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From risk reduction to risk elimination by conditional mean risk sharing of independent losses

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ABSTRACT

This paper studies diversification effects resulting from pooling insurance losses according to the risk allocation rule proposed by Denuit and Dhaene (2012). General comparison results are established for conditional expectations given sums of independent random variables. It is shown that these expectations decrease in the number of terms comprised in the conditioning sums. Additional inequalities are obtained under regression dependence in the sum. These general results are used to derive the monotonicity of the respective contributions of the participants with respect to the convex order, showing that increasing the number of participants is always beneficial under conditional mean risk sharing. New convergence results are obtained, showing that the variance of individual contributions tends to zero in many interesting cases. This provides actuaries with conditions ensuring that the risk can be fully eliminated within the pool, at the limit.

1. Introduction and motivation

In this paper, we consider the conditional mean risk allocation of losses, as defined by Denuit and Dhaene (2012). According to this rule, each participant to an insurance pool contributes the conditional expectation of the loss brought to the pool, given the total loss experienced by the entire pool. This risk sharing mechanism is regarded as beneficial by all risk-averse economic agents. If all the conditional expectations involved are non-decreasing functions of the total loss then the conditional mean risk sharing is Pareto-optimal and all participants have an interest to keep total losses as small as possible. Denuit and Robert (2020, 2021a, 2021b, 2021c, 2022a, 2021d, 2021e) studied this risk sharing mechanism and established several attractive properties including linear approximations when total losses or the number of participants get large. In particular, Denuit and Robert (2021d) proved that the conditional expectation defining the conditional mean risk sharing is asymptotically increasing in the total loss (under mild technical assumptions).

This allocation rule turns out to possess many desirable properties, making it an attractive risk sharing mechanism within insurance pools. Further properties are derived here, as direct consequences of general results obtained for conditional expectations. Some results of independent interest are also derived for conditional tail expectations.

This paper concentrates on independent individual losses. The analysis conducted here applies in the context of peer-to-peer (P2P) insurance which refers to risk sharing networks where a group of individuals pool their resources together to insure against a given peril. See, e.g., Abdikerimova and Feng (2022) and the references therein. The independence assumption appears to be reasonable for many existing P2P insurance schemes. The next few examples help to figure out the context where the results derived in this paper apply:

Medishare is a non-profit health care sharing membership program offering an alternative to traditional health insurance. Administered on behalf of its members by Christian Care Ministry, it is the US largest health care sharing community, with more than 400 thousands members. Medishare is built on pure risk sharing. Members remain responsible for a certain amount of medical care each year (out of pocket) and they contribute every month to a pool paying benefits to sick participants. Every member transfers his or her monthly share amount (replacing the insurance premium) to a credit union account and this money is used to cover

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eligible medical expenses. Medishare is operated through an online platform coordinating the direct sharing of medical costs between members. We refer the reader to the website medishare.com for more details.

Eusoh is a P2P pet health plan, operated through an online community sharing platform. See the website eusoh.com for a thorough description. Members of the Eusoh community share costs after they happen. A monthly fee is charged and members are asked to put down an additional deposit to pay for the cost of treatments. Eusoh distributes the total veterinary costs among all members. Portions of the deposit are drawn when expenses are submitted and the deposit balance must be refilled back to its initial amount at the end of each month.

Laka (previously branded "Insure A Thing") has developed a P2P insurance platform to insure bicycles and cycling equipment against theft or damage. It is presented as a "digital mutual" charging customers retrospectively according to claims experience. A thorough presentation can be found on laka.co. The monthly bill explains how many claims were paid in the previous month and sets the contribution accordingly. As Eusoh, Laka charges a fee on top of claims it pays. However, there is no deposit securing members' contributions so that Laka is exposed to counterparty credit risk (i.e. the risk that customers do not pay at the end of the month). Claims are handled by a team of cyclists, ensuring that members are helped by peers.

For each scheme, aggregate losses must be distributed among participants in such a way that joining the pool is beneficial. There are many possible allocation rules. Some of them are documented in Denuit et al. (2022) where a list of properties is proposed. For instance, the willingness-to-join property formalizes the superiority of pooling over the stand-alone position. As it is commonly the case with traditional insurance products, a certain degree of standardization is required for managing large P2P insurance pools. Thus, individual preferences are only partially taken into account in the sense that, even if the proposed coverage is not optimal for each individual participant, given his or her particular preferences, it must be attractive to all members of a reasonable class (e.g. risk-averse) of economic agents. If willingness to join is expressed with the help of the stochastic dominance relations expressing the common preferences shared by all risk-averse economic agents in the expected utility setting for choice under risk then it turns out that only the conditional mean risk-sharing rule (or some variants of it) satisfies this requirement among the many rules considered in Denuit et al. (2022). This is why the present paper concentrates on that particular allocation rule. We come back to its practical applicability in the concluding section.

When the losses comprised in the pool are mutually independent, it is known from Denuit and Robert (2021c) that increasing the number of participants is always beneficial under conditional mean risk sharing. This comes from the fact that each participant's contribution forms a reverse martingale when the size of the pool increases. As a consequence, there exists a limiting contribution when the number of participants tends to infinity. The limit may be the pure premium or remains random in case risk cannot be fully diversified inside the pool. These results demonstrate that conditional mean risk sharing is tailored to insurance applications: provided losses are allocated accordingly, it is always beneficial to increase the size of the pool to ensure diversification. The limiting case also identifies residual risk.

Since increasing the number of participants to an insurance pool favors diversification benefits when individual losses are independent, it is therefore interesting to study the asymptotic behavior of individual contributions when the size of the pool tends to infinity. Zabell (1980, 1993) studied the behavior of the conditional expectations defining the conditional mean risk sharing rule when the contribution of each individual loss to the aggregate loss of the entire pool is asymptotically negligible. These results have been extended by Denuit and Robert (2021a) who established the convergence of individual contributions to the corresponding pure premiums under appropriate technical conditions. In the present paper, we provide the reader with several criteria ensuring that the convergence to the expected value takes place. This helps to identify pools where the risk per participant can be fully eliminated at the limit. This approach is relevant for the P2P insurance schemes described earlier which purpose to gather large numbers of participants. It is shown that the conditional mean risk-sharing rule provides the actuary with a theoretically sound way to distribute losses among pool members. Additional comments on the practical applicability of this rule are given in the concluding section.

The remainder of this paper is organized as follows. Section 2 is devoted to general comparison results for expectations conditional on sums. Section 3 then applies the results to insurance risk pooling. The final Section 4 discusses the results and concludes. Mutual independence is assumed in the largest part of the paper. This is not problematic for the P2P insurance schemes described earlier but restricts the applicability of the results derived in the next sections. For instance, natural catastrophes or major industrial risks (e.g., induced by nuclear plants) are also typically covered by funds or pools where risk sharing operates. The Caribbean Catastrophe Risk Insurance Facility (CCRIF) or the Florida Hurricane Catastrophe Fund are two examples of catastrophe risk pools. The results derived in this paper do not apply to these risk-sharing schemes because losses cannot be considered as being mutually independent.

2. Ordering expectations conditional on sums

2.1. Convex orders

Given two random variables X and Y, X is said to be smaller than Y in the increasing convex order, henceforth denoted by $X \preceq_{\text{ICX}} Y$ if the inequality $\mathrm{E}[g(X)] \leq \mathrm{E}[g(Y)]$ holds for all non-decreasing and convex functions g for which the expectations exist. The convex order $\preceq_{\text{CX}}$ is then defined as

$$X \preceq_{\text{CX}} Y \Leftrightarrow X \preceq_{\text{ICX}} Y \text{ and } \mathrm{E}[X] = \mathrm{E}[Y].$$

Thus, $\preceq_{\text{CX}}$ only applies to random variables with the same expected value. The term "convex" is used since $X \preceq_{\text{CX}} Y \Leftrightarrow \mathrm{E}[g(X)] \leq \mathrm{E}[g(Y)]$ for all convex functions g for which the expectations exist. The stochastic inequality $X \preceq_{\text{CX}} Y$ intuitively means that X and Y have the same "size" (as $\mathrm{E}[X] = \mathrm{E}[Y]$ holds) but that Y is "more variable" than X. For instance, the variance of Y is larger than the variance of X.

The convex order expresses the common preferences for all risk-averse economic agents about losses with the same expected value. Formally, given two losses X and Y,

$$X \preceq_{\text{CX}} Y \Leftrightarrow \mathrm{E}[u(w - X)] \geq \mathrm{E}[u(w - Y)] \text{ for all concave utility function } u \text{ and wealth level } w,$$

provided the expectations exist. Hence, $\preceq_{\text{CX}}$ corresponds to the mean-preserving increase in risk in economics when gains are replaced with losses. The next property will be useful in the remainder of this text:

$$X \preceq_{\text{CX}} Y \Leftrightarrow aX + b \preceq_{\text{CX}} aY + b \text{ for all } a, b \in \mathbb{R}. \tag{2.1}$$

For a thorough description of the convex order and its applications in an actuarial context, we refer the reader to Denuit et al. (2005). A general treatment of this order relation can be found in Shaked and Shanthikumar (2007).

