for* $(\mathrm{VaR}^{\lambda_1}_{\alpha_1}, \mathrm{VaR}^{\lambda_2}_{\alpha_2}, \ldots, \mathrm{VaR}^{\lambda_n}_{\alpha_n})$, *which has the form*

$$X_k = \left(X - \mathrm{VaR}^L_\alpha(X)\right) I_{A_k} + \frac{1}{n}\mathrm{VaR}^L_\alpha(X), \quad k = 1, \ldots, n-1, \tag{3.5}$$

$$X_n = \left(X - \mathrm{VaR}^L_\alpha(X)\right) (I_{A^c} + I_{A_n}) + \frac{1}{n}\mathrm{VaR}^L_\alpha(X), \tag{3.6}$$

where $A = \left\{X \geq \mathrm{VaR}^R_\alpha(X)\right\}$, *and* $\{A_1, \ldots, A_n\}$ *is a partition of* A *with* $\mathrm{P}(A_k) = \alpha_k$ *for* $k = 1, \ldots, n$.

Proof. Obviously, $\{X_1, \ldots, X_n\}$ defined by (3.5) and (3.6) is an allocation of X. In view of translation invariance of VaR, without loss of generality, assume that $\mathrm{VaR}^L_\alpha(X) = 0$. In the sequel, it requires to consider the following two cases: $\mathrm{P}\left(X = \mathrm{VaR}^R_\alpha(X)\right) > 0$ and $\mathrm{P}\left(X = \mathrm{VaR}^R_\alpha(X)\right) = 0$.

Case 1: Suppose that $\theta := \mathrm{P}\left(X = \mathrm{VaR}^R_\alpha(X)\right) > 0$. Denote $K = \left\{X = \mathrm{VaR}^R_\alpha(X)\right\}$. In the atomless probability space, there exists a partition $\{K_1, \ldots K_n\}$ of K, satisfying that $\mathrm{P}(K_i) = \theta\alpha_i/\alpha$ for $i = 1, \ldots, n$. Similarly, there exists a partition $\{J_1, \ldots, J_n\}$ of $A \setminus K$ such that

$$\mathrm{P}(J_i) = (\alpha - \theta)\frac{\alpha_i}{\alpha}, \quad i = 1, \ldots, n.$$

Set $A_i = K_i \cup J_i$ for $i = 1, \ldots, n$. Then $\{A_1, \ldots, A_n\}$ is a partition of A. Define n random variables

$$X_i = X I_{A_i}, \quad i = 1, \ldots, n-1; \tag{3.7}$$

$$X_n = X - \sum_{i=1}^{n-1} X_i = X I_{A^c} + X I_{A_n}. \tag{3.8}$$

Note that $\mathrm{VaR}^R_\alpha(X) > 0$. It is easy to check that, for any $i = 1, \ldots, n$,

$$\mathrm{P}\left(X_i \geq \mathrm{VaR}^R_\alpha(X)\right) = \mathrm{P}(K_i \cup J_i) = \theta\frac{\alpha_i}{\alpha} + (\alpha - \theta)\frac{\alpha_i}{\alpha} = \alpha_i,$$

$$\mathrm{P}\left(X_i > \mathrm{VaR}^R_\alpha(X)\right) = \mathrm{P}(J_i) = (\alpha - \theta)\frac{\alpha_i}{\alpha} < \alpha_i,$$

$$\mathrm{P}(X_i \leq 0) = 1 - \alpha_i,$$

$$\mathrm{P}(X_n = 0) \geq \mathrm{P}(A \setminus (K_n \cup J_n)) = \alpha - \alpha_1 > 0.$$

