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Equilibria and efficiency in a reinsurance market ☆

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ABSTRACT

We study equilibria in a reinsurance market with multiple reinsurers that are endowed with heterogeneous beliefs, where preferences are given by distortion risk measures, and pricing is done via Choquet integrals. We construct a model in the form of a sequential economic game, where the reinsurers have the first-mover advantage over the insurer, as in the Stackelberg setting. However, unlike the Stackelberg setting, which assumes a single monopolistic reinsurer on the supply side, our model accounts for strategic price competition between reinsurers. We argue that the notion of a Subgame Perfect Nash Equilibrium (SPNE) is the appropriate solution concept for analyzing equilibria in the reinsurance market, and we characterize SPNEs using a set of sufficient conditions. We then examine efficiency properties of the contracts induced by an SPNE, and show that these contracts result in Pareto-efficient allocations. Additionally, we show that under mild conditions, the insurer realizes a strict welfare gain, which addresses the concerns of Boonen and Ghossoub (2022) with the Stackelberg model and thereby ultimately reflects the benefit to the insurer of competition on the supply side. We illustrate this point with a numerical example.

1. Introduction

Reinsurance is an important risk management tool for insurance companies. An insurer, subject to a state-contingent risk at the end of a given time horizon, can enter into a reinsurance contractual agreement whereby it can cede part of its future liabilities to a reinsurer, in exchange for a premium payment. The problem faced by the insurer is straightforward: the insurance company wishes to choose the level of reinsurance that minimizes their ultimate, end-of-period risk exposure. On the other hand, the reinsurer wishes to maximize its own profit from the sale of reinsurance. These two questions lie at the heart of the optimal (re)insurance literature, and their study has been extensive and thorough.

The underlying approach to problems of optimal (re)insurance can be broadly classified into two main categories: (i) those that study *Pareto efficiency* of contractual agreements, and (ii) those that study *equilibria* in the (re)insurance market. The literature examining efficiency focuses on the properties of each agent's welfare, after the terms of the contract have been specified (i.e., the indemnification function and the pricing functional). The primary solution concept in this setting is that of Pareto efficiency (PE). That is, one seeks those allocations in which no one agent can strictly improve their welfare without negatively affecting the welfare of another. On the other hand, the literature on equilibria takes a different approach: market mechanisms are specified *a priori*, and the focus is on identifying allocations that result from this underlying structure. The primary solution concept of interest is that of a market equilibrium, which in general does not have to be Pareto efficient. While the literature on PE in reinsurance markets is vast, market equilibria have been comparatively less

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studied. We note that the term "equilibria" is sometimes invoked in the cooperative sense to refer to the study of efficient allocations. For the sake of expositional clarity, the terms "equilibrium" and "equilibria" refer only to market equilibria in the remainder of this paper.

The theory of efficient optimal (re)insurance has its roots in seminal work by Borch (1960) and Arrow (1974), who showed that stop-loss contracts are efficient when both the insurer's and reinsurer's preferences admit an Expected-Utility (EU) representation, the insurer is risk-averse, and the reinsurer is risk neutral. It is shown in this case that premia are priced via the equivalence principle. That is, premia are determined by the expected indemnified loss (e.g., as in Arrow, 1974), sometimes multiplied by a loading factor to represent variable costs to the reinsurer (e.g., as in Raviv, 1979). The preferences of the risk-neutral reinsurer imply that in a PE allocation, the premium is a function of the indemnified risk (henceforth referred to as a premium principle). Under this premium principle, the efficient indemnification is obtained by solving a problem of demand for the insurer. This problem has been extensively studied in the two-agent case, and we refer to Gollier (2013) and Schlesinger (2000), for instance, for a review of the literature on optimal insurance with EU preferences. Of the numerous generalizations of the original EU model, we highlight the two that are most relevant to the present project: the use of distortion risk measures (DRMs) or Rank-Dependent Utility (RDU) preferences, and the consideration of heterogeneous beliefs.

In contrast to the literature on efficiency in reinsurance, the literature on market equilibria in reinsurance contracting is fairly sparse. This approach has been fruitful in broader problems of risk sharing, due to the close link with the classical literature on competitive equilibria. Each agent participates in a reinsurance market by buying or selling reinsurance with the objective of minimizing a measure of their end-of-period risk exposure, where the price of reinsurance is itself endogenously determined at an equilibrium. Competitive equilibria have been characterized in the case of EU (Aase, 1993), the case of VaR and ES (Embrechts et al., 2020, 2018), the case of convex DRMs (Boonen, 2015), and the case of general DRMs for comonotonic indemnities (Boonen et al., 2021a).

However, the problem of reinsurance contracting (i.e., between an insurer subject to risk and its reinsurer counterparties) has been comparatively less studied from a competitive perspective. The closest example is the Stackelberg setting,[1] an example of a sequential economic model first examined by Chan and Gerber (1985) in a reinsurance context. This setting returns to the case where there is only one counterparty (the reinsurer), but this counterparty now has the first-mover advantage: the reinsurer is able to fix a premium principle before reinsurance is purchased, which allows the reinsurer to profit if they correctly anticipate the actions of the insurer. Chan and Gerber (1985) formulate this problem for EU preferences, and they characterize solutions under the special case of exponential utility. Cheung et al. (2019) solve for Stackelberg equilibria when both the insurer and the reinsurer have DRM preferences, and some recent papers examine the dynamic context of reinsurance in continuous time (e.g., Chen and Shen, 2018 and Cao et al., 2022). The link between Stackelberg equilibria and PE is first examined by Boonen and Ghossoub (2022) for the DRM case, where Stackelberg equilibria are shown to be a strict subset of all PE allocations. However, they highlight a caveat in the Stackelberg setting: in equilibrium, the insurer has no incentive to purchase insurance, which suggests a shortcoming of the Stackelberg framework and its applications to optimal reinsurance. Furthermore, the Stackelberg model assumes monopolistic control of the reinsurance market by one entity, and it does not generalize directly to the case of multiple reinsurers. While the interaction between insurer and reinsurer can be interpreted as a form of competition, it is difficult to argue that this model reflects a truly competitive setting under the shadow of a monopolistic reinsurer.

In the present paper, we seek to address these latter issues by formulating a sequential model of a reinsurance market with multiple reinsurers. Similar to the Stackelberg model, the reinsurers collectively have the first-mover advantage, and they can choose their premium principles in anticipation of the insurer's behaviour. However, the reinsurers are in strategic competition when it comes to setting the price of reinsurance. This model is, to the best of our knowledge, the first model that examines reinsurance contracting between an insurer and multiple reinsurers from the perspective of market equilibrium. We assume that the preferences of each agent are given by DRMs under heterogeneous beliefs, which allows for a high level of flexibility. Furthermore, we allow the premium principles posted by each reinsurer to be fully general. In particular, a reinsurer's premium principle is not required to be related to their true preferences, thereby reflecting the strategic nature of interaction on the supply side of the market. This is a significant departure from the efficiency literature. In our market model, we propose the notion of a Subgame Perfect Nash Equilibrium (SPNE), a refinement of the Nash Equilibrium (NE), as the primary solution concept. These concepts emphasize competition within the reinsurance market. We assume that each reinsurer acts in order to maximize their own welfare, which is in line with the literature on competitive equilibria (notably, this is in contrast to the cooperative model proposed by Asimit and Boonen, 2018). We argue in this paper that our model, as well as our suggested solution concept, acts as a natural extension of the Stackelberg setting to multiple reinsurers. In particular, we show that if we consider a special case of our model in which there is only one reinsurer ($n = 1$), we can recover the Stackelberg model as a special case, in which all Stackelberg equilibria are SPNEs.

A standard result allows us to characterize SPNEs through the process of backward induction, which splits the problem into two steps: (i) the decision problem of the insurer, and (ii) that of the reinsurers. The insurer's decision problem reduces to a demand problem under fixed premium principles, and indemnities are fully characterized through the Marginal Indemnity Function (MIF) approach, as in Assa (2015) or Zhuang et al. (2016), for instance. To address the reinsurers' decision problem, we provide a set of sufficient conditions that explicitly characterize a class of SPNEs. In these equilibria, the insurer is subject to prices corresponding to the second-lowest true preferences of the reinsurers as measured by distorted subjective survival probabilities, similarly to the premium principle suggested by Boonen et al. (2021b) to represent the maximum possible premium that preserves coalitional stability. Our results expand on this by showing that this premium principle plays a central role in determining stability of SPNEs as well.

We then use our characterization to examine efficiency properties of the resulting equilibrium contracts. Since SPNEs are market equilibria by definition, it is not true in general that these coincide with PE allocations. However, we show that all allocations resulting from our class of SPNEs are indeed PE, thereby providing a link to the literature on efficient reinsurance contracting. We emphasize that PE in our setting results from the structure of the reinsurance market, and it is not assumed *a priori*. Furthermore, it is also important to examine allocations in terms of their relationship to the status quo – market participants can only be incentivised to enter into contracts that induce a welfare gain, which is not true in the Stackelberg model (as concluded by Boonen and Ghossoub, 2022). We find that within our class of SPNEs, each agent is able to realize a welfare gain under mild conditions, a consequence of both the competition in the

[1] Some authors refer to this setting as the Bowley setting, and to their solutions as Bowley optima. We use the Stackelberg terminology; solutions are referred to as Stackelberg equilibria.

reinsurance market and of the heterogeneity of beliefs. Moreover, one would expect that the welfare gain of the insurer increases as the level of competition in the market increases. We formalize this notion by showing that the insurer experiences an improvement in welfare if an additional reinsurer is added to the market. A consequence is that the presence of just two reinsurers is enough to realize a strict gain for the insurer compared to the Stackelberg case. This point is illustrated by a numerical example.

The remainder of this paper is organized as follows. Section 2 presents our sequential model for the reinsurance market, and it introduces SPNEs in this context. Section 3 provides a characterization of SPNEs and specific examples thereof. An analysis of SPNE contracts is given in Section 4, in which we examine efficiency of these contracts, as well as the relationship between SPNE contracts and the status quo. We also provide a numeral example, exhibiting an SPNE that achieves a welfare gain for all agents. Section 5 concludes. The proofs of most of this paper's results are given in Appendix A. Additional illustrative constructions of equilibria are presented in Appendix B.

2. Setup and definitions

We examine a model of optimal reinsurance, in a market represented by a measurable space (S, Σ). Let $\mathcal{X}$ denote the collection of all measurable real-valued functions on (S, Σ). An insurer is subject to a random insurable loss X, which is represented by a non-negative random variable $X \in \mathcal{X}$. Throughout this paper, the principal method for evaluating the risk of positions $Y \in \mathcal{X}$ is the Choquet expectation, which rigorously defines integration with respect to general set functions.

Definition 2.1. A (finite, non-negative) set function $\nu : \Sigma \to [0, M]$, for some $M \in \mathbb{R}_+$, is called a *capacity* if:

(1) $\nu(\varnothing) = 0$ and $\nu(S) = M$; and,
(2) If $A, B \in \Sigma$ such that $A \subseteq B$, then $\nu(A) \leqslant \nu(B)$.

We denote the set of all capacities by $\mathcal{C}$.

Definition 2.2. The *Choquet expectation* of $Y \in \mathcal{X}$ with respect to $\nu \in \mathcal{C}$ is defined as

$$\rho^{\nu}(Y) = \int Y \, d\nu := \int_{0}^{+\infty} \nu(Y > t) \, dt + \int_{-\infty}^{0} [\nu(Y > t) - \nu(S)] \, dt \,. \tag{2.1}$$

For a detailed treatment of capacities and Choquet integration, we refer to Denneberg (1994) and Marinacci and Montrucchio (2004). A special case of a capacity is a distorted probability measure; if T is a monotone function on $[0, 1]$ such that $T(0) = 0$ and $T(1) = 1$, and μ is a probability measure, then $T \circ \mu$ is a capacity. In this case, $\rho^{T\circ\mu}$ is called a distortion risk measure (DRM), a general class of risk measures that satisfy several desirable properties (e.g., Wang et al., 1997). One such property is comonotonic additivity – distortion risk measures are additive over comonotonic random variables.

Definition 2.3. Two random variables $X_1, X_2 \in \mathcal{X}$ are *comonotonic* if for all $s_1, s_2 \in S$,

$$(X_1(s_1) - X_1(s_2)) \, (X_2(s_1) - X_2(s_2)) \geqslant 0 \,.$$

It follows directly from Definition 2.3 that if $X \in \mathcal{X}$ and $I : \mathbb{R} \to \mathbb{R}$ is a non-decreasing function, then X and $I(X)$ are comonotonic and $\rho^{T\circ\mu}(X + I(X)) = \rho^{T\circ\mu}(X) + \rho^{T\circ\mu}(I(X))$, for any DRM $\rho^{T\circ\mu}$. We refer to Denneberg (1994) for an extended characterization of comonotonicity.

2.1. The market participants' preferences

The insurer is imbued with beliefs on the space (S, Σ), represented by a probability measure $\mathbb{P}$. We assume that the insurer's preferences are represented by a DRM consisting of the measure $\mathbb{P}$ distorted by a distortion function g. That is, for every risk $Y \in \mathcal{X}$, the risk measure of the insurer is given by

$$\rho_{IN}(Y) := \rho^{g\circ\mathbb{P}}(Y) = \int Y \, dg \circ \mathbb{P} \,. \tag{2.2}$$

We assume that there are n reinsurers in this market, denoted by the set $\mathcal{N} := \{1, \ldots, n\}$. Each reinsurer has heterogeneous beliefs on the states of nature, represented by probability measures $\mathbb{Q}_1, \ldots, \mathbb{Q}_n$ for reinsurers $1, \ldots, n$ respectively. We do not assume any relationship between $\mathbb{P}, \mathbb{Q}_1, \ldots, \mathbb{Q}_n$, allowing for a flexible setup such as that in Boonen and Ghossoub (2021). Like the insurer, each reinsurer evaluates risk according to a DRM. For each $i \in \mathcal{N}$, denote by g_i the distortion function of reinsurer i. Then the preference of reinsurer i is represented by the risk measure

$$\rho_i(Y) := \rho^{g_i\circ\mathbb{Q}_i}(Y) = \int Y \, dg_i \circ \mathbb{Q}_i \,. \tag{2.3}$$

To avoid pathological cases, we impose the restriction that all agents attribute finite risk to the initial position. Namely, for all $i \in \mathcal{N}$,

$$\rho_i(X) < \infty \text{ and } \rho_{IN}(X) < \infty \,, \tag{2.4}$$

which can be interpreted as an assumption of well-posedness of the problem. Under this restriction, all admissible premia are finite.

2.2. Reinsurance contracts

We assume that the premium principles charged by each reinsurer also follow a Choquet expectation. That is, the premium charged by reinsurer i to ensure a risk Y is

$$\pi_i(Y) := \int Y \, d\nu_i \,, \tag{2.5}$$

where $\nu_i \in \mathcal{C}$. Similar to (2.1), we also use the notation

$$\pi^{\nu_i}(Y) := \int Y \, d\nu_i \,. \tag{2.6}$$

Here, we use π to refer to premia and ρ to refer to risk measures. Note here that we do not further restrict the set of possible premium principles in $\mathcal{C}$. In particular, we do not require $\nu \in \mathcal{C}$ to be normalized such that $\nu(S) = 1$. Hence, the reinsurer is free to introduce a loading factor if they so choose, to represent any frictional cost of reinsurance.

In this market, the insurer may cede a portion of risk $I_i(X)$ to reinsurer i by paying the premium $\pi_i(I_i(X))$. Let the vector $\vec{I} := (I_1, \ldots, I_n)$ denote the indemnity schedule chosen by the insurer. We impose the following assumption, which restricts the available indemnities to those that satisfy the so-called *no-sabotage* condition (e.g., Carlier and Dana, 2003):

Assumption 2.4. $I_i \in \mathcal{I}$ for each $i \in \mathcal{N}$, and $\sum_{i\in\mathcal{N}} I_i \in \mathcal{I}$, where

$$\mathcal{I} := \left\{ f : \mathbb{R}_+ \to \mathbb{R}_+ \;\middle|\; f(0) = 0,\ f \text{ is absolutely continuous}, 0 \leqslant f'(x) \leqslant 1 \text{ for a.e. } x \in \mathbb{R}_+ \right\}.$$

That is, each indemnity function, as well as the aggregate indemnity, is monotone and 1-Lipschitz. We refer to the class of indemnity schedules $\vec{I}$ satisfying Assumption 2.4 as

$$\vec{\mathcal{I}} \subset \mathcal{I}^n.$$

Restricting the set of admissible indemnities *a priori* to those in the above class rules out any potential moral hazard that might arise from the insurer's misreporting of the true value of the loss in a given state of the world.

2.3. A sequential game framework

We model this reinsurance market as a sequential game. First, all reinsurers simultaneously select pricing capacities $\nu_i \in \mathcal{C}$. The insurer then views these pricing rules and selects an indemnity vector $(I_1, \ldots, I_n)$. The remaining random loss of the insurer is $X - \sum_{i\in\mathcal{N}} I_i(X)$ and the insurer pays the premium $\pi_i(I_i(X)) = \pi^{\nu_i}(I_i(X))$ to reinsurer i. Hence, the resulting risk exposure of the insurer is $X - \sum_{i\in\mathcal{N}} I_i(X) + \sum_{i\in\mathcal{N}} \pi_i(I_i(X))$, and the insurer evaluates this end-of-period risk exposure via

$$\rho_{IN}\left(X - \sum_{i\in\mathcal{N}} I_i(X) + \sum_{i\in\mathcal{N}} \pi_i(I_i(X))\right) = \rho_{IN}(X) - \rho_{IN}\left(\sum_{i\in\mathcal{N}} I_i(X)\right) + \sum_{i\in\mathcal{N}} \pi^{\nu_i}(I_i(X)) \,, \tag{2.7}$$

where the simplification follows from comonotonicity, and comonotonic additivity and translation invariance of the risk measure ρ_{IN}. Each reinsurer i will have assumed the risk $I_i(X)$ upon payment of the premium $\pi_i(I_i(X))$. Their resulting risk exposure is $I_i(X) - \pi_i(I_i(X))$, and they evaluate it via

$$\rho_i\left(I_i(X) - \pi_i(I_i(X))\right) = \rho_i(I_i(X)) - \pi^{\nu_i}(I_i(X)) \,. \tag{2.8}$$

The goal of each agent is to minimize their risk exposure – as such, (2.7) and (2.8) represent the payoffs of the game to the insurer and reinsurers respectively. We note that as a consequence of (2.4), the expressions (2.7) and (2.8) are always well-defined and finite.