2.2. Risk reduction by conditioning on sums

In this section, we derive conditions on a triplet (U_1, U_2, W) of random variables ensuring that $\mathrm{E}[U_1|U_2 + W]$ is smaller than $\mathrm{E}[U_1|U_2]$ in the convex order. This is not always true. For instance, with $W = U_1 - U_2$ we get $\mathrm{E}[U_1|U_2 + W] = U_1$ which dominates $\mathrm{E}[U_1|U_2]$ in the convex order. The next result shows that "conditional mean independence" provides us with a condition ensuring that conditioning on a sum decreases the conditional expectation in the convex order. Recall that conditional mean independence of U_1 given W conditioning on U_2 holds if

$$\mathrm{E}[U_1|U_2, W] = \mathrm{E}[U_1|U_2]. \tag{2.2}$$

Here, and throughout this paper, all equalities between random variables are assumed to hold almost surely (that is, with probability 1). Concept (2.2) is used in econometrics to decide whether additional features are needed in a regression model. See for instance Shao and Zhang (2014) and Jin et al. (2018).

Proposition 2.1. *Let (U_1, U_2, W) be such that (2.2) holds true. Then,*

$$\mathrm{E}[U_1|U_2 + W] \preceq_{\text{CX}} \mathrm{E}[U_1|U_2].$$

Proof. The sigma-algebra $\sigma(U_2, W)$ generated by U_2 and W coincides with $\sigma(U_2, U_2 + W)$ generated by U_2 and $U_2 + W$ because the two random pairs (U_2, W) and $(U_2, U_2 + W)$ are in linear one-to-one correspondence and thus measurable functions of each other. Considering (2.2), this allows us to write

$$\mathrm{E}[U_1|U_2] = \mathrm{E}[U_1|U_2, W] = \mathrm{E}[U_1|U_2, U_2 + W].$$

This implies that

$$\begin{aligned} \mathrm{E}[U_1|U_2 + W] &= \mathrm{E}\Big[\mathrm{E}\big[U_1|U_2, U_2 + W\big]\big|U_2 + W\Big] \\ &= \mathrm{E}\Big[\mathrm{E}\big[U_1|U_2\big]\big|U_2 + W\Big]. \end{aligned}$$

Considering a convex function g, Jensen's inequality then allows us to write

$$\begin{aligned} \mathrm{E}\Big[g\big(\mathrm{E}[U_1|U_2]\big)\Big] &= \mathrm{E}\Big[\mathrm{E}\big[g\big(\mathrm{E}[U_1|U_2]\big)\big|U_2 + W\big]\Big] \\ &\geq \mathrm{E}\Big[g\big(\mathrm{E}\big[\mathrm{E}[U_1|U_2]\big|U_2 + W\big]\big)\Big] \\ &= \mathrm{E}\Big[g\big(\mathrm{E}[U_1|U_2 + W]\big)\Big], \end{aligned}$$

which ends the proof. □

Clearly, (2.2) is valid when W is independent of (U_1, U_2). Adding an independent term to the conditioning variable thus always reduces the conditional expectation in the convex order. More generally, this remains true for dependent random variables fulfilling (2.2). Condition (2.2) holds when U_1 and W are independent given U_2. This is the case in particular if $W = h(U_2, Z)$ with Z independent of (U_1, U_2) and h a measurable function. Consider for instance a first-order autoregressive structure with $U_{i+1} = \rho U_i + Z_{i+1}$, $i \in \{2, 3\}$, starting from U_1 with independent innovations Z_2 and Z_3. With $W = (\rho - 1)U_2 + Z_3$ we get $U_2 + W = U_3$ and $\mathrm{E}[U_1|U_3] \preceq_{\text{CX}} \mathrm{E}[U_1|U_2]$.

2.3. Partial sums of independent random variables

We are now ready to state the following result, which shows that the conditional expectation of one term in a sum, given the sum, decreases with the number of terms in the convex order.

Proposition 2.2. *Let U, V, and W be independent random variables. Then,*

$$\mathrm{E}[U|U + V + W] \preceq_{\text{CX}} \mathrm{E}[U|U + V].$$

Proof. Since the random variables U, V, and W are mutually independent, we can write

$$\mathrm{E}[U|U + V] = \mathrm{E}[U|U + V, W]$$

so that (2.2) holds true with $U_1 = U$ and $U_2 = U + V$. Proposition 2.1 then shows that the announced result holds true. □

Proposition 2.2 can be extended to the bivariate case as follows.

Proposition 2.3. *Let U, V, and W be independent random variables. Then,*

$$\big(\mathrm{E}[U|U+V+W], \mathrm{E}[V|U+V+W]\big) \preceq_{\mathrm{CX}} \big(\mathrm{E}[U|U+V], \mathrm{E}[V|U+V]\big),$$

that is, $\mathrm{E}[g(\mathrm{E}[U|U+V+W], \mathrm{E}[V|U+V+W])] \leq \mathrm{E}[g(\mathrm{E}[U|U+V], \mathrm{E}[V|U+V])]$ *for all convex functions* $g:\mathbb{R}^2 \to \mathbb{R}$, *provided that the expectations exist.*

Proof. Proceeding as in the proof of Proposition 2.2, let us apply the bivariate version of Jensen inequality to the convex function g to obtain

$$\begin{aligned}\mathrm{E}\Big[g\big(\mathrm{E}[U|U+V], \mathrm{E}[V|U+V]\big)\Big] &= \mathrm{E}\Big[\mathrm{E}\big[g\big(\mathrm{E}[U|U+V], \mathrm{E}[V|U+V]\big)\big|U+V+W\big]\Big] \\ &\geq \mathrm{E}\Big[g\big(\mathrm{E}\big[\mathrm{E}[U|U+V]\big|U+V+W\big], \mathrm{E}\big[\mathrm{E}[V|U+V]\big|U+V+W\big]\big)\Big] \\ &= \mathrm{E}\Big[g\big(\mathrm{E}[U|U+V+W], \mathrm{E}[V|U+V+W]\big)\Big].\end{aligned}$$

This ends the proof. □

2.4. Ordered sums

In this section, we aim to order the conditional expectation of one term in a sum, given the sum, with respect to the characteristics of the other terms entering the sum. Recall that V is smaller than W in the convolution order, which is denoted as $V \preceq_{\mathrm{CONV}} W$, if W is distributed as $V+Z$ for some non-negative random variable Z, independent of V. See Section 1.D in Shaked and Shanthikumar (2007) for a discussion of the convolution order. In this case, it is easy to see that $V \preceq_{\mathrm{ICX}} W$ also holds true and that W is both “larger” and “more variable” compared to V. The next result shows that the conditional expectation of one term in a sum decreases in the convex order when the remaining terms increase in the convolution order.

Proposition 2.4. *Let U, V, and W be independent random variables. Then,*

$$V \preceq_{\mathrm{CONV}} W \Rightarrow \mathrm{E}[U|U+W] \preceq_{\mathrm{CX}} \mathrm{E}[U|U+V].$$

Proof. The announced result directly follows from Proposition 2.2. □

Remark 2.5. Proposition 2.4 still holds true when $W \stackrel{\mathrm{d}}{=} V+Z$ for some real-valued random variable Z, independent of V and W.

Proposition 2.4 does not hold in general with the convex order, as shown by the following counter-example. Consider a random variable V with a positive probability density function over the interval $[0,10]$ such that $\mathrm{E}[V]=5$, a random variable

$$W = \begin{cases} 0 \text{ with probability } 0.5 \\ 10 \text{ with probability } 0.5 \end{cases}$$

and a random variable U uniformly distributed over the interval $[4,6]$. The three random variables are assumed to be mutually independent. Clearly, $V \preceq_{\mathrm{CX}} W$. Since $U+W \in [4,6] \Rightarrow W=0$ and $U+W \in [14,16] \Rightarrow W=10$, the sum $U+W$ reveals the value of W and thus also the value of U. Hence, $\mathrm{E}[U|U+W]=U$ and $\mathrm{E}[U|U+V] \preceq_{\mathrm{CX}} \mathrm{E}[U|U+W]$ despite $V \preceq_{\mathrm{CX}} W$.

However, when $V \preceq_{\mathrm{CX}} W$, it is possible to identify a (centered) risk $\mathcal{E}$ (not independent on U and V) such that $U+V+\mathcal{E}$ has the same distribution as $U+W$ and

$$\mathrm{E}[U|U+V+\mathcal{E}] \preceq_{\mathrm{CX}} \mathrm{E}[U|U+V].$$

More precisely, we have the following result.

