Insurance: Mathematics and Economics 111 (2023) 84–101

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Cumulative Parisian ruin in finite and infinite time horizons for a renewal risk process with exponential claims

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ARTICLE INFO

Article history:
Received November 2022
Received in revised form March 2023
Accepted 22 March 2023
Available online 29 March 2023

JEL classification:
G22
C44

Keywords:
Cumulative Parisian ruin
Occupation time
Renewal risk model
Finite-time ruin probability
Erlangization

ABSTRACT

The Parisian ruin time, which is the first time the insurer's surplus process has an excursion below level zero that exceeds a prescribed time length, has been extensively analyzed in recent years mainly in the Lévy model and its special cases. However, the cumulative Parisian ruin time, which is the first time the total time spent by the surplus process below level zero exceeds a certain time length, has been rarely considered in the literature. In this paper, we study the cumulative Parisian ruin problem in a renewal risk model with general interclaim times and exponential claims. Explicit formulas for the infinite-time cumulative Parisian ruin probability is first derived under a deterministic Parisian clock and then under an Erlang clock, where the latter case can also serve as an approximation of the former. The finite-time cumulative Parisian ruin probability is subsequently analyzed as well when the time horizon is another Erlang random variable. Our formulas are applied in various numerical examples where the interclaim times follow gamma, Weibull, or Pareto distribution. Consequently, we demonstrate that the choice of the interclaim distribution does have a significant impact on the cumulative Parisian ruin probabilities when one deviates from the exponential assumption.

1. Introduction

In this paper, the insurance risk surplus process $\{U(t)\}_{t\geq 0}$ is assumed to follow the dynamics

$$U(t)=u+ct-\sum_{i=1}^{N(t)}Y_i,$$

where $U(0)=u\geq 0$ is the initial surplus level and $c>0$ is the incoming premium rate. Moreover, the number of claims process $\{N(t)\}_{t\geq 0}$ is assumed to be a renewal process with independent and identically distributed (i.i.d.) inter-arrival times $\{V_i\}_{i=1}^{\infty}$ possessing common density $k(\cdot)$ and Laplace transform $\widetilde{k}(s)=\int_0^{\infty}e^{-st}k(t)\,\mathrm{d}t$. The claim amounts $\{Y_i\}_{i=1}^{\infty}$ are assumed to form a sequence of i.i.d. random variables independent of $\{N(t)\}_{t\geq 0}$. Under the above specifications, $\{U(t)\}_{t\geq 0}$ is known as a renewal or Sparre Andersen risk process (Sparre Andersen (1957)). For convenience, we shall use $\mathbb{P}_u$ and $\mathbb{E}_u$ to denote, respectively, the probability and the expectation taken under the initial condition $U(0)=u$. Traditionally, the ruin time is defined by $\tau_{\mathrm{CL}}=\inf\{t\geq 0: U(t)<0\}$ (with the convention $\inf\emptyset=+\infty$). We assume the positive security loading condition $c\mathbb{E}[V_1]>\mathbb{E}[Y_1]$, which guarantees that $\{U(t)\}_{t\geq 0}$ drifts to positive infinity in the long run and the ultimate ruin probability $\mathbb{P}_u\{\tau_{\mathrm{CL}}<\infty\}$ is less than one for $u\geq 0$ (see Prabhu (1998, Part I, Theorems 2 and 7)). If each V_i is assumed to be exponentially distributed, then $\{U(t)\}_{t\geq 0}$ reduces to the classical compound Poisson or Cramér-Lundberg risk process. For $n=1,2,\ldots$, the n-th fold convolution of $k(\cdot)$ is denoted by $k^{*n}(\cdot)$ (which is the density of the n-th arrival time $\sum_{i=1}^{n}V_i$ of $\{N(t)\}_{t\geq 0}$). In the remainder of the paper, we assume each Y_i follows an exponential distribution with mean $1/\mu$.

The renewal risk process, being a natural generalization of the compound Poisson model, has been studied extensively in the literature (see Asmussen and Albrecher (2010) and Willmot and Woo (2017) for review). Quantities such as the ruin probability, the distribution of

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https://doi.org/10.1016/j.insmatheco.2023.03.003
0167-6687/

the deficit at ruin and more generally Gerber-Shiu type functions (see Gerber and Shiu (1998)) have been analyzed by e.g. Drekic et al. (2004), Willmot (2007), Landriault and Willmot (2008), Cheung et al. (2010) and Frostig et al. (2012). In these works, the distribution of the inter-arrival times is left unspecified. In particular, as far as explicit results are concerned, distributional assumption (such as exponential, phase-type, and Coxian) only needs to be made on the claim amounts. This is contrary to the analysis of the classical compound Poisson risk model where the inter-arrival times are specifically exponential while the claim amounts can possibly be more general.

In modifying the traditional definition of ruin, Dassios and Wu (2008a,b) proposed (in the compound Poisson risk model) the idea of Parisian ruin time which is the first time the surplus process has continuously remained negative for a prescribed amount of time (say $r > 0$). Since then, the Parisian ruin problem and its modifications (such as replacement of the deterministic grace period r by a random time; further incorporation of a lower bankruptcy barrier; consideration of Parisian drawdown) have received a lot of attention in spectrally negative Lévy risk process and its special cases. See e.g. Czarna and Palmowski (2011), Loeffen et al. (2013, 2018), Landriault et al. (2014, 2019), Baurdoux et al. (2016), Czarna (2016), Czarna et al. (2017), Lkabous et al. (2017) and Li and Zhou (2022), where the results are typically expressed in terms of scale functions. Outside the class of Lévy risk processes, the Parisian ruin problem was studied in the present renewal risk process with exponential claims by Wong and Cheung (2015). As mentioned by Dassios and Wu (2008a,b), such a concept of Parisian ruin was motivated by Parisian options in finance, where an option can be knocked in or knocked out if, before the maturity date, there is an excursion of the underlying asset price above or below a certain level that exceeds a prescribed length of time (see Chesney et al. (1997)). In an insurance context, the interpretation is that the insurer is given a grace period by the regulator for debt restructuring in attempt to recover from a negative surplus. Compared to the classical ruin time τ_{CL}, Parisian ruin is more consistent with the notion of bankruptcy and liquidation described by Chapters 7 and 11 of the US Bankruptcy Code. On average, the rehabilitation process lasts for about 2.5 years as the business undergoes reorganization (e.g. Broadie et al. (2007, footnote 14)). Moreover, Antill and Grenadier (2019) argued that modelling the 'conversion (from Chapter 11 (reorganization) to Chapter 7 (liquidation)) as exogenous and random is a reasonable approximation of reality'. Therefore, the consideration of a random grace period may make sense as well. Further motivation and justification of Parisian ruin problem in connection to Chapters 7 and 11 can be found in e.g. Li et al. (2014, 2020).

Chesney et al. (1997) also discussed an alternative type of Parisian option, namely cumulative Parisian option, where knock-in or knock-out is based on the cumulative time spent by the underlying asset above or below a fixed level. This has led to analogous definition of ruin in an actuarial context as well. From Guérin and Renaud (2017), cumulative Parisian ruin is said to occur when the total time the surplus process has spent in the red (i.e. below zero) reaches r time units for the first time, where $r > 0$ is a prescribed constant known as the (cumulative) Parisian clock. That is, as opposed to the previously discussed Parisian ruin which is based on excursion of the process below level zero, cumulative Parisian ruin is defined based on the occupation time below zero. Defining $\mathrm{OT}_t = \int_0^t 1_{\{U(s)<0\}}\,\mathrm{d}s$ to be the occupation time in the red by time $t \ge 0$, the event that cumulative Parisian ruin occurs by time t is equivalent to the event that OT_t exceeds r. Then, the cumulative Parisian ruin time is $\tau_r = \inf\{t \ge 0 : \mathrm{OT}_t > r\}$, and the finite-time cumulative Parisian ruin probability is

$$\psi_r(u,t) = \mathbb{P}_u\{\tau_r \le t\} = \mathbb{P}_u\{\mathrm{OT}_t > r\}, \tag{1.1}$$

whereas the infinite-time or ultimate cumulative Parisian ruin probability is

$$\psi_r(u) = \mathbb{P}_u\{\tau_r < \infty\} = \mathbb{P}_u\{\mathrm{OT}_\infty > r\}. \tag{1.2}$$

As commented by Li et al. (2020), the descriptions of the feature of rehabilitation are mainly qualitative and far from unified in essentially all global insurance prudential regulatory frameworks. We believe that cumulative Parisian ruin provides an attractive alternative to the usual definition of Parisian ruin. In particular, if the surplus process has fallen below zero, this possibly signals substantial underlying problems with the insurance portfolio, and the insurer's reputation has already taken a hit even it subsequently survives the first grace period. The regulators should be hesitant to grant the business another grace period of the same length (or distribution) if its surplus falls below zero again in the future. Therefore, it makes more sense to sum the durations of negative surplus in determining the event of ruin (as in cumulative Parisian ruin) instead of starting a new clock whenever an excursion below zero begins (as in the usual Parisian ruin). While Parisian ruin has been extensively analyzed in the literature, the study of the cumulative Parisian ruin problem is scarce. The limited references include Guérin and Renaud (2017) who primarily considered the classical compound Poisson and the Brownian motion risk models, and Bladt et al. (2019) who applied techniques from Markov-modulated fluid flow processes to a phase-type renewal process with phase-type claims. Lkabous and Renaud (2018) has subsequently looked into a VaR-type risk measure based on cumulative Parisian ruin in the compound Poisson risk model. Specifically, they defined such a risk measure as $\inf\{u \ge 0 : \psi_r(u,t) \le \varepsilon\}$ for some tolerance level $\varepsilon > 0$. Thanks to the relationship (1.1), this is equivalent to $\inf\{u \ge 0 : \mathbb{P}_u\{\mathrm{OT}_t > r\} \le \varepsilon\}$ which is expressed in terms of the distribution of OT_t. They have shown that this risk measure satisfies attractive properties such as translation invariance, positive homogeneity and monotonicity, which facilitate actuarial applications. Our focus here is to develop exact formulas and/or approximations for $\psi_r(u)$ and $\psi_r(u,t)$, and our work serves to fill an important literature gap in the cumulative Parisian ruin problem in allowing for the interclaim times to be arbitrary. It is important to note that finite-time ruin probabilities defined for the classical ruin time τ_{CL} are in general not easy to obtain even in the simple cases of the compound Poisson model (e.g. Dickson and Willmot (2005)) and the present renewal risk model with exponential claims (e.g. Borovkov and Dickson (2008) and Landriault et al. (2011)). Our results on $\psi_r(u,t)$ can in principle be applied to calculate or approximate the afore-mentioned risk measure as well.

This paper is organized as follows. Section 2 starts by providing some technical preliminaries that will be used throughout. In Section 3, the ultimate cumulative Parisian ruin probability (1.2) is first derived under a deterministic Parisian clock r, and then an approximation based on replacing r by an Erlang random variable (see Section 2.2 for justification) is also developed. Section 4 is concerned with approximations of the finite-time cumulative Parisian ruin probability (1.1) via replacing the time horizon t by an Erlang random variable. The formulas developed can be easily programmed in software such as *Mathematica*. Numerical examples are given in Section 5 to illustrate the impact of the choice of the interclaim time distribution (gamma, Weibull, or Pareto) on the cumulative Parisian ruin probabilities when one deviates from the exponential interclaim times implicitly assumed in the compound Poisson model. Section 6 ends the paper with some concluding remarks.

2. Preliminaries

2.1. Lundberg's fundamental equation

For each $y \geq 0$, the Lundberg's fundamental equation (in ξ) of the renewal risk model with exponential claims is given by (e.g. Willmot (2007, Equation (3.16)))

$$1 - \widetilde{k}(c\xi + y)\frac{\mu}{\mu - \xi} = 0, \tag{2.1}$$

which has a unique positive real root that is denoted by R_y (commonly known as the 'adjustment coefficient'). The root R_y and its derivatives evaluated at various values of y will be crucial to our analysis. In particular, the Laplace transform of the classical ruin time is

$$\mathbb{E}_u[e^{-\delta\tau_{\mathrm{CL}}}1_{\{\tau_{\mathrm{CL}}<\infty\}}] = \frac{\mu - R_\delta}{\mu}e^{-R_\delta u} \tag{2.2}$$

under a Laplace transform argument of $\delta \geq 0$ (e.g. Willmot (2007, Equation (3.15))). The classical ruin probability can be retrieved by letting $\delta = 0$. Moreover, for an exponential random variable with mean $1/\gamma$ (denoted by $\mathcal{E}_{1,\gamma}$) that is independent of the surplus process $\{U(t)\}_{t\geq 0}$, it is known from Wong and Cheung (2015, Equations (5.2) and (5.5)) that the Laplace transform of the occupation time $\mathrm{OT}_{\mathcal{E}_{1,\gamma}}$ in red until time $\mathcal{E}_{1,\gamma}$ is

$$\mathbb{E}_u[e^{-\omega \mathrm{OT}_{\mathcal{E}_{1,\gamma}}}] = 1 - \frac{c(R_{\omega+\gamma} - R_\gamma) + \omega}{cR_{\omega+\gamma} + \omega + \gamma}\frac{\mu - R_\gamma}{\mu}e^{-R_\gamma u}, \tag{2.3}$$

where $\omega \geq 0$ is the Laplace transform argument.

2.2. Erlangization

While the Parisian clock r and the finite-time horizon t in (1.1) and (1.2) are preferably deterministic, it is often mathematically more tractable to assume a random clock or a random time horizon so that a Markovian structure of the process may be exploited to ease the analysis. A popular choice to replace a deterministic amount of $h > 0$ is an Erlang(n) distribution with rate parameter $\chi > 0$ due to the Erlangization technique: if one sets $\chi = n/h$ and lets $n \to \infty$ then the Erlang variable converges in distribution to a point mass at h. Erlangization was first proposed in option pricing by e.g. Carr (1998) and Kyprianou and Pistorius (2003). It was subsequently utilized by e.g. Asmussen et al. (2002), Stanford et al. (2005) and Ramaswami et al. (2008) in finite-time ruin problems (to replace the finite-time horizon), by e.g. Albrecher et al. (2011) and Cheung and Zhang (2019) in risk models with periodic dividend decisions (to mimic the intervals between dividend decision times), and by e.g. Landriault et al. (2014) and Cheung and Wong (2017) in Parisian ruin and dividend problems (to mimic the Parisian implementation delays). The good performance of Erlangization has been well documented in these works and references therein. For later use, an Erlang(n) random variable with rate parameter χ will be denoted by $\mathcal{E}_{n,\chi}$.

2.3. Faà di Bruno's formula

It will be seen that in various steps of our analysis we need to take higher order derivatives of composite functions (such as differentiating $e^{-R_y u}$ with respect to y). This can be done with the help of Faà di Bruno's formula (e.g. Johnson (2002, Equation (2.2))) which states that

$$\frac{\mathrm{d}^i}{\mathrm{d}x^i}g_1(g_2(x)) = \sum_{j=0}^{i} g_1^{(j)}(g_2(x))B_{ij}(g_2^{(1)}(x), g_2^{(2)}(x)\ldots, g_2^{(i-j+1)}(x)), \qquad i = 0, 1, 2, \ldots, \tag{2.4}$$

as long as the relevant derivatives of $g_1(\cdot)$ and $g_2(\cdot)$ above exist. In the above formula, $B_{ij}(x_1, \ldots, x_{i-j+1})$ represents the Bell polynomial

$$B_{ij}(x_1, \ldots, x_{i-j+1}) = \sum \frac{i!}{k_1!k_2!\cdots k_{i-j+1}!}\left(\frac{x_1}{1!}\right)^{k_1}\left(\frac{x_2}{2!}\right)^{k_2}\cdots\left(\frac{x_{i-j+1}}{(i-j+1)!}\right)^{k_{i-j+1}}, \tag{2.5}$$

where the sum is taken over non-negative integers $k_1, k_2 \ldots, k_{i-j+1}$ such that $k_1 + k_2 + \ldots + k_{i-j+1} = i$ and $k_1 + 2k_2 + \ldots + (i-j+1)k_{i-j+1} = i$. The Faà di Bruno's formula (2.4) will be applied under specific choices of $g_1(\cdot)$ and $g_2(\cdot)$ later on (see (3.17) and (4.9)). The software *Mathematica* has the built-in function 'BellY' to compute Bell polynomials conveniently. We remark that if i in (2.4) is non-zero, then the lower limit of the summation on the right-hand side can start at $j = 1$ because $B_{i0}(x_1, \ldots, x_{i+1})$ is identical to zero.