Similar to the Stackelberg setting, reinsurers have the first-mover advantage, and their strategy is determined by their collective simultaneous choice of premium principles (which, by (2.6), depends only on pricing capacities). Therefore, the reinsurers' strategy can be identified by the choice of a vector of capacities $(\nu_1, \ldots, \nu_n) \in \mathcal{C}^n$. Here, the strategy of reinsurer i is denoted by $\nu_i \in \mathcal{C}$. The insurer's strategy is determined by its choice of indemnity schedule, after viewing the pricing rules selected by the reinsurers. This can be identified by a function

$$\begin{aligned} \mathfrak{I} : \quad \mathcal{C}^n &\to \vec{\mathcal{I}} \\ (\nu_1, \ldots, \nu_n) &\mapsto \mathfrak{I}(\nu_1, \ldots, \nu_n) \,, \end{aligned} \tag{2.9}$$

which maps the observed pricing capacities to a feasible indemnity schedule. A strategy (in the formal game-theoretical sense) is therefore represented by the tuple $(\nu_1, \ldots, \nu_n, \mathfrak{I}) \in \mathcal{C}^n \times (\vec{\mathcal{I}})^{\mathcal{C}^n}$.

2.4. Additional notational conventions

We use the following conventions in the remainder of this paper. Recall that $\mathcal{N} = \{1, \ldots, n\}$ denotes the set of reinsurers. For each reinsurer $i \in \mathcal{N}$, we define the capacity

$$\tau_i := g_i \circ \mathbb{Q}_i \tag{2.10}$$

to represent the true preferences of the reinsurer. For ease of notation, we use the index 0 to represent the insurance company. Hence, all agents in the reinsurance market are given by $\mathcal{N} \cup \{0\}$. Following this convention, we define

$$\rho_0 := \rho_{IN} \tag{2.11}$$

to represent the risk measure of the insurer, and

$$I_0(X) := X - \sum_{i \in \mathcal{N}} I_i(X) \tag{2.12}$$

to represent the retained loss of the insurer. Note that as a consequence of Assumption 2.4, we have $I_0 \in \mathcal{I}$. Additionally, we define

$$\tau_0 := g \circ \mathbb{P} \tag{2.13}$$

as the true preferences of the insurer. It will also be convenient to define

$$\nu_0 := g \circ \mathbb{P} \tag{2.14}$$

as the "pricing rule" used by the insurer in the reinsurance market. While (2.13) and (2.14) are different notational conventions for the same capacity, this will be convenient for the exposition of our main results. Note that we use the letter τ to represent true preferences of market agents, and we use the letter ν to represent pricing rules in the market. The interpretation for the insurer is as follows: the insurer uses their own true preferences as a premium principle when deciding which contracts to participate in (and which contracts to decline).

We use the unstyled capital letter I to refer to indemnity functions, i.e., elements of $\mathcal{I}$. Vectors of indemnities are notated with the vector symbol. For example, $\vec{I}$ is an element of $\vec{\mathcal{I}}$.

In the context of the economic game, the notation $\mathfrak{I}$ denotes a strategy, and it is therefore a map from $\mathcal{C}^n$ to $\vec{\mathcal{I}}$, as defined in (2.9). Hence, for any fixed reinsurance strategies $(\nu_1, \dots, \nu_n) \in \mathcal{C}^n$, the choice of indemnities $\mathfrak{I}(\nu_1, \dots, \nu_n)$ is a vector in $\vec{\mathcal{I}}$. For a reinsurer $i \in \mathcal{N}$, we use the notation $\mathfrak{I}_i(\nu_1, \dots, \nu_n)$ to refer to the i-th component of $\mathfrak{I}(\nu_1, \dots, \nu_n)$. That is, $\mathfrak{I}_i(\nu_1, \dots, \nu_n)$ is the indemnity ceded to the i-th insurer under the strategy $\mathfrak{I}$, when the pricing capacities of the reinsurers are given by $(\nu_1, \dots, \nu_n)$. When pricing capacities $(\nu_1, \dots, \nu_n)$ are fixed in context and we are only concerned with the indemnities $\mathfrak{I}(\nu_1, \dots, \nu_n)$, we drop the capacities from the argument of the function and write $I_i = \mathfrak{I}_i(\nu_1, \dots, \nu_n)$.

2.5. Equilibria in sequential games

A common solution concept for sequential games is that of the *Subgame Perfect Nash Equilibrium* (SPNE), which is given in the following definitions.

Definition 2.5. A strategy $(\nu_1^*, \dots, \nu_n^*, \mathfrak{I}^*)$ is a *Nash Equilibrium* (NE) if:

(1) There does not exist a capacity $\tilde{\nu}$ and $i \in \mathcal{N}$ such that

$$\rho_i(\mathfrak{I}_i^*(\tilde{\nu}, \nu_{-i}^*)(X)) - \pi^{\tilde{\nu}}(\mathfrak{I}_i^*(\tilde{\nu}, \nu_{-i}^*)(X))) < \rho_i(\mathfrak{I}_i^*(\nu_1^*, \dots, \nu_n^*)(X)) - \pi^{\nu_i^*}(\mathfrak{I}_i^*(\nu_1^*, \dots, \nu_n^*)(X)),$$

where $(\tilde{\nu}, \nu_{-i}^*)$ denotes the vector $(\nu_1^*, \dots, \nu_n^*)$ with the i-th component replaced by $\tilde{\nu}$.

(2) There does not exist an indemnity selection $\tilde{\mathfrak{I}}$ such that

$$\begin{aligned}\rho_{IN}(X) - &\sum_{i \in \mathcal{N}} \rho_{IN}\left(\tilde{\mathfrak{I}}_i(\nu_1^*, \dots, \nu_n^*)(X)\right) + \sum_{i \in \mathcal{N}} \pi^{\nu_i^*}(\tilde{\mathfrak{I}}_i(\nu_1^*, \dots, \nu_n^*)(X)) \\ &< \rho_{IN}(X) - \rho_{IN}(\mathfrak{I}_i^*(\nu_1^*, \dots, \nu_n^*)(X)) + \sum_{i \in \mathcal{N}} \pi^{\nu_i^*}(\mathfrak{I}_i^*(\nu_1^*, \dots, \nu_n^*)(X)).\end{aligned}$$

Definition 2.6. In our setting, a strategy $(\nu_1^*, \dots, \nu_n^*, \mathfrak{I}^*)$ is a *Subgame Perfect Nash Equilibrium* (SPNE) if:

(1) It is an NE.

(2) For any choice of capacities $(\nu_1, \dots, \nu_n) \in \mathcal{C}^n$, there does not exist an indemnity selection $\tilde{\mathfrak{I}}$ such that

$$\begin{aligned}\rho_{IN}(X) - &\sum_{i \in \mathcal{N}} \rho_{IN}\left(\tilde{\mathfrak{I}}_i(\nu_1, \dots, \nu_n)(X)\right) + \sum_{i \in \mathcal{N}} \pi^{\nu_i}(\tilde{\mathfrak{I}}_i(\nu_1, \dots, \nu_n)(X)) \\ &< \rho_{IN}(X) - \sum_{i \in \mathcal{N}} \rho_{IN}(\mathfrak{I}_i^*(\nu_1, \dots, \nu_n)(X)) + \sum_{i \in \mathcal{N}} \pi^{\nu_i}(\mathfrak{I}_i^*(\nu_1, \dots, \nu_n)(X)).\end{aligned}$$

A subgame is defined as a subset of a sequential game that is induced by viewing previous decisions as fixed. A strategy is an SPNE if it induces a NE in every subgame. In the context of the present paper, each strict subgame consists of the insurer's decision problem after observing the premium principles of the reinsurers. The main difference is that in an SPNE, the insurer behaves optimally for *any* choice of capacities $(\nu_1, \dots, \nu_n) \in \mathcal{C}^n$. For the strategy to be an NE, it is only required that the insurer behaves optimally for the *specific* choice of capacities $(\nu_1^*, \dots, \nu_n^*)$. For a formal definition of subgames and SPNEs and an extensive discussion thereof, we refer to Osborne and Rubinstein (1994).

While it is clear that every SPNE is an NE, the converse does not hold: in general, there exist NEs that are not SPNEs. This is true even in our setting, where the structure of the game is simple. We provide an example in Appendix B.2, and we briefly comment on the advantages of SPNEs over NEs.

3. A characterization of SPNEs

In this section, we provide sufficient conditions to characterize SPNEs in the sequential market outlined in Section 2. It is well known that all SPNEs can be found via the process backward induction (see, e.g., Osborne and Rubinstein, 1994, Proposition 99.2). We restate this result in our setting in the following proposition.

Proposition 3.1. *A strategy $(\nu_1^*, \dots, \nu_n^*, \mathfrak{I}^*)$ is an SPNE if and only if it can be found through backward induction. In other words, for any choice of pricing capacities $(\nu_1, \dots, \nu_n) \in \mathcal{C}^n$, the indemnity structure $\mathfrak{I}^*(\nu_1, \dots, \nu_n)$ solves*

$$\min_{I \in \vec{\mathcal{I}}} \left\{ \rho_{IN}(X) - \sum_{i \in \mathcal{N}} \rho_{IN}\left(I_i(X)\right) + \sum_{i \in \mathcal{N}} \pi^{\nu_i}\left(I_i(X)\right) \right\}, \tag{3.1}$$

and $(\nu_1^*, \dots, \nu_n^*)$ is a NE for the reduced game formed by fixing the insurer's strategy $\mathfrak{I}^*$. That is, there does not exist $i \in \mathcal{N}$ and $\tilde{\nu} \in \mathcal{C}$ such that

$$\rho_i(\mathfrak{I}_i^*(\tilde{\nu}, \nu_{-i}^*)(X)) - \pi^{\tilde{\nu}}(\mathfrak{I}_i^*(\tilde{\nu}, \nu_{-i}^*)(X)) < \rho_i(\mathfrak{I}_i^*(\nu_1^*, \dots, \nu_n^*)(X)) - \pi^{\nu_i^*}(\mathfrak{I}_i^*(\nu_1^*, \dots, \nu_n^*)(X)). \tag{3.2}$$

We will see from the result of Proposition 3.3 that Problem (3.1) always has a solution. Therefore, by Proposition 3.1, to characterize an SPNE, we will:

(1) Find Nash Equilibria in the subgames where the insurer selects reinsurance. That is, for any pricing capacities $(\nu_1, \dots, \nu_n) \in \mathcal{C}^n$, we will determine the optimal strategy $\mathfrak{I}^*$ of the insurer. This reduces to a demand problem for the insurer, and we can characterize solutions via the MIF approach, as in Boonen and Ghossoub (2021). This is outlined in Subsection 3.1.
(2) For the selected indemnity strategy $\mathfrak{I}^*$, we will find Nash Equilibria for the simultaneous selection of pricing capacities. We provide sufficient conditions in Subsection 3.2.

Remark 3.2. In the special case when $n = 1$, we have only one reinsurer. Then Proposition 3.1 implies that a strategy $(\nu_1^*, \mathfrak{I}^*)$ is an SPNE if and only if (3.1) and (3.2) hold. The condition for the reduced game (3.1) simplifies to

$$\mathfrak{I}^*(\nu_1^*) = \min_{I \in \vec{\mathcal{I}}} \left\{ \rho_{IN}(X) - \rho_{IN}\left(I(X)\right) + \pi^{\nu_1^*}\left(I(X)\right) \right\}.$$

Furthermore, the second condition (3.2) implies that

$$\rho_1(\mathfrak{I}^*(\nu_1^*)(X)) - \pi^{\nu_1^*}(\mathfrak{I}^*(\nu_1^*)(X)) = \min_{\tilde{\nu} \in \mathcal{C}} \left\{ \rho_1(\mathfrak{I}^*(\tilde{\nu})(X)) - \pi^{\tilde{\nu}}(\mathfrak{I}^*(\tilde{\nu})(X)) \right\}.$$

It follows directly from the definition of Stackelberg equilibria (e.g., Definition 2.7 of Boonen and Ghossoub, 2022) that every Stackelberg equilibrium is an SPNE. Therefore, it is natural to interpret the SPNE in our model as a generalization of the Stackelberg setting to multiple reinsurers. We elaborate on this special case in Subsection 4.1.1.

3.1. Backward induction step one – an optimal reinsurance problem

In the first step of the backwards induction, we solve Problem (3.1) given any fixed strategy of the reinsurers $(\nu_1, \dots, \nu_n) \in \mathcal{C}^n$. A characterization is given in the following proposition, which can be found in Boonen and Ghossoub (2021). We provide a slightly condensed proof in Appendix A.1.

Proposition 3.3. *Define the capacity*

$$\underline{\nu}(X > z) := \min_{j \in \mathcal{N}} \nu_j(X > z),$$

and define the set

$$\mathcal{N}_z := \left\{ i \in \mathcal{N} : \nu_i(X > z) = \underline{\nu}(X > z) \right\}. \tag{3.3}$$

The selection of indemnities $\vec{I}^* = (I_1^*, \dots, I_n^*)$ is optimal for Problem (3.1) if and only if for each $i \in \mathcal{N}$ and all $x \in \mathbb{R}_+$, there exists $[0,1]$-valued measurable functions h, h_i such that for almost all $z \in \mathbb{R}_+$,

$$\begin{aligned} I_i^*(x) &= \int_0^x \gamma_i^*(z)\, dz, \\ \gamma_i^*(z) &= h_i(z) \mathbb{1}_{\{i \in \mathcal{N}_z\}}, \\ \sum_{i \in \mathcal{N}} \gamma_i^*(z) &= \mathbb{1}_{\{g(\mathbb{P}(X>z)) > \underline{\nu}(X>z)\}} + h(z) \mathbb{1}_{\{g(\mathbb{P}(X>z)) = \underline{\nu}(X>z)\}}. \end{aligned}$$

We see that Proposition 3.3 gives a full characterization of optimal solutions to Problem (3.1). However, optimal indemnities are not unique in general: from the statement of Proposition 3.3, there is some flexibility in choosing the functions h_i when $i \in \mathcal{N}_z$. That is, the insurer has many decisions that are optimal in the sense of Problem (3.1) when they are indifferent between their choice of reinsurer. We narrow the set of optimal solutions by imposing the following condition.

Definition 3.4. For each $z \in \mathbb{R}_+$, let $\mathcal{N}_z$ be as defined in (3.3). An optimal indemnity $\vec{I}^*$ *distributes generously* if for almost all $z \in \mathbb{R}_+$, we have:

(1) If $g(\mathbb{P}(X > z)) = \underline{\nu}(X > z)$, and there exist $j \in \mathcal{N}_z$ such that $\tau_j(X > z) < g(\mathbb{P}(X > z))$, then

$$\sum_{i \in \mathcal{N}} \gamma_i^*(z) = 1\,.$$

(2) If $i \in \mathcal{N}_z$ and either:
- there exist $k \in \mathcal{N}_z \setminus \{i\}$ such that $\tau_k(X > z) < \tau_i(X > z)$; or,
- $g(\mathbb{P}(X > z)) = \underline{\nu}(X > z)$ and $g(\mathbb{P}(X > z)) < \tau_i(X > z)$,

then

$$\gamma_i^*(z) = 0\,.$$

Recall from (2.13) and (2.14) the conventions $\tau_0 = g \circ \mathbb{P}$ and $\nu_0 = g \circ \mathbb{P}$. Additionally, we may define γ_0^* such that for all $z \in \mathbb{R}_+$,

$$\gamma_0^*(z) := 1 - \sum_{i \in \mathcal{N}} \gamma_i^*(z)\,. \tag{3.4}$$

Under these conventions, Definition 3.4 can be stated more succinctly. An optimal indemnity $\vec{I}^*$ distributes generously if for almost all $z \in \mathbb{R}_+$ and all $i \in \mathcal{N} \cup \{0\}$ we have $\gamma_i^*(z) = 0$ if both of the following conditions hold:

- $\nu_i(X > z) = \min_{j \in \mathcal{N} \cup \{0\}}\{\nu_j(X > z)\}$; and,
- there exist $k \in (\mathcal{N} \cup \{0\}) \setminus \{i\}$ such that $\nu_k(X > z) = \nu_i(X > z)$ but $\tau_k(X > z) < \tau_i(X > z)$.

These conditions state that when the insurer is indifferent between reinsurers, it chooses the reinsurer that has the most to gain from the contract. This is interpreted as an act of good faith – it is in the insurer's best interest to maintain good business relationships with its counterparties. For a fixed set of pricing capacities $(\nu_1, \ldots, \nu_n)$, we denote the set of generously distributed optimal indemnities in the sense of Definition 3.4 by $\mathcal{I}(\nu_1, \ldots, \nu_n)$. In the following, we assume that the insurer always selects an optimal indemnity that distributes generously. We denote the set of such strategies by

$$\aleph := \left\{ \mathfrak{I} \in (\mathcal{C}^n)^{\vec{\mathcal{I}}} : \mathfrak{I}(\nu_1, \ldots, \nu_n) \in \mathcal{I}(\nu_1, \ldots, \nu_n),\ \forall\, (\nu_1, \ldots, \nu_n) \in \mathcal{C}^n \right\}. \tag{3.5}$$

In Section 3.2, we address the decision problem for the reinsurers, while fixing the insurer's strategy $\mathfrak{I} \in \aleph$. In doing so, we show that generous distribution is sufficient to guarantee the existence of an SPNE.

Remark 3.5. The notion of generous distribution outlined in Definition 3.4 is, to the best of our knowledge, a new concept. However, we note that a similar assumption has been applied in the Stackelberg setting by Cheung et al. (2019). They assume that the marginal indemnity function $h(z) \equiv 1$ in order to reduce the set of optimal insurer strategies to a singleton. It can be verified that in the Stackelberg case, their assumption is stronger than ours, in the sense that $h(z) \equiv 1$ implies generous distribution, but that the converse does not hold.

Remark 3.6. Note that the set $\aleph$ is in general a strict subset of the optimal solutions given by Proposition 3.3. Hence, maximizing coverage and generous distribution are not necessary conditions for the existence of SPNE, since there may exist SPNEs when the insurer chooses a strategy in the solution set of Proposition 3.3 but not in $\aleph$.