Then, for each i, we have $\mathrm{VaR}^L_{\alpha_i}(X_i) = 0$ and $\mathrm{VaR}^R_{\alpha_i}(X_i) = \mathrm{VaR}^R_\alpha(X)$. Thus,

$$\sum_{i=1}^{n} \mathrm{VaR}^{\lambda_i}_{\alpha_i}(X_i) = \lambda \mathrm{VaR}^R_\alpha(X) = \mathrm{VaR}^{\lambda_1}_{\alpha_1} \square \mathrm{VaR}^{\lambda_2}_{\alpha_2} \square \cdots \mathrm{VaR}^{\lambda_n}_{\alpha_n}(X), \tag{3.9}$$

that is, $(X_1, \ldots, X_n)$ is an optimal allocation of X for $(\mathrm{VaR}^{\lambda_1}_{\alpha_1}, \ldots, \mathrm{VaR}^{\lambda_n}_{\alpha_n})$.

Case 2: Suppose that $\mathrm{P}\left(X = \mathrm{VaR}^R_\alpha(X)\right) = 0$. Denote $B_\epsilon = \left\{\mathrm{VaR}^R_\alpha(X) \leq X < \mathrm{VaR}^R_\alpha(X) + \epsilon\right\}$ for any $\epsilon > 0$. Then $\mathrm{P}(B_\epsilon) > 0$. Note that $\mathrm{P}(B_\epsilon) \to \mathrm{P}\left(X = \mathrm{VaR}^R_\alpha(X)\right) = 0$ when $\epsilon \downarrow 0$. Thus, there exists a sequence $\{\epsilon_k\}$ such that $\epsilon_k \downarrow 0$, $\mathrm{P}(B_{\epsilon_k}) < \mathrm{P}(B_{\epsilon_{k-1}})$, and $\{B_{\epsilon_k}\}$ is a deceasing sequence of sets, where $\epsilon_0 = +\infty$. Denote $\tau_i = \mathrm{P}\left(B_{\epsilon_{i-1}} \setminus B_{\epsilon_i}\right) = \mathrm{P}\left(B_{\epsilon_{i-1}}\right) - \mathrm{P}\left(B_{\epsilon_i}\right)$. Obviously, we have $\sum_{i=1}^{\infty} \tau_i = \mathrm{P}\left(B_{\epsilon_0}\right) = \alpha$. In the atomless probability space, let $\{E^i_1, \ldots, E^i_n\}$ be a partition of the set

$$B_{\epsilon_{i-1}} \setminus B_{\epsilon_i} = \left\{\mathrm{VaR}^R_\alpha(X) + \epsilon_i \leq X < \mathrm{VaR}^R_\alpha(X) + \epsilon_{i-1}\right\},$$

satisfying that $\mathrm{P}(E^i_k) = \tau_i\alpha_k/\alpha$ for $k = 1, \ldots, n$. Define $A_n = \left(\cup_{i\in\mathbb{N}} E^i_n\right) \cup \left\{X = \mathrm{VaR}^R_\alpha(X)\right\}$, and $A_k = \cup_{i\in\mathbb{N}} E^i_k$ for $k = 1, \ldots, n-1$. Then $\{A_1, \ldots, A_n\}$ is a partition of set A. According to (3.7) and (3.8), construct random variables $X_1, \ldots, X_n$. Clearly, $(X_1, \ldots, X_n) \in \mathbb{A}_n(X)$. Also, it is easy to check that, for $k = 1, \ldots, n$,

$$\mathrm{P}\left(X_k \geq \mathrm{VaR}^R_\alpha(X)\right) = \mathrm{P}(A_k) = \sum_{i=1}^{n} \tau_i \frac{\alpha_k}{\alpha} = \alpha_k,$$

$$\mathrm{P}(X_k \leq 0) = 1 - \alpha_k,$$

$$\mathrm{P}(X_n = 0) \geq \alpha - \alpha_n > 0.$$

Moreover, for any $\eta > 0$, there exists ϵ_n such that $\epsilon_n < \eta$ and, thus,

$$\mathsf{P}\left(X_k > \mathrm{VaR}^R_\alpha(X) + \eta\right) \le \mathsf{P}\left(X_k \ge \mathrm{VaR}^R_\alpha(X) + \epsilon_n\right) = \frac{\alpha_k}{\alpha}\sum_{i=1}^n \tau_i < \alpha_k.$$