2.4. Lagrange's implicit function theorem

In insurance risk processes, the Laplace transforms of the ruin time and the occupation time are often expressed in terms of the roots of the Lundberg's equation. However, the roots typically depend on the Laplace transform argument in an implicit manner (like how R_y depends on y via (2.1)), which make it far from trivial to perform Laplace transforms inversion to obtain the related probability densities. The Lagrange's implicit function theorem has proved to be useful in this regard (e.g. Dickson and Willmot (2005) and Landriault et al. (2011)). From e.g. Cohen (1982, pp.652-653), it is known that for analytic functions $a(x)$ and $\varphi(x)$ in x on and inside a contour D surrounding a point w and for ζ satisfying

$$|\zeta\varphi(x)| < |x - w| \tag{2.6}$$

for every x on D, the function $x - w - \zeta\varphi(x)$ has exactly one zero, say v, inside D so that

$$v = w + \zeta\varphi(v), \tag{2.7}$$

and moreover one has

$$a(v) = a(w) + \sum_{n=1}^{\infty} \frac{\zeta^n}{n!} \frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}} \{a'(x)[\varphi(x)]^n\}\Big|_{x=w}. \tag{2.8}$$

3. Ultimate cumulative Parisian ruin probability

3.1. Deterministic Parisian clock r

In this subsection, we will find the ultimate cumulative Parisian ruin probability $\psi_r(u)$ under a deterministic Parisian clock r. The key to our approach is to invert the Laplace transform of the occupation time $\mathrm{OT}_\infty = \int_0^\infty 1_{\{U(s)<0\}}\,\mathrm{d}s$ to obtain its density function, and then $\psi_r(u)$ can be obtained by integrating the tail via (1.2). It is important to note that, because the (classical) ruin probability is strictly less than one under the positive loading condition, the occupation time OT_∞ in the red has a positive probability mass at zero. Consequently, OT_∞ follows a mixed distribution. In particular, the event $\{\mathrm{OT}_\infty = 0\}$ is equivalent to the event that the surplus process never falls below zero and therefore its probability of occurrence is simply the classical ultimate survival probability so that

$$\mathbb{P}_u\{\mathrm{OT}_\infty = 0\} = 1 - \frac{\mu - R_0}{\mu} e^{-R_0 u} \tag{3.1}$$

according to (2.2). With probability $1 - \mathbb{P}_u\{\mathrm{OT}_\infty = 0\} = [(\mu - R_0)/\mu]e^{-R_0 u}$, the occupation time OT_∞ has a proper density on $(0,\infty)$ which will be denoted by $f_{\mathrm{OT}_\infty}(\cdot)$. It will be seen that $f_{\mathrm{OT}_\infty}(\cdot)$ does not depend on the initial surplus u, and therefore u is omitted from the notation. Intuitively, it is well known that, conditional on (classical) ruin occurrence, the deficit always follows the same exponential distribution with mean $1/\mu$ regardless of the initial surplus u. Consequently, the dependence of the distribution of OT_∞ on u only appears via the classical ruin probability. The Laplace transform of OT_∞ (with Laplace transform argument $\omega > 0$) can now be written as

$$\mathbb{E}_u[e^{-\omega \mathrm{OT}_\infty}] = \mathbb{P}_u\{\mathrm{OT}_\infty = 0\} + (1 - \mathbb{P}_u\{\mathrm{OT}_\infty = 0\}) \int_0^\infty e^{-\omega t} f_{\mathrm{OT}_\infty}(t)\,\mathrm{d}t. \tag{3.2}$$

On the other hand, from the results in Wong and Cheung (2015, Section 5), we can put $\gamma = 0$ into (2.3) to obtain the Laplace transform of OT_∞ as

$$\mathbb{E}_u[e^{-\omega \mathrm{OT}_\infty}] = 1 - \frac{c(R_\omega - R_0) + \omega}{cR_\omega + \omega} \frac{\mu - R_0}{\mu} e^{-R_0 u} = 1 - \frac{\mu - R_0}{\mu} e^{-R_0 u} + \frac{\mu - R_0}{\mu} e^{-R_0 u} \frac{cR_0}{cR_\omega + \omega}. \tag{3.3}$$

Utilizing (3.1), comparison of (3.2) and (3.3) reveals the Laplace transform relationship

$$\int_0^\infty e^{-\omega t} f_{\mathrm{OT}_\infty}(t)\,\mathrm{d}t = \frac{cR_0}{cR_\omega + \omega} = cR_0 \int_0^\infty e^{-\omega t} e^{-ctR_\omega}\,\mathrm{d}t, \tag{3.4}$$

which indicates that $f_{\mathrm{OT}_\infty}(t)$ can be obtained by analytic inversion of Laplace transforms with respect to ω. Note that the right-hand side does not involve u.

Since R_ω on the right-hand side of (3.4) depends on ω via the Lundberg's equation (2.1), we shall apply the Lagrange's implicit function theorem from Section 2.4. To this end, by first rewriting (2.1) (with $y = \omega$ and $\xi = R_\omega$) as

$$R_\omega = \mu - \mu\widetilde{k}(cR_\omega + \omega),$$

we observe that (2.7) is satisfied with $v = R_\omega$, $w = \mu$, $\zeta = -\mu$ and $\varphi(x) = \widetilde{k}(cx + \omega)$. Although it is known from the literature that the adjustment coefficient R_ω is the only root of (2.1) (under $y = \omega$) with positive real part, for completeness we would still verify (2.6) holds true by considering a contour D that consists of (i) the semi-circle of radius s running clockwise from is to $-$is (where $\mathrm{i} = \sqrt{-1}$); and (ii) part of the imaginary axis from $-$is to is. For x on the semi-circle, it is clear that $|\zeta\varphi(x)| = \mu|\widetilde{k}(cx + \omega)| \le \mu\widetilde{k}(\omega)$ since x has non-negative real part, and with $\omega > 0$ we must have $|\zeta\varphi(x)| < \mu$. Moreover, on the semi-circle one also has $|x - w| = |x - \mu|$, and therefore for s sufficiently large (say for $s > 2\mu$) one has $\mu < |x - \mu|$ so that (2.6) is satisfied. For x on part of the imaginary axis, we use the parametric expression $x = \mathrm{i}s\epsilon$ for $-1 \le \epsilon \le 1$. Then $|\zeta\varphi(x)| = \mu|\widetilde{k}(\mathrm{i}s\epsilon + \omega)| \le \mu\widetilde{k}(\omega) < \mu$ and $|x - w| = |\mathrm{i}s\epsilon - \mu| > \mu$ and (2.6) is satisfied as well. From the statement of the Lagrange's implicit function theorem, we conclude that $x - w - \zeta\varphi(x) = 0$, or equivalently $x - \mu + \mu\widetilde{k}(cx + \omega) = 0$, has exactly one root inside the contour D (for sufficiently large s), meaning that $x - \mu + \mu\widetilde{k}(cx + \omega)$ only has one zero with positive real part (which is R_ω). Letting $a(x) = e^{-ctx}$, application of the result (2.8) gives

$$e^{-ctR_\omega} = e^{-ct\mu} + \sum_{n=1}^{\infty} \frac{(-\mu)^n}{n!} \frac{\partial^{n-1}}{\partial x^{n-1}} \{(-ct)e^{-ctx}[\widetilde{k}(cx + \omega)]^n\}\Big|_{x=\mu}.$$

Substitution into the right-hand side of (3.4) along with the use of the fact that $[\widetilde{k}(s)]^n = \int_0^\infty e^{-sz} k^{*n}(z)\,\mathrm{d}z$ leads to

$$\int_0^\infty e^{-\omega t} f_{\mathrm{OT}_\infty}(t)\,\mathrm{d}t = cR_0 \int_0^\infty e^{-\omega t}\left\{ e^{-c\mu t} + \sum_{n=1}^\infty \frac{(-\mu)^n}{n!}\frac{\partial^{n-1}}{\partial x^{n-1}}\left((-ct)e^{-ctx}\int_0^\infty e^{-(cx+\omega)z}k^{*n}(z)\,\mathrm{d}z \right)\Bigg|_{x=\mu}\right\}\mathrm{d}t. \tag{3.5}$$

We can simplify the integral as

$$\begin{aligned}
&\int_0^\infty e^{-\omega t}\sum_{n=1}^\infty \frac{(-\mu)^n}{n!}\frac{\partial^{n-1}}{\partial x^{n-1}}\left((-ct)e^{-ctx}\int_0^\infty e^{-(cx+\omega)z}k^{*n}(z)\,\mathrm{d}z \right)\Bigg|_{x=\mu}\mathrm{d}t\\
&= \int_0^\infty e^{-\omega t}\sum_{n=1}^\infty \frac{(-\mu)^n}{n!}(-ct)\frac{\partial^{n-1}}{\partial x^{n-1}}\left(\int_0^\infty e^{-(t+z)cx}e^{-\omega z}k^{*n}(z)\,\mathrm{d}z \right)\Bigg|_{x=\mu}\mathrm{d}t\\
&= \int_0^\infty e^{-\omega t}\sum_{n=1}^\infty \frac{(-\mu)^n}{n!}(-ct)\int_0^\infty (-c)^{n-1}(t+z)^{n-1}e^{-(t+z)c\mu}e^{-\omega z}k^{*n}(z)\,\mathrm{d}z\,\mathrm{d}t\\
&= \int_0^\infty\int_t^\infty \sum_{n=1}^\infty \frac{(c\mu)^n}{n!} te^{-\omega z}z^{n-1}e^{-c\mu z}k^{*n}(z-t)\,\mathrm{d}z\,\mathrm{d}t\\
&= \int_0^\infty\int_0^z \sum_{n=1}^\infty \frac{(c\mu)^n}{n!} te^{-\omega z}z^{n-1}e^{-c\mu z}k^{*n}(z-t)\,\mathrm{d}t\,\mathrm{d}z\\
&= \int_0^\infty e^{-\omega t}\left(e^{-c\mu t}\sum_{n=1}^\infty \frac{(c\mu)^n}{n!}t^{n-1}\int_0^t zk^{*n}(t-z)\,\mathrm{d}z \right)\mathrm{d}t.
\end{aligned}$$

By substituting the above expression into (3.5), the uniqueness of Laplace transform implies

$$f_{\mathrm{OT}_\infty}(t) = cR_0 e^{-c\mu t}\left(1 + \sum_{n=1}^\infty \frac{(c\mu)^n}{n!}t^{n-1}\int_0^t zk^{*n}(t-z)\,\mathrm{d}z \right), \qquad t>0. \tag{3.6}$$

Finally, with the relationship (1.2), the ultimate cumulative Parisian ruin probability under a deterministic clock $r>0$ is given by

$$\psi_r(u) = \frac{\mu - R_0}{\mu}e^{-R_0 u}\int_r^\infty f_{\mathrm{OT}_\infty}(t)\,\mathrm{d}t = \frac{\mu - R_0}{\mu}e^{-R_0 u}cR_0\int_r^\infty e^{-c\mu t}\left(1 + \sum_{n=1}^\infty \frac{(c\mu)^n}{n!}t^{n-1}\int_0^t zk^{*n}(t-z)\,\mathrm{d}z \right)\mathrm{d}t. \tag{3.7}$$

It is clear that explicit evaluation of the above probability heavily depends on whether the convolution $k^{*n}(\cdot)$ in (3.6) is of closed-form. In what follows, we present an example when the interclaim time follows an Erlang distribution.

Example 1 (**Erlang renewal process**). In this example, it is assumed that the interclaim time is Erlang(m) distributed with rate parameter $\beta > 0$, so that $k^{*n}(\cdot)$ corresponds to an Erlang(mn) density with the same rate for $n = 1, 2, \ldots$. We first evaluate the integral in (3.6) as

$$\begin{aligned}
\int_0^t zk^{*n}(t-z)\,\mathrm{d}z &= \int_0^t (t-z)k^{*n}(z)\,\mathrm{d}z = t\int_0^t \frac{\beta^{mn}z^{mn-1}e^{-\beta z}}{(mn-1)!}\,\mathrm{d}z - \int_0^t z\frac{\beta^{mn}z^{mn-1}e^{-\beta z}}{(mn-1)!}\,\mathrm{d}z\\
&= t\left(1 - \sum_{i=0}^{mn-1}\frac{(\beta t)^i}{i!}e^{-\beta t} \right) - \frac{mn}{\beta}\left(1 - \sum_{i=0}^{mn}\frac{(\beta t)^i}{i!}e^{-\beta t} \right).
\end{aligned}$$

In Section 4.1, it will be necessary to evaluate an integral (see (4.10)) that is slightly more general than the one in (3.7). Therefore, for $\ell = 0, 1, 2, \ldots$ and $\gamma \geq 0$, using the above result we consider

$$\begin{aligned}
&\int_r^\infty t^\ell e^{-\gamma t}e^{-c\mu t}\left(1 + \sum_{n=1}^\infty \frac{(c\mu)^n}{n!}t^{n-1}\int_0^t zk^{*n}(t-z)\,\mathrm{d}z \right)\mathrm{d}t\\
&= \int_r^\infty t^\ell e^{-(\gamma+c\mu)t}\,\mathrm{d}t + \sum_{n=1}^\infty \frac{(c\mu)^n}{n!}\left(\int_r^\infty t^{\ell+n}e^{-(\gamma+c\mu)t}\,\mathrm{d}t - \sum_{i=0}^{mn-1}\frac{\beta^i}{i!}\int_r^\infty t^{\ell+n+i}e^{-(\gamma+c\mu+\beta)t}\,\mathrm{d}t \right)\\
&\quad - \sum_{n=1}^\infty \frac{(c\mu)^n}{n!}\frac{mn}{\beta}\left(\int_r^\infty t^{\ell+n-1}e^{-(\gamma+c\mu)t}\,\mathrm{d}t - \sum_{i=0}^{mn}\frac{\beta^i}{i!}\int_r^\infty t^{\ell+n+i-1}e^{-(\gamma+c\mu+\beta)t}\,\mathrm{d}t \right)
\end{aligned} \tag{3.8}$$

$$= \frac{\ell!}{(\gamma+c\mu)^{\ell+1}}\sum_{j=0}^{\ell}\frac{[(\gamma+c\mu)r]^j}{j!}e^{-(\gamma+c\mu)r}+\sum_{n=1}^{\infty}\frac{(c\mu)^n}{n!}\left(\frac{(\ell+n)!}{(\gamma+c\mu)^{\ell+n+1}}\sum_{j=0}^{\ell+n}\frac{[(\gamma+c\mu)r]^j}{j!}e^{-(\gamma+c\mu)r}\right.$$
$$\left.-\sum_{i=0}^{mn-1}\frac{\beta^i}{i!}\frac{(\ell+n+i)!}{(\gamma+c\mu+\beta)^{\ell+n+i+1}}\sum_{j=0}^{\ell+n+i}\frac{[(\gamma+c\mu+\beta)r]^j}{j!}e^{-(\gamma+c\mu+\beta)r}\right)$$
$$-\sum_{n=1}^{\infty}\frac{(c\mu)^n}{(n-1)!}\frac{m}{\beta}\left(\frac{(\ell+n-1)!}{(\gamma+c\mu)^{\ell+n}}\sum_{j=0}^{\ell+n-1}\frac{[(\gamma+c\mu)r]^j}{j!}e^{-(\gamma+c\mu)r}\right.$$
$$\left.-\sum_{i=0}^{mn}\frac{\beta^i}{i!}\frac{(\ell+n+i-1)!}{(\gamma+c\mu+\beta)^{\ell+n+i}}\sum_{j=0}^{\ell+n+i-1}\frac{[(\gamma+c\mu+\beta)r]^j}{j!}e^{-(\gamma+c\mu+\beta)r}\right). \tag{3.9}$$

Alternatively, the integrals in (3.8) can also be expressed in terms of the incomplete gamma function defined by $\Gamma(a,x)=\int_x^\infty y^{a-1}e^{-y}\,\mathrm{d}y$ for $x\geq 0$ and $a>0$, which can be conveniently computed using built-in function in e.g. *Mathematica*. This leads to

$$\int_r^\infty t^\ell e^{-\gamma t}e^{-c\mu t}\left(1+\sum_{n=1}^{\infty}\frac{(c\mu)^n}{n!}t^{n-1}\int_0^t zk^{*n}(t-z)\,\mathrm{d}z\right)\mathrm{d}t$$
$$=\frac{\Gamma(\ell+1,(\gamma+c\mu)r)}{(\gamma+c\mu)^{\ell+1}}+\sum_{n=1}^{\infty}\frac{(c\mu)^n}{n!}\left(\frac{\Gamma(\ell+n+1,(\gamma+c\mu)r)}{(\gamma+c\mu)^{\ell+n+1}}-\sum_{i=0}^{mn-1}\frac{\beta^i}{i!}\frac{\Gamma(\ell+n+i+1,(\gamma+c\mu+\beta)r)}{(\gamma+c\mu+\beta)^{\ell+n+i+1}}\right)$$
$$-\sum_{n=1}^{\infty}\frac{(c\mu)^n}{(n-1)!}\frac{m}{\beta}\left(\frac{\Gamma(\ell+n,(\gamma+c\mu)r)}{(\gamma+c\mu)^{\ell+n}}-\sum_{i=0}^{mn}\frac{\beta^i}{i!}\frac{\Gamma(\ell+n+i,(\gamma+c\mu+\beta)r)}{(\gamma+c\mu+\beta)^{\ell+n+i}}\right). \tag{3.10}$$