3.2. Backward induction step two – the reinsurers' strategies

We now exhibit a characterization of SPNEs by completing the second step of the backward induction in Proposition 3.1. We begin by fixing an optimal strategy for the insurer $\mathfrak{I}^* \in \aleph$. Then, we characterize capacities $(\nu_1^*, \ldots, \nu_n^*)$ such that there does not exist $\hat{\nu}$ that satisfies, for some $i \in \mathcal{N}$,

$$\rho_i(\mathfrak{I}_i^*(\hat{\nu}, \nu_{-i}^*)(X)) - \pi^{\hat{\nu}}(\mathfrak{I}_i^*(\hat{\nu}, \nu_{-i}^*)(X)) < \rho_i(\mathfrak{I}_i^*(\nu_1^*, \ldots, \nu_n^*)(X)) - \pi^{\nu_i}(\mathfrak{I}_i^*(\nu_1^*, \ldots, \nu_n^*)(X))\,.$$

We show that capacities satisfying certain conditions constitute an SPNE strategy. These conditions are closely related to the *second-lowest true preferences* of all $n+1$ agents in the reinsurance market model. The second-lowest preferences are suggested by Boonen et al. (2021b) as an upper bound on premium principles that are coalitionally stable. In this paper, SPNEs can be interpreted as an alternate notion of stability, in which the second-lowest preferences again play an important role.

Definition 3.7. Let $\bar{\tau}$ be constructed as the pointwise second-lowest function of the set of capacities $\{\tau_0, \ldots, \tau_n\}$. Hence, for all $z \in \mathbb{R}_+$, there exist $i, j \in \mathcal{N} \cup \{0\}$ such that $i \neq j$, $\bar{\tau}(X > z) = \tau_j(X > z)$, $\tau_i(X > z) \leqslant \tau_j(X > z)$, and $\tau_k(X > z) \geqslant \tau_j(X > z)$ for all $k \neq i, j$. We refer to $\bar{\tau}$ as the *second-lowest true preferences*.

Definition 3.8. For each $z \in \mathbb{R}_+$, let $\mathcal{N}_z$ be as defined in (3.3). Define the class of reinsurer strategies $\beth \subseteq \mathcal{C}^n$ as the set of all $(\nu_1, \ldots, \nu_n)$ such that for almost all $z \in \mathbb{R}_+$:

(a) We have

$$\min_{j \in \mathcal{N}} \nu_j(X > z) = \bar{\tau}(X > z)\,. \tag{3.6}$$

(b) There exist $i, k \in \mathcal{N} \cup \{0\}$, $i \neq k$ such that

$$\min_{j \in \mathcal{N}} \nu_j(X > z) = \nu_i(X > z) = \nu_k(X > z)\,. \tag{3.7}$$

Recall from (2.14) that we use the convention $\nu_0 = g \circ \mathbb{P}$.

(c) We have

$$\mathcal{T}_z \neq \varnothing \implies \mathcal{T}_z \cap \mathcal{N}_z \neq \varnothing, \tag{3.8}$$

where

$$\mathcal{T}_z := \left\{ i \in \mathcal{N} : \tau_i(X > z) = \min_{j \in \mathcal{N} \cup \{0\}} \tau_j(X > z) \right\}.$$

That is, $\mathcal{T}_z$ are the indices in $\mathcal{N}$ that minimize $\tau_j(X > z)$ for $j \in \mathcal{N} \cup \{0\}$.

The first condition states that the lowest price for the loss layer $X > z$ is equal to the second-lowest true preferences $\bar{\tau}$. The second condition guarantees that there are always at least two agents charging this price. Finally, the third condition states that out of the reinsurers who have the lowest true preferences, at least one is posting the price based on $\bar{\tau}$.

We now provide the first main result of this paper: if $\mathfrak{I}^* \in \aleph$ and $(\nu_1^*, \ldots, \nu_n^*) \in \beth$, then the strategy $(\nu_1^*, \ldots, \nu_n^*, \mathfrak{I}^*)$ is an SPNE. We begin with the following two results, whose proofs can be found in Appendix A.1. Proposition 3.9 states that any such strategy $(\nu_1^*, \ldots, \nu_n^*, \mathfrak{I}^*)$ achieves the same risk for each reinsurer. The following result, Proposition 3.10, completes the second step of backward induction, by showing that every reinsurer strategy $(\nu_1^*, \ldots, \nu_n^*) \in \beth$ satisfies (3.2).

Proposition 3.9. *Let $\mathfrak{I}^* \in \aleph$ and $(\nu_1^*, \ldots, \nu_n^*) \in \beth$. Then for all $i \in \mathcal{N}$,*

$$\rho_i(I_i^*(X) - \pi_i) = \int_{\mathcal{Z}_i} \tau_i(X > z) - \bar{\tau}(X > z)\, dz\,,$$

where

$$\mathcal{Z}_i := \{z \in \mathbb{R}_+ : \tau_i(X > z) < \bar{\tau}(X > z)\}\,.$$

Proposition 3.10. *Let $\hat{\nu} \in \mathcal{C}$ be any pricing capacity, and let $\mathfrak{I}^* \in \aleph$ and $(\nu_1^*, \ldots, \nu_n^*) \in \beth$. Then for each $i \in \mathcal{N}$, we have*

$$\rho_i(\mathfrak{I}_i^*(\hat{\nu}, \nu_{-i}^*)(X)) - \pi^{\hat{\nu}}(\mathfrak{I}_i^*(\hat{\nu}, \nu_{-i}^*)(X)) \geqslant \rho_i(\mathfrak{I}_i^*(\nu_1^*, \ldots, \nu_n^*)(X)) - \pi^{\nu_i^*}(\mathfrak{I}_i^*(\nu_1^*, \ldots, \nu_n^*)(X))\,.$$

That is, when the strategies $(\nu_1^*, \ldots, \nu_n^*)$ are chosen by the reinsurers, no reinsurer has an incentive to deviate.

As a direct corollary of Propositions 3.1, 3.3, and 3.10, we obtain the following.

Theorem 3.11. *Let $\mathfrak{I}^* \in \aleph$ and $(\nu_1^*, \ldots, \nu_n^*) \in \beth$. Then the strategy $(\nu_1^*, \ldots, \nu_n^*, \mathfrak{I}^*)$ is an SPNE.*

We provide a few examples of such strategies in the following subsection. Note that while Theorem 3.11 provides sufficient conditions for a strategy to be an SPNE, these conditions are not necessary. An example of an SPNE *not* characterized by Theorem 3.11 is provided in Appendix B.1.

3.2.1. Some examples of SPNEs characterized by Theorem 3.11

We now provide explicit constructions of reinsurer strategies $(\nu_1^*, \ldots, \nu_n^*) \in \beth$. This results in an SPNE when combined with any insurer strategy $\mathfrak{I}^* \in \aleph$, as a consequence of Theorem 3.11. First, recall that

$$\mathcal{T}_z := \left\{ i \in \mathcal{N} : \tau_i(X > z) = \min_{j \in \mathcal{N} \cup \{0\}} \tau_j(X > z) \right\}.$$

Definition 3.12. For each $i \in \mathcal{N}$, define ν_i^* by

$$\nu_i^*(X > z) := \begin{cases} \bar{\tau}(X > z), & i \in \mathcal{T}_z \\ \tau_i(X > z) & \text{otherwise} \end{cases}.$$

In the strategy, the reinsurer with the lowest true preferences for every layer $X > z$ chooses the second-lowest true preferences as their premium principle. Otherwise, the reinsurer quotes a price consistent with its underlying risk measure. The following result shows that this strategy is an SPNE, and its proof is given in Appendix A.1.

Proposition 3.13. *The capacities $(\nu_1^*, \ldots, \nu_n^*)$ given in Definition 3.12 are in $\daleth$.*

Therefore by Theorem 3.11, for any $\mathfrak{I}^* \in \aleph$, the strategy $(\nu_1^*, \ldots, \nu_n^*, \mathfrak{I}^*)$ is an SPNE. By increasing prices to match the preferences of their nearest competitors, reinsurers are able to profit if they are able to correctly identify the heterogeneous beliefs of the agents in the market. Note that in this case, each reinsurer quotes prices that is at least as high as their true preferences. However, this is not necessary for all SPNEs, as shown by the following example.

Remark 3.14. It follows easily from the definition of $\daleth$ that $(\bar{\tau}, \bar{\tau}, \ldots, \bar{\tau}) \in \daleth$.

Again by Theorem 3.11, for any $\mathfrak{I}^* \in \aleph$, $(\bar{\tau}, \bar{\tau}, \ldots, \bar{\tau}, \mathfrak{I}^*)$ is an SPNE. This shows that there exists a pricing mechanism that every reinsurer can adopt for the market to achieve stability. That is, there is no incentive for any reinsurer to deviate from the pricing capacity $\bar{\tau}$, if every other reinsurer is also using the same premium principle.

However, this requires some reinsurers to quote prices that are below their true risk preference. The reinsurers do not experience a loss, since even when their price is too low, the insurer cedes this portion of the risk to another reinsurer. This perhaps places too much hope in the insurer's strategy, which is assumed to distribute generously. Nevertheless, the special form of this SPNE is worth mentioning, and further emphasizes the important role of the "second-lowest" pricing capacity $\bar{\tau}$.

This example is also an illustration of the wide range of pricing capacities that can yield an SPNE. On the levels $X > z$ that the reinsurer does not expect to receive any business, pricing can be fairly arbitrary. In this sense, $\daleth$ is a non-trivial class that allows for many different selections of pricing capacities.

4. Welfare analysis of SPNEs

It is clear that every strategy $(\nu_1, \ldots, \nu_n, \mathfrak{I})$ in this economic game determines a unique premium principle for each reinsurer and a unique indemnity structure demanded by the insurer. This therefore determines the premia that the insurer pays to each reinsurer, which determines the allocation of wealth (or risk) to each agent. In this section, we examine efficiency properties of the contracts induced by SPNEs. We consider both individual rationality (IR) and PE.

Definition 4.1. An *allocation* is a pair $(\vec{I}, \vec{\pi}) \in \vec{\mathcal{I}} \times \mathbb{R}^n$, where $\vec{I}$ denotes the structure of the indemnities and π denotes the vector of premia paid by the insurer. The resulting risk measures under this allocation are

$$\rho_{IN}\left(X - \sum_{i \in \mathcal{N}} I_i(X) + \pi_i\right) \quad \text{and} \quad \rho_i(I_i(X) - \pi_i), \ \forall i \in \mathcal{N}.$$

We say that an allocation $(\vec{I}, \vec{\pi})$ results from (or is induced by) a strategy $(\nu_1, \ldots, \nu_n, \mathfrak{I})$ if for all $i \in \mathcal{N}$, $I_i = \mathfrak{I}_i(\nu_1, \ldots, \nu_n)$ and $\pi_i = \pi_i^{\nu_i}(I_i(X)) = \int I_i(X)\, d\nu_i$.

4.1. Individual rationality of SPNEs

Definition 4.2. An allocation $(\vec{I}, \vec{\pi}) \in \vec{\mathcal{I}} \times \mathbb{R}^n$ is IR if

$$\rho_{IN}\left(X - \sum_{i \in \mathcal{N}} I_i(X) + \sum_{i \in \mathcal{N}} \pi_i\right) \leqslant \rho_{IN}(X) \ \text{and} \ \rho_i(I_i(X) - \pi_i) \leqslant \rho_i(0) = 0, \ \forall i \in \mathcal{N}.$$

That is, the allocation $(\vec{I}, \vec{\pi})$ is not worse than the status quo for any agent – hence, the insurer and all reinsurers are willing to participate with these transactions. Note that if an allocation is IR, then each premium is non-negative, since by translation invariance the condition $\rho_i(I_i(X) - \pi_i) \leqslant \rho_i(0) = 0$ yields

$$\pi_i \geqslant \rho_i(I_i(X)),$$

and by monotonicity $\rho_i(I_i(X)) \geqslant 0$ since $I_i(X) \geqslant 0$ and τ_i is non-negative. We show that for every $\mathfrak{I}^* \in \aleph$ and $(\nu_1^*, \ldots, \nu_n^*) \in \daleth$, the strategy $(\nu_1^*, \ldots, \nu_n^*, \mathfrak{I}^*)$ induces an IR allocation. The result follows readily from Proposition 3.9 and the following lemma.

Lemma 4.3. *Suppose $\mathfrak{I}^* \in \aleph$ and $(\nu_1^*, \ldots, \nu_n^*) \in \daleth$. Then*

$$\rho_{IN}\left(X - \sum_{i \in N} I_i^*(X) + \sum_{i \in \mathcal{N}} \pi^{\nu_i^*}(I_i^*(X))\right) = \int_{\mathcal{Z}_{IN}} g(\mathbb{P}(X > z))\, dz + \int_{\mathcal{Z}_{IN}^C} \bar{\tau}(X > z)\, dz,$$

where

$$\mathcal{Z}_{IN} := \{z \in \mathbb{R}_+ : g(\mathbb{P}(X > z)) < \bar{\tau}(X > z)\}.$$

Theorem 4.4. *Suppose $\mathfrak{I}^* \in \aleph$ and $(\nu_1^*, \ldots, \nu_n^*) \in \beth$. Then the strategy $(\nu_1^*, \ldots, \nu_n^*, \mathfrak{I}^*)$ induces an IR allocation.*

Proof. By Lemma 4.3,

$$\begin{aligned}\rho_{IN}\left(X - \sum_{i \in N} I_i^*(X) + \sum_{i \in \mathcal{N}} \pi^{\nu_i}(I_i^*(X))\right) &= \int_{\mathcal{Z}_{IN}} g(\mathbb{P}(X > z))\, dz + \int_{\mathcal{Z}_{IN}^C} \bar{\tau}(X > z)\, dz \\ &\leqslant \int_{\mathcal{Z}_{IN}} g(\mathbb{P}(X > z))\, dz + \int_{\mathcal{Z}_{IN}^C} g(\mathbb{P}(X > z))\, dz \\ &= \int_0^\infty g(\mathbb{P}(X > z))\, dz = \rho_{IN}(X)\,.\end{aligned}$$

Also, by Proposition 3.9, we have for each $i \in \mathcal{N}$,

$$\rho_i(I_i^*(X) - \pi_i) = \int_{\mathcal{Z}_i} \tau_i(X > z) - \bar{\tau}(X > z)\, dz \leqslant \int_{\mathcal{Z}_i} 0\, dz = 0. \quad \square$$

4.1.1. Market competition and the insurer's welfare

Within the literature on Stackelberg equilibria (i.e., $n = 1$), solutions to the Stackelberg problem imply that the insurer has no incentive to purchase insurance – that is, the insurer is indifferent between reinsurance and the status quo. This argument is emphasized in Boonen and Ghossoub (2022), which concludes that the Stackelberg setting is problematic in problems of optimal reinsurance. It is also straightforward to confirm that the optimal indemnification proposed in Cheung et al. (2019) does not provide a welfare gain for the insurer. The following result shows that in our setting, the same conclusion applies: if there is only one reinsurer, then the insurer does not improve upon the status quo.

Proposition 4.5. *Let $n = 1$, and suppose $\mathfrak{I}^* \in \aleph$ and $\nu_1^* \in \beth$. Then*

$$\rho_{IN}\left(X - I_1(X) + \pi^{\nu_1^*}(I_1(X))\right) = \rho_{IN}(X)\,.$$

However, as the number of reinsurers increases, the resulting risk measure for the insurer decreases, with the decrease being strict under some mild conditions. This result is intuitive from an economic perspective: increased competition in the market lowers prices and benefits the consumer. Hence, by extending the Stackelberg model to include multiple reinsurers, we address an apparent weakness of the Stackelberg setting by capturing value for the insurer.

Fix a value of n, and suppose that the $(n+1)$-st reinsurer has preferences given by the capacity $g_{n+1} \circ \mathbb{Q}_{n+1} := \tau_{n+1}$. We wish to compare the SPNEs in the market with n reinsurers to that with $n+1$ reinsurers. Let $\aleph^n$ and $\aleph^{n+1}$ denote the sets of generously distributing optimal strategies for the market with n and $n+1$ reinsurers, respectively. Similarly, let $\beth^n$ and $\beth^{n+1}$ be the sets of reinsurer strategies satisfying the properties of Definition 3.8 for the market with n and $n+1$ reinsurers, respectively.

Proposition 4.6. *Let $\mathfrak{I}^{*,n} \in \aleph^n$, $\mathfrak{I}^{*,n+1} \in \aleph^{n+1}$, $(\nu_1^{*,n}, \ldots, \nu_n^{*,n}) \in \beth^n$, and $(\nu_1^{*,n+1}, \ldots, \nu_{n+1}^{*,n+1}) \in \beth^{n+1}$. We use the convention that $(I_1^{*,n}, \ldots, I_n^{*,n}) := \mathfrak{I}^{*,n}(\nu_1^{*,n}, \ldots, \nu_n^{*,n})$, and similarly for $n+1$. Then*

$$\rho_{IN}\left(X - \sum_{i=1}^{n+1} I_i^{*,n+1}(X) + \sum_{i=1}^{n+1} \pi^{\nu_i^{*,n+1}}(I_i^{*,n+1}(X))\right) \leqslant \rho_{IN}\left(X - \sum_{i=1}^{n} I_i^{*,n}(X) + \sum_{i=1}^{n} \pi^{\nu_i^{*,n}}(I_i^{*,n}(X))\right).$$

That is, when a reinsurer is added to the market, the welfare of the insurer can only increase.

We can see that the inequality of Proposition 4.6 can be strict when, for example, $\bar{\tau}_{n+1} < \bar{\tau}_n$. Hence, when a reinsurer joins the market by offering a cheaper or differentiated product, the insurer can realize a strict welfare gain. This point is explicitly demonstrated by a numerical example in Subsection 4.4.

4.2. Pareto efficiency of SPNEs

We now show that allocations that result from the SPNE characterized in Section 3 are PE. In this section, we use the conventions $\rho_0 = \rho_{IN}$ and $I_0(X) = X - \sum_{i \in \mathcal{N}} I_i(X)$ as defined in (2.11) and (2.12). As a direct consequence, $X = \sum_{i=0}^n I_i(X)$. Recall that by Assumption 2.4, $I_0 \in \mathcal{I}$.

Definition 4.7. An allocation $(\vec{I}, \vec{\pi}) \in \vec{\mathcal{I}} \times \mathbb{R}^n$ is PE if there does not exist another allocation $(\tilde{I}, \tilde{\pi}) \in \vec{\mathcal{I}} \times \mathbb{R}^n$ such that

$$\rho_{IN}\left(X - \sum_{i \in \mathcal{N}} \tilde{I}_i(X) + \sum_{i \in \mathcal{N}} \tilde{\pi}_i\right) \leqslant \rho_{IN}\left(X - \sum_{i \in \mathcal{N}} I_i(X) + \sum_{i \in \mathcal{N}} \pi_i\right),$$

$$\rho_i(\tilde{I}_i(X) - \tilde{\pi}_i) \leqslant \rho_i(I_i(X) - \pi_i), \quad i \in \mathcal{N},$$

with at least one of these inequalities being strict.