Proposition 2.6. *Let U, V, and W be independent random variables such that $V \preceq_{\mathrm{CX}} W$. Then, there exists a random variable $\mathcal{E}$ such that $\mathrm{E}[\varepsilon|U+V]=0$, $U+V+\mathcal{E} \stackrel{\mathrm{d}}{=} U+W$ and*

$$\mathrm{E}[U|U+V+\mathcal{E}] \preceq_{\mathrm{CX}} \mathrm{E}[U|U+V].$$

Proof. Since U, V, and W are independent random variables, we have that $U+V \preceq_{\mathrm{CX}} U+W$. Let us now show that there exists a Markov chain $\{Z_1, Z_2, Z_3, \ldots\}$ with $Z_1 = U+V$, $Z_n \preceq_{\mathrm{CX}} Z_{n+1}$ for all $n \geq 1$, $\mathrm{E}[Z_{n+1}|Z_n] = Z_n$ for all $n \geq 1$, and such that the limiting distribution of Z_n is the distribution of $U+W$. Let $\pi_X(\cdot)$ be the stop-loss transform of the random variable X, defined for any real t as $\pi_X(t) = \mathrm{E}[(X-t)_+]$. The function $\pi_X(\cdot)$ is known to be decreasing and convex. Now, $U+V \preceq_{\mathrm{CX}} U+W \Leftrightarrow \pi_{U+W}(t) \geq \pi_{U+V}(t)$ for any real t and π_{U+V} is the supremum of countable set of affine functions $l_1(\cdot)$, $l_2(\cdot)$,... which can be chosen, for example, as the support functions of π_{U+V} in all rational points. Define $\pi_1 = \pi_{U+V}$ and recursively $\pi_{n+1} = \max\{\pi_n, l_n\}$. We denote by P_n the distribution corresponding to π_n. Assume that there is an interval (a_n, b_n) such that

$$\pi_{n+1}(t) = \begin{cases} l_n(t) & \text{if } t \in (a_n, b_n) \\ \pi_n(t) & \text{otherwise} \end{cases}.$$

Thus, P_{n+1} is obtained from P_n by removing all mass from the interval (a_n, b_n) and moving it to the endpoints in such a way that the mean is preserved. Now, define the kernel Q_n as

$$Q_n(x,\cdot) = \begin{cases} \delta_x & \text{for } x \notin (a_n, b_n) \\ \frac{x-a_n}{b_n-a_n}\delta_{b_n} + \frac{b_n-x}{b_n-a_n}\delta_{a_n} & \text{otherwise} \end{cases},$$

where δ_x is the one-point distribution with mass in x. It is easy to see that Q_n is a Markov kernel with $\int y Q_n(x, dy) = x$ and

$$\int Q_n(x, A) P_n(dx) = P_{n+1}(A), \qquad \text{for all measurable } A.$$

Let $\{U_1, U_2, U_3, \ldots\}$ be a sequence of independent random variables uniformly distributed on the interval $[0, 1]$, and independent on U and V. We can then define the sequence $\{Z_1, Z_2, Z_3, \ldots\}$ as

$$Z_{n+1} = Z_n + h_n(Z_n, U_{n+1})$$

where the function h_n is defined as

$$h_n(Z_n, U_{n+1}) = \left(a_n \mathrm{I}\left[U_{n+1} \le \frac{b_n - Z_n}{b_n - a_n}\right] + b_n \mathrm{I}\left[U_{n+1} > \frac{b_n - Z_n}{b_n - a_n}\right] - Z_n\right) \mathrm{I}[Z_n \in (a_n, b_n)].$$

Since Z_n is a function of $U, V, U_1, \ldots, U_n$, we have that U and $h_n(Z_n, U_{n+1})$ are independent given Z_n and therefore we can write

$$\mathrm{E}[U|Z_n] = \mathrm{E}[U|Z_n, h_n(Z_n, U_{n+1})].$$

The sigma-algebra $\sigma(Z_n, h_n(Z_n, U_{n+1}))$ generated by Z_n and $h_n(Z_n, U_{n+1})$ coincides with $\sigma(Z_n, Z_{n+1})$ generated by Z_n and $Z_{n+1} = Z_n + h_n(Z_n, U_{n+1})$. This allows us to write

$$\mathrm{E}[U|Z_n] = \mathrm{E}[U|Z_n, Z_{n+1}].$$

This implies that

$$\mathrm{E}[U|Z_{n+1}] = \mathrm{E}\Big[\mathrm{E}\big[U|Z_n, Z_{n+1}\big]\big|Z_{n+1}\Big] = \mathrm{E}\Big[\mathrm{E}\big[U|Z_n\big]\big|Z_{n+1}\Big].$$

Considering a convex function g, Jensen's inequality then allows us to write

$$\begin{aligned} \mathrm{E}\Big[g\big(\mathrm{E}[U|Z_n]\big)\Big] &= \mathrm{E}\Big[\mathrm{E}\big[g\big(\mathrm{E}[U|Z_n]\big)\big|Z_{n+1}\big]\Big] \\ &\ge \mathrm{E}\Big[g\big(\mathrm{E}\big[\mathrm{E}[U|Z_n]\big|Z_{n+1}\big]\big)\Big] \\ &= \mathrm{E}\Big[g\big(\mathrm{E}[U|Z_{n+1}]\big)\Big]. \end{aligned}$$

We therefore deduce that

$$\mathrm{E}[U|Z_{n+1}] \preceq_{\mathrm{CX}} \mathrm{E}[U|Z_n].$$

Moreover P_n weakly converges to the distribution of $U + W$, and it follows that $\mathrm{E}[U|Z] \preceq_{\mathrm{CX}} \mathrm{E}[U|U + V]$ where $Z = \lim_{n\to\infty} Z_n$. To end the proof, it suffices to let $\mathcal{E} = \sum_{n=2}^{\infty} h_n(Z_n, U_{n+1})$. □

2.5. Regression dependence and large pools

Often, the monotonicity of $\mathrm{E}[X|S]$ in the conditioning variable S is imposed as a condition or established to allow for further developments. The increasingness of $\mathrm{E}[X|S]$ in the conditioning variable S is referred to as (positive) regression dependence in the literature. This can be related to a problem investigated by Efron (1965) who established that log-concavity is a sufficient condition for one term to be stochastically increasing in a sum of independent random variables. This problem has attracted a lot of attention and we refer the reader to Saumard and Wellner (2018) and Denuit and Robert (2021d) for a detailed treatment. In this section, we assume regression dependence in the sum.

Under regression dependence, Mizuno (2006) derived a result similar to Proposition 2.4 with the dispersive order. Recall that V is smaller than W in the dispersive order, which is denoted as $V \preceq_{\mathrm{DISP}} W$, if the difference $F_W^{-1} - F_V^{-1}$ of their respective quantile functions F_W^{-1} and F_V^{-1} is non-decreasing. Precisely, Remark 3 in Mizuno (2006) ensures that

$$V \preceq_{\mathrm{DISP}} W \Rightarrow \mathrm{E}[U|U + W] \preceq_{\mathrm{CX}} \mathrm{E}[U|U + V] \text{ under regression dependence.}$$

In this section, we consider large pools so that total losses can be modeled by continuous random variables with positive probability density functions, neglecting the probability masses at 0. As shown by Denuit and Robert (2021d), regression dependence is generally fulfilled within large pools of independent losses. Throughout this section, individual losses are still allowed to have a probability mass at zero, as it is generally the case in insurance studies. It is interesting to notice that when V and W are continuous random variables, (3.B.14) in Shaked and Shanthikumar (2007) ensures that $V \preceq_{\text{DISP}} W \Leftrightarrow W \stackrel{d}{=} V + \psi(V)$ for some non-decreasing function ψ. This shows that Proposition 2.2 remains true when V and W are comonotonic. Specifically, let (V, W) be a pair of continuous comonotonic random variables independent of U, that is, V and W are non-decreasing functions of some random variable Z, independent of U. Then, $\text{E}[U|U + V + W] \preceq_{\text{CX}} \text{E}[U|U + V]$ holds true under regression dependence. This shows that despite V and W are comonotonic and thus cannot serve as a hedge against each other, the conditional expectation of U gets smaller when W is added to V because $U + W$ gets more variable (in the dispersive order). If $U + V$ and $U + W$ are continuous random variables both possessing positive probability density functions, their respective distribution functions are one-to-one and we have

$$\text{E}[U|U + V] = \text{E}[U|F_{U+V}(U + V)] \text{ and } \text{E}[U|U + W] = \text{E}[U|F_{U+W}(U + W)].$$

We then deduce from Denuit (2010) that under regression dependence, $\text{E}[U|F_{U+V}(U + V) \ge p] \le \text{E}[U|F_{U+W}(U + W) \ge p]$ for all probability levels p implies $\text{E}[U|U + V] \preceq_{\text{CX}} \text{E}[U|U + W]$. These results suggest that regression dependence is the key assumption to derive comparison results for conditional expectations.