The special case of (3.9) or (3.10) where $\ell=\gamma=0$ corresponds to the integral in (3.7), and therefore the ultimate cumulative Parisian ruin probability $\psi_r(u)$ is fully characterized. $\square$

3.2. Erlang(m_1,ω) Parisian clock

It is important to note that the ultimate cumulative Parisian ruin probability (3.7) under a deterministic clock in Section 3.1 requires closed-form expression for the convolution of the interclaim time density, namely $k^{*n}(\cdot)$, to work well. This somewhat limits its use. In this subsection, the deterministic Parisian clock is replaced by an Erlang(m_1,ω) random variable $\mathcal{E}_{m_1,\omega}$ that is independent of the surplus process $\{U(t)\}_{t\geq 0}$. We shall determine the resulting cumulative Parisian ruin probability $\mathbb{P}_u\{\mathrm{OT}_\infty>\mathcal{E}_{m_1,\omega}\}=1-\mathbb{P}_u\{\mathrm{OT}_\infty<\mathcal{E}_{m_1,\omega}\}$ that can be more widely applicable to other interclaim time distributions (see comments before Example 2). As we shall see, the survival probability $\mathbb{P}_u\{\mathrm{OT}_\infty<\mathcal{E}_{m_1,\omega}\}$ is closely related to the Laplace transform and the discounted moments of OT_∞. First, we substitute (3.1) into (3.2) so that

$$\mathbb{E}_u[e^{-\omega \mathrm{OT}_\infty}]=1-\frac{\mu-R_0}{\mu}e^{-R_0u}+\frac{\mu-R_0}{\mu}e^{-R_0u}\int_0^\infty e^{-\omega t}f_{\mathrm{OT}_\infty}(t)\,\mathrm{d}t. \tag{3.11}$$

Taking the i-th derivative with respect to ω results in the discounted moments

$$\mathbb{E}_u[\mathrm{OT}_\infty^i e^{-\omega \mathrm{OT}_\infty}]=\frac{\mu-R_0}{\mu}e^{-R_0u}\int_0^\infty t^i e^{-\omega t}f_{\mathrm{OT}_\infty}(t)\,\mathrm{d}t,\qquad i=1,2,\ldots. \tag{3.12}$$

Now, we proceed to analyze $\mathbb{P}_u\{\mathrm{OT}_\infty<\mathcal{E}_{m_1,\omega}\}$ by conditioning on whether the occupation time OT_∞ is zero or not and comparing it with $\mathcal{E}_{m_1,\omega}$ which is independent of OT_∞. Note that $\mathbb{P}_u\{\mathrm{OT}_\infty<\mathcal{E}_{m_1,\omega}|\mathrm{OT}_\infty=0\}=1$ since $\mathcal{E}_{m_1,\omega}$ is positive almost surely. Therefore, we arrive at

$$\mathbb{P}_u\{\mathrm{OT}_\infty<\mathcal{E}_{m_1,\omega}\}$$
$$=1-\frac{\mu-R_0}{\mu}e^{-R_0u}+\frac{\mu-R_0}{\mu}e^{-R_0u}\int_0^\infty \mathbb{P}\{\mathcal{E}_{m_1,\omega}>t\}f_{\mathrm{OT}_\infty}(t)\,\mathrm{d}t$$
$$=1-\frac{\mu-R_0}{\mu}e^{-R_0u}+\frac{\mu-R_0}{\mu}e^{-R_0u}\int_0^\infty\left(\sum_{i=0}^{m_1-1}\frac{(\omega t)^i}{i!}e^{-\omega t}\right)f_{\mathrm{OT}_\infty}(t)\,\mathrm{d}t$$
$$=1-\frac{\mu-R_0}{\mu}e^{-R_0u}+\frac{\mu-R_0}{\mu}e^{-R_0u}\int_0^\infty e^{-\omega t}f_{\mathrm{OT}_\infty}(t)\,\mathrm{d}t+\frac{\mu-R_0}{\mu}e^{-R_0u}\sum_{i=1}^{m_1-1}\frac{\omega^i}{i!}\int_0^\infty t^ie^{-\omega t}f_{\mathrm{OT}_\infty}(t)\,\mathrm{d}t$$

$$= \mathbb{E}_u[e^{-\omega \mathrm{OT}_\infty}] + \sum_{i=1}^{m_1-1} \frac{\omega^i}{i!} \mathbb{E}_u[\mathrm{OT}_\infty^i e^{-\omega \mathrm{OT}_\infty}]$$

$$= \sum_{i=0}^{m_1-1} \frac{(-\omega)^i}{i!} \frac{\partial^i}{\partial \omega^i} \mathbb{E}_u[e^{-\omega \mathrm{OT}_\infty}], \tag{3.13}$$

where the second last equality follows from (3.11) and (3.12). Meanwhile, we also have from (3.3) an expression for the Laplace transform of OT_∞ in terms of the root R_ω of the Lundberg's equation, and therefore the above equation becomes

$$\mathbb{P}_u\{\mathrm{OT}_\infty < \mathcal{E}_{m_1,\omega}\} = 1 - \frac{\mu - R_0}{\mu} e^{-R_0 u} + \frac{\mu - R_0}{\mu} e^{-R_0 u} c R_0 \sum_{i=0}^{m_1-1} \frac{(-\omega)^i}{i!} \frac{\mathrm{d}^i}{\mathrm{d}\omega^i} \frac{1}{cR_\omega + \omega}. \tag{3.14}$$

While we adopt the convention that the zeroth derivative is simply the function itself, the higher order derivatives appearing on the right-hand side can be evaluated by applying Faà di Bruno's formula discussed in Section 2.3 with $h_1(\cdot)$ and $h_2(\cdot)$ in place of $g_1(\cdot)$ and $g_2(\cdot)$, where

$$h_1(x) = \frac{1}{x}, \qquad h_2(x) = cR_x + x. \tag{3.15}$$

For convenience, we define $R_x^{(n)} = (\mathrm{d}^n/\mathrm{d}x^n)R_x$. Clearly, one has

$$h_2^{(n)}(x) = \begin{cases} cR_x^{(1)} + 1, & n = 1. \\ cR_x^{(n)}, & n = 2, 3, \ldots. \end{cases} \tag{3.16}$$

Then, according to Faà di Bruno's formula (2.4) we have

$$\frac{\mathrm{d}^i}{\mathrm{d}x^i} h_1(h_2(x)) = \frac{\mathrm{d}^i}{\mathrm{d}x^i} \frac{1}{cR_x + x} = \sum_{j=1}^{i} \frac{(-1)^j (j!)}{(cR_x + x)^{j+1}} B_{ij}(h_2^{(1)}(x), h_2^{(2)}(x) \ldots, h_2^{(i-j+1)}(x)), \qquad i = 1, 2, \ldots, \tag{3.17}$$

where the Bell polynomial $B_{ij}(x_1, \ldots, x_{i-j+1})$ is defined in (2.5). Moreover, the derivatives $R_x^{(n)}$ (for $n = 1, 2, \ldots$) appearing in (3.17) via (3.16) can be calculated recursively in an approach similar to that in Landriault et al. (2014, Appendix). To this end, it is noted that the Lundberg's equation (2.1) can be rewritten as, for $x \geq 0$,

$$R_x = \mu - \mu h_3(h_2(x)), \tag{3.18}$$

where $h_2(x) = cR_x + x$ is the same as before and $h_3(x) = \int_0^\infty e^{-xt} k(t)\,\mathrm{d}t$ so that $h_3^{(n)}(x) = \int_0^\infty (-t)^n e^{-xt} k(t)\,\mathrm{d}t$ for $n = 1, 2, \ldots$. Differentiating both sides of the above equation and solving for $R_x^{(1)}$ leads to

$$R_x^{(1)} = \frac{\mu \int_0^\infty t e^{-(cR_x+x)t} k(t)\,\mathrm{d}t}{1 - c\mu \int_0^\infty t e^{-(cR_x+x)t} k(t)\,\mathrm{d}t}. \tag{3.19}$$

For the higher order derivatives of R_x, we apply Faà di Bruno's formula to $h_3(h_2(x))$ in (3.18) to arrive at

$$R_x^{(n)} = -\mu \sum_{j=1}^{n} h_3^{(j)}(h_2(x)) B_{nj}(h_2^{(1)}(x), h_2^{(2)}(x), \ldots, h_2^{(n-j+1)}(x))$$

$$= -\mu h_3^{(1)}(h_2(x)) h_2^{(n)}(x) - \mu \sum_{j=2}^{n} h_3^{(j)}(h_2(x)) B_{nj}(h_2^{(1)}(x), h_2^{(2)}(x), \ldots, h_2^{(n-j+1)}(x)), \qquad n = 2, 3, \ldots,$$

where the last equality is due to the fact that $B_{n1}(x_1, \ldots, x_n) = x_n$. Because $h_2^{(n)}(x)$ above equals $cR_x^{(n)}$ according to (3.16), we can solve for $R_x^{(n)}$ and it is found that

$$R_x^{(n)} = \frac{-\mu}{1 + c\mu h_3^{(1)}(h_2(x))} \sum_{j=2}^{n} h_3^{(j)}(h_2(x)) B_{nj}(h_2^{(1)}(x), h_2^{(2)}(x), \ldots, h_2^{(n-j+1)}(x))$$

$$= \frac{-\mu}{1 - c\mu \int_0^\infty t e^{-(cR_x+x)t} k(t)\,\mathrm{d}t}$$

$$\times \sum_{j=2}^{n} (-1)^j \left(\int_0^\infty t^j e^{-(cR_x+x)t} k(t)\,\mathrm{d}t \right) B_{nj}(h_2^{(1)}(x), h_2^{(2)}(x), \ldots, h_2^{(n-j+1)}(x)), \qquad n = 2, 3, \ldots. \tag{3.20}$$

To conclude, the ultimate cumulative Parisian ruin probability under an Erlang(m_1, ω) Parisian clock can be computed as follows.

1. Solve the Lundberg's equation (2.1) (with $y = \omega$) to obtain the unique positive root R_ω.

2. Compute the derivatives $R_\omega^{(n)}$ for $n=1,2,\dots,m_1-1$ recursively via (3.20) with the starting point (3.19) (both with $x=\omega$). Note that the dependence of $R_\omega^{(n)}$ on $\{R_\omega^{(i)}\}_{i=1}^{n-1}$ is through the derivatives of $h_2(\omega)$ because of (3.16).
3. Compute the expression in (3.17) for $i=1,2,\dots,m_1-1$ (with $x=\omega$), where the derivatives of $h_2(\omega)$ are given in (3.16).
4. Calculate the ultimate cumulative Parisian survival probability using (3.14), where R_0 is the unique positive root of the Lundberg's equation (2.1) with $y=0$.

While the results concerning deterministic Parisian clock in Section 3.1 require the convolution $k^{*n}(\cdot)$ of the interclaim time, our analysis in the present case of Erlang Parisian clock requires computation of integral of the form $\int_0^\infty t^j e^{-st}k(t)\,dt$ for $s\geq 0$ and $j=1,2,\dots$ as in (3.19) and (3.20). (Note that when $j=0$, an expression for $\int_0^\infty t^j e^{-st}k(t)\,dt=\widetilde{k}(s)$ can be helpful for solving the Lundberg's equation (2.1).) While the evaluation of such an integral can clearly be done analytically for interclaim time distributions such as phase-type and gamma, this also works for distributions such as Weibull and Pareto where closed-form expressions for $k^{*n}(\cdot)$ are not available. Nevertheless, the key formula (3.7) in Section 3.1 for deterministic Parisian clock is still valuable because it can be used to assess the performance of Erlangization for interclaim time distributions such as exponential and Erlang (see Tables 3-4 in Section 5 and the comments therein). We end this section by illustrating how $\int_0^\infty t^j e^{-st}k(t)\,dt$ can be expressed in terms of special functions in the following examples of interclaim time distributions.

Example 2 (**Gamma renewal process**). Suppose that the interclaim times follow a Gamma(α,β) distribution with density

$$k(t)=\frac{\beta^\alpha t^{\alpha-1}e^{-\beta t}}{\Gamma(\alpha)},$$

where $\alpha,\beta>0$ are the parameters and $\Gamma(\cdot)$ is the gamma function. Then it is straightforward to see that

$$\int_0^\infty t^j e^{-st}k(t)\,dt=\frac{\Gamma(\alpha+j)}{\Gamma(\alpha)}\frac{\beta^\alpha}{(\beta+s)^{\alpha+j}},$$

for $s\geq 0$ and $j=0,1,2,\dots$. □

Example 3 (**Pareto renewal process**). In this example, we assume that the interclaim times are Pareto(α,β) distributed with density

$$k(t)=\frac{\alpha\beta^\alpha}{(\beta+t)^{\alpha+1}},$$

where $\alpha,\beta>0$ are the parameters. Then, by a change of variable $x=(\beta+t)/\beta$ and a binomial expansion on $(x-1)^j$, we find that

$$\begin{aligned}\int_0^\infty t^j e^{-st}k(t)\,dt&=\alpha\beta^\alpha\int_1^\infty e^{-s\beta(x-1)}\frac{[\beta(x-1)]^j}{(\beta x)^{\alpha+1}}\beta\,dx\\&=\alpha\beta^j e^{\beta s}\sum_{i=0}^j\binom{j}{i}(-1)^{j-i}\int_1^\infty e^{-\beta sx}\frac{1}{x^{\alpha+1-i}}\,dx\\&=\alpha\beta^j e^{\beta s}\sum_{i=0}^j\binom{j}{i}(-1)^{j-i}\mathrm{E}_{\alpha+1-i}(\beta s),\end{aligned}$$

for $s\geq 0$ and $j=0,1,2,\dots$, where

$$\mathrm{E}_n(z)=\int_1^\infty\frac{e^{-zx}}{x^n}\,dx$$

is the exponential integral function (and n does not necessarily need to be an integer). *Mathematica* has built-in function to compute exponential integrals to arbitrary numerical precision. When evaluating $\mathrm{E}_n(z)$ for negative n, one may express it in terms of the incomplete gamma function as $\mathrm{E}_n(z)=z^{n-1}\Gamma(1-n,z)$, where the incomplete gamma function can also be computed to arbitrary numerical precision by *Mathematica*. □

Example 4 (**Weibull renewal process**). Assume that the interclaim times are Weibull distributed with density

$$k(t)=\frac{\alpha}{\beta}\left(\frac{t}{\beta}\right)^{\alpha-1}e^{-\left(\frac{t}{\beta}\right)^\alpha},$$

where $\alpha,\beta>0$ are the parameters. Performing a change of variable $x=st$, it is seen that

$$\int_0^\infty t^j e^{-st} k(t)\,dt = \frac{\alpha}{\beta}\int_0^\infty \left(\frac{x}{s}\right)^j e^{-x}\left(\frac{x/s}{\beta}\right)^{\alpha-1} e^{-\left(\frac{x/s}{\beta}\right)^\alpha}\frac{1}{s}\,dx$$
$$= \frac{\alpha}{s^{j+\alpha}\beta^\alpha}\int_0^\infty x^{(j+\alpha)-1} e^{-x-\frac{1}{(\beta s)^\alpha}x^\alpha}\,dx,$$

for $s \geq 0$ and $j = 0, 1, 2, \ldots$. If α is rational such that $\alpha = q_1/q_2$ for some positive integers q_1 and q_2, then it is known from Sagias and Karagiannidis (2005, Equation (8)) that the integral above can be expressed in terms of the Meijer G-function such that

$$\int_0^\infty t^j e^{-st} k(t)\,dt = \frac{\alpha}{s^{j+\alpha}\beta^\alpha}\frac{q_1^{j+\alpha}\sqrt{1/\alpha}}{(\sqrt{2\pi})^{q_1+q_2-2}} G^{q_2,q_1}_{q_1,q_2}\left(\frac{q_1^{q_1}}{(\beta s)^{\alpha q_2} q_2^{q_2}}\,\middle|\, \begin{matrix} \frac{1-(j+\alpha)}{q_1} & \frac{2-(j+\alpha)}{q_1} & \ldots & \frac{q_1-(j+\alpha)}{q_1} \\ \frac{0}{q_2} & \frac{1}{q_2} & \ldots & \frac{q_2-1}{q_2}\end{matrix}\right),$$

for $s \geq 0$ and $j = 0, 1, 2, \ldots$. *Mathematica* has built-in function to evaluate the Meijer G-function to arbitrary numerical precision. For given specific parameters, *Mathematica* can also analytically simplify the Meijer G-function to other more well-known special functions such as the confluent hypergeometric function. □

4. Finite-time cumulative Parisian ruin probability

In this section, we will study the cumulative Parisian ruin problem over a finite time horizon. Because finite-time ruin problems are known to be challenging even under the classical definition of ruin, the deterministic time horizon will be replaced by an independent Erlang(m_2, γ) random variable for the entire section. Similar to Section 3, both deterministic and Erlang(m_1, ω) Parisian clocks will be considered.