It is well-known that when all risk measures are translation invariant, PE is related to the inf-convolution (e.g., Asimit and Boonen, 2018). For the sake of completeness, we provide a proof in Appendix A.2.

Definition 4.8. The *inf-convolution* of risk measures ρ_i, $i \in \mathcal{N} \cup \{0\}$ with respect to a random variable X is defined as

$$\square_{i=0}^{n} \rho_i(X) := \inf \left\{ \sum_{i=0}^{n} \rho_i(I_i(X)), I \in \vec{\mathcal{I}} \right\}.$$

Proposition 4.9. *An allocation $(\vec{I}, \vec{\pi})$ is PE if and only if $\sum_{i=0}^{n} \rho_i(I_i(X)) = \square_{i=0}^{n} \rho_i(X)$.*

In our case, we can obtain an explicit characterization of the indemnities structures that attain the inf-convolution.

Proposition 4.10. *For each $z \in \mathbb{R}_+$, let*

$$\mathcal{L}_z := \{i \in \mathcal{N} \cup \{0\} : \tau_i(X > z) = \min_{j \in \mathcal{N} \cup \{0\}} \{\tau_j(X > z)\}\},$$

and $\mathcal{L}_z^C := (\mathcal{N} \cup \{0\}) \setminus \mathcal{L}_z$. For each $i \in \mathcal{N} \cup \{0\}$, let γ_i take values in $[0, 1]$ such that for almost every z,

$$\sum_{i \in \mathcal{L}_z} \gamma_i(z) = 1, \sum_{i \in \mathcal{L}_z^C} \gamma_i(z) = 0,$$

and define $I_i(z) := \int_0^z \gamma_i(z)\, dz$. Then

$$\sum_{i=0}^{n} \rho_i(I_i(X)) = \square_{i=0}^{n} \rho_i(X).$$

That is, $\vec{I}$ achieves the inf-convolution. Furthermore, any indemnity schedule that achieves the inf-convolution is of this form.

We now provide the main result of this section: the SPNEs from Theorem 3.11 induce PE allocations. First, we show that for any $\mathfrak{I}^* \in \aleph$, the reinsurer strategy $(\bar{\tau}, \ldots, \bar{\tau})$ given in Remark 3.14 induces a PE allocation. We then extend this result to all reinsurer strategies in $\daleth$ by applying Proposition 3.9 and Lemma 4.3.

Lemma 4.11. *The strategy $(\bar{\tau}, \bar{\tau}, \ldots, \bar{\tau}, \mathfrak{I}^*)$ from Remark 3.14 induces an IR and PE allocation. That is, $(\vec{I}^*, \vec{\pi}^*)$ is IR and PE in the sense of Definitions 4.2 and 4.7, where $\vec{I}^*$ is an element of $\mathcal{I}(\bar{\tau}, \bar{\tau}, \ldots, \bar{\tau})$ and $\pi_i^* = \rho^{\bar{\tau}}(I_i^*(X))$.*

Theorem 4.12. *Let $\mathfrak{I}^* \in \aleph$ and $(\nu_1^*, \ldots, \nu_n^*) \in \daleth$. Then the strategy $(\nu_1^*, \ldots, \nu_n^*, \mathfrak{I}^*)$ induces an IR and PE allocation.*

As a direct corollary of Theorem 4.12, we obtain the following.

Corollary 4.13. *The strategy $(\nu_1^*, \ldots, \nu_n^*, \mathfrak{I}^*)$ with $\mathfrak{I}^* \in \aleph$ and $\nu_1^*, \ldots, \nu_n^*$ as in Definition 3.12 is IR and PE.*

4.3. Decentralization of Pareto-efficient allocations via an SPNE

We now provide a partial converse to Theorem 4.12. That is, given a PE allocation $(\vec{I}^*, \vec{\pi}^*)$, we show that this can be generated by an SPNE as characterized in Theorem 3.11.

However, by Proposition 3.9 and Lemma 4.3, we see that every allocation induced by such an SPNE has the same resulting risk measure for each agent. Therefore, we must first impose the following assumption on the allocations.

Assumption 4.14. The allocation $(\vec{I}^*, \vec{\pi}^*)$ satisfies

$$\rho_i(I_i^*(X) - \pi_i^*) = \int_{\mathcal{Z}_i} \tau_i(X > z) - \bar{\tau}(X > z)\, dz,$$

$$\rho_{IN}\left(X - \sum_{i \in \mathcal{N}} I_i^*(X) + \sum_{i \in \mathcal{N}} \pi_i^*\right) = \int_{\mathcal{Z}_{IN}} g(\mathbb{P}(X > z))\, dz + \int_{\mathcal{Z}_{IN}^C} \bar{\tau}(X > z)\, dz.$$

We show that if $(\vec{I}^*, \vec{\pi}^*)$ is an allocation satisfying Assumption 4.14, then it is induced by the choice of reinsurer strategies $(\nu_1^*, \ldots, \nu_n^*)$ given in Definition 3.12. This result follows from the following propositions.

Proposition 4.15. *Let $(\vec{I}^*, \vec{\pi}^*)$ be an allocation satisfying Assumption 4.14, and $(\nu_1^*, \dots, \nu_n^*)$ be as in Definition 3.12. Then for each $i \in \mathcal{N}$,*

$$\pi_i^* = \int I_i^*(X)\, d\nu_i^* .$$

Proposition 4.16. *Let $(\vec{I}^*, \vec{\pi}^*)$ be an allocation satisfying Assumption 4.14, and $(\nu_1^*, \dots, \nu_n^*)$ be the insurer strategies from Definition 3.12. Then the indemnity profile $\vec{I}^*$ is optimal in the sense of Proposition 3.3. That is, the profile $\vec{I}^*$ solves*

$$\min_{I \in \vec{\mathcal{I}}} \left\{ \rho_{IN} \left(X - \sum_{i \in \mathcal{N}} I_i(X) + \sum_{i \in \mathcal{N}} \pi^{\nu_i^*}(I_i(X)) \right) \right\} .$$

Proposition 4.17. *Let $(\vec{I}^*, \vec{\pi}^*)$ be an allocation satisfying Assumption 4.14. Then for any choice of reinsurer strategies $(\nu_1^*, \dots, \nu_n^*) \in \beth$, the indemnities $\vec{I}^*$ distribute generously the sense of Definition 3.4.*

Theorem 4.18. *Suppose $(\vec{I}^*, \vec{\pi}^*)$ is an allocation satisfying Assumption 4.14. Then this allocation results from an SPNE with capacities $(\nu_1^*, \dots, \nu_n^*)$ as given in Definition 3.12. That is, there exists an $\mathfrak{I}^* \in \aleph$ such that $\vec{I}^* = \mathfrak{I}^*(\nu_1^*, \dots, \nu_n^*)$, and for all $i \in \mathcal{N}$,*

$$\pi_i^* = \int I_i^*(X)\, d\nu_i^* .$$

Proof. By Propositions 4.16 and 4.17, we see that $\vec{I}^*$ is optimal and distributes generously, if reinsurer strategies are fixed to be those in Definition 3.12. Hence, there exists a strategy $\mathfrak{I}^* \in \aleph$ such that $\vec{I}^* = \mathfrak{I}^*(\nu_1^*, \dots, \nu_n^*)$. The rest follows directly from Proposition 4.15. □

4.4. A numerical example

We now illustrate our main results with a numerical example, where the insurer is subject to a risk X distributed according to a censored exponential distribution. By applying Theorem 3.11, we explicitly characterize an SPNE in this market. We also show that in this SPNE, each agent realizes a strict welfare gain, which implies that each agent has an incentive to participate in this contract. This illustrates the result of Proposition 4.6, by showing that introducing competition on the supply side benefits the insurer. Furthermore, by Theorems 4.4 and 4.12, the allocation resulting from the SPNE is IR and PE.

We assume that the insurer's beliefs of the risk X are represented by the survival function

$$\mathbb{P}(X > z) = \begin{cases} \exp(-\beta_0 \cdot z)\,, & 0 \leqslant z \leqslant 5\,, \\ 0\,, & z \geqslant 5\,. \end{cases}$$

Here, we take $\beta_0 = 2.5$. We assume that there are two reinsurers in the market: i.e. $n = 2$. For $i = 1, 2$, let

$$\mathbb{Q}_i(X > z) = \begin{cases} \exp(-\beta_i \cdot z)\,, & 0 \leqslant z \leqslant 5\,, \\ 0\,, & z \geqslant 5\,, \end{cases}$$

with $\beta_1 = 2$ and $\beta_2 = 1.7$. These different parameters are due to the heterogeneity in beliefs among agents: while the reinsurers agree with the insurer that X is distributed according to a censored exponential distribution and that this distribution is censored at 5, they differ in their belief of the parameter of the distribution.

We also assume that each agent measures risk according to the Tail Value-at-Risk (TVaR). That is, we have

$$g(t) = \min\left\{ \frac{t}{1-\alpha_0}, 1 \right\}, \qquad g_i(t) = \min\left\{ \frac{t}{1-\alpha_i}, 1 \right\}, i = 1, 2,$$

where $\alpha_0 = 0.99, \alpha_1 = 0.95$ and $\alpha_2 = 0.90$.

By the results of Section 3, we are primarily interested in the distorted subjective survival probabilities $\tau_i(X > z)$. These probabilities for τ_i (where $i = 0, 1, 2$) are displayed in Fig. 1a. The second-lowest capacity $\bar{\tau}$, as given in Definition 3.7, is displayed in Fig. 1b.

Let ν_1^*, ν_2^* be as in Definition 3.12. Then, the indemnity structure for an optimal insurer strategy $\mathfrak{I}^* \in \aleph$ is shown in Fig. 2. It can be seen that the indemnity function for reinsurer i is strictly increasing only if $\tau_i(X > z) = \min_{j=0,1,2} \tau_j(X > z)$.

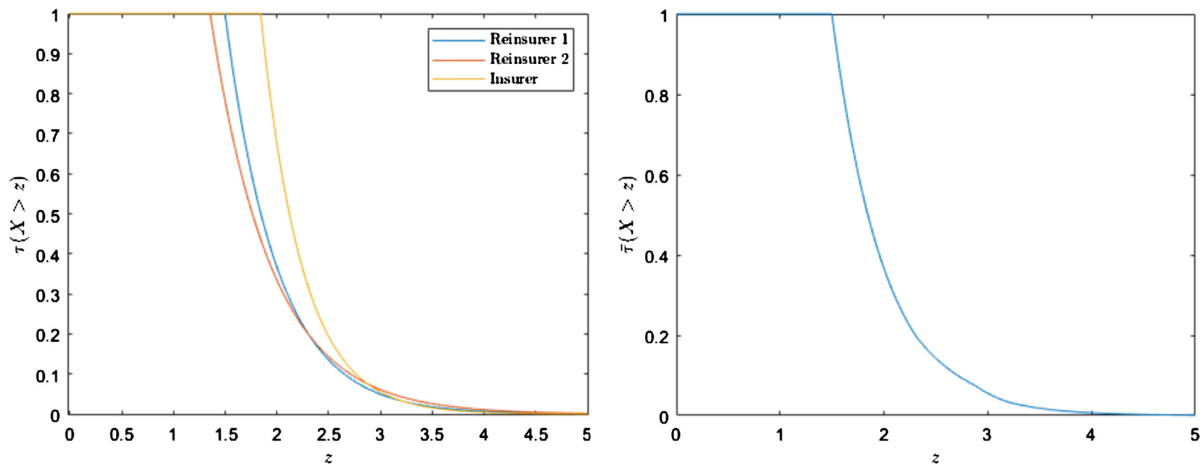
Based on the reinsurers' strategies ν_1^{**} and ν_2^*, numerical calculation yields the following premia:

$$\pi_1^*(I_1^*(X)) = \pi^{\nu_1^*}(I_1^*(X)) = 1.4433 \text{ and } \pi_2^*(I_2^*(X)) = 0.5450.$$

The risk measures for the agents resulting from these strategies are,

$$\rho_{IN}\left(X - \sum_{i=1}^{2} I_i^*(X) + \sum_{i=1}^{2} \pi_i^* \right) = 2.001, \quad \rho_1(I_1^*(X) - \pi_1^*) = -0.0065, \quad \rho_2(I_2^*(X) - \pi_2^*) = -0.0725.$$

From the calculations, it is evident that $\rho_{IN}(X) \approx 2.2419$, indicating that the insurer realizes a strict welfare gain from this allocation. Additionally, the initial risk of both reinsurers is 0, whereas the risk of both reinsurers are negative under the SPNE contracts. Therefore, both reinsurers also decrease their risk as a result of this allocation.



(A) Distorted survival probabilities τ. (B) Second-lowest true preferences $\bar{\tau}$.

Fig. 1. Survival functions of true preferences.

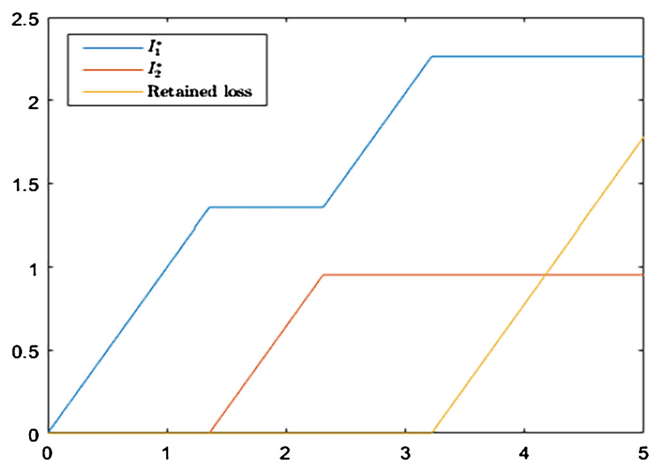


Fig. 2. Optimal indemnity structure $\vec{I}^*$.

5. Conclusion

In this paper, we provide a novel reinsurance market mechanism with multiple reinsurers, in which the reinsurers have the first-mover advantage. We assume distortion risk measure preferences for each agent and allow for heterogeneity in beliefs. Premium principles are nonlinear and are taken as Choquet expectations with respect to general capacities. Within this general setup, we argue that the notion of a Subgame Perfect Nash Equilibrium (SPNE) is the appropriate solution concept. Our main results characterize SPNEs, identify their properties, and provide a welfare analysis of resulting equilibrium allocations.

We first provide sufficient conditions which lead to an SPNE in Theorem 3.11. By applying backward induction, we isolate the decision problem of the insurer and the decision problem of the reinsurers. The former is addressed by applying the so-called marginal indemnification function approach to the case of heterogeneous beliefs, as done in Boonen and Ghossoub (2021). The latter is addressed by constructing strategies satisfying the conditions of Definition 3.8. In these equilibria, the insurer faces prices induced by the second-lowest true preferences as measured by distorted subjective survival probabilities.

Additionally, we examine the Pareto efficiency properties of contracts resulting from SPNEs in Section 4. Since market equilibria are not efficient in general, we separately analyze the welfare of each agent. In Theorem 4.12, we demonstrate that such equilibria result in Pareto-efficient contracts in our market model Conversely, we show in Theorem 4.18 that certain efficient allocations can be decentralized: that is, they can be induced by market forces. Since we identify market equilibria before considering efficiency, we do not need to assume that the agents in the market cooperate with each other, or that there exists a central planner influencing decisions.

Finally, our setting and results could be extended in several potential directions. For example, we consider a market with only one (representative) insurer, whereas typically multiple insurers, with heterogeneous risk preferences and risk exposures, participate in reinsurance markets. Moreover, reinsurers often operate with a background risk arising from other operational and financial decisions (e.g., Dana and Scarsini, 2007, Balbás et al., 2022). Another interesting extension would be to examine the effect of information asymmetry between the insurer and the reinsurer. For instance, in markets with multiple insurers with hidden types, a reinsurer can offer a menu of different premia to the insurers. We refer to Liang et al. Liang et al. (2022) for a recent discussion of this topic.

Declaration of competing interest

None declared.

Data availability

No data was used for the research described in the article.

Appendix A. Proofs of main results

A.1. Characterization of SPNEs

Proof. Suppose first that the strategy $(\nu_1^*, \ldots, \nu_n^*, \mathfrak{I}^*)$ was found through backward induction. Then it satisfies condition (2) of Definition 2.6. It remains to check that this is an NE. For the sake of contradiction, suppose $(\nu_1^*, \ldots, \nu_n^*, \mathfrak{I}^*)$ is not an NE. By (3.1), there does not exist a strategy $\tilde{\mathfrak{I}}$ such that

$$\rho_{IN}(X) - \sum_{i \in \mathcal{N}} \rho_{IN}\left(\tilde{\mathfrak{I}}_i(\nu_1^*, \ldots, \nu_n^*)(X)\right) + \sum_{i \in \mathcal{N}} \pi^{\nu_i^*}(\tilde{\mathfrak{I}}_i(\nu_1^*, \ldots, \nu_n^*)(X))$$
$$< \rho_{IN}(X) - \sum_{i \in \mathcal{N}} \rho_{IN}(\mathfrak{I}_i^*(\nu_1^*, \ldots, \nu_n^*)(X)) + \sum_{i \in \mathcal{N}} \pi^{\nu_i}(\mathfrak{I}_i^*(\nu_1^*, \ldots, \nu_n^*)(X)).$$

Furthermore, by (3.2), there does not exist $i \in \mathcal{N}$ and $\tilde{\nu} \in \mathcal{C}$ such that

$$\rho_i(\mathfrak{I}_i^*(\tilde{\nu}, \nu_{-i}^*)(X)) - \pi^{\tilde{\nu}}(\mathfrak{I}_i^*(\tilde{\nu}, \nu_{-i}^*)(X)) < \rho_i(\mathfrak{I}_i^*(\nu_1^*, \ldots, \nu_n^*)(X)) - \pi^{\nu_i^*}(\mathfrak{I}_i^*(\nu_1^*, \ldots, \nu_n^*)(X)).$$

Therefore, $\left(\nu_1^*, \ldots, \nu_n^*, \mathfrak{I}^*\right)$ is an NE.