Based on these observations, we conclude that $\mathrm{VaR}^L_{\alpha_k}(X_k) = 0$ and $\mathrm{VaR}^R_{\alpha_k}(X_k) = \mathrm{VaR}^R_\alpha(X)$ for each k. This implies that

$$\mathrm{VaR}^{\lambda_k}_{\alpha_k}(X_k) = \lambda_k \mathrm{VaR}^R_\alpha(X), \quad k = 1, \ldots, n$$

Therefore, (3.9) holds, that is, $(X_1, \ldots, X_n)$ is an optimal allocation of X for $(\mathrm{VaR}^{\lambda_1}_{\alpha_1}, \ldots, \mathrm{VaR}^{\lambda_n}_{\alpha_n})$. □

Remark 3.2. The optimal allocation $(X_1, \ldots, X_n) \in \mathbb{A}_n(X)$ given by (3.5) and (3.6) has a similar form to that given by (3.1) or (3.2). However, the constructing methods of partition $\{A_k\}$ are different. The construction of $\{A_k\}$ in (3.5) and (3.6) is more technical. A significant feature of the $\{X_k\}$ constructed in Theorem 3.1 is

$$\mathrm{VaR}^L_{\alpha_k}(X_k) = \frac{1}{n}\mathrm{VaR}^L_\alpha(X), \quad \mathrm{VaR}^R_{\alpha_k}(X_k) = \mathrm{VaR}^R_\alpha(X) - \left(1 - \frac{1}{n}\right)\mathrm{VaR}^L_\alpha(X);$$

while the $\{X_k\}$ in (3.2) has the following feature

$$\mathrm{VaR}^R_{\alpha_k}(X_k) = \frac{1}{n}\mathrm{VaR}^R_\alpha(X), \quad k = 1, \ldots, n. \quad \triangleleft$$

It should be mentioned that when only one λ_i is non-zero, $(X_1, \ldots, X_n)$ given by (3.1) is also an optimal allocation (see Example 3.4), while when two or more λ_i are non-zero, $(X_1, \ldots, X_n)$ is in general not an optimal allocation (see Example 3.5). To see it, we need the following lemma, whose proof is straightforward and hence omitted.

Lemma 3.3. *Let $X = F_X^{-1}(U_X)$, a.s.. For $\lambda \in [0, 1]$, and $\alpha_1 > 0$, $\alpha_2 > 0$ such that $\alpha = \alpha_1 + \alpha_2 < 1$, let $m \in (-\infty, \mathrm{VaR}^L_\alpha(X)]$ and define a random variable $Y = (X - m)I_{\{U_X \le 1-\alpha_1\}}$. Then*

$$\mathrm{VaR}^\lambda_{\alpha_2}(Y) = \mathrm{VaR}^\lambda_\alpha(X) - m. \tag{3.10}$$

In Lemma 3.3, the condition $m \in \left(-\infty, \mathrm{VaR}^L_\alpha(X)\right]$ can not be replaced by $m \in \left(-\infty, \mathrm{VaR}^R_\alpha(X)\right]$ because $F(m+y) \ge 1-\alpha$ does not imply $y \ge 0$ when $m = \mathrm{VaR}^R_\alpha(X)$.