4.1. Erlang(m_2, γ) time horizon and deterministic Parisian clock r

With the 'finite' time horizon being Erlang(m_2, γ) distributed (denoted by $\mathcal{E}_{m_2,\gamma}$) and the Parisian clock being deterministic at $r > 0$, our goal is to find the 'finite-time' cumulative Parisian ruin probability $\mathbb{P}_u\{\mathrm{OT}_{\mathcal{E}_{m_2,\gamma}} > r\} = \mathbb{P}_u\{\int_0^{\mathcal{E}_{m_2,\gamma}} 1_{\{U(s)<0\}}\,ds > r\}$. This can also be regarded as $\mathbb{E}[\psi_r(u, \mathcal{E}_{m_2,\gamma})]$ by recalling that $\psi_r(u,t)$ is the finite-time cumulative Parisian ruin probability with deterministic time horizon t. If we can determine the distribution of the occupation time $\mathrm{OT}_{\mathcal{E}_{m_2,\gamma}} = \int_0^{\mathcal{E}_{m_2,\gamma}} 1_{\{U(s)<0\}}\,ds$ over an $\mathcal{E}_{m_2,\gamma}$ horizon, then the finite-time cumulative Parisian ruin probability is simply its survival function. Similar to OT_∞ discussed Section 3, $\mathrm{OT}_{\mathcal{E}_{m_2,\gamma}}$ also has a mixed distribution consisting of a probability mass at zero and a density $f_{\mathrm{OT}_{\mathcal{E}_{m_2,\gamma}}}(\cdot|u)$. However, different from Section 3, here $f_{\mathrm{OT}_{\mathcal{E}_{m_2,\gamma}}}(\cdot|u)$ is defined to be a defective density which includes the event $\{\mathrm{OT}_{\mathcal{E}_{m_2,\gamma}} > 0\}$ and it depends on the initial surplus u. Then, with the above definitions the Laplace transform of $\mathrm{OT}_{\mathcal{E}_{m_2,\gamma}}$ admits the representation

$$\mathbb{E}_u[e^{-\omega \mathrm{OT}_{\mathcal{E}_{m_2,\gamma}}}] = \mathbb{P}_u\{\mathrm{OT}_{\mathcal{E}_{m_2,\gamma}} = 0\} + \int_0^\infty e^{-\omega t} f_{\mathrm{OT}_{\mathcal{E}_{m_2,\gamma}}}(t|u)\,dt. \tag{4.1}$$

Now we analyze $\mathbb{E}_u[e^{-\omega \mathrm{OT}_{\mathcal{E}_{m_2,\gamma}}}]$ by conditioning on the Erlang time horizon $\mathcal{E}_{m_2,\gamma}$ to get

$$\mathbb{E}_u[e^{-\omega \mathrm{OT}_{\mathcal{E}_{m_2,\gamma}}}] = \int_0^\infty \mathbb{E}_u[e^{-\omega \mathrm{OT}_t}]\frac{\gamma^{m_2} t^{m_2-1} e^{-\gamma t}}{(m_2-1)!}\,dt$$
$$= \frac{(-1)^{m_2-1}\gamma^{m_2}}{(m_2-1)!}\frac{\partial^{m_2-1}}{\partial\gamma^{m_2-1}}\left(\frac{1}{\gamma}\int_0^\infty \mathbb{E}_u[e^{-\omega \mathrm{OT}_t}]\gamma e^{-\gamma t}\,dt\right)$$
$$= \frac{(-1)^{m_2-1}\gamma^{m_2}}{(m_2-1)!}\frac{\partial^{m_2-1}}{\partial\gamma^{m_2-1}}\left(\frac{1}{\gamma}\mathbb{E}_u[e^{-\omega \mathrm{OT}_{\mathcal{E}_{1,\gamma}}}]\right)$$
$$= \sum_{i=0}^{m_2-1}\frac{(-\gamma)^i}{i!}\frac{\partial^i}{\partial\gamma^i}\mathbb{E}_u[e^{-\omega \mathrm{OT}_{\mathcal{E}_{1,\gamma}}}], \tag{4.2}$$

where the Leibniz's rule for differentiating products has been used. The above result expresses the Laplace transform of the occupation time over an Erlang time horizon in terms of the derivatives of the Laplace transform of the occupation time over an exponential time horizon. Because $\mathbb{E}_u[e^{-\omega \mathrm{OT}_{\mathcal{E}_{1,\gamma}}}]$ on the right-hand side of (4.2) can be written in the form of (4.1) (which is also valid for $m_2 = 1$), (4.2) now becomes

$$\mathbb{E}_u[e^{-\omega \mathrm{OT}_{\mathcal{E}_{m_2,\gamma}}}] = \sum_{i=0}^{m_2-1}\frac{(-\gamma)^i}{i!}\frac{\partial^i}{\partial\gamma^i}\mathbb{P}_u\{\mathrm{OT}_{\mathcal{E}_{1,\gamma}} = 0\} + \int_0^\infty e^{-\omega t}\sum_{i=0}^{m_2-1}\frac{(-\gamma)^i}{i!}\left(\frac{\partial^i}{\partial\gamma^i}f_{\mathrm{OT}_{\mathcal{E}_{1,\gamma}}}(t|u)\right)dt.$$

By the uniqueness of Laplace transforms, comparison of the above equation with (4.1) shows that the point mass of $\mathrm{OT}_{\mathcal{E}_{m_2,\gamma}}$ at zero is

$$\mathbb{P}_u\{\mathrm{OT}_{\mathcal{E}_{m_2,\gamma}}=0\}=\sum_{i=0}^{m_2-1}\frac{(-\gamma)^i}{i!}\frac{\partial^i}{\partial\gamma^i}\mathbb{P}_u\{\mathrm{OT}_{\mathcal{E}_{1,\gamma}}=0\}, \tag{4.3}$$

and the density part is

$$f_{\mathrm{OT}_{\mathcal{E}_{m_2,\gamma}}}(t|u)=\sum_{i=0}^{m_2-1}\frac{(-\gamma)^i}{i!}\frac{\partial^i}{\partial\gamma^i}f_{\mathrm{OT}_{\mathcal{E}_{1,\gamma}}}(t|u), \qquad t>0. \tag{4.4}$$

The above results again connects the distribution of $\mathrm{OT}_{\mathcal{E}_{m_2,\gamma}}$ to that of $\mathrm{OT}_{\mathcal{E}_{1,\gamma}}$.

To find the distribution of $\mathrm{OT}_{\mathcal{E}_{1,\gamma}}$, we start with the point mass $\mathbb{P}_u\{\mathrm{OT}_{\mathcal{E}_{1,\gamma}}=0\}$ which is the probability that classical ruin does not occur within the exponential time horizon $\mathcal{E}_{1,\gamma}$. With the help of (2.2), it is found that

$$\mathbb{P}_u\{\mathrm{OT}_{\mathcal{E}_{1,\gamma}}=0\}=1-\mathbb{P}_u\{\tau_{\mathrm{CL}}\le\mathcal{E}_{1,\gamma}\}=1-\int_0^\infty e^{-\gamma t}\mathbb{P}_u\{\tau_{\mathrm{CL}}\in \mathrm{d}t\}=1-\frac{\mu-R_\gamma}{\mu}e^{-R_\gamma u}. \tag{4.5}$$

For the density $f_{\mathrm{OT}_{\mathcal{E}_{1,\gamma}}}(t|u)$, we rearrange (4.1) (at $m_2=1$) and make use of the above expression and (2.3) to arrive at its Laplace transform

$$\begin{aligned}\int_0^\infty e^{-\omega t}f_{\mathrm{OT}_{\mathcal{E}_{1,\gamma}}}(t|u)\,\mathrm{d}t&=\left(1-\frac{c(R_{\omega+\gamma}-R_\gamma)+\omega}{cR_{\omega+\gamma}+\omega+\gamma}\frac{\mu-R_\gamma}{\mu}e^{-R_\gamma u}\right)-\left(1-\frac{\mu-R_\gamma}{\mu}e^{-R_\gamma u}\right)\\&=\frac{\mu-R_\gamma}{\mu}e^{-R_\gamma u}\frac{cR_\gamma+\gamma}{cR_{\omega+\gamma}+\omega+\gamma}\\&=\frac{\mu-R_\gamma}{\mu}e^{-R_\gamma u}(cR_\gamma+\gamma)\int_0^\infty e^{-(cR_{\omega+\gamma}+\omega+\gamma)t}\,\mathrm{d}t.\end{aligned}$$

We shall perform Laplace transform inversion with respect to ω to get $f_{\mathrm{OT}_{\mathcal{E}_{1,\gamma}}}(t|u)$. It is instructive to note that the integral on the right-hand side is structurally identical to that in (3.4) except that $\omega+\gamma$ is in place of ω. Hence, following the same procedure leading to (3.6) gives rise to

$$f_{\mathrm{OT}_{\mathcal{E}_{1,\gamma}}}(t|u)=\frac{\mu-R_\gamma}{\mu}e^{-R_\gamma u}(cR_\gamma+\gamma)e^{-\gamma t}\left[e^{-c\mu t}\left(1+\sum_{n=1}^\infty\frac{(c\mu)^n}{n!}t^{n-1}\int_0^t zk^{*n}(t-z)\,\mathrm{d}z\right)\right], \qquad t>0. \tag{4.6}$$

When differentiating (4.6) with respect to γ to obtain $f_{\mathrm{OT}_{\mathcal{E}_{m_2,\gamma}}}(t|u)$ via (4.4), we note that the components inside the square bracket on the right-hand side of (4.6) do not depend on γ, and the derivatives of the component outside the bracket with respect to γ can be evaluated by Leibniz's differentiation rule for products. Thus, we rewrite (4.6) as

$$f_{\mathrm{OT}_{\mathcal{E}_{1,\gamma}}}(t|u)=h_2(\gamma)h_4(h_5(\gamma)|u)h_6(\gamma|t)\left[e^{-c\mu t}\left(1+\sum_{n=1}^\infty\frac{(c\mu)^n}{n!}t^{n-1}\int_0^t zk^{*n}(t-z)\,\mathrm{d}z\right)\right], \qquad t>0, \tag{4.7}$$

where

$$h_2(x)=cR_x+x, \qquad h_4(x|u)=\frac{\mu-x}{\mu}e^{-xu}, \qquad h_5(x)=R_x, \qquad h_6(x|t)=e^{-xt}. \tag{4.8}$$

Note that the definition of $h_2(x)$ is the same as that in (3.15) and its derivatives are already given in (3.16), where the derivatives $h_5^{(n)}(x)=R_x^{(n)}$ for $n=1,2,\ldots$ are available in (3.19) and (3.20). It is straightforward to find that

$$\frac{\partial^j}{\partial x^j}h_4(x|u)=(-1)^ju^{j-1}e^{-xu}\left(u+\frac{j-xu}{\mu}\right), \qquad \frac{\partial^j}{\partial x^j}h_6(x|t)=(-t)^je^{-xt}, \qquad j=0,1,2,\ldots.$$

Faà di Bruno's formula (2.4) implies

$$\frac{\partial^n}{\partial\gamma^n}h_4(h_5(\gamma)|u)=\sum_{j=1}^n(-1)^ju^{j-1}e^{-R_\gamma u}\left(u+\frac{j-R_\gamma u}{\mu}\right)B_{nj}(R_\gamma^{(1)},R_\gamma^{(2)},\ldots,R_\gamma^{(n-j+1)}), \qquad n=1,2,\ldots. \tag{4.9}$$

Consolidating the above results, Leibniz's differentiation rule leads us to

$$\frac{\partial^i}{\partial\gamma^i}[h_2(\gamma)h_4(h_5(\gamma)|u)h_6(\gamma|t)]=\sum_{j=0}^i\sum_{\ell=0}^{i-j}\frac{i!}{j!\,\ell!\,(i-j-\ell)!}h_2^{(j)}(\gamma)\left(\frac{\partial^{i-j-\ell}}{\partial\gamma^{i-j-\ell}}h_4(h_5(\gamma)|u)\right)(-t)^\ell e^{-\gamma t}, \qquad i=0,1,2,\ldots,$$

and therefore substitution of (4.7) into (4.4) yields

$$f_{\mathrm{OT}_{\mathcal{E}_{m_2,\gamma}}}(t|u) = \sum_{i=0}^{m_2-1}\sum_{j=0}^{i}\sum_{\ell=0}^{i-j}\frac{(-\gamma)^i(-1)^\ell}{j!\,\ell!\,(i-j-\ell)!}h_2^{(j)}(\gamma)\left(\frac{\partial^{i-j-\ell}}{\partial\gamma^{i-j-\ell}}h_4(h_5(\gamma)|u)\right)$$
$$\times t^\ell e^{-\gamma t}e^{-c\mu t}\left(1+\sum_{n=1}^{\infty}\frac{(c\mu)^n}{n!}t^{n-1}\int_0^t zk^{*n}(t-z)\,\mathrm{d}z\right), \qquad t>0.$$

Finally, similar to (3.7), the finite-time cumulative Parisian ruin probability $\mathbb{P}_u\{\mathrm{OT}_{\mathcal{E}_{m_2,\gamma}} > r\}$ under deterministic clock $r>0$ and Erlang time horizon $\mathcal{E}_{m_2,\gamma}$ is given by

$$\mathbb{P}_u\{\mathrm{OT}_{\mathcal{E}_{m_2,\gamma}} > r\} = \sum_{i=0}^{m_2-1}\sum_{j=0}^{i}\sum_{\ell=0}^{i-j}\frac{(-\gamma)^i(-1)^\ell}{j!\,\ell!\,(i-j-\ell)!}h_2^{(j)}(\gamma)\left(\frac{\partial^{i-j-\ell}}{\partial\gamma^{i-j-\ell}}h_4(h_5(\gamma)|u)\right)$$
$$\times\int_r^\infty t^\ell e^{-\gamma t}e^{-c\mu t}\left(1+\sum_{n=1}^{\infty}\frac{(c\mu)^n}{n!}t^{n-1}\int_0^t zk^{*n}(t-z)\,\mathrm{d}z\right)\mathrm{d}t, \tag{4.10}$$

where the integral has been evaluated in (3.9) (or (3.10)) for the Erlang(m) risk process.