Conversely, suppose that $(\nu_1^*, \ldots, \nu_n^*, \mathfrak{I}^*)$ is an SPNE. Then by condition (2) of Definition 2.6, for any choice of capacities $(\nu_1, \ldots, \nu_n) \in \mathcal{C}^n$, the indemnity structure $\mathfrak{I}^*(\nu_1, \ldots, \nu_n)$ solves (3.1). It remains to show that $(\nu_1^*, \ldots, \nu_n^*)$ is a Nash Equilibrium in the reduced game. Since the strategy $(\nu_1^*, \ldots, \nu_n^*, \mathfrak{I}^*)$ is an NE, there does not exist $i \in \mathcal{N}$ and $\tilde{\nu} \in \mathcal{C}$ such that

$$\rho_i(\mathfrak{I}_i^*(\tilde{\nu}, \nu_{-i}^*)(X)) - \pi^{\tilde{\nu}}(\mathfrak{I}_i^*(\tilde{\nu}, \nu_{-i}^*)(X)) < \rho_i(\mathfrak{I}_i^*(\nu_1^*, \ldots, \nu_n^*)(X)) - \pi^{\nu_i}(\mathfrak{I}_i^*(\nu_1^*, \ldots, \nu_n^*)(X)).$$

Hence, $\left(\nu_1^*, \ldots, \nu_n^*\right)$ is an NE for the reduced game formed by fixing $\mathfrak{I}^*$, which completes the proof. □

Proof of Proposition 3.3. First we check feasibility. Suppose $(I_1^*, \ldots, I_n^*)$ is of the given form – then $I_i^* = \int_0^x \gamma_i^*(z)\,dz$ where $0 \leqslant \gamma_i^* \leqslant 1$. Then for all z, $(I_i^*)'(z) = \gamma_i^*(z) \geqslant 0$. Furthermore, $\sum_{i \in \mathcal{N}} (I_i^*)'(z) = \sum_{i \in \mathcal{N}} \gamma_i^*(z) \leqslant 1$. Hence, $(I_1^*, \ldots, I_n^*)$ is a feasible choice of indemnities.

Let $\vec{I} = (I_1, \ldots, I_n)$ be any feasible indemnity, and write $I_i(x) = \int_0^x \gamma_i(z)\,dz$. Then,

$$\begin{aligned}
\rho_{IN}\left(X - \sum_{i \in \mathcal{N}} I_i(X) + \sum_{i \in \mathcal{N}} \pi^{\nu_i}(I_i(X))\right) &= \rho_{IN}\left(X - \sum_{i \in \mathcal{N}} I_i(X)\right) + \sum_{i \in \mathcal{N}} \pi^{\nu_i}(I_i(X)) \\
&= \rho_{IN}(X) - \rho_{IN}\left(\sum_{i \in \mathcal{N}} I_i(X)\right) + \sum_{i \in \mathcal{N}} \pi^{\nu_i}(I_i(X)) \\
&= \rho_{IN}(X) - \sum_{i \in \mathcal{N}} \rho_{IN}(I_i(X)) + \sum_{i \in \mathcal{N}} \pi^{\nu_i}(I_i(X)) \\
&= \rho_{IN}(X) + \sum_{i \in \mathcal{N}} \int_0^\infty (\nu_i(X > z) - g(\mathbb{P}(X > z)))\gamma_i(z)\,dz \\
&= \rho_{IN}(X) + \int_0^\infty \sum_{i \in \mathcal{N}} (\nu_i(X > z) - g(\mathbb{P}(X > z)))\gamma_i(z)\,dz,
\end{aligned}$$

where the second equality follows from comonotonic additivity of ρ_{IN}. First we show that $(I_1^*, \ldots, I_n^*)$ solves Problem (3.1). We have

$$\begin{aligned}
\rho_{IN}\left(X - \sum_{i \in \mathcal{N}} I_i^*(X) + \sum_{i \in \mathcal{N}} \pi^{\nu_i}(I_i^*(X))\right) &= \rho_{IN}(X) + \int_0^\infty \sum_{i \in \mathcal{N}} (\nu_i(X > z) - g(\mathbb{P}(X > z)))\gamma_i^*(z)\,dz \\
&= \rho_{IN}(X) + \int_0^\infty \sum_{i \in \mathcal{N}_z} (\nu_i(X > z) - g(\mathbb{P}(X > z)))\gamma_i^*(z)\,dz \\
&= \rho_{IN}(X) + \int_0^\infty \sum_{i \in \mathcal{N}_z} (\underline{\nu}(X > z) - g(\mathbb{P}(X > z)))\gamma_i^*(z)\,dz
\end{aligned}$$

$$= \rho_{IN}(X) + \int_0^\infty (\underline{\nu}(X > z) - g(\mathbb{P}(X > z))) \sum_{i \in \mathcal{N}_z} h_i(z)\, dz$$

$$= \rho_{IN}(X) + \int_{\mathcal{A}} (\underline{\nu}(X > z) - g(\mathbb{P}(X > z)))\, dz\,,$$

where we define

$$\mathcal{A} := \{z \in \mathbb{R}_+ : g(\mathbb{P}(X > z)) > \underline{\nu}(X > z)\}\,.$$

Then we have

$$\rho_{IN}(X) + \int_{\mathcal{A}} (\underline{\nu}(X > z) - g(\mathbb{P}(X > z)))\, dz$$

$$\leqslant \rho_{IN}(X) + \int_{\mathcal{A}} (\underline{\nu}(X > z) - g(\mathbb{P}(X > z))) \sum_{i \in \mathcal{N}} \gamma_i(z)\, dz \tag{A.1}$$

$$\leqslant \rho_{IN}(X) + \int_{\mathcal{A}} (\underline{\nu}(X > z) - g(\mathbb{P}(X > z))) \sum_{i \in \mathcal{N}} \gamma_i(z)\, dz \tag{A.2}$$

$$+ \int_{\mathcal{A}^C} (\underline{\nu}(X > z) - g(\mathbb{P}(X > z))) \sum_{i \in \mathcal{N}} \gamma_i(z)\, dz$$

$$= \rho_{IN}(X) + \int_0^1 (\underline{\nu}(X > z) - g(\mathbb{P}(X > z))) \sum_{i \in \mathcal{N}} \gamma_i(z)\, dz$$

$$\leqslant \rho_{IN}(X) + \int_0^1 \sum_{i \in \mathcal{N}} (\nu_i(X > z) - g(\mathbb{P}(X > z)))\gamma_i(z)\, dz \tag{A.3}$$

$$= \rho_{IN}\left(X - \sum_{i \in \mathcal{N}} \tilde{I}_i(X) + \sum_{i \in \mathcal{N}} \pi^{\nu_i}(\tilde{I}_i(X))\right),$$

and hence $(I_1^*, \ldots, I_n^*)$ is optimal. We now show the converse. Suppose that $(I_1, \ldots, I_n)$ is an allocation that does not satisfy the above form. Then there are three possibilities:

i) There exists a set $\mathcal{A}_1$ of positive measure such that for each $z \in \mathcal{A}_1$,

$$\gamma_i(z) > 0 \text{ for at least one } i \notin \mathcal{N}_z\,.$$

ii) There exists a set $\mathcal{A}_2$ of positive measure such that for each $z \in \mathcal{A}_2$,

$$\sum_{i \in \mathcal{N}} \gamma_i(z) < 1, \quad g(\mathbb{P}(X > t)) > \underline{\nu}(X > z)\,.$$

Note that $\mathcal{A}_2 \subseteq \mathcal{A}$.

iii) There exists a set $\mathcal{A}_3$ of positive measure such that for each $z \in \mathcal{A}_3$,

$$\sum_{i \in \mathcal{N}} \gamma_i(z) > 0, \quad g(\mathbb{P}(X > t)) < \underline{\nu}(X > z)\,.$$

Note that $\mathcal{A}_3 \subseteq \mathcal{A}^C$.

Suppose that we are in the first case. Then for each $z \in \mathcal{A}_1$, we have

$$\sum_{i \in \mathcal{N}} \underline{\nu}(X > z)\gamma_i(z) < \sum_{i \in \mathcal{N}} \nu_i(X > z)\gamma_i(z)\,.$$

Hence,

$$\int_{\mathcal{A}_1} \sum_{i \in \mathcal{N}} (\underline{\nu}(X > z) - g(\mathbb{P}(X > z)))\gamma_i(z)\, dz < \int_{\mathcal{A}_1} \sum_{i \in \mathcal{N}} (\nu_i(X > z) - g(\mathbb{P}(X > z)))\gamma_i(z)\, dz\,,$$

where this inequality is strict. Also, since $\underline{\nu} \leqslant \nu_i$ for all i, we have

$$\int_{\mathcal{A}_1^C} \sum_{i\in\mathcal{N}} (\underline{\nu}(X > z) - g(\mathbb{P}(X > z)))\gamma_i(z)\,dz \leqslant \int_{\mathcal{A}_1^C} \sum_{i\in\mathcal{N}} (\nu_i(X > z) - g(\mathbb{P}(X > z)))\gamma_i(z)\,dz\,,$$

where $\mathcal{A}_1^C = \mathbb{R}_+ \setminus \mathcal{A}_1$. Adding these inequalities gives

$$\int_0^\infty \sum_{i\in\mathcal{N}} (\underline{\nu}(X > z) - g(\mathbb{P}(X > z)))\gamma_i(z)\,dz < \int_0^\infty \sum_{i\in\mathcal{N}} (\nu_i(X > z) - g(\mathbb{P}(X > z)))\gamma_i(z)\,dz\,,$$

and so inequality (A.3) is strict. Therefore, $(I_1, \ldots, I_n)$ is not optimal.

The remaining two cases are similar. Suppose that (ii) is true. Then for each $z \in \mathcal{A}_2$, we have

$$\underline{\nu}(X > z) - g(\mathbb{P}(X > z)) < \big(\underline{\nu}(X > z) - g(\mathbb{P}(X > z))\big) \sum_{i\in\mathcal{N}} \gamma_i(z)\,.$$

Hence,

$$\int_{\mathcal{A}_2} \underline{\nu}(X > z) - g(\mathbb{P}(X > z))\,dz < \int_{\mathcal{A}_2} \big(\underline{\nu}(X > z) - g(\mathbb{P}(X > z))\big) \sum_{i\in\mathcal{N}} \gamma_i(z)\,dz\,.$$

Also, since $g(\mathbb{P}(X > z)) > \underline{\nu}(X > z)$ on $\mathcal{A}$, and $\sum_{i\in\mathcal{N}} \gamma_i(z)\,dz \geqslant 0$, we have

$$\int_{\mathcal{A}\setminus\mathcal{A}_2} \underline{\nu}(X > z) - g(\mathbb{P}(X > z))\,dz \leqslant \int_{\mathcal{A}\setminus\mathcal{A}_2} \big(\underline{\nu}(X > z) - g(\mathbb{P}(X > z))\big) \sum_{i\in\mathcal{N}} \gamma_i(z)\,dz\,.$$

Adding these inequalities gives

$$\int_{\mathcal{A}} \underline{\nu}(X > z) - g(\mathbb{P}(X > z))\,dz < \int_{\mathcal{A}} \big(\underline{\nu}(X > z) - g(\mathbb{P}(X > z))\big) \sum_{i\in\mathcal{N}} \gamma_i(z)\,dz\,,$$

and so inequality (A.1) is strict. Hence, $(I_1, \ldots, I_n)$ is not optimal.

Finally, suppose that (iii) is true. Then for each $z \in \mathcal{A}_3$, we have

$$\big(\underline{\nu}(X > z) - g(\mathbb{P}(X > z))\big) \sum_{i\in\mathcal{N}} \gamma_i(z) > 0\,,$$

so

$$\int_{\mathcal{A}_3} \big(\underline{\nu}(X > z) - g(\mathbb{P}(X > z))\big) \sum_{i\in\mathcal{N}} \gamma_i(z)\,dz > 0\,.$$

Also, since $g(\mathbb{P}(X > z)) \leqslant \underline{\nu}(X > z)$ on $\mathcal{A}^C$ and $\sum_{i\in\mathcal{N}} \gamma_i(z) \geqslant 0$,

$$\int_{\mathcal{A}^C\setminus\mathcal{A}_3} \big(\underline{\nu}(X > z) - g(\mathbb{P}(X > z))\big) \sum_{i\in\mathcal{N}} \gamma_i(z)\,dz > 0\,.$$

Adding these inequalities gives

$$\int_{\mathcal{A}^C} \big(\underline{\nu}(X > z) - g(\mathbb{P}(X > z))\big) \sum_{i\in\mathcal{N}} \gamma_i(z)\,dz > 0\,,$$

and therefore inequality (A.2) is strict. Thus, $(I_1, \ldots, I_n)$ is not optimal. □

Proof of Proposition 3.9. Let $i \in \mathcal{N}$ and $\mathfrak{I}^* \in \aleph$. First, we have

$$\begin{aligned}\rho_i(I_i^*(X) - \pi_i) &= \int_0^\infty \tau_i(X > z)\gamma_i^*(z)\,dz - \int_0^\infty \nu_i^*(X > z)\gamma_i^*(z)\,dz \\ &= \int_0^\infty (\tau_i(X > z) - \nu_i^*(X > z))\gamma_i^*(z)\,dz\,.\end{aligned}$$

Let

$$\mathcal{A}_1 = \{z : \gamma_i^*(z) = 1\} \text{ and } \mathcal{A}_2 = \{z : \gamma_i^*(z) \in (0, 1)\},$$

so that

$$\int_0^\infty (\tau_i(X>z)-\nu_i^*(X>z))\gamma_i^*(z)\,dz = \int_{\mathcal{A}_1} \tau_i(X>z)-\nu_i^*(X>z)\,dz + \int_{\mathcal{A}_2} (\tau_i(X>z)-\nu_i^*(X>z))\gamma_i^*(z)\,dz.$$

We first show that for almost all $z\in\mathcal{A}_2$, $\tau_i(X>z)=\nu_i^*(X>z)$, so the second integral vanishes. We then identify the first integral with $\int_0^\infty(\tau_i(X>z)-\nu_i^*(X>z))\gamma_i^*(z)\,dz$ as desired. Note that on $\mathcal{A}_1\cup\mathcal{A}_2$, we have $g(\mathbb{P}(X>z))\geqslant \underline{\nu}(X>z)$.

Suppose that $z\in\mathcal{A}_2$, and z satisfies (3.6) and (3.8). Since $\sum_{j=0}^n\gamma_j^*(z)=1$, there must exist $k\in(\mathcal{N}\cup\{0\})\setminus\{i\}$ for which $\gamma_k^*(z)\in(0,1)$. This implies that $\nu_i^*(X>z)=\nu_k^*(X>z)=\underline{\nu}(X>z)=\bar{\tau}(X>z)$ by (3.6), where we use the convention that $\nu_0^*=g\circ\mathbb{P}$. Since the indemnity distributes generously in the sense of Definition 3.4, there does not exist any $j\in\mathcal{N}\cup\{0\}$ such that $\nu_j^*(X>z)=\min\{g(\mathbb{P}(X>z),\underline{\nu}(X>z)\}=\underline{\nu}(X>z)$ and $\tau_j(X>z)<\tau_i(X>z)$. Therefore, it must be true that $\tau_k(X>z)\geqslant\tau_i(X>z)$. A symmetric argument by switching the indices i,k implies $\tau_k(X>z)\leqslant\tau_i(X>z)$. Hence, $\tau_k(X>z)=\tau_i(X>z)$ for any k with $\gamma_k^*(z)\in(0,1)$.

It remains to show that $\bar{\tau}(X>z)=\tau_i(X>z)$, which combined with the above, imply that $\tau_i(X>z)=\nu_i^*(X>z)$. Since $\tau_i(X>z)=\tau_k(X>z)$, it suffices to show that $\tau_i(X>z)=\tau_k(X>z)=\min_{j\in\mathcal{N}}\tau_j(X>z)$. To this end, note that $\gamma_i^*(z)>0$ implies $g(\mathbb{P}(X>z))\geqslant\bar{\tau}(X>z)$. Hence, we must have $g(\mathbb{P}(X>z))\geqslant\min_{j\in\mathcal{N}}\tau_j(X>z)$, which implies that $\mathcal{T}_z\neq\varnothing$. Therefore (3.8) implies that there exists a $k'\in\mathcal{T}_z\cap\mathcal{N}_z$: that is, $\tau_{k'}(X>z)=\min_{j\in\mathcal{N}}\tau_j(X>z)$ and $\nu_{k'}^*(X>z)=\bar{\tau}(X>z)=\nu_i^*(X>z)$.

Assume for the sake of contradiction that $\tau_i(X>z)>\min_{j\in\mathcal{N}}\tau_j(X>z)$. Then since $\tau_{k'}=\min_{j\in\mathcal{N}}\tau_j(X>z)<\tau_i(X>z)$ and $\nu_{k'}^*(X>z)=\bar{\tau}(X>z)$, generous distribution implies that $\gamma_i^*(z)=0$, a contradiction. Hence, $\tau_i(X>z)=\min_{j\in\mathcal{N}}\tau_j(X>z)$, and so $\bar{\tau}(X>z)=\tau_i(X>z)=\tau_k(X>z)$ as desired. Since $\bar{\tau}(X>z)=\nu_i^*(X>z)$ by the above, we have $\tau_i(X>z)=\nu_i^*(X>z)$ for almost all $z\in\mathcal{A}_2$, which implies that the integral $\int_{\mathcal{A}_2}(\tau_i(X>z)-\nu_i^*(X>z))\gamma_i^*(z)\,dz$ vanishes.

Consider now the first integral $\int_{\mathcal{A}_1}\tau_i(X>z)-\nu_i^*(X>z)\,dz$. Let $z\in\mathcal{A}_1$, and suppose z satisfies (3.6) and (3.8). Since $\gamma_i^*(z)=1$, we must have $i\in\mathcal{N}_z$, so $\nu_i^*(X>z)=\underline{\nu}(X>z)=\bar{\tau}(X>z)$ by (3.6). Hence, we have

$$\int_{\mathcal{A}_1}\tau_i(X>z)-\nu_i^*(X>z)\,dz=\int_{\mathcal{A}_1}\tau_i(X>z)-\bar{\tau}(X>z)\,dz\,.$$

Note that on $\mathcal{A}_1$, we must have $\tau_i(X>z)\leqslant\nu_i^*(X>z)=\bar{\tau}(X>z)$. Suppose for the sake of contradiction that $\tau_i(X>z)>\nu_i^*(X>z)$. Since $\gamma_i^*(z)=1$, we have $g(\mathbb{P}(X>z))\geqslant\bar{\tau}(X>z)$, so there must exist some reinsurer $k\in\mathcal{N}$ such that $\tau_k(X>z)=\min_{j\in\mathcal{N}\cup\{0\}}\tau_j(X>z)$. Then $k\in\mathcal{T}_z$, so there exist $k'\in\mathcal{T}_z\cap\mathcal{N}_z$ by (3.8). Therefore $\tau_{k'}(X>z)\leqslant\bar{\tau}(X>z)<\tau_i(X>z)$, and $\nu_{k'}^*(X>z)=\bar{\tau}(X>z)=\nu_i^*(X>z)$. However, since the indemnity distributes generously, this implies $\gamma_i^*(z)=0$, which contradicts $z\in\mathcal{A}_1$.