We need to recall the following concepts used below for comparing conditional expectations. Given a non-negative random variable X, any random variable $\widetilde{X}$ with distribution function

$$\text{P}[\widetilde{X} \le t] = \frac{1}{\text{E}[X]} \int_0^t x \text{d}F_X(x), \quad t \ge 0,$$

is called a size-biased version of X. We refer the reader to Arratia et al. (2019) for a general presentation of this concept and a survey of its application in probability. In the next result, we also need the usual stochastic order. Given two random variables X and Y, X is said to be smaller than Y in the usual stochastic order, henceforth denoted by $X \preceq_{\text{ST}} Y$ if the inequality $\text{E}[g(X)] \le \text{E}[g(Y)]$ holds for all non-decreasing functions g for which the expectations exist.

We are now ready to state the main result of this section. Here, U and V represent individual losses. As it is often the case in insurance studies, U and V have a probability mass at 0 and a positive probability density function over $(0, \infty)$. The random variable W stands for the total losses of some large insurance pool. Hence, the probability mass at zero can be neglected and we assume that W possesses a positive probability density function over $(0, \infty)$.

Proposition 2.7. *Let U, V, and W be non-negative, independent random variables having a positive probability density function over $(0, \infty)$. Random variables U and V may have a positive probability mass at 0. Assume that the functions $t \mapsto \text{E}[U|U + V + W = t]$ and $t \mapsto \text{E}[V|U + V + W = t]$ are both continuous and (strictly) increasing. Let $\widetilde{U}$ and $\widetilde{V}$ denote size-biased versions of U and V, respectively, assumed to be independent of U, V and W. Then,*

$$\text{E}[U] \le \text{E}[V] \text{ and } \widetilde{U} + V \preceq_{\text{ST}} U + \widetilde{V} \Rightarrow \text{E}[U|U + V + W] \preceq_{\text{ICX}} \text{E}[V|U + V + W].$$

Proof. Considering Theorem 4.A.3 in Shaked and Shanthikumar (2007), $Y \preceq_{\text{ICX}} Z$ if, and only if,

$$\int_p^1 F_Y^{-1}(u)\text{d}u \le \int_p^1 F_Z^{-1}(u)\text{d}u \text{ for all } p \in [0, 1].$$

$$\Leftrightarrow \text{E}[Y|Y \ge F_Y^{-1}(p)] \le \text{E}[Z|Z \ge F_Z^{-1}(p)] \text{ for all } p \in (0, 1).$$

Since the function $t \mapsto \text{E}[U|U + V + W = t]$ is continuous and (strictly) increasing, we have

$$\begin{aligned}
&\text{E}\Big[\text{E}[U|U + V + W]\Big|\text{E}[U|U + V + W] \ge F^{-1}_{\text{E}[U|U+V+W]}(p)\Big] \\
&= \text{E}\Big[\text{E}[U|U + V + W]\Big|U + V + W \ge F^{-1}_{U+V+W}(p)\Big] \\
&= \frac{1}{1-p}\text{E}\Big[\text{E}[U|U + V + W]\text{I}\big[U + V + W \ge F^{-1}_{U+V+W}(p)\big]\Big] \\
&= \frac{1}{1-p}\text{E}\Big[\text{E}\Big[U\text{I}\big[U + V + W \ge F^{-1}_{U+V+W}(p)\big]\Big|U + V + W\Big]\Big] \\
&= \text{E}[U|U + V + W \ge F^{-1}_{U+V+W}(p)].
\end{aligned}$$

This shows that $\text{E}[U|U + V + W] \preceq_{\text{ICX}} \text{E}[V|U + V + W]$ holds true if, and only if, the inequality

$$\text{E}[U|U + V + W \ge F^{-1}_{U+V+W}(p)] \le \text{E}[V|U + V + W \ge F^{-1}_{U+V+W}(p)] \text{ is valid for all } p \in (0, 1),$$

or

$$\text{E}[U|U + V + W \ge t] \le \text{E}[V|U + V + W \ge t] \text{ is valid for all } t \ge 0.$$

Since

$$\text{E}[U|U + V + W > t] = \text{E}[U]\frac{\text{P}[\widetilde{U} + V + W > t]}{\text{P}[U + V + W > t]}, \tag{2.3}$$

we see that $\mathrm{E}[U|U+V+W] \preceq_{\mathrm{ICX}} \mathrm{E}[V|U+V+W]$ holds true if $\mathrm{E}[U] \leq \mathrm{E}[V]$ and

$$\widetilde{U}+V+W \preceq_{\mathrm{ST}} U+\widetilde{V}+W.$$

The latter stochastic inequality is valid if $\widetilde{U}+V \preceq_{\mathrm{ST}} U+\widetilde{V}$, as announced. □

Remark 2.8. Compared to previous results, the random variables U, V, and W are assumed to be non-negative in Proposition 2.7. This requirement is necessary to use the size-biased transform which is only defined for non-negative random variables.

The next result shows that under the conditions of Proposition 2.7, we can find a sequence of functions allowing us to move from $t \mapsto \mathrm{E}[U|U+V+W=t]$ to $t \mapsto \mathrm{E}[V|U+V+W=t]$, each of them crossing the preceding one only once.

Property 2.9. *Let U, V, and W be random variables satisfying the assumptions of Proposition 2.7. Let $h_U(t) = \mathrm{E}[U|U+V+W=t]$ and $h_V(t) = \mathrm{E}[V|U+V+W=t]$. Then, there exists a sequence of continuous and (strictly) increasing function $\left(h_j\right)_{j\geq 0}$ and a sequence of positive constants $\left(t_j\right)_{j\geq 0}$ such that $h_0 = h_U$, $\lim_{j\to\infty} h_j(t) = h_V(t)$ for any $t \geq 0$, $\mathrm{E}[h_j(U+V+W)] \leq \mathrm{E}[V]$, $\lim_{j\to\infty} \mathrm{E}[h_j(U+V+W)] = \mathrm{E}[V]$ and, for $j = 0, 1, \ldots$, $h_j(t) \geq h_{j+1}(t)$ for $0 \leq t \leq t_j$ and $h_j(t) \leq h_{j+1}(t)$ for $t \geq t_j$.*

Proof. Let $X = \mathrm{E}[U|U+V+W]$ and $Y = \mathrm{E}[V|U+V+W]$. By Proposition 2.7, $X \preceq_{\mathrm{ICX}} Y$. We know from Theorem 4.A.23 in Shaked and Shanthikumar (2007) that there exist random variables $Z_0, Z_1, \ldots$ with continuous and (strictly) increasing distribution functions $F_0, F_1, \ldots$ such that $Z_0 \stackrel{\mathrm{d}}{=} X$, $\mathrm{E}[Z_j] \leq \mathrm{E}[Y]$, $j = 0, 1, \ldots$, $Z_j \stackrel{\mathrm{d}}{\to} Y$ as $j \to \infty$, $\mathrm{E}[Z_j] \to \mathrm{E}[Y]$ as $j \to \infty$, and the number of sign changes of $F_j - F_{j+1}$ is equal to 1 and the sign sequence is $+$, $-$. It suffices to choose $h_j = F_j^{-1} \circ F_{U+V+W}$ and t_j such that $F_j^{-1}\left(F_{U+V+W}\left(t_j\right)\right) = F_{j+1}^{-1}\left(F_{U+V+W}\left(t_j\right)\right)$ to conclude. □

Proposition 2.2 allows us to derive new results about individual contributions to conditional tail expectation under regression dependence. This is formally stated next. Recall that the conditional tail expectation of a random variable Z is defined as $\mathrm{E}[Z|Z \geq F_Z^{-1}(p)]$ where $F_Z^{-1}(p)$ is the quantile of Z at probability level p. If $Z = Z_1 + Z_2$ then individual contributions of each term to the conditional tail expectation of Z are given by $\mathrm{E}[Z_1|Z \geq F_Z^{-1}(p)]$ and $\mathrm{E}[Z_2|Z \geq F_Z^{-1}(p)]$, respectively.

Property 2.10. *Let U, V, and W be non-negative, independent random variables having a positive probability density function over $(0, \infty)$. Random variables U and V may have a positive probability mass at 0. Assume that the functions $t \mapsto \mathrm{E}[U|U+V+W=t]$ and $t \mapsto \mathrm{E}[V|U+V+W=t]$ are both continuous and (strictly) increasing. Then,*

$$\mathrm{E}[U|U+V+W \geq F_{U+V+W}^{-1}(p)] \leq \mathrm{E}[U|U+V \geq F_{U+V}^{-1}(p)] \text{ for all } p \in (0,1).$$

Proof. We know from Proposition 2.2 that the conditional expectations of U given $U+V$ dominate the conditional expectation of U given $U+V+W$. Now, since the functions $t \mapsto \mathrm{E}[U|U+V=t]$ and $t \mapsto \mathrm{E}[U|U+V+W=t]$ are both continuous and (strictly) increasing, we obtain from (3.A.41) in Shaked and Shanthikumar (2007), following the lines of the proof of Proposition 2.7 that

$$\mathrm{E}[U|U+V+W] \preceq_{\mathrm{CX}} \mathrm{E}[U|U+V] \Leftrightarrow \mathrm{E}[U|U+V+W \geq F_{U+V+W}^{-1}(p)] \leq \mathrm{E}[U|U+V \geq F_{U+V}^{-1}(p)] \text{ for all } p \in (0,1),$$

as announced. □

Translated to an insurance risk management context, where U, V and W are monetary losses, Property 2.10 shows that conditional tail expectation always rewards diversification of independent losses under positive regression dependence: the contribution of U to the conditional tail expectation decreases when the pool made of independent losses U and V is supplemented with a third independent loss W.