Example 3.4. [Optimal allocation for $\left(\mathrm{VaR}^L_{\alpha_1}, \ldots, \mathrm{VaR}^L_{\alpha_{n-1}}, \mathrm{VaR}^\lambda_{\alpha_n}\right)$] For $\lambda \in [0, 1]$ and $\alpha_1 > 0, \ldots, \alpha_n > 0$ such that $\alpha = \sum_{i=1}^n \alpha_i < 1$, it follows from Theorem 2.3 that

$$\mathrm{VaR}^L_{\alpha_1} \square \cdots \square \mathrm{VaR}^L_{\alpha_{n-1}} \square \mathrm{VaR}^\lambda_{\alpha_n}(X) = \mathrm{VaR}^\lambda_\alpha(X), \quad X \in \mathcal{X}. \tag{3.11}$$

For any $X \in \mathcal{X}$, let $(X_1, \ldots, X_n)$ be defined by (3.1). Obviously, $(X_1, \ldots, X_n) \in \mathbb{A}_n(X)$. Observe the following facts:

- For any $k = 1, \ldots, n-1$, since $X_k \ge 0$ and
$$\mathsf{P}(X_k > 0) \le \mathsf{P}\left(1 - \sum_{i=1}^k \alpha_i < U_X < 1 - \sum_{i=1}^{k-1}\alpha_i\right) = \alpha_k,$$
we have $\mathrm{VaR}^L_{\alpha_k}(X_k) = 0$.
- By Lemma 3.3,
$$\mathrm{VaR}^\lambda_{\alpha_n}\left((X - m)I_{\{U_X \le 1 - \sum_{i=1}^{n-1}\alpha_i\}}\right) = \mathrm{VaR}^\lambda_{\alpha_n}(X) - m,$$
which implies that $\mathrm{VaR}^\lambda_{\alpha_n}(X_n) = \mathrm{VaR}^\lambda_\alpha(X)$ for $\lambda \in [0, 1]$.

Thus, we have

$$\mathrm{VaR}^\lambda_{\alpha_n}(X_n) + \sum_{k=1}^{n-1}\mathrm{VaR}^L_{\alpha_k}(X_k) = \mathrm{VaR}^\lambda_\alpha(X).$$

Therefore, $(X_1, \ldots, X_n)$ is an optimal allocation of X for $\left(\mathrm{VaR}^L_{\alpha_1}, \ldots, \mathrm{VaR}^L_{\alpha_{n-1}}, \mathrm{VaR}^\lambda_{\alpha_n}\right)$. $\triangleleft$

Example 3.5. Let $\alpha_1 > 0, \alpha_2 > 0$ such that $\alpha = \alpha_1 + \alpha_2 < 1$, and let $\beta \in (\alpha_1, \alpha)$ be a constant. Define a random variable X and its one allocation (X_1, X_2) as follows,

$$X = I_{\{1-\beta < U < 1\}}, \quad X_1 = I_{\{1-\alpha_1 \le U < 1\}}, \quad X_2 = I_{\{1-\beta \le U < 1-\alpha_1\}},$$

where U is a uniform random variable on $[0, 1]$. Obviously, $\mathrm{VaR}^R_\alpha(X) = \mathrm{VaR}^L_\alpha(X) = 0$, $\mathrm{VaR}^R_{\alpha_1}(X_1) = 1$, $\mathrm{VaR}^L_{\alpha_1}(X_1) = 0$ and $\mathrm{VaR}^R_{\alpha_2}(X_2) = \mathrm{VaR}^L_{\alpha_2}(X_2) = 0$. Then for any $\lambda_1 > 0$ and $\lambda_2 > 0$, we have $\mathrm{VaR}^{\lambda_1}_{\alpha_1}(X_1) + \mathrm{VaR}^{\lambda_2}_{\alpha_2}(X_2) = \lambda_1 > \mathrm{VaR}^\lambda_\alpha(X)$, where $\lambda = \min\{\lambda_1 + \lambda_2, 1\}$. So (X_1, X_2) given by (3.1) is not an optimal allocation of X for risk measures $(\mathrm{VaR}^{\lambda_1}_{\alpha_1}, \mathrm{VaR}^{\lambda_2}_{\alpha_2})$. $\triangleleft$