Therefore $\tau_i(X>z)\leqslant\nu_i^*(X>z)$, so the integrand is non-zero only when $\tau_i(X>z)<\nu_i^*(X>z)=\bar{\tau}(X>z)$. Let $\mathcal{A}_3:=\{z\in\mathbb{R}_+:\tau_i(X>z)<\nu_i^*(X>z)\}$. Then the above simplifies to

$$\begin{aligned}\int_{\mathcal{A}_1}\tau_i(X>z)-\bar{\tau}(X>z)\,dz&=\int_{\mathcal{A}_1\cap\mathcal{A}_3}\tau_i(X>z)-\bar{\tau}(X>z)\,dz+\int_{\mathcal{A}_1\setminus\mathcal{A}_3}\tau_i(X>z)-\bar{\tau}(X>z)\,dz\\&=\int_{\mathcal{A}_1\cap\mathcal{A}_3}\tau_i(X>z)-\bar{\tau}(X>z)\,dz+\int_{\mathcal{A}_1\setminus\mathcal{A}_3}0\,dz\\&=\int_{\mathcal{A}_1\cap\mathcal{A}_3}\tau_i(X>z)-\bar{\tau}(X>z)\,dz\,.\end{aligned}$$

For $z\in\mathcal{A}_1\cap\mathcal{A}_3$, we have $\tau_i(X>z)<\nu_i^*(X>z)=\bar{\tau}(X>z)$, so $\mathcal{A}_1\cap\mathcal{A}_3\subseteq\mathcal{Z}_i$. We now show that $\mathcal{Z}_i\subseteq\mathcal{A}_1$. Let $z\in\mathcal{Z}_i$, and suppose for the sake of contradiction that $\gamma_i^*(z)<1$. Then since $\sum_{j=0}^n\gamma_j^*(z)=1$, we must have $\gamma_k^*(z)>0$ for some $k\in(\mathcal{N}\cup\{0\})\setminus\{i\}$, which implies $\nu_k^*(X>z)=\bar{\tau}(X>z)$. However, since $z\in\mathcal{Z}_i$, we have $\tau_i(X>z)<\bar{\tau}(X>z)$, which implies that $\mathcal{T}_z=\{i\}$. Then by (3.8), $i\in\mathcal{N}_z$, so $\nu_i^*(X>z)=\bar{\tau}(X>z)=\nu_k^*(X>z)$. Since the indemnity distributes generously and $\tau_i(X>z)<\tau_k(X>z)$, $\gamma_k^*(z)=0$, which is a contradiction – hence, $\mathcal{Z}_i\subseteq\mathcal{A}_1$. Putting the above together, we have

$$\begin{aligned}&\int_{\mathcal{A}_1}\tau_i(X>z)-\bar{\tau}(X>z)\,dz\\&\quad=\int_{\mathcal{A}_1\cap\mathcal{A}_3}\tau_i(X>z)-\bar{\tau}(X>z)\,dz\\&\quad=\int_{\mathcal{A}_1\cap\mathcal{A}_3}\tau_i(X>z)-\bar{\tau}(X>z)\,dz+\int_{\mathcal{Z}_i\cap(\mathcal{A}_1\setminus\mathcal{A}_3)}0\,dz\\&\quad=\int_{\mathcal{A}_1\cap\mathcal{A}_3}\tau_i(X>z)-\bar{\tau}(X>z)\,dz+\int_{\mathcal{Z}_i\cap(\mathcal{A}_1\setminus\mathcal{A}_3)}\tau_i(X>z)-\bar{\tau}(X>z)\,dz\\&\quad=\int_{\mathcal{Z}_i}\tau_i(X>z)-\bar{\tau}(X>z)\,dz\,,\end{aligned}$$

which completes the proof. □

Proof of Proposition 3.10. Let $i \in \mathcal{N}$. For notational convenience, we define $\hat{\gamma}_i$ such that for each $x \in \mathbb{R}_+$, we have $\mathfrak{I}_i^*(\hat{\nu}, \nu_{-i}^*)(x) = \int_0^x \hat{\gamma}_i(z)\,dz$. We now fix the pricing capacities $(\hat{\nu}, \nu_{-i}^*)$, and use our previous notation as introduced in Proposition 3.3. For example, $\mathcal{N}_z$ is the set of indices that minimize over the set $\{\hat{\nu}(X>z), \nu_j^*(X>z)\}$, for $j \in \mathcal{N} \setminus \{i\}$. Then we can write

$$\rho_i(\mathfrak{I}_i^*(\hat{\nu}, \nu_{-i}^*)(X)) - \pi^{\hat{\nu}}(\mathfrak{I}_i^*(\hat{\nu}, \nu_{-i}^*)(X)) = \int_0^\infty (\tau_i(X>z) - \hat{\nu}(X>z))\hat{\gamma}_i(z)\,dz.$$

On the other hand, by Proposition 3.9, we have that

$$\rho_i(\mathfrak{I}_i^*(\nu_1^*, \ldots, \nu_n^*)(X)) - \pi^{\nu_i}(\mathfrak{I}_i^*(\nu_1^*, \ldots, \nu_n^*)(X)) = \int_{\mathcal{Z}_i} \tau_i(X>z) - \bar{\tau}(X>z)\,dz\,,$$

where

$$\mathcal{Z}_i := \{z \in \mathbb{R}_+ : \tau_i(X>z) < \bar{\tau}(X>z)\}\,.$$

We first show that for all $z \in \mathcal{Z}_i$, $(\tau_i(X>z) - \hat{\nu}(X>z))\hat{\gamma}_i(z) \geqslant \tau_i(X>z) - \bar{\tau}(X>z)$. Then for all $z \in \mathcal{Z}_i^C$, we show that $(\tau_i(X>z) - \hat{\nu}(X>z))\hat{\gamma}_i(z) \geqslant 0$. Combining these equations yields the desired relation.

(1) First, suppose that $z \in \mathcal{Z}_i$. Then $\tau_i(X>z) < \bar{\tau}(X>z)$, and we can distinguish three cases.

(a) $\hat{\nu}(X>z) > \bar{\tau}(X>z)$: In this case, by (3.7), there exist some $k \neq i$, $k \in \mathcal{N} \cup \{0\}$ such that $\nu_k^*(X>z) = \bar{\tau}(X>z)$, where we use the convention $\nu_0^*(X>z) := g(\mathbb{P}(X>z))$. If there exists any such $k \neq 0$, then $i \notin \mathcal{N}_z$, so $\hat{\gamma}_i(z) = 0$, so $(\tau_i(X>z) - \hat{\nu}(X>z))\hat{\gamma}_i(z) = 0$. On the other hand, if the only $k \neq i$ such that $\nu_k^*(X>z) = \bar{\tau}(X>z)$ is $k=0$, then $\nu_0^*(X>z) < \nu_j^*(X>z)$ for all $j \neq i$. This, along with the assumption $\hat{\nu}(X>z) > \bar{\tau}(X>z) = \nu_0^*(X>z)$, implies that $\hat{\gamma}_i(z) = 0$, so again $(\tau_i(X>z) - \hat{\nu}(X>z))\hat{\gamma}_i(z) = 0$. Note that $0 \geqslant \tau_i(X>z) - \bar{\tau}(X>z)$ on $\mathcal{Z}_i$.

(b) $\hat{\nu}(X>z) < \bar{\tau}(X>z)$: Recall that we have $\bar{\tau}(X>z) = \min_{j\in\mathcal{N}} \nu_j^*(X>z)$ by (3.6). Then by assumption, $\hat{\nu}(X>z) < \min_{j\neq i} \nu_j^*(X>z)$ and $\hat{\nu}(X>z) < g(\mathbb{P}(X>z))$, since $g(\mathbb{P}(X>z)) \geqslant \bar{\tau}(X>z)$ whenever $\tau_i(X>z) < \bar{\tau}(X>z)$. This implies that $\mathcal{N}_z = \{i\}$ and $\sum_{j=1}^n \hat{\gamma}_j(z) = 1$. This implies $\hat{\gamma}_j(z) = 0$ for $j \neq i$, so it must be true that $\hat{\gamma}_i(z) = 1$. Hence,

$$(\tau_i(X>z) - \hat{\nu}(X>z))\hat{\gamma}_i(z) = \tau_i(X>z) - \hat{\nu}(X>z) \geqslant \tau_i(X>z) - \bar{\tau}(X>z)\,.$$

(c) $\hat{\nu}(X>z) = \bar{\tau}(X>z)$: In this case, note that $\tau_i(X>z) - \bar{\tau}(X>z) < 0$, so $\tau_i(X>z) - \hat{\nu}(X>z) < 0$. Since $\hat{\gamma}_i(z) \leqslant 1$,

$$(\tau_i(X>z) - \hat{\nu}(X>z))\hat{\gamma}_i(z) \geqslant (\tau_i(X>z) - \hat{\nu}(X>z)) \cdot 1 = \tau_i(X>z) - \bar{\tau}(X>z)\,.$$

(2) Now, suppose that $z \in \mathcal{Z}_i^C$. We distinguish two cases.

(a) $\hat{\nu}(X>z) > \bar{\tau}(X>z)$: By the same logic as Case (1-a), we have $(\tau_i(X>z) - \hat{\nu}(X>z))\hat{\gamma}_i(z) = 0$.

(b) $\hat{\nu}(X>z) \leqslant \bar{\tau}(X>z)$: In this case, since $z \in \mathcal{Z}_i^C$, $\bar{\tau}(X>z) \leqslant \tau_i(X>z)$. Therefore,

$$\left(\tau_i(X>z) - \hat{\nu}(X>z)\right)\hat{\gamma}_i(z) \geqslant 0\,.$$

Hence, we have

$$\begin{aligned}
\rho_i\left(\mathfrak{I}_i^*(\hat{\nu}, \nu_{-i}^*)(X)\right) - \pi^{\hat{\nu}}\left(\mathfrak{I}_i^*(\hat{\nu}, \nu_{-i}^*)(X)\right) &= \int_0^\infty (\tau_i(X>z) - \hat{\nu}(X>z))\hat{\gamma}_i(z)\,dz \\
&= \int_{\mathcal{Z}_i} (\tau_i(X>z) - \hat{\nu}(X>z))\hat{\gamma}_i(z)\,dz \\
&\qquad + \int_{\mathcal{Z}_i^C} (\tau_i(X>z) - \hat{\nu}(X>z))\hat{\gamma}_i(z)\,dz \\
&\geqslant \int_{\mathcal{Z}_i} \tau_i(X>z) - \bar{\tau}(X>z)\,dz + \int_{\mathcal{Z}_i^C} 0\,dz \\
&= \int_{\mathcal{Z}_i} \tau_i(X>z) - \bar{\tau}(X>z)\,dz \\
&= \rho_i(\mathfrak{I}_i^*(\nu_1^*, \ldots, \nu_n^*)(X)) - \pi^{\nu_i^*}(\mathfrak{I}_i^*(\nu_1^*, \ldots, \nu_n^*)(X))\,,
\end{aligned}$$

as desired. □

Proof of Proposition 3.13. We check properties (3.6), (3.7), and (3.8) respectively. First, let $z \in \mathbb{R}_+$. Then by definition of $\bar{\tau}$, there exist $i, j \in \mathcal{N} \cup \{0\}$, $i \neq j$ such that

$$\tau_i(X > z) \leqslant \tau_j(X > z) = \bar{\tau}(X > z) \leqslant \tau_k(X > z)\,,$$

for all $k \in \mathcal{N}$, $k \neq i, j$. This implies that either $i \in \tilde{\mathcal{N}}_z$, or $i = 0$. In the first case, we have $\underline{\nu}(X > z) \leqslant \nu_i^*(X > z) = \bar{\tau}(X > z)$. In the second case, $j \in \tilde{\mathcal{N}}_z$, so $\underline{\nu}(X > z) = \tau_j^*(X > z) = \bar{\tau}(X > z)$. Therefore $\underline{\nu}(X > z) \leqslant \bar{\tau}(X > z)$.

On the other hand, we have $\bar{\tau}(X > z) = \tau_j(X > z)$, and so for $k \neq i, j$, $k \in \mathcal{N}$, $\bar{\tau}(X > z) \leqslant \tau_k(X > z)$. Then we have $\bar{\tau}(X > z) = \nu_j^*(X > z)$ and $\bar{\tau}(X > z) \leqslant \nu_k^*(X > z)$ by definition of ν_j^*, ν_k^*. If $i \neq 0$, then $\nu_i^*(X > z) = \bar{\tau}(X > z)$ by the above. This gives $\bar{\tau}(X > z) \leqslant \nu_l^*(X > z)$ for all $l \in \mathcal{N}$. That is, $\bar{\tau}(X > z) \leqslant \underline{\nu}(X > z)$. Therefore $\underline{\nu}(X > z) = \bar{\tau}(X > z)$ as desired, so (3.6) holds.

Next, we check (3.7). Fix a $z \in \mathbb{R}_+$. By definition of $\bar{\tau}$, there exist $i, j \in \mathcal{N}$, $i \neq j$ such that $\bar{\tau}(X > z) = \tau_j(X > z)$, $\tau_i(X > z) \leqslant \tau_j(X > z)$, and $\tau_k(X > z) \geqslant \tau_j(X > z)$ for all $k \neq i, j$. Then by construction of $(\nu_1^*, \ldots, \nu_n^*)$, $\nu_j^*(X > z) = \bar{\tau}(X > z) = \tau_j(X > z)$, and $\bar{\tau} \leqslant \nu_k^*(X > z)$ for all $k \neq i, j$. Assume for the sake of contradiction that $\nu_i^*(X > z) < \bar{\tau}(X > z)$. Then by definition of ν_i^*, we must have $\nu_i^*(X > z) = \tau_i(X > z)$, so $\tau_i(X > z) \leqslant \tau_k(X > z)$ for all $k \neq i$ as above. But then $i \in \tilde{\mathcal{N}}_z$, so $\nu_i^*(X > z) = \bar{\tau}(X > z)$ – a contradiction. Therefore $\nu_i^*(X > z) = \bar{\tau}(X > z) = \nu_j^*(X > z)$ as desired, so (3.7) holds.

Finally, we check (3.8). For each $z \in \mathbb{R}_+$, if $\mathcal{T}_z \neq \varnothing$, take $i \in \mathcal{T}_z$. Then we have $\nu_i^*(X > z) = \bar{\tau}(X > z) = \min_{j \in \mathcal{N}} \nu_j^*(X > z)$, so $i \in \mathcal{N}_z$. Hence, $\mathcal{T}_z \cap \mathcal{N}_z \neq \varnothing$, so (3.8) holds. □

A.2. Welfare analysis of SPNEs

Proof of Lemma 4.3. Note that for $z \in \mathcal{Z}_{IN}$, we have $\sum_{i \in \mathcal{N}} \gamma_i^*(z) = 0$, and so $\gamma_i^*(z) = 0$ for all $i \in \mathcal{N}$. Then we have

$$\rho_{IN}\left(X - \sum_{i \in N} I_i^*(X) + \sum_{i \in \mathcal{N}} \pi^{\nu_i^*}(I_i^*(X))\right) = \int_0^\infty g(\mathbb{P}(X > z))\left(1 - \sum_{i \in \mathcal{N}} \gamma_i^*(z)\right) dz + \sum_{i \in \mathcal{N}} \int_0^\infty \nu_i^*(X > z)\gamma_i^*(z)\, dz\,.$$

Define the sets

$$\mathcal{Z}_{IN}^{=} := \{z \in \mathbb{R}_+ : g(\mathbb{P}(X > z)) = \bar{\tau}(X > z)\}\,, \quad \mathcal{Z}_{IN}^{+} := \{z \in \mathbb{R}_+ : g(\mathbb{P}(X > z)) > \bar{\tau}(X > z)\}\,.$$

Since $\vec{I}^*$ is optimal, the first term simplifies to

$$\begin{aligned}
\int_0^\infty g(\mathbb{P}(X > z))\left(1 - \sum_{i \in \mathcal{N}} \gamma_i^*(z)\right) dz &= \int_{\mathcal{Z}_{IN}} g(\mathbb{P}(X > z))\, dz + \int_{\mathcal{Z}_{IN}^{=}} g(\mathbb{P}(X > z))\gamma_0^*(z)\, dz \\
&= \int_{\mathcal{Z}_{IN}} g(\mathbb{P}(X > z))\, dz + \int_{\mathcal{Z}_{IN}^{=}} \bar{\tau}(X > z)\gamma_0^*(z)\, dz\,.
\end{aligned}$$

For second term, since $\gamma_i^*(z) > 0$ implies $\nu_i^*(X > z) = \bar{\tau}(X > z)$, we have

$$\begin{aligned}
\sum_{i \in \mathcal{N}} \int_0^\infty \nu_i^*(X > z)\gamma_i^*(z)\, dz &= \sum_{i \in \mathcal{N}} \int_{\mathcal{Z}_{IN}^C} \nu_i^*(X > z)\gamma_i^*(z)\, dz \\
&= \int_{\mathcal{Z}_{IN}^C} \sum_{i \in \mathcal{N}} \nu_i^*(X > z)\gamma_i^*(z)\, dz = \int_{\mathcal{Z}_{IN}^C} \bar{\tau}(X > z) \sum_{i \in \mathcal{N}} \gamma_i^*(z)\, dz \\
&= \int_{\mathcal{Z}_{IN}^{=}} \bar{\tau}(X > z) \sum_{i \in \mathcal{N}} \gamma_i^*(z)\, dz + \int_{\mathcal{Z}_{IN}^{-}} \bar{\tau}(X > z)\, dz\,.
\end{aligned}$$

Adding this to the above, we have

$$\begin{aligned}
&\int_0^\infty g(\mathbb{P}(X > z))\left(1 - \sum_{i \in \mathcal{N}} \gamma_i^*(z)\right) dz + \sum_{i \in \mathcal{N}} \int_0^\infty \nu_i^*(X > z)\gamma_i^*(z)\, dz \\
&\quad = \int_{\mathcal{Z}_{IN}} g(\mathbb{P}(X > z))\, dz + \int_{\mathcal{Z}_{IN}^{=}} \bar{\tau}(X > z)\gamma_0^*(z)\, dz \\
&\qquad + \int_{\mathcal{Z}_{IN}^{=}} \bar{\tau}(X > z) \sum_{i \in \mathcal{N}} \gamma_i^*(z)\, dz + \int_{\mathcal{Z}_{IN}^{-}} \bar{\tau}(X > z)\, dz
\end{aligned}$$

$$= \int_{\mathcal{Z}_{IN}} g(\mathbb{P}(X>z))\,dz + \int_{\mathcal{Z}_{IN}^{=}} \bar{\tau}(X>z)\,dz + \int_{\mathcal{Z}_{IN}^{-}} \bar{\tau}(X>z)\,dz$$
$$= \int_{\mathcal{Z}_{IN}} g(\mathbb{P}(X>z))\,dz + \int_{\mathcal{Z}_{IN}^{C}} \bar{\tau}(X>z)\,dz. \quad \square$$