We remark that, although the point mass $\mathbb{P}_u\{\mathrm{OT}_{\mathcal{E}_{m_2,\gamma}} = 0\}$ is not needed in the calculation of the ruin probability $\mathbb{P}_u\{\mathrm{OT}_{\mathcal{E}_{m_2,\gamma}} > r\}$ and can indeed be obtained as $1-\mathbb{P}_u\{\mathrm{OT}_{\mathcal{E}_{m_2,\gamma}} > 0\}$ by putting $r=0$ above, it can readily be expressed in terms of simpler quantities. To see this, we can substitute (4.5) into (4.3) to get

$$\mathbb{P}_u\{\mathrm{OT}_{\mathcal{E}_{m_2,\gamma}} = 0\} = 1-\frac{\mu-R_\gamma}{\mu}e^{-R_\gamma u}-\sum_{i=1}^{m_2-1}\frac{(-\gamma)^i}{i!}\frac{\partial^i}{\partial\gamma^i}h_4(h_5(\gamma)|u).$$

4.2. Erlang(m_2,γ) time horizon and Erlang(m_1,ω) Parisian clock

Similar to (3.7), we need the convolution $k^{*n}(\cdot)$ when applying (4.10) to calculate $\mathbb{P}_u\{\mathrm{OT}_{\mathcal{E}_{m_2,\gamma}} > r\}$ under deterministic Parisian clock. In order to obtain approximation formulas that are applicable to more general interclaim time distributions, in this subsection both the Parisian clock and the finite time horizon are assumed to be Erlang distributed with the corresponding random variables denoted by $\mathcal{E}_{m_1,\omega}$ and $\mathcal{E}_{m_2,\gamma}$ respectively. The surplus process $\{U(t)\}_{t\ge0}$ and these variables are mutually independent. The finite-time cumulative Parisian ruin probability is thus $\mathbb{P}_u\{\mathrm{OT}_{\mathcal{E}_{m_2,\gamma}} > \mathcal{E}_{m_1,\omega}\} = \mathbb{P}_u\{\int_0^{\mathcal{E}_{m_2,\gamma}} 1_{\{U(s)<0\}}\,\mathrm{d}s > \mathcal{E}_{m_1,\omega}\}$, which may be written as $\mathbb{E}[\psi_{\mathcal{E}_{m_1,\omega}}(u,\mathcal{E}_{m_2,\gamma})]$. First, we recall from Section 3.2 that when the Parisian clock is Erlang, the ultimate cumulative Parisian survival probability can be expressed in terms of the derivatives of the Laplace transform of the occupation time OT_∞ as in (3.13). The same connection can also be established for the finite-time cumulative Parisian survival probability $\mathbb{P}_u\{\mathrm{OT}_{\mathcal{E}_{m_2,\gamma}} < \mathcal{E}_{m_1,\omega}\}$. We omit the rather repetitive details and state that

$$\mathbb{P}_u\{\mathrm{OT}_{\mathcal{E}_{m_2,\gamma}} < \mathcal{E}_{m_1,\omega}\} = \sum_{i=0}^{m_1-1}\frac{(-\omega)^i}{i!}\frac{\partial^i}{\partial\omega^i}\mathbb{E}_u[e^{-\omega\mathrm{OT}_{\mathcal{E}_{m_2,\gamma}}}], \tag{4.11}$$

which is in terms of the Laplace transform of $\mathrm{OT}_{\mathcal{E}_{m_2,\gamma}}$ studied in Section 4.1. Substitution of (2.3) into (4.2) gives rise to

$$\mathbb{E}_u[e^{-\omega\mathrm{OT}_{\mathcal{E}_{m_2,\gamma}}}] = \sum_{j=0}^{m_2-1}\frac{(-\gamma)^j}{j!}\frac{\partial^j}{\partial\gamma^j}\left(1-\frac{c(R_{\omega+\gamma}-R_\gamma)+\omega}{cR_{\omega+\gamma}+\omega+\gamma}\frac{\mu-R_\gamma}{\mu}e^{-R_\gamma u}\right)$$
$$=1-\sum_{j=0}^{m_2-1}\frac{(-\gamma)^j}{j!}\frac{\partial^j}{\partial\gamma^j}\left(\frac{\mu-R_\gamma}{\mu}e^{-R_\gamma u}\right)+\sum_{j=0}^{m_2-1}\frac{(-\gamma)^j}{j!}\frac{\partial^j}{\partial\gamma^j}\left(\frac{cR_\gamma+\gamma}{cR_{\omega+\gamma}+\omega+\gamma}\frac{\mu-R_\gamma}{\mu}e^{-R_\gamma u}\right).$$

Since the first summation does not depend on ω, it will disappear upon taking derivatives with respect to ω. Thus, we substitute the above equation into (4.11) to yield

$$\mathbb{P}_u\{\mathrm{OT}_{\mathcal{E}_{m_2,\gamma}} < \mathcal{E}_{m_1,\omega}\}$$
$$=1-\sum_{j=0}^{m_2-1}\frac{(-\gamma)^j}{j!}\frac{\partial^j}{\partial\gamma^j}\left(\frac{\mu-R_\gamma}{\mu}e^{-R_\gamma u}\right)+\sum_{i=0}^{m_1-1}\sum_{j=0}^{m_2-1}\frac{(-\omega)^i}{i!}\frac{(-\gamma)^j}{j!}\frac{\partial^i}{\partial\omega^i}\frac{\partial^j}{\partial\gamma^j}\left(\frac{cR_\gamma+\gamma}{cR_{\omega+\gamma}+\omega+\gamma}\frac{\mu-R_\gamma}{\mu}e^{-R_\gamma u}\right)$$
$$=1-\sum_{j=0}^{m_2-1}\frac{(-\gamma)^j}{j!}\frac{\partial^j}{\partial\gamma^j}h_4(h_5(\gamma)|u)+\sum_{i=0}^{m_1-1}\sum_{j=0}^{m_2-1}\frac{(-\omega)^i}{i!}\frac{(-\gamma)^j}{j!}\frac{\partial^j}{\partial\gamma^j}\left(h_2(\gamma)h_4(h_5(\gamma)|u)\frac{\partial^i}{\partial\omega^i}h_1(h_2(\omega+\gamma))\right), \tag{4.12}$$

where the definitions (3.15) and (4.8) have been used. Note that the derivatives of $h_2(\cdot)$, $h_1(h_2(\cdot))$ and $h_4(h_5(\cdot)|u)$ have already been identified in (3.16), (3.17) and (4.9) (with the help of (3.19) and (3.20)). Moreover, concerning $h_1(h_2(\cdot))$, the two arguments ω and γ always appear together so that

Table 1
Examples of interclaim time distributions and their variances.

Interclaim time distribution	Variance
Weilbull($\alpha = 2, \beta = 2/\sqrt{\pi} = 1.128379$)	0.273240
Erlang(2) with $\beta = 2$	0.5
Exponential with mean 1	1
Pareto($\alpha = 3, \beta = 2$)	3
Gamma($\alpha = 1/3, \beta = 1/3$)	3
Pareto($\alpha = 5/2, \beta = 3/2$)	5
Weilbull($\alpha = 1/2, \beta = 1/2$)	5
Gamma($\alpha = 1/5, \beta = 1/5$)	5

Table 2
Ultimate cumulative Parisian ruin probability for Weibull interclaim times with variance 0.273240.

m_1	$\mathbb{E}[\mathcal{E}_{m_1,\omega}] = m_1/\omega = 1$				$\mathbb{E}[\mathcal{E}_{m_1,\omega}] = m_1/\omega = 5$				$\mathbb{E}[\mathcal{E}_{m_1,\omega}] = m_1/\omega = 10$			
	$u = 0$	$u = 5$	$u = 10$	$u = 20$	$u = 0$	$u = 5$	$u = 10$	$u = 20$	$u = 0$	$u = 5$	$u = 10$	$u = 20$
1	0.704272	0.260553	0.096394	0.013194	0.566811	0.209698	0.077580	0.010618	0.482301	0.178432	0.066013	0.009035
5	0.691421	0.255799	0.094635	0.012953	0.534500	0.197744	0.073158	0.010013	0.437776	0.161960	0.059919	0.008201
10	0.689394	0.255049	0.094358	0.012915	0.530001	0.196080	0.072542	0.009929	0.431640	0.159690	0.059079	0.008086
20	0.688340	0.254659	0.094214	0.012895	0.527725	0.195238	0.072230	0.009886	0.428539	0.158543	0.058655	0.008028
30	0.687983	0.254527	0.094165	0.012888	0.526963	0.194956	0.072126	0.009872	0.427501	0.158159	0.058513	0.008009
40	0.687803	0.254460	0.094140	0.012885	0.526581	0.194815	0.072074	0.009865	0.426981	0.157966	0.058441	0.007999
50	0.687695	0.254420	0.094126	0.012883	0.526352	0.194730	0.072042	0.009861	0.426669	0.157851	0.058399	0.007993
60	0.687623	0.254394	0.094116	0.012882	0.526199	0.194673	0.072021	0.009858	0.426461	0.157774	0.058370	0.007989
70	0.687571	0.254374	0.094109	0.012881	0.526090	0.194633	0.072007	0.009856	0.426312	0.157719	0.058350	0.007986
80	0.687532	0.254360	0.094103	0.012880	0.526008	0.194602	0.071995	0.009854	0.426201	0.157678	0.058335	0.007984

Table 3
Ultimate cumulative Parisian ruin probability for Erlang interclaim times with variance 0.5.

m_1	$\mathbb{E}[\mathcal{E}_{m_1,\omega}] = m_1/\omega = 1$				$\mathbb{E}[\mathcal{E}_{m_1,\omega}] = m_1/\omega = 5$				$\mathbb{E}[\mathcal{E}_{m_1,\omega}] = m_1/\omega = 10$			
	$u = 0$	$u = 5$	$u = 10$	$u = 20$	$u = 0$	$u = 5$	$u = 10$	$u = 20$	$u = 0$	$u = 5$	$u = 10$	$u = 20$
1	0.739956	0.314322	0.133519	0.024092	0.606748	0.257737	0.109483	0.019755	0.522642	0.222010	0.094306	0.017017
5	0.728762	0.309567	0.131499	0.023728	0.576244	0.244779	0.103978	0.018762	0.479858	0.203836	0.086586	0.015624
10	0.727010	0.308822	0.131183	0.023671	0.571943	0.242952	0.103202	0.018622	0.473919	0.201313	0.085515	0.015430
20	0.726100	0.308436	0.131019	0.023641	0.569764	0.242027	0.102809	0.018551	0.470915	0.200037	0.084972	0.015333
30	0.725792	0.308305	0.130963	0.023631	0.569034	0.241717	0.102677	0.018527	0.469909	0.199610	0.084791	0.015300
40	0.725637	0.308239	0.130935	0.023626	0.568669	0.241561	0.102611	0.018515	0.469405	0.199396	0.084700	0.015283
50	0.725544	0.308199	0.130918	0.023623	0.568449	0.241468	0.102572	0.018508	0.469102	0.199267	0.084645	0.015274
60	0.725481	0.308173	0.130907	0.023621	0.568303	0.241406	0.102545	0.018503	0.468900	0.199181	0.084609	0.015267
70	0.725437	0.308154	0.130899	0.023620	0.568198	0.241361	0.102526	0.018500	0.468756	0.199120	0.084583	0.015262
80	0.725403	0.308140	0.130893	0.023618	0.568119	0.241328	0.102512	0.018497	0.468648	0.199074	0.084563	0.015259
Exact	0.725168	0.308040	0.130850	0.023611	0.567569	0.241094	0.102413	0.018480	0.467890	0.198752	0.084427	0.015234

$$\frac{\partial^{i+j}}{\partial\omega^i\partial\gamma^j}h_1(h_2(\omega+\gamma)) = \frac{\mathrm{d}^{i+j}}{\mathrm{d}x^{i+j}}h_1(h_2(x))\bigg|_{x=\omega+\gamma}, \qquad i, j = 0, 1, 2, \ldots.$$

Then, applying the Leibniz's rule to (4.12) results in

$$\begin{aligned}
&\mathbb{P}_u\{\mathrm{OT}_{\mathcal{E}_{m_2,\gamma}} < \mathcal{E}_{m_1,\omega}\} \\
&= 1 - \sum_{j=0}^{m_2-1}\frac{(-\gamma)^j}{j!}\frac{\partial^j}{\partial\gamma^j}h_4(h_5(\gamma)|u) \\
&\quad + \sum_{i=0}^{m_1-1}\sum_{j=0}^{m_2-1}\sum_{\ell=0}^{j}\sum_{n=0}^{j-\ell}\frac{(-\omega)^i(-\gamma)^j}{i!\,\ell!\,n!\,(j-\ell-n)!}h_2^{(\ell)}(\gamma)\left(\frac{\partial^n}{\partial\gamma^n}h_4(h_5(\gamma)|u)\right)\frac{\mathrm{d}^{i+j-\ell-n}}{\mathrm{d}x^{i+j-\ell-n}}h_1(h_2(x))\bigg|_{x=\omega+\gamma},
\end{aligned} \tag{4.13}$$

which can be used to compute the finite-time cumulative Parisian survival probability.

5. Numerical illustrations

In this section, the theoretical results from the previous sections are applied to illustrate the impact of the interclaim time distributions on the cumulative Parisian ruin probabilities. To demonstrate the versatility of our approach, we consider eight interclaim distributions which have the same mean of 1 but possibly possess different variances. The parameters and the variances of these distributions are summarized in Table 1 in increasing (i.e. non-decreasing) order of the variance. In all numerical examples, the claim amounts are exponentially distributed with parameter $\mu = 1$ (and hence mean 1) and the premium rate is $c = 1.15$ (so that the relative security loading is 15%).

Tables 2-9 show the ultimate cumulative Parisian ruin probabilities for the eight interclaim time distributions described in Table 1. In each table, we follow Section 3.2 to compute the values of $\mathbb{P}_u\{\mathrm{OT}_\infty > \mathcal{E}_{m_1,\omega}\}$ under initial surplus levels $u = 0, 5, 10, 20$ and Erlang

Table 4
Ultimate cumulative Parisian ruin probability for exponential interclaim times with variance 1.

m_1	$\mathbb{E}[\mathcal{E}_{m_1,\omega}]=m_1/\omega=1$				$\mathbb{E}[\mathcal{E}_{m_1,\omega}]=m_1/\omega=5$				$\mathbb{E}[\mathcal{E}_{m_1,\omega}]=m_1/\omega=10$			
	$u=0$	$u=5$	$u=10$	$u=20$	$u=0$	$u=5$	$u=10$	$u=20$	$u=0$	$u=5$	$u=10$	$u=20$
1	0.794105	0.413659	0.215480	0.058470	0.670499	0.349271	0.181940	0.049369	0.589082	0.306860	0.159847	0.043374
5	0.785347	0.409097	0.213104	0.057826	0.643461	0.335187	0.174603	0.047378	0.550057	0.286531	0.149258	0.040501
10	0.783979	0.408384	0.212732	0.057725	0.639578	0.333164	0.173549	0.047092	0.544576	0.283676	0.147771	0.040097
20	0.783267	0.408013	0.212539	0.057672	0.637606	0.332136	0.173014	0.046947	0.541800	0.282230	0.147017	0.039893
30	0.783026	0.407888	0.212474	0.057655	0.636944	0.331792	0.172834	0.046899	0.540870	0.281746	0.146765	0.039825
40	0.782904	0.407824	0.212441	0.057646	0.636612	0.331619	0.172744	0.046874	0.540404	0.281503	0.146638	0.039790
50	0.782831	0.407786	0.212421	0.057640	0.636413	0.331515	0.172690	0.046859	0.540124	0.281357	0.146562	0.039770
60	0.782782	0.407761	0.212407	0.057637	0.636280	0.331446	0.172654	0.046850	0.539937	0.281260	0.146512	0.039756
70	0.782747	0.407742	0.212398	0.057634	0.636185	0.331396	0.172628	0.046843	0.539804	0.281190	0.146475	0.039746
80	0.782721	0.407729	0.212391	0.057632	0.636114	0.331359	0.172609	0.046837	0.539703	0.281138	0.146448	0.039739
Exact	0.782536	0.407633	0.212341	0.057619	0.635614	0.331099	0.172474	0.046801	0.539002	0.280773	0.146258	0.039687

Table 5
Ultimate cumulative Parisian ruin probability for Pareto interclaim times with variance 3.

m_1	$\mathbb{E}[\mathcal{E}_{m_1,\omega}]=m_1/\omega=1$				$\mathbb{E}[\mathcal{E}_{m_1,\omega}]=m_1/\omega=5$				$\mathbb{E}[\mathcal{E}_{m_1,\omega}]=m_1/\omega=10$			
	$u=0$	$u=5$	$u=10$	$u=20$	$u=0$	$u=5$	$u=10$	$u=20$	$u=0$	$u=5$	$u=10$	$u=20$
1	0.869715	0.585269	0.393852	0.178357	0.774734	0.521351	0.350840	0.158879	0.706138	0.475190	0.319776	0.144811
5	0.864549	0.581792	0.391513	0.177297	0.756468	0.509060	0.342568	0.155133	0.677913	0.456197	0.306994	0.139023
10	0.863754	0.581257	0.391153	0.177134	0.753802	0.507266	0.341361	0.154586	0.673865	0.453473	0.305161	0.138193
20	0.863341	0.580979	0.390966	0.177050	0.752441	0.506350	0.340744	0.154307	0.671804	0.452086	0.304228	0.137770
30	0.863202	0.580885	0.390903	0.177021	0.751983	0.506042	0.340537	0.154213	0.671112	0.451620	0.303914	0.137628
40	0.863131	0.580838	0.390871	0.177007	0.751753	0.505887	0.340433	0.154166	0.670764	0.451386	0.303757	0.137557
50	0.863089	0.580810	0.390852	0.176998	0.751615	0.505794	0.340371	0.154138	0.670556	0.451246	0.303663	0.137514
60	0.863061	0.580791	0.390839	0.176992	0.751523	0.505732	0.340329	0.154119	0.670417	0.451152	0.303600	0.137486
70	0.863040	0.580777	0.390830	0.176988	0.751457	0.505688	0.340299	0.154105	0.670317	0.451085	0.303555	0.137465
80	0.863025	0.580767	0.390823	0.176985	0.751408	0.505655	0.340277	0.154095	0.670243	0.451035	0.303521	0.137450