Under regression dependence, we also have a bivariate comparison complementing Proposition 2.3, in terms of directionally convex order whose definition is recalled next. Supermodular functions are often used in applied probability in order to express the fact that the components of one random vector are "more positively dependent" than those of another random vector. Precisely, a function $g : \mathbb{R}^d \to \mathbb{R}$ is said to be supermodular if the inequality

$$\begin{aligned} & g(x_1, \ldots, x_i+\epsilon, \ldots, x_j+\delta, \ldots, x_d) - g(x_1, \ldots, x_i+\epsilon, \ldots, x_j, \ldots, x_d) \\ & \geq g(x_1, \ldots, x_i, \ldots, x_j+\delta, \ldots, x_d) - g(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_d) \end{aligned}$$

holds for all $\boldsymbol{x} \in \mathbb{R}^d$, $1 \leq i < j \leq d$ and all $\epsilon, \delta > 0$. If the function is regular enough then supermodularity corresponds to $\frac{\partial^2}{\partial x_i \partial x_j} g \geq 0$ for every $i \neq j \in \{1, \ldots, d\}$. To account for positive dependence and different marginal behavior, we need to restrict the class of supermodular functions to the subset of directionally convex ones. Recall that the function g is said to be directionally convex if it is supermodular and coordinatewise convex. If g is twice differentiable then it is directionally convex if, and only if, $\frac{\partial^2}{\partial x_i \partial x_j} g \geq 0$ for all $i, j \in \{1, \ldots, d\}$. Now, the d-dimensional random vectors $\boldsymbol{Y}$ and $\boldsymbol{Z}$ are said to be ordered in the directionally convex order, which is denoted by $\boldsymbol{Y} \preceq_{\mathrm{DIR-CX}} \boldsymbol{Z}$, if $\mathrm{E}[g(\boldsymbol{Y})] \leq \mathrm{E}[g(\boldsymbol{Z})]$ for all directionally convex functions $g : \mathbb{R}^d \to \mathbb{R}$, provided the expectations exist. The directionally convex order is closely related to the supermodular order with main difference that supermodular order compares only dependence structures of random vectors with fixed marginals, whereas the directionally convex order additionally takes into account the variability of the marginals, which may then be different (and ordered in the univariate $\preceq_{\text{CX}}$-sense). A ranking in the $\preceq_{\text{DIR-CX}}$-sense is useful in applications since Theorem 7.A.30 from Shaked and Shanthikumar (2007) ensures that

$$\boldsymbol{Y} \preceq_{\text{DIR-CX}} \boldsymbol{Z} \Rightarrow \Psi(\boldsymbol{Y}) \preceq_{\text{ICX}} \Psi(\boldsymbol{Z}) \text{ for all } \Psi : \mathbb{R}^d \to \mathbb{R} \text{ non-decreasing and directionally convex.} \tag{2.4}$$

We are now ready to state the bivariate extension of Proposition 2.2 under positive regression dependence.

Proposition 2.11. *Let U, V, and W be non-negative, independent random variables such that the functions $t \mapsto \mathrm{E}[U|U+V=t]$, $t \mapsto \mathrm{E}[V|U+V=t]$, $t \mapsto \mathrm{E}[U|U+V+W=t]$ and $t \mapsto \mathrm{E}[V|U+V+W=t]$ are all non-decreasing. Then,*

$$\big(\mathrm{E}[U|U+V+W], \mathrm{E}[V|U+V+W]\big) \preceq_{\text{DIR-CX}} \big(\mathrm{E}[U|U+V], \mathrm{E}[V|U+V]\big).$$

Proof. The components of the random vector $(\mathrm{E}[U|U+V+W], \mathrm{E}[V|U+V+W])$ are non-decreasing functions of the random variable $U+V+W$. Those of $(\mathrm{E}[U|U+V], \mathrm{E}[V|U+V])$ are non-decreasing functions of the random variable $U+V$. Hence, they have a common conditionally increasing copula and Theorem 7.A.38 in Shaked and Shanthikumar (2007) ensures that the announced $\preceq_{\text{DIR-CX}}$-ranking holds true. □

3. Application to pooling insurance losses

3.1. Conditional mean risk sharing rule

Consider n individuals, numbered $i = 1, 2, \ldots, n$. Each of them faces a risk X_i. By risk, we mean a non-negative random variable representing the monetary loss caused by some insurable peril over one period. The losses $X_1, X_2, \ldots, X_n$ are assumed to be independent, with finite mean and variance.

A risk sharing rule is a way to distribute the total losses among participants. According to the conditional mean risk sharing rule proposed by Denuit and Dhaene (2012), participant i contributes $\mathrm{E}[X_i|S_n]$ to the total loss $S_n = \sum_{i=1}^n X_i$. In words, each participant contributes the expected value of the risk he or she brings to the pool, given the total loss experienced by the entire group. Stated differently, the contribution paid by each participant is the average part of the total loss that can be attributed to the risk he or she brings to the pool. Notice that participants can be informed when they enter the pool about the amount they will have to contribute as a function of the total realized loss, that is, the function $s \mapsto \mathrm{E}[X_i|S_n = s]$ can be communicated to participant i beforehand. The calculation of $\mathrm{E}[X_i|S_n = s]$ can be performed in a direct way or by exploiting representations in terms of size-biasing derived by Denuit (2019). Numerical approximations using orthogonal polynomials have been derived by Denuit and Robert (2022a). These properties make conditional mean risk sharing attractive for peer-to-peer insurance models, as shown in Denuit (2020).

3.2. Risk reduction in the convex order

Recall that when risks X_i are independent and identically distributed, every participant contributes $\overline{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ under the conditional mean risk sharing rule. This is because

$$X_1, X_2, \ldots, X_n \text{ independent and identically distributed} \Rightarrow \mathrm{E}[X_i|S_n] = \frac{S_n}{n} \tag{3.1}$$

for every $i \in \{1, \ldots, n\}$. We know from Example 3.A.29 in Shaked and Shanthikumar (2007) that

$$\overline{X}_{n+1} \preceq_{\text{CX}} \overline{X}_n \tag{3.2}$$

holds for all n when risks X_i are independent and identically distributed. Thus, recruiting more members is always beneficial in that case. Denuit and Robert (2021c) established that a similar result holds true for independent but not identically distributed risks, replacing $\overline{X}_n$ with $\mathrm{E}[X_1|S_n]$. The decreasingness of the individual contributions to the total loss S_n with the number n of participants, in the convex order $\preceq_{\text{CX}}$, is obtained here as a direct consequence of Proposition 2.2.

Proposition 3.1. *Consider independent losses $X_1, X_2, \ldots$ with partial sums $S_n = X_1 + \ldots + X_n$. Then, the stochastic inequality*

$$\mathrm{E}[X_i|S_{n+1}] \preceq_{\text{CX}} \mathrm{E}[X_i|S_n]$$

holds for every integer n and $i \in \{1, 2, \ldots, n\}$.

Proof. It suffices to take $U = X_i$, $V = S_n$ and $W = X_{n+1}$ in Proposition 2.2 to obtain the announced result. □

In the expected utility setting, Proposition 3.1 shows that increasing the number of participants is always regarded as beneficial by all risk-averse economic agents, whatever the distribution of the risks they bring to the pool as long as these risks are mutually independent and the sharing is operated according to the conditional mean risk allocation rule. The conditional mean risk sharing rule thus provides the appropriate extension to simple averaging in the homogeneous case.

3.3. Residual risk

Proposition 3.1 shows that the variance of $\mathrm{E}[X_i|S_n]$ decreases with the number n of participants. The question is thus to check whether it tends to 0, under suitable conditions. In that respect, Proposition 3.1 shows that the sequence of conditional expectations forms a backward, or reverse martingale. Hence, $\mathrm{E}[X_1|S_n]$ converges to the conditional expectation $\mathrm{E}[X_1|\mathcal{T}]$ where $\mathcal{T}$ is the tail sigma-field

$$\mathcal{T} = \cap_{n=1}^{\infty}\sigma(S_n, S_{n+1}, S_{n+2}, \ldots).$$

If $\mathcal{T}$ is trivial, that is, $\mathcal{T} = \{\emptyset, \Omega\}$, then all the conditional expectations $\mathrm{E}[X_i|S_n]$ tend to the corresponding expected values $\mathrm{E}[X_i]$. However, if $\mathcal{T}$ is not trivial then limits $\mathrm{E}[X_i|\mathcal{T}]$ remain generally random (that is, with positive variance). Some of them may nevertheless reduce to $\mathrm{E}[X_i]$. The next example illustrates the variety of situations encountered when $\mathcal{T}$ is not trivial (with some conditional expectations converging to the corresponding pure premiums and others not).