4. Worst-case VaR under model uncertainty

Let $\mathcal{P}$ be the set of all probability measures that are absolutely continuous with respect to P, where P is a common benchmark for all agents. For any $Q \in \mathcal{P}$, let $\mathrm{VaR}_\alpha^{L,Q}$, $\mathrm{VaR}_\alpha^{R,Q}$ and $\mathrm{VaR}_\alpha^{\lambda,Q}$ be the VaR_α^L, VaR_α^R and $\mathrm{VaR}_\alpha^\lambda$ evaluated under the probability measure Q instead of P. Model uncertainty is prevalent in risk management. In a risk sharing problem, model uncertainty means that the agents are uncertain about the distributions of the random losses allocated to them. A popular approach to incorporate model uncertainty into decision is through a worst-case approach (see Gilboa and Schmeidler, 1989; El Ghaoui et al., 2003; Zhu and Fukushima, 2009). We consider the worst-case mixed-VaR risk measure

$$\overline{\mathrm{VaR}}_\alpha^{\lambda,\mathcal{Q}} = \sup_{Q \in \mathcal{Q}} \mathrm{VaR}_\alpha^{\lambda,Q},$$

where $\mathcal{Q}$ is the subset of $\mathcal{P}$, describing model uncertainty. We call $\mathcal{Q}$ an *uncertainty set* of probability measures. A particular choice of $\mathcal{Q}$ is the following set of probability measures whose Randon-Nikodym derivatives with respect to P do not exceed a constant, that is,

$$\mathcal{P}_\beta = \left\{ Q \in \mathcal{P} : \frac{\mathrm{d}Q}{\mathrm{d}P} \le \frac{1}{\beta} \right\} \text{ for } \beta \in (0,1].$$

The set $\mathcal{P}_\beta$ was also used by Embrechts et al. (2020) and Liu et al. (2022) to describe model uncertainty. $\mathcal{P}_\beta$ is also used in the characterization of the Expected Shortfall at level β, ES_β, via the following dual representation (see, for example, Embrechts and Wang, 2018, Lemma 3.14):

$$\mathrm{ES}_\beta(X) := \mathrm{RVaR}_{0,\beta} = \sup_{Q \in \mathcal{P}_\beta} \mathbb{E}^Q[X], \quad X \in L^1.$$

Liu et al. (2022) considered the special cases VaR_α^L and VaR_α^R under uncertainty set $\mathcal{P}_\beta$, and obtained that

$$\overline{\mathrm{VaR}}_\alpha^{L,\mathcal{P}_\beta} = \mathrm{VaR}_{\alpha\beta}^L, \qquad \overline{\mathrm{VaR}}_\alpha^{R,\mathcal{P}_\beta} = \mathrm{VaR}_{\alpha\beta}^R. \tag{4.1}$$

Proposition 4.1. *For $\alpha \in (0,1)$, $\lambda \in [0,1]$ and $\beta \in (0,1]$, we have*

$$\overline{\mathrm{VaR}}_\alpha^{\lambda,\mathcal{P}_\beta}(X) = \mathrm{VaR}_{\alpha\beta}^\lambda(X), \quad X \in L^0. \tag{4.2}$$

Proof. Note that for any $X \in \mathcal{X}$,

$$\begin{aligned}
\overline{\mathrm{VaR}}_\alpha^{\lambda,\mathcal{P}_\beta}(X) &= \sup_{Q \in \mathcal{P}_\beta} \mathrm{VaR}_\alpha^{\lambda,Q}(X) \\
&= \sup_{Q \in \mathcal{P}_\beta} \left[(1-\lambda)\mathrm{VaR}_\alpha^{L,Q}(X) + \lambda \mathrm{VaR}_\alpha^{R,Q}(X) \right] \\
&\le (1-\lambda) \sup_{Q \in \mathcal{P}_\beta} \mathrm{VaR}_\alpha^{L,Q}(X) + \lambda \sup_{Q \in \mathcal{P}_\beta} \mathrm{VaR}_\alpha^{R,Q}(X) \\
&= (1-\lambda)\mathrm{VaR}_{\alpha\beta}^L(X) + \lambda \mathrm{VaR}_{\alpha\beta}^R(X) \\
&= \mathrm{VaR}_{\alpha\beta}^\lambda(X),
\end{aligned} \tag{4.3}$$