Table 6
Ultimate cumulative Parisian ruin probability for gamma interclaim times with variance 3.

m_1	$\mathbb{E}[\mathcal{E}_{m_1,\omega}]=m_1/\omega=1$				$\mathbb{E}[\mathcal{E}_{m_1,\omega}]=m_1/\omega=5$				$\mathbb{E}[\mathcal{E}_{m_1,\omega}]=m_1/\omega=10$			
	$u=0$	$u=5$	$u=10$	$u=20$	$u=0$	$u=5$	$u=10$	$u=20$	$u=0$	$u=5$	$u=10$	$u=20$
1	0.885431	0.634164	0.454201	0.232993	0.791110	0.566609	0.405817	0.208173	0.722835	0.517709	0.370794	0.190207
5	0.880995	0.630987	0.451926	0.231825	0.772921	0.553582	0.396487	0.203387	0.694475	0.497397	0.356246	0.182744
10	0.880313	0.630498	0.451576	0.231646	0.770211	0.551641	0.395097	0.202673	0.690370	0.494457	0.354140	0.181664
20	0.879958	0.630244	0.451394	0.231552	0.768821	0.550645	0.394384	0.202308	0.688278	0.492959	0.353067	0.181114
30	0.879838	0.630158	0.451332	0.231521	0.768353	0.550310	0.394143	0.202184	0.687576	0.492456	0.352707	0.180929
40	0.879777	0.630114	0.451301	0.231505	0.768118	0.550142	0.394023	0.202123	0.687224	0.492204	0.352527	0.180836
50	0.879741	0.630088	0.451282	0.231495	0.767976	0.550040	0.393950	0.202085	0.687012	0.492052	0.352418	0.180781
60	0.879716	0.630071	0.451270	0.231489	0.767882	0.549973	0.393902	0.202061	0.686871	0.491951	0.352346	0.180743
70	0.879699	0.630058	0.451261	0.231484	0.767815	0.549925	0.393867	0.202043	0.686770	0.491879	0.352294	0.180717
80	0.879686	0.630049	0.451254	0.231481	0.767764	0.549888	0.393841	0.202030	0.686695	0.491825	0.352255	0.180697

Table 7
Ultimate cumulative Parisian ruin probability for Pareto interclaim times with variance 5.

m_1	$\mathbb{E}[\mathcal{E}_{m_1,\omega}]=m_1/\omega=1$				$\mathbb{E}[\mathcal{E}_{m_1,\omega}]=m_1/\omega=5$				$\mathbb{E}[\mathcal{E}_{m_1,\omega}]=m_1/\omega=10$			
	$u=0$	$u=5$	$u=10$	$u=20$	$u=0$	$u=5$	$u=10$	$u=20$	$u=0$	$u=5$	$u=10$	$u=20$
1	0.890912	0.641871	0.462446	0.240042	0.806956	0.581384	0.418867	0.217422	0.744514	0.536397	0.386455	0.200598
5	0.886692	0.638831	0.460256	0.238905	0.791552	0.570286	0.410871	0.213271	0.720256	0.518920	0.373864	0.194062
10	0.886044	0.638364	0.459919	0.238731	0.789299	0.568663	0.409702	0.212664	0.716763	0.516403	0.372051	0.193120
20	0.885709	0.638122	0.459745	0.238640	0.788147	0.567833	0.409104	0.212354	0.714982	0.515120	0.371126	0.192641
30	0.885595	0.638040	0.459686	0.238609	0.787759	0.567553	0.408903	0.212249	0.714384	0.514689	0.370816	0.192479
40	0.885538	0.637999	0.459656	0.238594	0.787565	0.567413	0.408802	0.212197	0.714084	0.514473	0.370660	0.192399
50	0.885503	0.637975	0.459639	0.238585	0.787448	0.567329	0.408741	0.212165	0.713904	0.514343	0.370566	0.192350
60	0.885480	0.637958	0.459627	0.238579	0.787370	0.567273	0.408700	0.212144	0.713783	0.514256	0.370504	0.192318
70	0.885464	0.637946	0.459618	0.238574	0.787314	0.567233	0.408672	0.212129	0.713697	0.514194	0.370459	0.192294
80	0.885452	0.637937	0.459612	0.238571	0.787272	0.567203	0.408650	0.212118	0.713633	0.514148	0.370426	0.192277

Parisian clocks with means $\mathbb{E}[\mathcal{E}_{m_1,\omega}]=1,5,10$. Moreover, Erlangization is performed by increasing m_1 to up to 80 down each column while keeping the mean $\mathbb{E}[\mathcal{E}_{m_1,\omega}]=m_1/\omega$ fixed. Tables 3-4 further contain an additional row, labelled as 'Exact', which is concerned with $\psi_r(u)=\mathbb{P}_u\{\mathrm{OT}_\infty>r\}$ under deterministic clocks $r=1,5,10$ (see Section 3.1). These values are obtained using the results in Example 1 for the class of Erlang renewal processes (which contains compound Poisson processes).

We begin by examining Table 4 regarding exponential interclaim times, as this situation corresponds to the standard compound Poisson case. First, it can be seen from each row that the ultimate cumulative Parisian ruin probability decreases in u for each fixed $\mathbb{E}[\mathcal{E}_{m_1,\omega}]$ (or fixed r in the final row). This is because a higher initial surplus keeps the surplus process further away from level zero, thereby

Table 8
Ultimate cumulative Parisian ruin probability for Weibull interclaim times with variance 5.

m_1	$\mathbb{E}[\mathcal{E}_{m_1,\omega}]=m_1/\omega=1$				$\mathbb{E}[\mathcal{E}_{m_1,\omega}]=m_1/\omega=5$				$\mathbb{E}[\mathcal{E}_{m_1,\omega}]=m_1/\omega=10$			
	$u=0$	$u=5$	$u=10$	$u=20$	$u=0$	$u=5$	$u=10$	$u=20$	$u=0$	$u=5$	$u=10$	$u=20$
1	0.917311	0.724059	0.571519	0.356078	0.841969	0.664589	0.524578	0.326832	0.784052	0.618874	0.488494	0.304350
5	0.914364	0.721733	0.569683	0.354934	0.828895	0.654269	0.516432	0.321757	0.762553	0.601904	0.475099	0.296004
10	0.913916	0.721379	0.569404	0.354760	0.826940	0.652726	0.515215	0.320998	0.759401	0.599416	0.473135	0.294781
20	0.913683	0.721195	0.569259	0.354669	0.825936	0.651934	0.514589	0.320608	0.757789	0.598144	0.472131	0.294155
30	0.913604	0.721133	0.569210	0.354639	0.825597	0.651666	0.514378	0.320477	0.757247	0.597716	0.471793	0.293945
40	0.913565	0.721101	0.569185	0.354623	0.825427	0.651532	0.514272	0.320410	0.756975	0.597501	0.471624	0.293839
50	0.913541	0.721082	0.569170	0.354614	0.825324	0.651451	0.514208	0.320371	0.756811	0.597372	0.471522	0.293776
60	0.913525	0.721070	0.569160	0.354608	0.825256	0.651397	0.514165	0.320344	0.756702	0.597285	0.471454	0.293733
70	0.913513	0.721061	0.569153	0.354604	0.825207	0.651358	0.514135	0.320325	0.756624	0.597224	0.471405	0.293703
80	0.913505	0.721054	0.569148	0.354600	0.825171	0.651330	0.514112	0.320311	0.756565	0.597178	0.471369	0.293680

Table 9
Ultimate cumulative Parisian ruin probability for gamma interclaim times with variance 5.

m_1	$\mathbb{E}[\mathcal{E}_{m_1,\omega}]=m_1/\omega=1$				$\mathbb{E}[\mathcal{E}_{m_1,\omega}]=m_1/\omega=5$				$\mathbb{E}[\mathcal{E}_{m_1,\omega}]=m_1/\omega=10$			
	$u=0$	$u=5$	$u=10$	$u=20$	$u=0$	$u=5$	$u=10$	$u=20$	$u=0$	$u=5$	$u=10$	$u=20$
1	0.919719	0.734966	0.587325	0.375061	0.842720	0.673434	0.538154	0.343661	0.784001	0.626510	0.500657	0.319715
5	0.916900	0.732713	0.585525	0.373911	0.829115	0.662562	0.529466	0.338113	0.761659	0.608656	0.486389	0.310604
10	0.916472	0.732371	0.585252	0.373737	0.827057	0.660917	0.528152	0.337274	0.758367	0.606026	0.484287	0.309262
20	0.916250	0.732193	0.585110	0.373647	0.825996	0.660070	0.527474	0.336841	0.756683	0.604680	0.483212	0.308575
30	0.916175	0.732133	0.585062	0.373616	0.825638	0.659783	0.527246	0.336695	0.756117	0.604228	0.482850	0.308344
40	0.916137	0.732103	0.585038	0.373600	0.825458	0.659640	0.527131	0.336621	0.755832	0.604000	0.482668	0.308228
50	0.916114	0.732085	0.585023	0.373591	0.825350	0.659553	0.527062	0.336577	0.755662	0.603864	0.482559	0.308159
60	0.916099	0.732073	0.585014	0.373585	0.825277	0.659495	0.527015	0.336548	0.755548	0.603773	0.482487	0.308112
70	0.916088	0.732064	0.585007	0.373580	0.825226	0.659454	0.526982	0.336527	0.755466	0.603708	0.482435	0.308079
80	0.916080	0.732057	0.585001	0.373577	0.825187	0.659423	0.526958	0.336511	0.755405	0.603659	0.482395	0.308054

decreasing the occupation time in the red. Second, for fixed u, the ruin probability in each row decreases in the mean $\mathbb{E}[\mathcal{E}_{m_1,\omega}]$ of the Erlang Parisian clock (or decreases in r in the case of deterministic Parisian clock). This is also expected since it is less likely for the occupation time OT_∞ to exceed the clock when the clock is longer. Third, we look at the performance of Erlangization by moving down each column of Table 4. As m_1 increases, the ruin probability under an Erlang Parisian clock approaches that under a deterministic clock. We notice that Erlangization produces very good performance even for moderate values of m_1. We have indeed separately computed the percentage error by comparing the Erlangian approximation with the corresponding exact value in the last row. While the table of percentage errors is omitted here for brevity, we would like to report that (i) for fixed pair of (m_1,ω), the percentage errors are almost identical across different initial surplus levels; (ii) for fixed m_1, the percentage error increases with the mean $\mathbb{E}[\mathcal{E}_{m_1,\omega}]=m_1/\omega$ (or equivalently, decreases with ω). The latter observation is also evident in the Erlangization demonstration in Cheung and Wong (2017), and similar explanations are applicable and given as follows. Recall that the idea of Erlangization is to successively reduce the variance of the Erlang distribution to mimic the deterministic clock via increasing m_1. Note that the variance of the Erlang Parisian clock is $\mathrm{Var}(\mathcal{E}_{m_1,\omega})=m_1/\omega^2=(\mathbb{E}[\mathcal{E}_{m_1,\omega}])^2/m_1$. If $\mathbb{E}[\mathcal{E}_{m_1,\omega}]$ is large, then a larger value of m_1 is required to keep the variance and hence the percentage error small. In particular, when $\mathbb{E}[\mathcal{E}_{m_1,\omega}]=1$ the use of $m_1=2$ leads to a percentage error of 0.831% (i.e. less than 1%) while for $\mathbb{E}[\mathcal{E}_{m_1,\omega}]=10$ we require $m_1=10$ to give a comparable percentage error of 1.034%. For further reference, as m_1 increases from 1 to 80, when $\mathbb{E}[\mathcal{E}_{m_1,\omega}]=1$ the percentage error decreases from 1.478% to 0.024%. For $\mathbb{E}[\mathcal{E}_{m_1,\omega}]=5$ (resp. $\mathbb{E}[\mathcal{E}_{m_1,\omega}]=10$), the percentage error decreases from 5.488% to 0.079% (resp. from 9.291% to 0.130%).

Moving from Table 4 to Tables 2-3 and 5–9, we observe the same pattern in terms of the monotonicity of the ultimate cumulative Parisian ruin probability in the initial surplus level and the mean Parisian clock within each table (and also regarding Erlangization in Table 3). Now, we are interested in comparing the ruin probabilities within the same class of distributions that have different variances. Specifically, Tables 3, 4, 6 and 9 feature gamma distributions with shape parameters $\alpha=2,1,1/3,1/5$, respectively. It can be seen that the ultimate cumulative Parisian ruin probability increases with the variance of the gamma distribution. Intuitively, as the variance of the interclaim times increases, the increments $\{cV_i-Y_i\}_{i=1}^{\infty}$ of the surplus process between successive claim arrival times also have a larger variance. This means that the surplus is more likely to downcross level zero (see the numerical illustrations in Landriault and Willmot (2008) for similar explanations), which in turn increases the chance that the occupation time in the red exceeds the Parisian clock. For the same reason, we can also compare Tables 2, 4 and 8 which concern Weibull distributions with shape parameters $\alpha=2,1,1/2$, respectively, and the same observation can be made. Likewise, from Tables 5 and 7 the ruin probability also increases with the variance of the Pareto interclaim times.

It is worthwhile to point out that when one moves away from the compound Poisson model (Table 4), the ultimate cumulative Parisian ruin probability can become very different especially when the initial surplus level gets large. For example, when $u=20$ and the mean Parisian clock is 10, such probability is around 4%-4.3% in the compound Poisson model. This drastically reduces to about 0.8%-0.9% when the interclaim times are Weibull with small variance (Table 2) and significantly increases to over 30% for gamma interclaim times with higher variance (Table 9). This demonstrates the importance of allowing for interclaim time distributions other than the classic exponential assumption.

Next, we would like to compare the ultimate cumulative Parisian ruin probability across different classes of interclaim time distributions with the same variance. In particular, we inspect Tables 7-9 which correspond to Pareto, Weibull and gamma distributions having the same variance of 5. Interestingly, we note that the case of Pareto interclaim times produces significantly lower ultimate cumulative Parisian ruin probabilities. This can be attributed to the heaviest tail of the Pareto distribution among the three distributions. Under heavy-tail

Table 10
Finite-time ($\mathbb{E}[\mathcal{E}_{m_2,\gamma}]=50$) cumulative Parisian ruin probability for Weibull interclaim times with variance 0.273240: Erlang Parisian clock $\mathcal{E}_{m_1,\omega}$.

$m_1=m_2$	$\mathbb{E}[\mathcal{E}_{m_1,\omega}]=m_1/\omega=1$				$\mathbb{E}[\mathcal{E}_{m_1,\omega}]=m_1/\omega=5$				$\mathbb{E}[\mathcal{E}_{m_1,\omega}]=m_1/\omega=10$			
	$u=0$	$u=5$	$u=10$	$u=20$	$u=0$	$u=5$	$u=10$	$u=20$	$u=0$	$u=5$	$u=10$	$u=20$
1	0.598570	0.151848	0.038522	0.002479	0.428222	0.108634	0.027559	0.001774	0.333373	0.084572	0.021455	0.001381
5	0.647112	0.182729	0.045970	0.002255	0.465259	0.123898	0.029640	0.001348	0.349927	0.088308	0.020222	0.000862
10	0.650115	0.187582	0.047502	0.002182	0.468907	0.127193	0.030232	0.001240	0.354628	0.090145	0.020114	0.000740
20	0.651345	0.190033	0.048364	0.002139	0.470262	0.128968	0.030614	0.001176	0.356454	0.091344	0.020123	0.000667
30	0.651718	0.190849	0.048667	0.002123	0.470653	0.129572	0.030757	0.001153	0.356949	0.091778	0.020141	0.000641
40	0.651898	0.191256	0.048822	0.002115	0.470838	0.129876	0.030831	0.001141	0.357178	0.092000	0.020154	0.000627
50	0.652004	0.191501	0.048916	0.002110	0.470945	0.130059	0.030877	0.001134	0.357310	0.092135	0.020162	0.000619

Table 11
Finite-time ($\mathbb{E}[\mathcal{E}_{m_2,\gamma}]=50$) cumulative Parisian ruin probability for Erlang interclaim times with variance 0.5: deterministic Parisian clock r.

m_2	$r=1$				$r=5$				$r=10$			
	$u=0$	$u=5$	$u=10$	$u=20$	$u=0$	$u=5$	$u=10$	$u=20$	$u=0$	$u=5$	$u=10$	$u=20$
1	0.611581	0.179636	0.052763	0.004552	0.403019	0.118376	0.034770	0.003000	0.282699	0.083036	0.024390	0.002104
5	0.677895	0.221307	0.065003	0.004425	0.490640	0.150777	0.041990	0.002632	0.368422	0.106534	0.028187	0.001634
10	0.683101	0.227714	0.067454	0.004361	0.499546	0.156770	0.043620	0.002514	0.380815	0.111900	0.029148	0.001493
20	0.685465	0.230936	0.068809	0.004323	0.503511	0.159885	0.044569	0.002442	0.386226	0.114853	0.029759	0.001406
30	0.686220	0.232008	0.069281	0.004310	0.504771	0.160933	0.044909	0.002416	0.387920	0.115866	0.029987	0.001374
40	0.686592	0.232543	0.069520	0.004303	0.505390	0.161459	0.045083	0.002403	0.388750	0.116377	0.030107	0.001357
50	0.686814	0.232864	0.069665	0.004298	0.505758	0.161775	0.045188	0.002395	0.389241	0.116685	0.030180	0.001347
60	0.686960	0.233077	0.069762	0.004296	0.506002	0.161986	0.045260	0.002389	0.389567	0.116890	0.030229	0.001340
70	0.687065	0.233230	0.069832	0.004294	0.506175	0.162136	0.045311	0.002385	0.389799	0.117037	0.030265	0.001335
80	0.687143	0.233344	0.069885	0.004292	0.506305	0.162249	0.045349	0.002383	0.389972	0.117147	0.030292	0.001331

Table 12
Finite-time ($\mathbb{E}[\mathcal{E}_{m_2,\gamma}]=50$) cumulative Parisian ruin probability for Erlang interclaim times with variance 0.5: Erlang Parisian clock $\mathcal{E}_{m_1,\omega}$.