Example 3.2. Assume $\mathrm{P}[X_1 = 0.5] = \mathrm{P}[X_1 = 1] = 0.5$ and that $X_2, X_3, X_4, \ldots$ are independent and identically distributed, all valued in $\{0, 1, 2, \ldots\}$. In this artificial example, if S_n is not integer valued then $X_1 = 0.5$ because $X_2, X_3, X_4, \ldots$ are all integer valued. Thus, S_n reveals the value of X_1: if S_n is not integer valued then $X_1 = 0.5$ and $X_1 = 1$ otherwise. Hence, $\mathrm{E}[X_1|S_n] = X_1$ for all n since $X_1 = 1$ when S_n assumes an integer value and $X_1 = 0.5$ otherwise. But all the remaining $\mathrm{E}[X_i|S_n]$, $i = 2, 3, \ldots$, tend to $\mathrm{E}[X_2]$. This is because for any $i \geq 2$,

$$\mathrm{E}[X_i|S_n] = \mathrm{E}[X_i|S_n - X_1, X_1] = \mathrm{E}[X_i|S_n - X_1] = \frac{X_2 + \ldots + X_n}{n-1}$$

by (3.1) so that $\mathrm{E}[X_i|S_n]$ indeed tends to $\mathrm{E}[X_2]$ by the Law of Large Numbers. Thus, all conditional expectations tend to the corresponding expected value except $\mathrm{E}[X_1|S_n]$. The tail sigma-field $\mathcal{T}$ is not trivial in that case.

In the next sections, we derive several sufficient conditions ensuring that individual contributions tend to the corresponding pure premiums so that the risk per participant can be fully eliminated at the limit, within an infinitely large pool.

3.4. Risk elimination within subsets

Proposition 3.1 allows us to derive several results about the variance of individual contributions. The next result considers pools with a subclass where risk can be fully eliminated at the limit. It states that this is still the case in any larger pool, for the losses within that subclass.

Proposition 3.3. *Consider a sequence of random variables $T_1, T_2, T_3, \ldots$ independent of the losses $X_1, X_2, X_3, \ldots$. Then,*

$$\mathrm{Var}\big[\mathrm{E}[X_i|S_n]\big] \to 0 \text{ as } n \to \infty \Rightarrow \mathrm{Var}\big[\mathrm{E}[X_i|S_n + T_n]\big] \to 0 \text{ as } n \to \infty.$$

Proof. Without loss of generality, let us establish the result for $i = 1$. Proposition 2.2 ensures that

$$\mathrm{E}[X_1|S_n + T_n] \preceq_{\mathrm{CX}} \mathrm{E}[X_1|S_n] \Rightarrow \mathrm{Var}\big[\mathrm{E}[X_1|S_n + T_n]\big] \leq \mathrm{Var}\big[\mathrm{E}[X_1|S_n]\big] \text{ for all } n.$$

This ends the proof. □

Proposition 3.3 shows that if there is a subset of the pool where conditional mean risk allocations converge to the corresponding expected values, this remains true in the entire pool. Letting new participants bringing losses T_j enter the pool does not impact on the risk elimination for individuals bringing losses X_i, whatever the losses T_j. In particular, assume that the existing pool is homogeneous, that is, that $X_1, X_2, \ldots, X_n$ are independent and identically distributed. Considering (3.1), we have $\mathrm{E}[X_i|S_n] = \frac{S_n}{n}$ and it is then clear that $\mathrm{Var}\big[\mathrm{E}[X_1|S_n]\big] \to 0$ as $n \to \infty$. This leads to the following result.

Property 3.4. *Assume that the pool is partitioned into p homogeneous classes of size $n_j = \lfloor \alpha_j n \rfloor$, where $\lfloor\cdot\rfloor$ denotes the integer part, with $\alpha_j \geq 0$ for all j and $\sum_{j=1}^{p}\alpha_j = 1$. Then,*

$$\mathrm{Var}\big[\mathrm{E}[X_i|S_n]\big] \to 0 \text{ as } n \to \infty \text{ for all } i.$$

Proof. Let us denote respectively as μ_j and σ_j^2 the mean and variance common to all losses in class j. Also, denote as $\mathcal{C}(i) \in \{1, 2, \ldots, p\}$ the class to which participant i belongs. Proposition 3.3 allows us to write

$$\mathrm{E}[X_i|S_n] \preceq_{\mathrm{CX}} \mathrm{E}\left[X_i \Big| \sum_{k\in\mathcal{C}(i)} X_k\right] = \frac{1}{n_{\mathcal{C}(i)}}\sum_{k\in\mathcal{C}(i)} X_k$$

since (3.1) applies within each homogeneous class. Hence, we get

$$\mathrm{Var}\big[\mathrm{E}[X_i|S_n]\big] \leq \frac{\sigma^2_{\mathcal{C}(i)}}{\lfloor \alpha_{\mathcal{C}(i)} n \rfloor} \to 0 \text{ as } n \to \infty,$$

as announced. □

3.5. Losses dominated within the pool

We know from Proposition 2.4 that combining a random variable with a larger one in the convolution order results in a smaller conditional expectation in the convex order. The next result exploits this fact, in pools containing infinitely many losses dominating the one under consideration, in the convolution order.

Property 3.5. *Consider independent losses $X_1, X_2, X_3, \dots$ and denote as $X_{1,1}, X_{1,2}, X_{1,3}, \dots$ independent random variables, all distributed as X_1. Define*

$$\nu_{1,n}^{\text{conv}} = \sup\left\{ i \geq 0 \middle| \sum_{j=1}^{i} X_{1,j} \preceq_{\text{CONV}} \sum_{j=2}^{n} X_j \right\}$$

where $\sum_{j=1}^{0} \ldots = 0$ by convention. Then,

$$\lim_{n\to\infty} \nu_{1,n}^{\text{conv}} = \infty \Rightarrow \text{Var}\big[\text{E}[X_1|S_n]\big] \to 0 \text{ as } n \to \infty.$$

Proof. Let $\mu_i = \text{E}[X_i]$ and $m_n = \text{E}[S_n] = \sum_{i=1}^{n} \mu_i$. Since the stochastic inequality

$$\sum_{j=1}^{\nu_{1,n}^{\text{conv}}} X_{1,j} \preceq_{\text{CONV}} \sum_{j=2}^{n} X_j$$

holds by definition, we have from Proposition 2.4 that

$$\text{E}[X_1 - \mu_1 | S_n - m_n] \preceq_{\text{CX}} \text{E}\left[X_1 - \mu_1 \middle| X_1 - \mu_1 + \sum_{j=1}^{\nu_{1,n}^{\text{conv}}} (X_{1,j} - \mu_1) \right]$$

where

$$\text{E}\left[X_1 - \mu_1 \middle| X_1 - \mu_1 + \sum_{j=1}^{\nu_{1,n}^{\text{conv}}} (X_{1,j} - \mu_1) \right] = \frac{1}{1+\nu_{1,n}^{\text{conv}}} \left(X_1 - \mu_1 + \sum_{j=1}^{\nu_{1,n}^{\text{conv}}} (X_{1,j} - \mu_1) \right)$$

by (3.1). Finally, we have

$$\begin{aligned}
\text{Var}\big[\text{E}[X_1|S_n]\big] &= \text{E}\Big[\big(\text{E}[X_1 - \mu_1|S_n - m_n]\big)^2\Big] \\
&\leq \text{E}\left[\left(\text{E}\left[X_1 - \mu_1 \middle| X_1 - \mu_1 + \sum_{j=1}^{\nu_{1,n}^{\text{conv}}} (X_{1,j} - \mu_1) \right] \right)^2 \right] \\
&= \text{E}\left[\left(\frac{1}{1+\nu_{1,n}^{\text{conv}}} \left(X_1 - \mu_1 + \sum_{j=1}^{\nu_{1,n}^{\text{conv}}} (X_{1,j} - \mu_1) \right) \right)^2 \right] \\
&= \frac{1}{1+\nu_{1,n}^{\text{conv}}} \text{Var}[X_1]
\end{aligned}$$

and the announced result follows. □

With the same approach we deduce the following property which ensures that provided there are infinitely many losses among $X_2, X_3, \dots$ in the pool dominating X_1 in the convolution order, the variance of the individual contribution $\text{E}[X_1|S_n]$ tends to 0 when the size n of the pool increases.