Proof of Proposition 4.5. Note that when $n=1$, if $g(\mathbb{P}(X>z)) \geqslant \bar{\tau}(X>z)$, then it must be true that $g(\mathbb{P}(X>z)) = \bar{\tau}(X>z)$. Therefore by Lemma 4.3, we have

$$\begin{aligned}\rho_{IN}\left(X - I_1(X) + \pi^{\nu_1^*}(I_1(X))\right) &= \int_{\mathcal{Z}_{IN}} g(\mathbb{P}(X>z))\,dz + \int_{\mathcal{Z}_{IN}^{C}} \bar{\tau}(X>z)\,dz \\ &= \int_{\mathcal{Z}_{IN}} g(\mathbb{P}(X>z))\,dz + \int_{\mathcal{Z}_{IN}^{C}} g(\mathbb{P}(X>z))\,dz \\ &= \int_0^{\infty} g(\mathbb{P}(X>z))\,dz = \rho_{IN}(X). \quad \square\end{aligned}$$

Proof of Proposition 4.6. Let $\bar{\tau}_n$ be the second-lowest function of the set of capacities $\{\tau_0,\ldots,\tau_n\}$, and let $\bar{\tau}_{n+1}$ be the second-lowest function of $\{\tau_0,\ldots,\tau_{n+1}\}$. Define

$$\begin{aligned}\mathcal{Z}_{IN}^{n} &:= \{z \in \mathbb{R}_+ : g(\mathbb{P}(X>z)) < \bar{\tau}_n(X>z)\}, \\ \mathcal{Z}_{IN}^{n+1} &:= \{z \in \mathbb{R}_+ : g(\mathbb{P}(X>z)) < \bar{\tau}_{n+1}(X>z)\}.\end{aligned}$$

Then $\bar{\tau}_{n+1} \leqslant \bar{\tau}_n$, and so $\mathcal{Z}_{IN}^{n+1} \subseteq \mathcal{Z}_{IN}^{n}$. By Lemma 4.3, we have

$$\begin{aligned}&\rho_{IN}\left(X - \sum_{i=1}^{n} I_i^{*,n}(X) + \sum_{i=1}^{n} \pi^{\nu_i^{*,n}}(I_i^{*,n}(X))\right) \\ &= \int_{\mathcal{Z}_{IN}^{n}} g(\mathbb{P}(X>z))\,dz + \int_{(\mathcal{Z}_{IN}^{n})^C} \bar{\tau}_n(X>z)\,dz \\ &= \int_{\mathcal{Z}_{IN}^{n+1}} g(\mathbb{P}(X>z))\,dz + \int_{\mathcal{Z}_{IN}^{n}\setminus\mathcal{Z}_{IN}^{n+1}} g(\mathbb{P}(X>z))\,dz + \int_{(\mathcal{Z}_{IN}^{n})^C} \bar{\tau}_n(X>z)\,dz \\ &\geqslant \int_{\mathcal{Z}_{IN}^{n+1}} g(\mathbb{P}(X>z))\,dz + \int_{\mathcal{Z}_{IN}^{n}\setminus\mathcal{Z}_{IN}^{n+1}} \bar{\tau}_{n+1}(X>z)\,dz + \int_{(\mathcal{Z}_{IN}^{n})^C} \bar{\tau}_{n+1}(X>z)\,dz \\ &= \int_{\mathcal{Z}_{IN}^{n+1}} g(\mathbb{P}(X>z))\,dz + \int_{\left(\mathcal{Z}_{IN}^{n+1}\right)^C} \bar{\tau}_{n+1}(X>z)\,dz \\ &= \rho_{IN}\left(X - \sum_{i=1}^{n+1} I_i^{*,n+1}(X) + \sum_{i=1}^{n+1} \pi^{\nu_i^{*,n+1}}(I_i^{*,n+1}(X))\right). \quad \square\end{aligned}$$

Proof of Proposition 4.9. Let $(\vec{I},\vec{\pi})$ be an allocation, and suppose for the sake of contradiction that $\sum_{i=0}^{n}\rho_i(I_i(X)) > \square_{i=0}^{n}\rho_i(X)$. Then there exist $\vec{\tilde{I}} \in \vec{\mathcal{I}}$ such that $\sum_{i=0}^{n}\rho_i(\tilde{I}_i(X)) < \sum_{i=0}^{n}\rho_i(I_i(X))$. For $i \in \mathcal{N}$, define $\tilde{\pi}_i$ by

$$\tilde{\pi}_i := \rho_i(\tilde{I}_i(X)) - \rho_i(I_i(X)) + \pi_i.$$

Then we have

$$\rho_i(I_i(X)) - \pi_i = \rho_i(\tilde{I}_i(X)) - \rho_i(\tilde{I}_i(X)) + \rho_i(I_i(X)) - \pi_i = \rho_i(\tilde{I}_i(X)) - \tilde{\pi}_i.$$

On the other hand, we have

$$\rho_{IN}\left(X - \sum_{i\in\mathcal{N}} I_i(X) + \sum_{i\in\mathcal{N}} \pi_i\right) = \rho_{IN}\left(X - \sum_{i\in\mathcal{N}} I_i(X)\right) + \sum_{i\in\mathcal{N}} \pi_i$$

$$
\begin{aligned}
&= \rho_{IN}\left(X - \sum_{i\in\mathcal{N}} I_i(X)\right) + \sum_{i\in\mathcal{N}} \tilde{\pi}_i - \sum_{i\in\mathcal{N}} \rho_i(\tilde{I}_i(X)) + \sum_{i\in\mathcal{N}} \rho_i(I_i(X)) \\
&= \sum_{i=0}^{n} \rho_i(I_i(X)) + \sum_{i\in\mathcal{N}} \tilde{\pi}_i - \sum_{i\in\mathcal{N}} \rho_i(\tilde{I}_i(X)) \\
&> \sum_{i=0}^{n} \rho_i(\tilde{I}_i(X)) + \sum_{i\in\mathcal{N}} \tilde{\pi}_i - \sum_{i\in\mathcal{N}} \rho_i(\tilde{I}_i(X)) \\
&= \rho_{IN}\left(X - \sum_{i\in\mathcal{N}} \tilde{I}_i(X)\right) + \sum_{i\in\mathcal{N}} \rho_i(\tilde{I}_i(X)) + \sum_{i\in\mathcal{N}} \tilde{\pi}_i - \sum_{i\in\mathcal{N}} \rho_i(\tilde{I}_i(X)) \\
&= \rho_{IN}\left(X - \sum_{i\in\mathcal{N}} \tilde{I}_i(X)\right) + \sum_{i\in\mathcal{N}} \tilde{\pi}_i = \rho_{IN}\left(X - \sum_{i\in\mathcal{N}} \tilde{I}_i(X) + \sum_{i\in\mathcal{N}} \tilde{\pi}_i\right),
\end{aligned}
$$

implying that the allocation $(\tilde{I}, \tilde{\pi})$ improves over $(\vec{I}, \vec{\pi})$, with a strict improvement for the insurer. Hence, $(\vec{I}, \vec{\pi})$ is not PE.

Conversely, if $(\vec{I}, \vec{\pi})$ is not PE, then there exist an allocation $\left(\tilde{I}_i, \tilde{\pi}\right)$ such that

$$
\rho_{IN}\left(X - \sum_{i\in\mathcal{N}} \tilde{I}_i(X) + \sum_{i\in\mathcal{N}} \tilde{\pi}_i\right) \leqslant \rho_{IN}\left(X - \sum_{i\in\mathcal{N}} I_i(X) + \sum_{i\in\mathcal{N}} \pi_i\right) \text{ and } \rho_i(\tilde{I}_i(X) - \tilde{\pi}_i) \leqslant \rho_i(I_i(X) - \pi_i),\ \forall i \in \mathcal{N},
$$

with at least one strict inequality. Summing these inequalities gives

$$
\rho_{IN}\left(X - \sum_{i\in\mathcal{N}} \tilde{I}_i(X) + \sum_{i\in\mathcal{N}} \tilde{\pi}_i\right) + \sum_{i\in\mathcal{N}}^{n} \rho_i(\tilde{I}_i(X) - \tilde{\pi}_i) < \rho_{IN}\left(X - \sum_{i\in\mathcal{N}} I_i(X) + \sum_{i\in\mathcal{N}} \pi_i\right) + \sum_{i\in\mathcal{N}}^{n} \rho_i(I_i(X) - \pi_i),
$$

which, by translation invariance, yields

$$
\rho_{IN}\left(X - \sum_{i\in\mathcal{N}} \tilde{I}_i(X)\right) + \sum_{i\in\mathcal{N}}^{n} \rho_i(\tilde{I}_i(X)) < \rho_{IN}\left(X - \sum_{i\in\mathcal{N}} I_i(X)\right) + \sum_{i\in\mathcal{N}} \rho_i(I_i(X)),
$$

and thus

$$
\sum_{i=0}^{n} \rho_i(\tilde{I}_i(X)) < \sum_{i=0}^{n} \rho_i(I_i(X)) .
$$

Therefore $\sum_{i=0}^{n} I_i(X) > \square_{i=0}^{n} \rho_i(X)$, and hence this allocation does not achieve the inf-convolution. □

Proof of Proposition 4.10. First we check feasibility. We have

$$
\sum_{i=0}^{n} I_i(z) = \sum_{i=0}^{n} \int_0^z \gamma_i(z)\, dz = \int_0^z \sum_{i=0}^{n} \gamma_i(z)\, dz = \int_0^z 1\, dz = z .
$$

Therefore $\sum_{i=0}^{n} X_i = \sum_{i=0}^{n} I_i(X) = X$. Also, since the derivative of I_i is non-negative, I_i is increasing, and so $I_i(z)$ is comonotonic with X – hence, $\vec{I} \in \vec{\mathcal{I}}$, so it is feasible.

Next, we show that $\vec{I}$ achieves the inf-convolution. Let $\vec{\tilde{I}}$ be any other profile of indemnities in $\vec{\mathcal{I}}$. We have

$$
\begin{aligned}
\sum_{i=0}^{n} \rho_i(\tilde{I}_i(z)) &= \sum_{i=0}^{n} \int_0^{\infty} \tau_i(X > z)\tilde{I}_i'(z)\, dz = \int_0^{\infty} \sum_{i=0}^{n} \tau_i(X > z)\tilde{I}_i'(z)\, dz \\
&\geqslant \int_0^{\infty} \sum_{i=0}^{n} \min_{j\in\mathcal{N}\cup\{0\}} \{\tau_j(X > z)\}\tilde{I}_i'(z)\, dz && \text{(A.4)} \\
&= \int_0^{\infty} \min_{j\in\mathcal{N}\cup\{0\}} \{\tau_j(X > z)\} \sum_{i=0}^{n} \tilde{I}_i'(z)\, dz \\
&= \int_0^{\infty} \min_{j\in\mathcal{N}\cup\{0\}} \{\tau_j(X > z)\}\, dz && \text{(A.5)}
\end{aligned}
$$

$$= \int_0^\infty \left[\min_{j \in \mathcal{N} \cup \{0\}} \{\tau_j(X > z)\} \cdot 1 + \sum_{i \in \mathcal{L}_z^C} \tau_i(X > z) \cdot 0 \right] dz$$
$$= \int_0^\infty \left[\min_{j \in \mathcal{N} \cup \{0\}} \{\tau_j(X > z)\} \cdot \sum_{i \in \mathcal{L}_z} \gamma_i(z) + \sum_{i \in \mathcal{L}_z^C} \tau_i(X > z) \cdot \gamma_i(z) \right] dz$$
$$= \int_0^\infty \left[\sum_{i \in \mathcal{L}_z} \min_{j \in \mathcal{N} \cup \{0\}} \{\tau_j(X > z)\} \cdot \gamma_i(z) + \sum_{i \in \mathcal{L}_z^C} \tau_i(X > z) \cdot \gamma_i(z) \right] dz$$
$$= \int_0^\infty \sum_{i \in \mathcal{L}_z} \tau_i(X > z) \cdot \gamma_i(z) + \sum_{i \in \mathcal{L}_z^C} \tau_i(X > z) \cdot \gamma_i(z) \, dz$$
$$= \int_0^\infty \sum_{i=0}^n \tau_i(X > z)\gamma_i(z) \, dz = \sum_{i=0}^n \int_0^\infty \tau_i(X > z)\gamma_i(z) \, dz = \sum_{i=0}^n \rho_i(I_i(z)) \, .$$

Therefore $\vec{I}$ attains the inf-convolution.

We now show the converse. Suppose that $\vec{\tilde{I}}$ is a feasible indemnity structure not of the specified form: that is, $\sum_{i \in \mathcal{L}_z^C} \tilde{I}_i'(z) > 0$ on a set $\mathcal{A}$ of positive measure. Then for every z in $\mathcal{A}$, we have

$$\begin{aligned}
\sum_{i=0}^n \tau_i(X > z)\tilde{I}_i'(z) &= \sum_{i \in \mathcal{L}_z} \tau_i(X > z)\tilde{I}_i'(z) + \sum_{i \in \mathcal{L}_z^C} \tau_i(X > z)\tilde{I}_i'(z) \\
&= \sum_{i \in \mathcal{L}_z} \min_{j \in \mathcal{N} \cup \{0\}} \{\tau_j(X > z)\}\tilde{I}_i'(z) + \sum_{i \in \mathcal{L}_z^C} \tau_i(X > z)\tilde{I}_i'(z) \\
&> \sum_{i \in \mathcal{L}_z} \min_{j \in \mathcal{N} \cup \{0\}} \{\tau_j(X > z)\}\tilde{I}_i'(z) + \sum_{i \in \mathcal{L}_z^C} \min_{j \in \mathcal{N} \cup \{0\}} \{\tau_j(X > z)\}\tilde{I}_i'(z) \\
&= \sum_{i=0}^n \min_{j \in \mathcal{N} \cup \{0\}} \{\tau_j(X > z)\}\tilde{I}_i'(z) \, ,
\end{aligned}$$

where the strict inequality follows because $\tilde{I}_i'(z)$ are not all zero. Taking the integral over the set $\mathcal{A}$ gives

$$\int_{\mathcal{A}} \sum_{i=0}^n \tau_i(X > z)\tilde{I}_i'(z) \, dz > \int_{\mathcal{A}} \sum_{i=0}^n \min_{j \in \mathcal{N} \cup \{0\}} \{\tau_j(X > z)\}\tilde{I}_i'(z) \, dz \, ,$$

where the inequality is strict since $\mathcal{A}$ has positive measure. Therefore in this case, the inequality (A.4) is strict, so $\vec{\tilde{I}}$ does not attain the inf-convolution. □

Proof of Lemma 4.11. Since $\mathfrak{I}^* \in \aleph$ and $(\nu_1^*, \ldots, \nu_n^*) \in \beth$, we know that $(\vec{I}^*, \vec{\pi}^*)$ is IR by Theorem 4.4.

For PE, by Proposition 4.9, it suffices to show that the indemnities $\vec{I}^*$ attain the inf-convolution. We have seen in (A.5) that

$$\square_{i=0}^n \rho_i(X) = \int_0^\infty \min_{j \in \mathcal{N} \cup \{0\}} \{\tau_j(X > z)\} \, dz \, .$$

Therefore it suffices to show that

$$\sum_{i=0}^n \rho_i(I_i^*(X)) = \int_0^\infty \min_{j \in \mathcal{N} \cup \{0\}} \{\tau_j(X > z)\} \, dz \, .$$

First recall that $I_0^*(x) = \int_0^x \gamma_0^*(z) \, dz$ as defined in (3.4). Then we can write

$$\sum_{i=0}^n \rho_i(I_i^*(X)) = \sum_{i=0}^n \int_0^\infty \tau_i(X > z)\gamma_i^*(z) \, dz = \int_0^\infty \sum_{i=0}^n \tau_i(X > z)\gamma_i^*(z) \, dz \, .$$

We show that for all $z \in \mathbb{R}_+$, we have $\sum_{i=0}^n \tau_i(X > z)\gamma_i^*(z) = \min_{j \in \mathcal{N} \cup \{0\}} \{\tau_j(X > z)\}$. It suffices to show that $\gamma_i^*(z) > 0$ implies that $\tau_i(X > z) = \min_{j \in \mathcal{N} \cup \{0\}} \tau_j(X > z)$. We consider two cases:

(1) $\underline{i=0}$: Then $\gamma_0^*(z) > 0$ implies that $g(\mathbb{P}(X > z)) \leqslant \bar{\tau}(X > z)$. If this inequality is strict, then we automatically have $g(\mathbb{P}(X > z)) = \min_{j\in\mathcal{N}\cup\{0\}} \tau_j(X > z)$. Otherwise, if $g(\mathbb{P}(X > z)) = \bar{\tau}(X > z)$, then since every reinsurer is quoting the same price $\bar{\tau}(X > z)$, generous distribution implies that there does not exist $k \in \mathcal{N}$ such that $\tau_k(X > z) < g(\mathbb{P}(X > z))$. Therefore $g(\mathbb{P}(X > z)) = \min_{j\in\mathcal{N}\cup\{0\}} \tau_j(X > z)$ as desired.

(2) $\underline{i\in\mathcal{N}}$: In this case, since every reinsurer is using the strategy $\bar{\tau}$, there does not exist $k \in \mathcal{N}$ such that $\tau_k(X > z) < \tau_i(X > z)$, so $\tau_i(X > z) = \min_{j\in\mathcal{N}} \tau_j(X > z)$. It remains to show that $g(\mathbb{P}(X > z)) \geqslant \tau_i(X > z)$.
To this end, note that $\gamma_i^*(z) > 0$ implies $g(\mathbb{P}(X > z)) \geqslant \bar{\tau}(X > z)$. Then there exist $k' \in \mathcal{N}$ such that $\tau_{k'}(X > z) \leqslant \bar{\tau}(X > z) \leqslant g(\mathbb{P}(X > z))$, which implies that $\tau_i(X > z) \leqslant g(\mathbb{P}(X > z))$. Therefore $\tau_i(X > z) = \min_{j\in\mathcal{N}\cup\{0\}} \tau_j(X > z)$ as desired.