$m_1=m_2$	$\mathbb{E}[\mathcal{E}_{m_1,\omega}]=m_1/\omega=1$				$\mathbb{E}[\mathcal{E}_{m_1,\omega}]=m_1/\omega=5$				$\mathbb{E}[\mathcal{E}_{m_1,\omega}]=m_1/\omega=10$			
	$u=0$	$u=5$	$u=10$	$u=20$	$u=0$	$u=5$	$u=10$	$u=20$	$u=0$	$u=5$	$u=10$	$u=20$
1	0.632585	0.185805	0.054575	0.004708	0.459418	0.134942	0.039636	0.003420	0.360598	0.105916	0.031110	0.002684
5	0.682082	0.222919	0.065545	0.004470	0.500576	0.154700	0.043330	0.002743	0.382086	0.112124	0.030095	0.001788
10	0.685216	0.228542	0.067740	0.004385	0.504511	0.158739	0.044311	0.002572	0.387494	0.114647	0.030130	0.001574
20	0.686527	0.231355	0.068955	0.004335	0.505992	0.160872	0.044920	0.002472	0.389589	0.116219	0.030254	0.001448
30	0.686930	0.232289	0.069379	0.004318	0.506424	0.161593	0.045143	0.002436	0.390166	0.116776	0.030318	0.001402
40	0.687125	0.232754	0.069594	0.004309	0.506629	0.161954	0.045259	0.002418	0.390434	0.117059	0.030355	0.001378
50	0.687240	0.233033	0.069725	0.004303	0.506749	0.162171	0.045330	0.002407	0.390590	0.117231	0.030379	0.001364

Table 13
Finite-time ($\mathbb{E}[\mathcal{E}_{m_2,\gamma}]=50$) cumulative Parisian ruin probability for exponential interclaim times with variance 1: deterministic Parisian clock r.

m_2	$r=1$				$r=5$				$r=10$			
	$u=0$	$u=5$	$u=10$	$u=20$	$u=0$	$u=5$	$u=10$	$u=20$	$u=0$	$u=5$	$u=10$	$u=20$
1	0.667717	0.245640	0.090366	0.012230	0.454247	0.167108	0.061476	0.008320	0.326189	0.119998	0.044145	0.005974
5	0.732032	0.298413	0.111530	0.012748	0.547995	0.211613	0.075334	0.007966	0.422405	0.154388	0.052425	0.005147
10	0.736928	0.305974	0.115511	0.012814	0.557020	0.219279	0.078331	0.007832	0.435584	0.161801	0.054506	0.004893
20	0.739158	0.309719	0.117655	0.012856	0.561034	0.223172	0.080021	0.007756	0.441268	0.165746	0.055774	0.004742
30	0.739872	0.310957	0.118392	0.012872	0.562310	0.224470	0.080614	0.007730	0.443047	0.167077	0.056235	0.004688
40	0.740223	0.311573	0.118764	0.012881	0.562937	0.225118	0.080916	0.007717	0.443917	0.167745	0.056473	0.004661
50	0.740433	0.311943	0.118989	0.012886	0.563310	0.225507	0.081099	0.007709	0.444433	0.168146	0.056618	0.004644
60	0.740571	0.312188	0.119139	0.012890	0.563557	0.225766	0.081222	0.007704	0.444775	0.168413	0.056716	0.004633
70	0.740670	0.312364	0.119247	0.012893	0.563733	0.225950	0.081310	0.007700	0.445018	0.168605	0.056786	0.004624
80	0.740744	0.312495	0.119328	0.012895	0.563865	0.226089	0.081376	0.007697	0.445200	0.168748	0.056839	0.004618

interclaim times, there is higher chance that there can be extended periods where the surplus process does not experience any claims. If this happens when the surplus process is in the red then it can be beneficial because the surplus process can recover quicker and the occupation time in the red (and hence the Parisian ruin probability) can be reduced. Another observation from Tables 8-9 is that the Weibull and the gamma cases have comparable ultimate cumulative Parisian ruin probabilities with the gamma case producing slightly higher ruin probabilities in most situations. The similarity in these two cases is possibly due to the fact that all the moments exist for both Weibull and gamma distributions, whereas the slightly lower figures in the Weibull case may be explained via the heavy-tailedness of a Weibull distribution with shape parameter $\alpha<1$ (as the moment generating function does not converge in such case).

We now turn our attention to the finite-time cumulative Parisian ruin probabilities which are provided in Tables 10-19 for the eight interclaim time distributions in Table 1, where the mean Erlang time horizon is taken to be $\mathbb{E}[\mathcal{E}_{m_2,\gamma}]=m_2/\gamma=50$. Specifically, Tables 11 and 13 provide the ruin probabilities $\mathbb{P}_u\{\mathrm{OT}_{\mathcal{E}_{m_2,\gamma}}>r\}$ under deterministic clocks $r=1,5,10$, where the computation is done using the results from Section 4.1 with Erlangization performed on $\mathcal{E}_{m_2,\gamma}$. In contrast, Tables 10, 12 and 14–19 are about the ruin probabilities $\mathbb{P}_u\{\mathrm{OT}_{\mathcal{E}_{m_2,\gamma}}>\mathcal{E}_{m_1,\omega}\}$ computed with the formulas in Section 4.2. Here Erlangization is performed simultaneously on both $\mathcal{E}_{m_1,\omega}$ and $\mathcal{E}_{m_2,\gamma}$ by setting $m_1=m_2$ and increasing them at the same time while keeping $m_1/\omega=1,5,10$ and $m_2/\gamma=50$ fixed.

Table 14
Finite-time ($\mathbb{E}[\mathcal{E}_{m_2,\gamma}]=50$) cumulative Parisian ruin probability for exponential interclaim times with variance 1: Erlang Parisian clock $\mathcal{E}_{m_1,\omega}$.

$m_1=m_2$	$\mathbb{E}[\mathcal{E}_{m_1,\omega}]=m_1/\omega=1$				$\mathbb{E}[\mathcal{E}_{m_1,\omega}]=m_1/\omega=5$				$\mathbb{E}[\mathcal{E}_{m_1,\omega}]=m_1/\omega=10$			
	$u=0$	$u=5$	$u=10$	$u=20$	$u=0$	$u=5$	$u=10$	$u=20$	$u=0$	$u=5$	$u=10$	$u=20$
1	0.685854	0.252312	0.092820	0.012562	0.509812	0.187549	0.068995	0.009338	0.405356	0.149122	0.054859	0.007424
5	0.735468	0.300065	0.112239	0.012847	0.557369	0.216241	0.077357	0.008252	0.435412	0.161119	0.055388	0.005559
10	0.738664	0.306820	0.115883	0.012867	0.561718	0.221596	0.079366	0.007983	0.441974	0.165072	0.056006	0.005112
20	0.740030	0.310147	0.117845	0.012883	0.563383	0.224332	0.080544	0.007834	0.444498	0.167372	0.056526	0.004855
30	0.740454	0.311243	0.118520	0.012890	0.563875	0.225244	0.080964	0.007782	0.445204	0.168161	0.056737	0.004764
40	0.740661	0.311788	0.118860	0.012895	0.564111	0.225699	0.081179	0.007756	0.445536	0.168558	0.056849	0.004718
50	0.740783	0.312115	0.119066	0.012897	0.564249	0.225971	0.081310	0.007740	0.445729	0.168796	0.056919	0.004690

Table 15
Finite-time ($\mathbb{E}[\mathcal{E}_{m_2,\gamma}]=50$) cumulative Parisian ruin probability for Pareto interclaim times with variance 3: Erlang Parisian clock $\mathcal{E}_{m_1,\omega}$.

$m_1=m_2$	$\mathbb{E}[\mathcal{E}_{m_1,\omega}]=m_1/\omega=1$				$\mathbb{E}[\mathcal{E}_{m_1,\omega}]=m_1/\omega=5$				$\mathbb{E}[\mathcal{E}_{m_1,\omega}]=m_1/\omega=10$			
	$u=0$	$u=5$	$u=10$	$u=20$	$u=0$	$u=5$	$u=10$	$u=20$	$u=0$	$u=5$	$u=10$	$u=20$
1	0.757371	0.364569	0.175489	0.040662	0.589329	0.283680	0.136553	0.031640	0.480857	0.231466	0.111419	0.025817
5	0.809562	0.431239	0.213768	0.044199	0.653311	0.332961	0.158528	0.030686	0.533208	0.260226	0.119345	0.021776
10	0.812911	0.439987	0.220534	0.044943	0.658952	0.341339	0.163391	0.030470	0.542702	0.267914	0.122148	0.020811
20	0.814378	0.444216	0.224081	0.045397	0.661183	0.345462	0.166105	0.030390	0.546321	0.272064	0.124022	0.020291
30	0.814840	0.445599	0.225283	0.045565	0.661859	0.346814	0.167047	0.030372	0.547355	0.273439	0.124717	0.020116
40	0.815066	0.446285	0.225888	0.045653	0.662186	0.347485	0.167524	0.030365	0.547846	0.274123	0.125076	0.020030
50	0.815200	0.446696	0.226251	0.045706	0.662378	0.347885	0.167812	0.030362	0.548134	0.274531	0.125296	0.019978

Table 16
Finite-time ($\mathbb{E}[\mathcal{E}_{m_2,\gamma}]=50$) cumulative Parisian ruin probability for gamma interclaim times with variance 3: Erlang Parisian clock $\mathcal{E}_{m_1,\omega}$.

$m_1=m_2$	$\mathbb{E}[\mathcal{E}_{m_1,\omega}]=m_1/\omega=1$				$\mathbb{E}[\mathcal{E}_{m_1,\omega}]=m_1/\omega=5$				$\mathbb{E}[\mathcal{E}_{m_1,\omega}]=m_1/\omega=10$			
	$u=0$	$u=5$	$u=10$	$u=20$	$u=0$	$u=5$	$u=10$	$u=20$	$u=0$	$u=5$	$u=10$	$u=20$
1	0.782167	0.420513	0.226079	0.065346	0.608878	0.327349	0.175991	0.050869	0.496662	0.267018	0.143556	0.041494
5	0.828806	0.486884	0.272205	0.075302	0.669281	0.379065	0.204896	0.053525	0.546820	0.298747	0.156429	0.038785
10	0.831802	0.494934	0.279502	0.077312	0.674256	0.386972	0.210568	0.054189	0.555644	0.306585	0.160370	0.038311
20	0.833126	0.498797	0.283223	0.078448	0.676200	0.390759	0.213576	0.054624	0.558928	0.310589	0.162756	0.038141
30	0.833543	0.500058	0.284470	0.078849	0.676787	0.391991	0.214598	0.054789	0.559862	0.311888	0.163598	0.038102
40	0.833748	0.500683	0.285094	0.079054	0.677070	0.392600	0.215111	0.054875	0.560306	0.312530	0.164027	0.038087
50	0.833869	0.501056	0.285468	0.079179	0.677236	0.392964	0.215420	0.054929	0.560566	0.312912	0.164285	0.038079

Table 17
Finite-time ($\mathbb{E}[\mathcal{E}_{m_2,\gamma}]=50$) cumulative Parisian ruin probability for Pareto interclaim times with variance 5: Erlang Parisian clock $\mathcal{E}_{m_1,\omega}$.

$m_1=m_2$	$\mathbb{E}[\mathcal{E}_{m_1,\omega}]=m_1/\omega=1$				$\mathbb{E}[\mathcal{E}_{m_1,\omega}]=m_1/\omega=5$				$\mathbb{E}[\mathcal{E}_{m_1,\omega}]=m_1/\omega=10$			
	$u=0$	$u=5$	$u=10$	$u=20$	$u=0$	$u=5$	$u=10$	$u=20$	$u=0$	$u=5$	$u=10$	$u=20$
1	0.777255	0.401750	0.207658	0.055480	0.613407	0.317060	0.163883	0.043784	0.504786	0.260915	0.134863	0.036031
5	0.830085	0.474409	0.253316	0.061243	0.682801	0.374012	0.192055	0.043528	0.565273	0.296850	0.146938	0.031412
10	0.833456	0.483737	0.261242	0.062490	0.688868	0.383467	0.198167	0.043504	0.575833	0.306062	0.150885	0.030325
20	0.834941	0.488223	0.265366	0.063243	0.691288	0.388076	0.201535	0.043559	0.579849	0.310945	0.153437	0.029764
30	0.835409	0.489688	0.266758	0.063519	0.692025	0.389581	0.202696	0.043594	0.581003	0.312550	0.154368	0.029582
40	0.835638	0.490414	0.267457	0.063662	0.692383	0.390327	0.203283	0.043615	0.581553	0.313345	0.154847	0.029493
50	0.835774	0.490847	0.267877	0.063749	0.692594	0.390772	0.203637	0.043629	0.581875	0.313820	0.155139	0.029441

Table 18
Finite-time ($\mathbb{E}[\mathcal{E}_{m_2,\gamma}]=50$) cumulative Parisian ruin probability for Weibull interclaim times with variance 5: Erlang Parisian clock $\mathcal{E}_{m_1,\omega}$.

$m_1=m_2$	$\mathbb{E}[\mathcal{E}_{m_1,\omega}]=m_1/\omega=1$				$\mathbb{E}[\mathcal{E}_{m_1,\omega}]=m_1/\omega=5$				$\mathbb{E}[\mathcal{E}_{m_1,\omega}]=m_1/\omega=10$			
	$u=0$	$u=5$	$u=10$	$u=20$	$u=0$	$u=5$	$u=10$	$u=20$	$u=0$	$u=5$	$u=10$	$u=20$
1	0.816087	0.491486	0.295996	0.107358	0.651917	0.392615	0.236451	0.085761	0.539566	0.324952	0.195702	0.070981
5	0.862904	0.566385	0.356450	0.126866	0.721622	0.458616	0.279958	0.094472	0.604287	0.371544	0.220149	0.070665
10	0.865780	0.574933	0.365527	0.130782	0.727116	0.468089	0.288077	0.096585	0.614666	0.382070	0.226865	0.070968
20	0.867059	0.578985	0.370073	0.132959	0.729283	0.472521	0.292263	0.097870	0.618444	0.387214	0.230715	0.071352
30	0.867464	0.580301	0.371584	0.133720	0.729943	0.473948	0.293664	0.098338	0.619521	0.388850	0.232036	0.071529
40	0.867663	0.580952	0.372338	0.134106	0.730263	0.474653	0.294365	0.098579	0.620035	0.389654	0.232700	0.071628
50	0.867781	0.581341	0.372789	0.134341	0.730451	0.475072	0.294786	0.098727	0.620335	0.390131	0.233100	0.071690

First, it can be seen that the finite-time cumulative Parisian ruin probabilities in Tables 10-19 show similar patterns to those in Tables 2-9: (i) within each table, the probabilities are decreasing in both the initial surplus u and the (mean) Parisian clock; (ii) the probabilities converge down each column due to Erlangization taking place; (iii) the probabilities increase with the variance of the interclaim times within the same class of distributions (such as gamma in Tables 12, 14, 16 and 19); and (iv) when comparing across different classes of distributions with the same variance, in most situations Pareto and gamma interclaim times lead to the smallest and the highest ruin probabilities, respectively, with the Weibull case falling in between (see Tables 17-19). A lot of the intuitive explanations provided

Table 19
Finite-time ($\mathbb{E}[\mathcal{E}_{m_2,\gamma}]=50$) cumulative Parisian ruin probability for gamma interclaim times with variance 5: Erlang Parisian clock $\mathcal{E}_{m_1,\omega}$.