Property 3.6. *Consider independent losses $X_1, X_2, X_3, \dots$ and define*

$$\rho_{1,n}^{\text{conv}} = \#\left\{ 2 \leq i \leq n \middle| X_1 \preceq_{\text{CONV}} X_i \right\}.$$

Then,

$$\lim_{n\to\infty} \rho_{1,n}^{\text{conv}} = \infty \Rightarrow \text{Var}\big[\text{E}[X_1|S_n]\big] \to 0 \text{ as } n \to \infty.$$

The next result provides us with a sufficient condition ensuring that variances of individual contributions tend to 0 for all participants when individual losses can be bounded from above and from below in the convolution order.

Property 3.7. *Consider independent losses $X_1, X_2, X_3, \dots$ and assume that there exist two random variables Y and Z such that*

$$Y \preceq_{\text{CONV}} X_i \preceq_{\text{CONV}} Z \text{ for all } i$$

where $Z \stackrel{d}{=} \sum_{j=1}^{m} Y_j$ for some integer $m \geq 2$ and $Y_1, Y_2, Y_3, \dots$ are independent random variables, all distributed as Y. Then, $\text{Var}\big[\text{E}[X_i|S_n]\big] \to 0$ as $n \to \infty$, for all i.

Proof. Let us consider the case $i = 1$. We have

$$\sum_{j=2}^{n} Y_j \preceq_{\text{CONV}} \sum_{j=2}^{n} X_j.$$

Let $p_n = \lfloor (n-1)/m \rfloor$. We have

$$\sum_{j=1}^{p_n} Z_j \preceq_{\text{CONV}} \sum_{j=2}^{n} Y_j$$

where $Z_1, Z_2, Z_3, \dots$ are independent random variables, all distributed as Z. We also have

$$\sum_{j=1}^{p_n} X_{1,j} \preceq_{\text{CONV}} \sum_{j=1}^{p_n} Z_j$$

where $X_{1,1}, X_{1,2}, X_{1,3}, \dots$ are independent random variables, all distributed as X_1. The result follows from Property 3.6 since $p_n \to \infty$ as $n \to \infty$. □

3.6. Losses in positive regression dependence with their sum

In this section, we assume that all individual contributions are positively regression dependent in the sum S_n, that is, every $\text{E}[X_i|S_n]$ is continuously increasing in S_n. As explained earlier, this assumption is generally valid provided the size n of the pool is large enough so that we neglect the probability mass of S_n at 0 and model it as a continuous random variable.

If $\text{E}[X_i|S_n] \preceq_{\text{CX}} \text{E}[X_j|S_n]$ holds for all n then the convergence of the latter to $\text{E}[X_j]$ implies the convergence for the former to $\text{E}[X_i]$. It is thus interesting to identify conditions under which this ranking holds. Interestingly, it only involves the characteristics of X_i and X_j, without reference to the remaining X_k, $k \neq i, j$. This is formally stated in the next result which directly follows from Proposition 2.7.

Property 3.8. *Assume that X_i and X_j have a positive probability density function over $(0, \infty)$ with a non-negative probability mass at zero, and that S_n has a positive probability density function over $(0, \infty)$. If the functions $s \mapsto \text{E}[X_k|S_n = s]$ are continuous and (strictly) increasing for $k \in \{i, j\}$ then the stochastic inequality $\text{E}[X_i|S_n] \preceq_{\text{ICX}} \text{E}[X_j|S_n]$ holds true if $\text{E}[X_i] \leq \text{E}[X_j]$ and $\widetilde{X}_i + X_j \preceq_{\text{ST}} X_i + \widetilde{X}_j$ where the size-biased versions $\widetilde{X}_i$ and $\widetilde{X}_j$ are assumed to be independent of X_i and X_j.*

Proof. It suffices to apply Proposition 2.7 with $U = X_i$, $V = X_j$, and $W = S_n - X_i - X_j$. □

Pakes et al. (1996, Theorem 2.1) established that the distributional equality

$$\widetilde{X} \stackrel{d}{=} X + \Delta \text{ for some random variable } \Delta \geq 0, \text{ independent of X,} \tag{3.3}$$

is valid if, and only if, the distribution of X is infinitely divisible. When X_i and X_j are infinitely divisible, $\widetilde{X}_i + X_j \preceq_{\text{ST}} X_i + \widetilde{X}_j$ holds true when $\Delta_i \preceq_{\text{ST}} \Delta_j$. The condition involved in Proposition 3.8 is thus easy to check. For instance, consider individual losses $X_1, X_2, \dots$ of the form

$$X_i = \sum_{k=1}^{N_i} C_{ik} \text{ with } N_i \sim \text{Poisson}(\lambda_i), \quad i = 1, 2, \dots, \tag{3.4}$$

where the claim severities C_{ik} are independent, distributed as C_i, and independent of N_i. The size-biased version of the compound sum X_i in (3.4) is given by $\widetilde{X}_i \stackrel{d}{=} X_i + \widetilde{C}_i$ where X_i and $\widetilde{C}_i$ are mutually independent. See, e.g. Denuit and Robert (2020) for a proof. Hence, we see that $\widetilde{X}_i + X_j \preceq_{\text{ST}} X_i + \widetilde{X}_j$ holds true if $\widetilde{C}_i \preceq_{\text{ST}} \widetilde{C}_j$. With Gamma distributed severities, that is, if $C_i \sim\text{Gamma}(\alpha_i, \tau)$ then $\widetilde{C}_i \sim\text{Gamma}(\alpha_i + 1, \tau)$ and $\widetilde{C}_i \preceq_{\text{ST}} \widetilde{C}_j$ reduces to $\alpha_i \leq \alpha_j \Leftrightarrow C_i \preceq_{\text{ST}} C_j$.

The next result shows that under the conditions of Proposition 3.8, it is possible to move from the contribution of participant i to the contribution of participant j by a sequence of functions crossing only once with each other. It directly results from Property 2.9.

Property 3.9. *Consider independent losses $X_1, \dots, X_n$ such that the functions $s \mapsto h_k(s) = \text{E}[X_k|S_n = s]$ are continuous and strictly increasing for all $k = 1, \dots, n$. If $\text{E}[X_i] \leq \text{E}[X_j]$ and $\widetilde{X}_i + X_j \preceq_{\text{ST}} X_i + \widetilde{X}_j$, then there exists a sequence of continuous and (strictly) increasing function $(h_k^{i\to j})_{k\geq 0}$ and a sequence of positive constants $(s_k)_{k\geq 0}$ such that $h_0^{i\to j} = h_i$, $\lim_{k\to\infty} h_k^{i\to j}(s) = h_j(s)$ for any $s \geq 0$, $\text{E}[h_k^{i\to j}(S_n)] \leq \text{E}[X_j]$, $\lim_{k\to\infty} \text{E}[h_k^{i\to j}(S_n)] = \text{E}[X_j]$ and, for $k = 0, 1, \dots$, $h_k^{i\to j}(s) \geq h_{k+1}^{i\to j}(s)$ for $0 \leq s \leq s_k$ and $h_k^{i\to j}(s) \leq h_{k+1}^{i\to j}(s)$ for $s \geq s_j$.*

Under regression dependence, we can also order the vector of individual contributions in the directionally convex order, as shown next.

Property 3.10. *Consider independent losses $X_1, X_2, X_3, \ldots$ such that the functions $s \mapsto \mathrm{E}[X_k|S_n = s]$ and $s \mapsto \mathrm{E}[X_k|S_{n+1} = s]$ are non-decreasing for all $k \in \{1, \ldots, n\}$. Then*

$$\big(\mathrm{E}[X_1|S_{n+1}], \mathrm{E}[X_2|S_{n+1}], \ldots, \mathrm{E}[X_n|S_{n+1}]\big) \preceq_{\text{DIR}-\text{CX}} \big(\mathrm{E}[X_1|S_n], \mathrm{E}[X_2|S_n], \ldots, \mathrm{E}[X_n|S_n]\big).$$

Proof. We know from Proposition 3.1 that $\mathrm{E}[X_i|S_{n+1}] \preceq_{\text{CX}} \mathrm{E}[X_i|S_n]$ holds for $i \in \{1, \ldots, n\}$. Since the components of the random vectors under consideration are non-decreasing functions of the same random variables S_{n+1} and S_n, respectively, Theorem 7.A.38 in Shaked and Shanthikumar (2007) ensures that the announced $\preceq_{\text{DIR}-\text{CX}}$-ranking holds true. □

Under the assumptions of Property 3.10, we have that

$$\mathrm{Cov}\Big[\mathrm{E}[X_i|S_{n+1}], \mathrm{E}[X_j|S_{n+1}]\Big] \le \mathrm{Cov}\Big[\mathrm{E}[X_i|S_n], \mathrm{E}[X_j|S_n]\Big] \text{ for all } i \ne j \in \{1, \ldots, n\}.$$

This illustrates the fact that the dependence between individual contributions tends to decrease with the size n of the pool. Also, (2.4) shows that the stochastic inequality

$$\Psi\big(\mathrm{E}[X_1|S_{n+1}], \mathrm{E}[X_2|S_{n+1}], \ldots, \mathrm{E}[X_n|S_{n+1}]\big) \preceq_{\text{ICX}} \Psi\big(\mathrm{E}[X_1|S_n], \mathrm{E}[X_2|S_n], \ldots, \mathrm{E}[X_n|S_n]\big)$$

holds true for any non-decreasing and directionally convex function $\Psi : \mathbb{R}^n \to \mathbb{R}$.