To conclude the proof, we see that since $\tau_i(X > z) = \min_{j\in\mathcal{N}\cup\{0\}}\{\tau_j(X > z)\}$ whenever $\gamma_i^*(z) > 0$, we have

$$\begin{aligned}\sum_{i=0}^{n} \tau_i(X > z)\gamma_i^*(z) &= \sum_{i\in\mathcal{N},\gamma_i^*(z)>0} \tau_i(X > z)\gamma_i^*(z) = \sum_{i\in\mathcal{N},\gamma_i^*(z)>0} \min_{j\in\mathcal{N}\cup\{0\}}\{\tau_j(X > z)\}\gamma_i^*(z)\\ &= \min_{j\in\mathcal{N}\cup\{0\}}\{\tau_j(X > z)\} \sum_{i\in\mathcal{N},\gamma_i^*(z)>0} \gamma_i^*(z) = \min_{j\in\mathcal{N}\cup\{0\}}\{\tau_j(X > z)\}.\end{aligned}$$

Therefore

$$\sum_{i=0}^{n} \rho_i(I_i^*(X)) = \int_0^\infty \min_{j\in\mathcal{N}\cup\{0\}}\{\tau_j(X > z)\}\,dz. \quad \square$$

Proof of Theorem 4.12. By Theorem 4.4, we know that $(\nu_1^*, \ldots, \nu_n^*, \mathfrak{I}^*)$ induces an IR allocation. It remains to show PE.
First, let $(\bar{I}_1^*, \ldots, \bar{I}_n^*) := \mathfrak{I}^*(\bar{\tau}, \bar{\tau}, \ldots, \bar{\tau})$. Then since $\mathfrak{I}^* \in \aleph$ and $(\bar{\tau}, \bar{\tau}, \ldots, \bar{\tau}) \in \daleth$, we have

$$\rho_i(\bar{I}_i^*(X) - \pi^{\bar{\tau}}(\bar{I}_i^*(X))) = \int_{\mathcal{Z}_i} \tau_i(X > z) - \bar{\tau}(X > z)\,dz,$$

$$\rho_{IN}\left(X - \sum_{i\in N} \bar{I}_i^*(X) + \sum_{i\in\mathcal{N}} \pi^{\bar{\tau}}(\bar{I}_i^*(X))\right) = \int_{\mathcal{Z}_{IN}} g(\mathbb{P}(X > z))\,dz + \int_{\mathcal{Z}_{IN}^C} \bar{\tau}(X > z)\,dz,$$

by Proposition 3.9 and Lemma 4.3. Summing these inequalities and applying Lemma 4.11, we have

$$\begin{aligned}&\int_{\mathcal{Z}_i} \tau_i(X > z) - \bar{\tau}(X > z)\,dz + \int_{\mathcal{Z}_{IN}} g(\mathbb{P}(X > z))\,dz + \int_{\mathcal{Z}_{IN}^C} \bar{\tau}(X > z)\,dz\\ &= \rho_{IN}\left(X - \sum_{i\in\mathcal{N}} \bar{I}_i^*(X) + \sum_{i\in\mathcal{N}} \pi^{\bar{\tau}}(\bar{I}_i^*(X))\right) + \sum_{i\in\mathcal{N}} \rho_i(\bar{I}_i^*(X) - \pi^{\bar{\tau}}(\bar{I}_i^*(X)))\\ &= \rho_{IN}\left(X - \sum_{i\in\mathcal{N}} \bar{I}_i^*(X)\right) + \sum_{i\in\mathcal{N}} \pi^{\bar{\tau}}(\bar{I}_i^*(X)) + \sum_{i\in\mathcal{N}} \rho_i(\bar{I}_i^*(X)) - \sum_{i\in\mathcal{N}} \pi^{\bar{\tau}}(\bar{I}_i^*(X))\\ &= \rho_{IN}\left(X - \sum_{i\in\mathcal{N}} \bar{I}_i^*(X)\right) + \sum_{i\in\mathcal{N}} \rho_i(\bar{I}_i^*(X)) = \sum_{i=0}^{n} \rho_i(\bar{I}_i^*(X))\\ &= \int_0^\infty \min_{j\in\mathcal{N}\cup\{0\}}\{\tau_j(X > z)\}\,dz = \square_{i=0}^{n}\,\rho_i(X).\end{aligned}$$

Now suppose that $\mathfrak{I}^* \in \aleph$ and $(\nu_1^*, \ldots, \nu_n^*) \in \daleth$. As before, we use the notation $(I_1^*, \ldots, I_n^*) = \mathfrak{I}^*(\nu_1^*, \ldots, \nu_n^*)$. By Proposition 3.9 and Lemma 4.3, we have

$$\begin{aligned}\sum_{i=0}^{n} \rho_i(I_i^*(X)) &= \rho_{IN}\left(X - \sum_{i\in\mathcal{N}} I_i^*(X) + \sum_{i\in\mathcal{N}} \pi^{\nu_i^*}(I_i^*(X))\right) + \sum_{i\in\mathcal{N}} \rho_i\left(I_i^*(X) - \pi^{\nu_i^*}(I_i^*(X))\right)\\ &= \int_{\mathcal{Z}_i} \tau_i(X > z) - \bar{\tau}(X > z)\,dz + \int_{\mathcal{Z}_{IN}} g(\mathbb{P}(X > z))\,dz + \int_{\mathcal{Z}_{IN}^C} \bar{\tau}(X > z)\,dz\\ &= \square_{i=0}^{n}\,\rho_i(X),\end{aligned}$$

so the resulting allocation achieves the inf-convolution. Therefore by Proposition 4.9, the strategy $(\nu_1^*, \ldots, \nu_n^*, \mathfrak{I}^*)$ induces a PE allocation, as desired. $\square$

Proof of Proposition 4.15. Recall that $\mathcal{Z}_i = \{z \in \mathbb{R}_+ : \tau_i(X > z) < \bar{\tau}(X > z)\}$. Note that this condition is equivalent to $\tau_i(X > z) < \tau_j(X > z)$ for $j \in (\mathcal{N} \cup \{0\}) \setminus \{i\}$ – that is, $\tau_i(X > z)$ attains the minimum $\min_j \tau_j(X > z)$ uniquely. Then for all $z \in \mathcal{Z}_i$, we have $\gamma_i^*(z) = 1$ by Proposition 4.10. Now define

$$\bar{\mathcal{Z}}_i := \{z \in \mathbb{R}_+ : \tau_i(X > z) = \min_j \tau_j(X > z)\} \setminus \mathcal{Z}_i\,.$$

That is, $\bar{\mathcal{Z}}_i$ is the set over which $\tau_i(X > z)$ attains the minimum, but not uniquely. Note that on this set, $\tau_i(X > z) = \bar{\tau}_i(X > z)$. By Proposition 4.10, we have that $\gamma_i^*(z) = 0$ on $(\mathcal{Z}_i \cup \bar{\mathcal{Z}}_i)^C$. By Assumption 4.14, we have

$$\begin{aligned}
\int_{\mathcal{Z}_i} \tau_i(X > z) - \bar{\tau}(X > z)\,dz = \rho_i(I_i^*(X) - \pi_i^*) \\
&= \int_0^\infty \tau_i(X > z)\gamma_i^*(z)\,dz - \pi_i^* \\
&= \int_{\mathcal{Z}_i} \tau_i(X > z)\cdot 1\,dz + \int_{\bar{\mathcal{Z}}_i} \tau_i(X > z)\gamma_i^*(z)\,dz \\
&\quad + \int_{(\mathcal{Z}_i \cup \bar{\mathcal{Z}}_i)^C} \tau_i(X > z)\cdot 0\,dz - \pi_i^* \\
&= \int_{\mathcal{Z}_i} \tau_i(X > z)\,dz + \int_{\bar{\mathcal{Z}}_i} \tau_i(X > z)\gamma_i^*(z)\,dz - \pi_i^*\,.
\end{aligned}$$

Subtracting $\int_{\mathcal{Z}_i} \tau_i(X > z)\,dz$ from both sides and rearranging yields

$$\begin{aligned}
\pi_i^* &= \int_{\bar{\mathcal{Z}}_i} \tau_i(X > z)\gamma_i^*(z)\,dz + \int_{\mathcal{Z}_i} \bar{\tau}(X > z)\,dz \\
&= \int_{\bar{\mathcal{Z}}_i} \bar{\tau}(X > z)\gamma_i^*(z)\,dz + \int_{\mathcal{Z}_i} \bar{\tau}(X > z)\,dz \\
&= \int_{\bar{\mathcal{Z}}_i} \bar{\tau}(X > z)\gamma_i^*(z)\,dz + \int_{\mathcal{Z}_i} \bar{\tau}(X > z)\cdot 1\,dz + \int_{(\mathcal{Z}_i \cup \bar{\mathcal{Z}}_i)^C} \tau_i(X > z)\cdot 0\,dz \\
&= \int_0^\infty \nu_i^*(X > z)\gamma_i^*(z)\,dz = \int I_i^*(X)\,d\nu_i^*\,,
\end{aligned}$$

as desired. □

Proof of Proposition 4.16. By Proposition 4.15 and Lemma 4.3, we have

$$\begin{aligned}
\rho_{IN}\left(X - \sum_{i\in\mathcal{N}} I_i^*(X) + \sum_{i\in\mathcal{N}} \pi^{\nu_i^*}(I_i^*(X))\right) &= \rho_{IN}\left(X - \sum_{i\in\mathcal{N}} I_i^*(X) + \sum_{i\in\mathcal{N}} \pi_i^*\right) \\
&= \int_{\mathcal{Z}_{IN}} g(\mathbb{P}(X > z))\,dz + \int_{\mathcal{Z}_{IN}^C} \bar{\tau}(X > z)\,dz \\
&= \min_{I\in\vec{\mathcal{I}}} \left\{\rho_{IN}\left(X - \sum_{i\in\mathcal{N}} I_i(X) + \sum_{i\in\mathcal{N}} \pi^{\nu_i^*}(I_i(X))\right)\right\}. \quad \square
\end{aligned}$$

Proof of Proposition 4.17. Suppose for the sake of contradiction that $\vec{I}^*$ does not distribute generously. Then there exists a $k \in \mathcal{N} \cup \{0\}$ and a set of positive measure $\mathcal{A}$ such that for $z \in \mathcal{Z}$, we have

$$\gamma_k^*(z) > 0\,,$$

and there exists a $k' \in (\mathcal{N} \cup \{0\}) \setminus \{k\}$ such that

$$\tau_{k'}(X > z) < \tau_k(X > z)\,.$$

Then $\sum_{i=0}^n \gamma_i^*(z)\tau_i(X > z) > \sum_{i=0}^n \gamma_i^*(z) \min_{j\in\mathcal{N}\cup\{0\}} \tau_j(X > z) = \min_{j\in\mathcal{N}\cup\{0\}} \tau_j(X > z)$, so

$$\sum_{i=0}^{n}\rho_i(I_i^*(X)) = \sum_{i=0}^{n}\int_0^\infty \tau_i(X>z)\gamma_i^*(z)\,dz = \int_0^\infty \sum_{i=0}^{n}\tau_i(X>z)\gamma_i^*(z)\,dz$$
$$= \int_{\mathcal{A}} \sum_{i=0}^{n}\tau_i(X>z)\gamma_i^*(z)\,dz + \int_{\mathcal{A}^C} \sum_{i=0}^{n}\tau_i(X>z)\gamma_i^*(z)\,dz$$
$$> \int_{\mathcal{A}} \min_{j\in\mathcal{N}\cup\{0\}}\tau_j(X>z)\,dz + \int_{\mathcal{A}^C} \sum_{i=0}^{n}\tau_i(X>z)\gamma_i^*(z)\,dz$$
$$\geqslant \int_{\mathcal{A}} \min_{j\in\mathcal{N}\cup\{0\}}\tau_j(X>z)\,dz + \int_{\mathcal{A}^C} \sum_{i=0}^{n}\min_{j\in\mathcal{N}\cup\{0\}}\tau_j(X>z)\gamma_i^*(z)\,dz$$
$$\geqslant \int_0^\infty \min_{j\in\mathcal{N}\cup\{0\}}\tau_j(X>z)\,dz = \square_{i=0}^{n}\rho_i(X)\,.$$

Hence, $\vec{I}^*$ does not achieve the inf-convolution, so $\vec{I}^*$ is not PE by Proposition 4.9, which contradicts our initial assumption. □

Appendix B. Additional examples of equilibria

B.1. An SPNE not characterized by Theorem 3.11

As an example of an SPNE that is not characterized by Theorem 3.11, consider the following example for $n=2$. We construct an SPNE such that the reinsurers' strategy does not satisfy (3.6), and therefore not in $\beth$. Suppose for all $z\in\mathbb{R}_+$, we have

$$\tau_1(X>z) < \tau_2(X>z) < g(\mathbb{P}(X>z))\,.$$

Let $\mathfrak{I}^*\in\aleph$, and define ν_1,ν_2 by

$$\nu_1=\nu_2=\tau_1\,.$$

Note that by construction, since $\bar{\tau}=\tau_2>\tau_1=\underline{\nu}$, this does not satisfy (3.6). We claim that $(\nu_1,\nu_2,\mathfrak{I}^*)$ is an SPNE. Indeed, for reinsurer 1, we have $\rho_1(I_1^*(X)-\pi^{\tau_1}(I_1^*(X)))=0$. It is impossible for reinsurer 1 to profit, as quoting a higher price than their true preferences τ_1 will result in being undercut by reinsurer 2. On the other hand, since the insurer distributes generously, $I_2^*(X)=0$. Hence, we have $\rho_2(I_2^*(X)-\pi^{\tau_1}(I_2^*(X)))=0$ as well. It is impossible for reinsurer 2 to profit, as quoting a higher price than their true preferences τ_2 will result in being undercut by reinsurer 1.

However, by Remark 3.14, we see that $\nu_1^*=\nu_2^*=\bar{\tau}=\tau_2$ is also an SPNE, which charges a strictly higher price for reinsurance. From the perspective of the reinsurers (in particular reinsurer 1), this SPNE would be preferable.

B.2. A remark on NE vs. SPNE

Throughout this paper, we focus on finding SPNEs, as opposed to the perhaps more familiar notion of NE. It is clear from Definition 2.6 that all SPNEs are NEs. However, we show in this section that the converse statement is not true, even within our model where the game structure is not overly complicated.

Suppose that the strategy $(\nu_1^*,\ldots,\nu_n^*,\mathfrak{I}^*)$ is an SPNE, as given in Theorem 3.11. We construct a strategy as follows. Define by $\hat{\nu}\in\mathcal{C}$ a capacity such that:

(1) $\hat{\nu}\neq\nu_i^*$ for all $i\in\mathcal{N}$; and,
(2) $\hat{\nu}(X>z)>g(\mathbb{P}(X>z))$ on a set of positive measure.

Now define by $\hat{\mathfrak{I}}$ a strategy in $(\vec{\mathcal{I}})^{\mathcal{C}^n}$ such that

$$\begin{aligned}&\hat{\mathfrak{I}}:\mathcal{C}^n\to\vec{\mathcal{I}}\\ &(\hat{\nu},\hat{\nu},\ldots,\hat{\nu})\mapsto \mathbf{0}\,,\\ &(\nu_1,\ldots,\nu_n)\mapsto\mathfrak{I}^*(\nu_1,\ldots,\nu_n),\quad\text{otherwise.}\end{aligned}$$

Under the strategy $\hat{\mathfrak{I}}$, the insurer refuses to do business with any reinsurer in the scenario that all reinsurers quote the price $\hat{\nu}$. This is admittedly strange behaviour, which is indeed not optimal for the insurer. Since $\hat{\nu}(X>z)>g(\mathbb{P}(X>z))$ on a set of positive measure, the insurer can benefit from reinsurance that is priced via $\hat{\nu}$. Hence, 0 does not solve

$$\min_{I\in\vec{\mathcal{I}}}\left\{\rho_{IN}\left(X-\sum_{i\in\mathcal{N}}I_i(X)+\sum_{i\in\mathcal{N}}\pi^{\hat{\nu}}(I_i(X))\right)\right\},$$

so $(\nu_1^*,\ldots,\nu_n^*,\hat{\mathfrak{I}})$ is not an SPNE.

However, we can see that $(\nu_1^*, \ldots, \nu_n^*, \hat{\mathfrak{I}})$ is an NE. Let $\tau \in \mathcal{C}$ be any non-negative capacity, and $i \in \mathcal{N}$. By Proposition 3.10, we have

$$\begin{aligned}\rho_i(\hat{\mathfrak{I}}_i(\tau, \nu_{-i}^*)(X)) - \rho^\tau(\hat{\mathfrak{I}}_i(\tau, \nu_{-i}^*)(X)) &= \rho_i(\mathfrak{I}_i^*(\tau, \nu_{-i})(X)) - \rho^\tau(\mathfrak{I}_i^*(\tau, \nu_{-i})(X)) \\ &\geqslant \rho_i(\mathfrak{I}_i^*(\nu_1^*, \ldots, \nu_n^*)(X)) - \pi^{\nu_i^*}(\mathfrak{I}_i^*(\nu_1^*, \ldots, \nu_n^*)(X)) \\ &= \rho_i(\hat{\mathfrak{I}}_i(\nu_1^*, \ldots, \nu_n^*)(X)) - \pi^{\nu_i^*}(\hat{\mathfrak{I}}_i(\nu_1^*, \ldots, \nu_n^*)(X)).\end{aligned}$$

Also, by construction, $\hat{\mathfrak{I}}(\nu_1^*, \ldots, \nu_n^*) \in \mathcal{I}(\nu_1^*, \ldots, \nu_n^*)$, so it solves

$$\min_{I \in \vec{\mathcal{I}}} \left\{ \rho_{IN}\left(X - \sum_{i \in \mathcal{N}} I_i(X) + \sum_{i \in \mathcal{N}} \pi^{\nu_i^*}(I_i(X)) \right) \right\}.$$

Hence, $(\nu_1^*, \ldots, \nu_n^*, \hat{\mathfrak{I}})$ is an NE.

Although this is an NE, we see that the strategy of the insurer is not a reasonable one. One possible interpretation is that the insurer's sub-optimal behaviour when the pricing capacity is $\hat{\nu}$ is a threat to the reinsurers, to dissuade them from actually choosing such a pricing rule. The notion of SPNE can be seen as a refinement that eliminates this possibility. That is, we assume that the insurer is unable or unwilling to issue such threats, or that the reinsurers correctly identify that the insurer will never follow through with such threats.

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