$m_1=m_2$	$\mathbb{E}[\mathcal{E}_{m_1,\omega}]=m_1/\omega=1$				$\mathbb{E}[\mathcal{E}_{m_1,\omega}]=m_1/\omega=5$				$\mathbb{E}[\mathcal{E}_{m_1,\omega}]=m_1/\omega=10$			
	$u=0$	$u=5$	$u=10$	$u=20$	$u=0$	$u=5$	$u=10$	$u=20$	$u=0$	$u=5$	$u=10$	$u=20$
1	0.822674	0.512368	0.319107	0.123778	0.654803	0.407816	0.253991	0.098520	0.540686	0.336744	0.209727	0.081351
5	0.866410	0.583085	0.379283	0.147168	0.721021	0.471260	0.298229	0.110245	0.601738	0.381748	0.235170	0.083045
10	0.869149	0.590998	0.387894	0.151547	0.726126	0.479947	0.305976	0.112865	0.611504	0.391590	0.241824	0.083899
20	0.870375	0.594759	0.392185	0.153921	0.728135	0.483990	0.309905	0.114382	0.615044	0.396304	0.245506	0.084580
30	0.870764	0.595983	0.393608	0.154740	0.728746	0.485292	0.311214	0.114920	0.616054	0.397798	0.246753	0.084853
40	0.870955	0.596589	0.394318	0.155155	0.729041	0.485934	0.311867	0.115196	0.616535	0.398530	0.247377	0.084998
50	0.871069	0.596951	0.394744	0.155406	0.729215	0.486317	0.312259	0.115363	0.616817	0.398966	0.247752	0.085088

for the ultimate cumulative Parisian ruin probabilities are applicable here as well. We also note that (except for small values of m_2 less than 10) the ruin probabilities across Tables 11-12 are reasonably close in the case of Erlang(2) interclaim time distribution, and the same comments apply to Tables 13-14 in the exponential case. It is interesting to note that, except in some cases with small initial surplus, the majority of the ruin probabilities in Table 17 for Pareto interclaim times with variance 5 are smaller than those in Table 16 concerning gamma interclaim times with variance 3. Therefore, in general one cannot simply rank the ruin probabilities according to the variance of the interclaim times when comparing across different classes of distributions, as the tail behaviour of the interclaim times can also impact the ruin probabilities.

We end this section by discussing some computational matters in relation to our algorithms. For reference, all the computations are done by *Mathematica* using a Dell Latitude 9420 with 16.0GB RAM and i7 CPU at 3.00 GHz. Regarding the calculation of the ultimate cumulative Parisian ruin probability $\psi_r(u)=\mathbb{P}_u\{\mathrm{OT}_\infty>r\}$ under a deterministic clock, we utilize (3.7) and (3.10) to obtain the last row of Tables 3-4. In doing so, we need to truncate the infinite sums in (3.10). Note that (3.10) does not depend on the initial surplus u, and $\psi_r(u)$ in (3.7) is simply (3.10) times an exponential term in u. This means that the afore-mentioned sums do not have to be recalculated when altering u. In the case of Erlang interclaim times, a truncation point of 2000 is more than sufficient to yield converging results for $\psi_r(u)$ at the 6 decimal places for the Parisian clocks $r=1,5,10$ considered. For each $r=1,5,10$, it took about 8 minutes to obtain the whole $\mathbb{P}_u\{\mathrm{OT}_\infty>r\}$ as a function of u. For the ultimate cumulative Parisian ruin probability $\mathbb{P}_u\{\mathrm{OT}_\infty>\mathcal{E}_{m_1,\omega}\}$ under an Erlang clock in Tables 2-9, the computational procedure stated after (3.20) is applied. The time-consuming part arises from Step 2 where the evaluation of Bell polynomials is needed to obtain the higher order derivatives of R_x recursively via (3.20). The values of these polynomials should be stored in *Mathematica*'s memory for the computation of (3.17) in Step 3. Once these polynomials have been calculated for a given pair of (m_1,ω), the whole ultimate cumulative Parisian survival probability $\mathbb{P}_u\{\mathrm{OT}_\infty<\mathcal{E}_{m_1,\omega}\}$ as a function of u is obtained thanks to (3.14). However, changing ω does require recalculations because the Bell polynomials need to be evaluated at $x=\omega$. When performing Erlangization by keeping the mean Parisian clock $\mathbb{E}[\mathcal{E}_{m_1,\omega}]=m_1/\omega$ fixed, it is found that $\mathbb{P}_u\{\mathrm{OT}_\infty>\mathcal{E}_{m_1,\omega}\}$ can be obtained instantly for m_1 up to 50. When $m_1=60,70,80$, the computational times increase quickly to about 20, 100 and 500 seconds, respectively, and these are not sensitive to the choice of interclaim time distribution or the mean Parisian clock considered. Moving to Tables 11 and 13 for the finite-time cumulative Parisian ruin probabilities $\mathbb{P}_u\{\mathrm{OT}_{\mathcal{E}_{m_2,\gamma}}>r\}$ under a deterministic clock, computation via (4.10) requires (i) truncation of the infinite sums in (3.10); and (ii) the use of Bell polynomials because of the derivative terms. A truncation point of about 400-600 is found to be sufficient, and the computation times for the whole Table 11 and the whole Table 13 are about 9300 and 4300 seconds, respectively. Lastly, the calculation of $\mathbb{P}_u\{\mathrm{OT}_{\mathcal{E}_{m_2,\gamma}}>\mathcal{E}_{m_1,\omega}\}$ in Tables 10, 12 and 14–19 using (the complement of) (4.13) requires the first m_1+m_2-2 derivatives of $h_1(h_2(\cdot))$. This in turn requires the evaluation of a large number of Bell polynomials which can be very time-consuming. While $\mathbb{P}_u\{\mathrm{OT}_{\mathcal{E}_{m_2,\gamma}}>\mathcal{E}_{m_1,\omega}\}$ as a function of u can be obtained instantly for $m_1=m_2$ up to 20, when $m_1=m_2=30,40,50$ the computational times are around 20, 400 and 7000 seconds, respectively, and these do not change much across different interclaim time distributions and different mean Parisian clocks. Nevertheless, the evaluation of Bell polynomials itself presents a complex problem in mathematics (see e.g. Cvijović (2011), Natalini and Ricci (2016) and references therein for further discussions), and our calculations can be possibly sped up if there are advances in such a field of mathematics.

6. Concluding remarks

In this paper, we consider the cumulative Parisian ruin problem in both finite and infinite time horizons, where the Parisian clock can be deterministic or Erlang distributed, and in the finite-time case the time horizon is replaced by an Erlang random variable. Our analysis does not require any specific distributional assumption on the interclaim times. Consequently, the present paper fills an important literature gap in the cumulative Parisian ruin problem where exponential or phase-type interclaim times are typically assumed. Our numerical illustrations (which include gamma, Weibull and Pareto interclaim times) demonstrate that the cumulative Parisian ruin probabilities can be quite different when one deviates from the classical compound Poisson risk model. Moreover, the heavy-tailedness of the interclaim time distributions can also have a significant impact on these probabilities when comparing across different classes of interclaim distributions with the same variance. We hope our findings can stimulate further research that allows for more general claim arrival processes.

Declaration of competing interest

The authors declare that there is no competing interest.

Data availability

No data was used for the research described in the article.

Acknowledgements

The authors would like to thank the anonymous reviewers for their comments and suggestions which have improved an earlier version of the manuscript. Eric Cheung gratefully acknowledges the support from The Australian Research Council Discovery Project (DP200100615). The views expressed herein are those of the authors and are not necessarily those of the Australian Research Council.

References

Albrecher, H., Cheung, E.C.K., Thonhauser, S., 2011. Randomized observation periods for the compound Poisson risk model: dividends. ASTIN Bulletin 41 (2), 645–672.
Antill, S., Grenadier, S.R., 2019. Optimal capital structure and bankruptcy choice: dynamic bargaining versus liquidation. Journal of Financial Economics 133 (1), 198–224.
Asmussen, S., Albrecher, H., 2010. Ruin Probabilities, second edition. World Scientific, New Jersey.
Asmussen, S., Avram, F., Usabel, M., 2002. Erlangian approximations for finite-horizon ruin probabilities. ASTIN Bulletin 32 (2), 267–281.
Bladt, M., Nielsen, B.F., Peralta, O., 2019. Parisian types of ruin probabilities for a class of dependent risk-reserve processes. Scandinavian Actuarial Journal 2019 (1), 32–61.
Borovkov, K.A., Dickson, D.C.M., 2008. On the ruin time distribution for a Sparre Andersen process with exponential claim sizes. Insurance. Mathematics & Economics 42 (3), 1104–1108.
Broadie, M., Chernov, M., Sundaresan, S., 2007. Optimal debt and equity values in the presence of Chapter 7 and Chapter 11. Journal of Finance 62 (3), 1341–1377.
Baurdoux, E.J., Pardo, J.C., Pérez, J.L., Renaud, J.-F., 2016. Gerber-Shiu distribution at Parisian ruin for Lévy insurance risk processes. Journal of Applied Probability 53 (2), 572–584.
Carr, P., 1998. Randomization and the American put. The Review of Financial Studies 11 (3), 597–626.
Chesney, M., Jeanblanc-Picqué, M., Yor, M., 1997. Brownian excursions and Parisian barrier options. Advances in Applied Probability 29 (1), 165–184.
Cheung, E.C.K., Landriault, D., Willmot, G.E., Woo, J.-K., 2010. Structural properties of Gerber-Shiu functions in dependent Sparre Andersen models. Insurance. Mathematics & Economics 46 (1), 117–126.
Cheung, E.C.K., Wong, J.T.Y., 2017. On the dual risk model with Parisian implementation delays in dividend payments. European Journal of Operational Research 257 (1), 159–173.
Cheung, E.C.K., Zhang, Z., 2019. Periodic threshold-type dividend strategy in the compound Poisson risk model. Scandinavian Actuarial Journal 2019 (1), 1–31.
Cohen, J.W., 1982. The Single Server Queue. Elsevier Science Publishers B.V., Netherlands.
Cvijović, D., 2011. New identities for the partial Bell polynomials. Applied Mathematics Letters 24 (9), 1544–1547.
Czarna, I., 2016. Parisian ruin probability with a lower ultimate bankrupt barrier. Scandinavian Actuarial Journal 2016 (4), 319–337.
Czarna, I., Li, Y., Palmowski, Z., Zhao, C., 2017. The joint distribution of the Parisian ruin time and the number of claims until Parisian ruin in the classical risk model. Journal of Computational and Applied Mathematics 313, 499–514.
Czarna, I., Palmowski, Z., 2011. Ruin probability with Parisian delay for a spectrally negative Lévy risk process. Journal of Applied Probability 48 (4), 984–1002.
Dassios, A., Wu, S., 2008a. Parisian ruin with exponential claims. Preprint, available at http://stats.lse.ac.uk/angelos/docs/exponentialjump.pdf.
Dassios, A., Wu, S., 2008b. Ruin probabilities of the Parisian type for small claims. Preprint, available at http://stats.lse.ac.uk/angelos/docs/paper5a.pdf.
Dickson, D.C.M., Willmot, G.E., 2005. The density of the time to ruin in the classical Poisson risk model. ASTIN Bulletin 35 (1), 45–60.
Drekic, S., Dickson, D.C.M., Stanford, D.A., Willmot, G.E., 2004. On the distribution of the deficit at ruin when claims are phase-type. Scandinavian Actuarial Journal 2004 (2), 105–120.
Frostig, E., Pitts, S.M., Politis, K., 2012. The time to ruin and the number of claims until ruin for phase-type claims. Insurance. Mathematics & Economics 51 (1), 19–25.
Gerber, H.U., Shiu, E.S.W., 1998. On the time value of ruin. North American Actuarial Journal 2 (1), 48–72.
Guérin, H., Renaud, J.-F., 2017. On the distribution of cumulative Parisian ruin. Insurance. Mathematics & Economics 73, 116–123.
Johnson, W.P., 2002. The curious history of Faà di Bruno's Formula. The American Mathematical Monthly 109 (3), 217–234.
Kyprianou, A.E., Pistorius, M.R., 2003. Perpetual options and Canadization through fluctuation theory. The Annals of Applied Probability 13 (3), 1077–1098.
Landriault, D., Li, B., Shi, T., Xu, D., 2019. On the distribution of classic and some exotic ruin times. Insurance. Mathematics & Economics 89, 38–45.
Landriault, D., Renaud, J.-F., Zhou, X., 2014. An insurance risk model with Parisian implementation delays. Methodology and Computing in Applied Probability 16 (3), 583–607.
Landriault, D., Shi, T., Willmot, G.E., 2011. Joint densities involving the time to ruin in the Sparre Andersen risk model under exponential assumptions. Insurance. Mathematics & Economics 49 (3), 371–379.
Landriault, D., Willmot, G.E., 2008. On the Gerber-Shiu discounted penalty function in the Sparre Andersen model with an arbitrary interclaim time distribution. Insurance. Mathematics & Economics 42 (2), 600–608.
Li, B., Tang, Q., Wang, L., Zhou, X., 2014. Liquidation risk in the presence of Chapters 7 and 11 of the US bankruptcy code. The Journal of Financial Engineering 1 (3), 1450023.
Li, S., Zhou, X., 2022. The Parisian and ultimate drawdowns of Lévy insurance models. Insurance. Mathematics & Economics 107, 140–160.
Li, X., Liu, H., Tang, Q., Zhu, J., 2020. Liquidation risk in insurance under contemporary regulatory frameworks. Insurance. Mathematics & Economics 93, 36–49.
Lkabous, M.A., Czarna, I., Renaud, J.-F., 2017. Parisian ruin for a refracted Lévy process. Insurance. Mathematics & Economics 74, 153–163.
Lkabous, M.A., Renaud, J.-F., 2018. A VaR-type risk measure derived from cumulative Parisian ruin for the classical risk model. Risks 6 (3), 85.
Loeffen, R., Czarna, I., Palmowski, Z., 2013. Parisian ruin probability for spectrally negative Lévy processes. Bernoulli 19 (2), 599–609.
Loeffen, R., Palmowski, Z., Surya, B.A., 2018. Discounted penalty function at Parisian ruin for Lévy insurance risk process. Insurance. Mathematics & Economics 83, 190–197.
Natalini, P., Ricci, P.E., 2016. Remarks on Bell and higher order Bell polynomials and numbers. Cogent Mathematics 3 (1), 1220670.
Prabhu, N.U., 1998. Stochastic Storage Processes: Queues, Insurance Risk, Dams, and Data Communication. Springer, New York.
Ramaswami, V., Woolford, D.G., Stanford, D.A., 2008. The Erlangization method for Markovian fluid flows. Annals of Operations Research 160 (1), 215–225.
Sagias, N.C., Karagiannidis, G.K., 2005. Gaussian class multivariate Weibull distributions: theory and applications in fading channels. IEEE Transactions on Information Theory 51 (10), 3608–3619.
Sparre Andersen, E., 1957. On the collective theory of risk in the case of contagion between claims. In: Proceedings of the Transactions of the XVth International Congress on Actuaries, vol. II, New York, pp. 219–229.
Stanford, D.A., Avram, F., Badescu, A.L., Breuer, L., Da Silva Soares, A., Latouche, G., 2005. Phase-type approximations to finite-time ruin probabilities in the Sparre-Anderson and stationary renewal risk models. ASTIN Bulletin 35 (1), 131–144.
Willmot, G.E., 2007. On the discounted penalty function in the renewal risk model with general interclaim times. Insurance. Mathematics & Economics 41 (1), 17–31.
Willmot, G.E., Woo, J.-K., 2017. Surplus Analysis of Sparre Andersen Insurance Risk Processes. Springer, Switzerland.
Wong, J.T.Y., Cheung, E.C.K., 2015. On the time value of Parisian ruin in (dual) renewal risk processes with exponential jumps. Insurance. Mathematics & Economics 65, 280–290.