4. Discussion

This paper considers risk sharing within insurance pools, where the respective losses X_1, X_2, ..., X_n for the n participants to the pool are distributed ex post according to the conditional mean risk sharing rule proposed by Denuit and Dhaene (2012). Throughout the paper, losses $X_1, X_2, \ldots, X_n$ are assumed to be independent. Diversification effects are assessed with the help of general comparison results in terms of the convex order for conditional expectations given sums. Letting the pool size tend to infinity, the convergence of individual contributions to the corresponding pure premiums is established under various sets of conditions. This helps actuaries to identify pools where the risk can be fully eliminated at the limit.

As explained in the introduction, the analysis conducted in this paper mostly confines to independent losses X_i. Correlated losses are considered in Denuit and Dhaene (2012), Denuit and Robert (2022b) and Denuit et al. (2022). Willingness-to-join property remains valid with correlated losses as $\mathrm{E}[X_i|S_n] \preceq_{\text{CX}} X_i$ still holds true whatever the dependence structure of $(X_1, X_2, \ldots, X_n)$. Thus, joining the pool remains beneficial in the correlated case, as long as total losses S_n are distributed among participants according to the conditional mean risk-sharing rule. However, pooling may not reduce risk compared to the stand-alone position when the dependence becomes perfect. This is for instance the case with comonotonic losses X_i, that is, with losses that are all increasing functions of the same underlying risk factor Z. In this case, $\mathrm{E}[X_i|S_n] = X_i$ and pooling does not lead to risk reduction. In other cases, increasing the size of the pool may remain beneficial but individual risks cannot be fully diversified at the limit. This is for instance the case with exchangeable risks. Recall that the random vector $(X_1, X_2, \ldots, X_n)$ is called exchangeable if its joint distribution function is symmetric in its arguments. This means that for any permutation π of $\{1, \ldots, n\}$, the random vector $(X_{\pi(1)}, X_{\pi(2)}, \ldots, X_{\pi(n)})$ is distributed as $(X_1, X_2, \ldots, X_n)$. In particular, X_1, X_2, ..., X_n are identically distributed and the pool is homogeneous. If $(X_1, X_2, \ldots, X_n)$ is exchangeable then we have for any $i \ne j$ in $\{1, 2, \ldots, n\}$ that

$$\mathrm{E}[X_i|S_n] = \mathrm{E}[X_j|S_n] = \frac{1}{n}\sum_{k=1}^{n} \mathrm{E}[X_k|S_n] = \frac{S_n}{n}.$$

The latter formula extends (3.1) to exchangeable risks. It can then be shown that $\mathrm{E}[X_i|S_n]$ decreases with n in the $\preceq_{\text{CX}}$-sense. Thus, joining the pool and welcoming new participants are beneficial when losses are exchangeable, as it is the case for independent losses. However, there remains some residual risk that cannot be diversified within the pool. Consider the typical construction leading to exchangeability: assume that individual losses X_i are independent and identically distributed given a risk factor Z. The conditional Law of Large Numbers then shows that $\mathrm{E}[X_i|S_n]$ tends to $\mathrm{E}[X_i|Z]$ in this case. These two simple examples illustrate the variety of situations when the independence assumption is relaxed.

Under pure P2P insurance, participants' contributions are theoretically unlimited. To avoid counterparty risk and to be able to deal with larger sums insured, Denuit (2020) replaced unlimited ex-post contributions with a deposit paid in advance combined with a bonus mechanism restoring fairness in arrear, with the guarantee that the final amount due never exceeds this down payment. Part of the deposit feeds a common fund, while the remaining part is paid to a partnering insurance or reinsurance company. If the common fund is insufficient to pay for the claims then the (re-)insurance carrier pays the excess. Conversely, if the pool has few claims then the surplus is given back to the participants or to a cause the pool members care about. Denuit (2020) designed this scheme under the conditional mean risk-sharing rule. Some P2P insurance schemes specify such a maximum contribution per period. Either benefits are reduced when claims exceed the cap or the risk is transferred to a partnering insurance company. This is for instance the case for Laka insured by Zurich UK to cover the risk of claims exceeding the cap. Risk sharing is therefore limited to the lower layer. The analysis conducted in this paper still applies to this layer.

P2P insurance may appear to be too complex to engage the mass market. This is often cited as the main cause of failure of Guevara. After a promising start in 2013, this pioneering Insurtech had to close down in 2017. Guevara targeted motor insurance, where many customers still feel more comfortable with established business model. The schemes taken as examples in the introduction have nevertheless found market niches, offering simple and transparent (at least compared to their commercial insurance competitors) alternatives to the dominant insurance model. In that respect, there is thus no hope to implement the conditional mean risk-sharing rule within large pools because conditional expectations are certainly regarded as obscure mathematical objects by the majority of participants. Fortunately, there are simple approximations to the conditional mean risk-sharing rule that are transparent enough for real-life applications. Within large pools, Denuit and Robert (2021a) established a simple linear rule based on means and variances that closely matches theoretical contributions resulting from the conditional mean risk-sharing rule. Precisely, define the linear regression rule as

$$h_{i,n}^{\text{reg}}(S_n) = \mathrm{E}[X_i] + \frac{\mathrm{Var}[X_i]}{\sum_{j=1}^n \mathrm{Var}[X_j]}\big(S_n - \mathrm{E}[S_n]\big), \quad i = 1, 2, \ldots, n.$$

Both the linear regression rule and the conditional mean risk-sharing rule optimally approximate X_i as a function of S_n, with respect to mean squared error, but the linear regression rule restricts the search to linear functions of S_n while the conditional mean risk-sharing rule allows for all measurable functions of S_n. Subject to some technical conditions, $h_{i,n}^{\text{reg}}(S_n)$ accurately approximates $\mathrm{E}[X_i|S_n]$ in large pools of independent losses. The conditional mean risk-sharing rule also simplifies into the elementary proportional rule, allocating $\frac{\mathrm{E}[X_i]}{\mathrm{E}[S_n]}S_n$ to participant i, in some particular cases (as the semi-homogeneous risk model, that is, compound Poisson losses with homogeneous severities but heterogeneous frequencies). See Section 2.4.2 in Denuit and Robert (2021c) for more details. These simplifications provide actuaries with practical solutions to make the distribution of total losses transparent.

The analysis conducted in this paper also appears to be relevant for the growing market of Islamic insurance. Takaful (translated as solidarity or mutual guarantee) is the Sharia-compliant alternative to conventional insurance. We refer the reader to Malik and Ullah (2019) or Billah (2019) for an introduction to the topic. In a nutshell, conventional insurance has elements of riba (usury), maysir (gambling) and gharar (excessive uncertainty), which make it non-permissible from the Shariah perspective. Risk transfer appears to be impossible and Islamic insurance is therefore characterized by risk sharing. Pure P2P insurance schemes generally comply with Takaful, which makes countries with significant Muslim populations an appealing area of expansion. The huge Takaful opportunity may significantly expand the scope of P2P insurance in the very near future. This development may also be boosted by the launch of decentralized insurance platforms. For instance, Nexus Mutual allows P2P insurance risk-sharing pools to be created in a cost-effective and scalable way, using blockchain technology. In the approach developed by Nexus Mutual, the insurance pool is decentralized and operates under a discretionary mutual structure (which means that all insurance claims are paid at the discretion of all other pool members). Other decentralized insurance providers include Risk Harbor and Unslashed Finance. P2P insurance technology is thus becoming available and the results derived in this paper can be considered as a proof that the P2P insurance concept effectively works, under some mild conditions.

Conditional expectations $\mathrm{E}[X_1|S_n]$ also arise in a variety of other applications. For instance, Van Bochove (2011) derived the conditional expectation of a random variable X_1 obeying a uniform distribution, given the sum S_n of this variable and of $n-1$ independent variables with a second uniform distribution, and applied it to the selection of the most talented young researchers for tenure track. The results derived in the present paper also apply to this setting. Let us also mention the link with renewal processes. Considering $X_1, X_2, \ldots$ as steps in a random walk, the results derived in this paper apply to the conditional expectation of a previous step given the current position of the process. In that respect, we have established that this conditional expectation decreases in the convex order with the number of previous steps. Bar-Lev et al. (2013) derived limit theorems for the conditional distribution of X_1 given $S_n = s_n$ when the random variables X_i are independent and identically distributed and s_n/n converges or is constant. In renewal theory, this corresponds to studying the asymptotic behavior of the conditional interarrival time distribution given that the nth renewal takes place at time $S_n = s_n$. In the present paper, we allow for random interarrival times X_i obeying different distributions, restricting our study to conditional expectations of the random variables X_i, given the time to the nth renewal.

Declaration of competing interest

None declared.

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