where the last but one equality follows from (4.1). To prove the inverse inequality of (4.3), we choose a special $Q_0 \in \mathcal{P}_\beta$ such that $\mathrm{d}Q_0/\mathrm{d}P = (1/\beta) I_{\{U_X > 1-\beta\}}$. Then

$$\begin{aligned}
\mathrm{VaR}_\alpha^{L,Q_0}(X) &= \inf\{x : Q_0(X \le x) \ge 1-\alpha\} \\
&= \inf\{x : P(X \le x, U_X > 1-\beta) \ge \beta(1-\alpha)\} \\
&= \inf\{x : P(1-\beta < U_X \le F_X(x)) \ge \beta(1-\alpha)\} \\
&= \inf\{x : F_X(x) \ge 1-\alpha\beta\} = \mathrm{VaR}_{\alpha\beta}^L(X), \\
\mathrm{VaR}_\alpha^{R,Q_0}(X) &= \inf\{x : Q_0(X \le x) > 1-\alpha\} \\
&= \inf\{x : P(1-\beta < U_X \le F_X(x)) > \beta(1-\alpha)\} \\
&= \inf\{x : F_X\,(x) > 1-\alpha\beta\} = \mathrm{VaR}_{\alpha\beta}^R(X).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\overline{\mathrm{VaR}}_\alpha^{\lambda,\mathcal{P}_\beta}(X) &\ge \mathrm{VaR}_\alpha^{\lambda,Q_0}(X) = (1-\lambda)\mathrm{VaR}_\alpha^{L,Q_0}(X) + \lambda\, \mathrm{VaR}_\alpha^{R,Q_0}(X) \\
&= (1-\lambda)\mathrm{VaR}_{\alpha\beta}^L(X) + \lambda\, \mathrm{VaR}_{\alpha\beta}^R(X) \\
&= \mathrm{VaR}_{\alpha\beta}^\lambda(X).
\end{aligned} \tag{4.4}$$

The desired result now follows from (4.3) and (4.4). □

By Proposition 4.1 and Theorem 2.3, we obtain the inf-convolution of X for mixed-VaRs in the setting of model uncertainty: For $\lambda_i \in [0,1]$, $\alpha_i \in (0,1)$, $\beta_i \in (0,1]$ such that $\alpha^* = \sum_{i=1}^n \alpha_i \beta_i < 1$, we have

$$\mathop{\square}_{i=1}^{n} \overline{\mathrm{VaR}}_{\alpha_i}^{\lambda_i, \mathcal{P}_{\beta_i}} = \mathop{\square}_{i=1}^{n} \mathrm{VaR}_{\alpha_i \beta_i}^{\lambda_i} = \mathrm{VaR}_{\alpha^*}^{\lambda}(X), \quad X \in L^0,$$

where $\lambda = \min\left\{\sum_{i=1}^n \lambda_i, 1\right\}$.

A reasonable specification of the uncertainty structure (set) is the key issue for successful practical applications. The other most often used uncertainty structures are mixture distribution uncertainty, box uncertainty and ellipsoidal uncertainty in probability measures (see, for example, Goldfarb and Iyengar, 2003; El Ghaoui et al., 2003; Zhu and Fukushima, 2009). The uncertain structure induced by Wasserstein metrics can be found in Liu et al. (2022). Optimal allocations for mixed-VaRs or other risk measures under these uncertainty structures are left for further investigation.

Declaration of competing interest

No potential conflict of interest was reported by the authors.

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