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Conditional mean risk sharing of losses at occurrence time in the compound Poisson surplus model

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ABSTRACT

This paper proposes a new risk-sharing procedure, framed into the classical insurance surplus process. Compared to the standard setting where total losses are shared at the end of the period, losses are allocated among participants at their occurrence time in the proposed model. The conditional mean risk-sharing rule proposed by Denuit and Dhaene (2012) is applied to this end. The analysis adopts two different points of views: a collective one for the pool and an individual one for sharing losses and adjusting the amounts of contributions among participants. These two views are compatible under the compound Poisson risk process. Guarantees can also be added by partnering with an insurer.

1. Introduction and motivation

Risk-sharing mechanisms have been studied for decades in the actuarial literature. In a risk-sharing pool, each participant is compensated from the pool for his or her individual losses. In return, he or she pays an ex-post contribution to the pool, which is determined so that the sum of all the individual contributions matches the aggregate loss of the pool.

In the standard setting, risk sharing thus operates ex post, by allocating total losses to participants at the end of the period. This is not in line with decentralized insurance (DeIn) models where claims are handled when they occur. Once approved, claims are settled without waiting until the end of the period. This is the case for example with Nexus Mutual (https://nexusmutual.io/). For a thorough presentation of DeIn models, we refer the interested reader to Chapter 9 in Feng (2023). This is why an alternative model where claims are shared when they are submitted is proposed in this paper. The conditional mean risk-sharing rule proposed by Denuit and Dhaene (2012) and axiomatized by Jiao et al. (2022) is used to that end, resulting in an intuitive allocation among participants.

Let us mention that the allocation of losses is performed using the conditional mean risk-sharing rule at every occurrence time. In that respect, the rule itself does not take into account the dynamic of the underlying risk process. This is in contrast with the approach developed by Abdikerimova et al. (2022), where dynamic risk-sharing rules are proposed in the sense that the allocation policy itself changes over time and adapts to previous losses. Here, the sharing rule remains static but it is applied repeatedly, each time a loss occurs within the pool and not only once at the end of the period. This aligns the analysis of the pooling mechanism with practice, since claims are settled when reported, without waiting until the end of the agreed coverage period.

According to Chapter 24 in Culp (2006), finite risk programs correspond to a range of risk management solutions in which the (re)insurer's downside risk is limited and the policyholders participate in case of favorable claims experience. Some degree of finite risk turns out to be useful to design successful DeIn schemes. In this paper, timing risk is transferred to a partnering financial institution, bank or insurance company. This refers to the risk that actual losses occur at a faster rate than expected and that accumulated surplus is too low to fund those claims when they occur, whereas on the long run balance is expected to be positive. The amount of deficit is then borrowed from the partnering institution and the community pays interest at an agreed rate. With finite risk, participants benefit from favorable claims experience while

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the partnering financial institution provides the loan in case of temporary deficits.

Participants may also buy additional guarantees from a partnering insurer. The model developed in this paper then makes explicit the diversification within the pool of participants and optional guarantees added by a partnering insurer. Instead of paying a fixed amount of premium, participants can then be granted bonuses in case of favorable experience, while benefiting from insurer's guarantee in case of adverse experience.

The remainder of this paper is organized as follows. In Section 2, we introduce individual and collective insurance accounts developing according to the compound Poisson surplus process of risk theory. Then we formalize risk sharing among participants, allocating losses at their occurrence time. Section 3 discusses the impact of this new risk-sharing mechanism based on different criteria. Considering infinite-time default probability, it is shown that pooling is beneficial. Over a finite-time horizon, the analysis is more contrasted. On the one hand, pooling is detrimental with respect to finite-time default probabilities with zero initial deposit but on the other hand, it remains beneficial under several other criteria. Under the no-sabotage condition, Section 4 gives formulas to share the burden of interest on temporary deficits as well as the accumulated surplus at maturity. Section 5 considers large pools. It is shown there that under mild assumptions, individual risk can be fully diversified when the number of participants tends to infinity. The asymptotic behavior of finite-time and infinite-time default probabilities is studied to assess the benefits of risk pooling. The final Section 6 summarizes the main findings of the paper. The inclusion of additional guarantees by partnering with an insurer is discussed there.

2. Dynamic risk sharing

2.1. Individual accounts

Each participant owns an individual insurance account reflecting his or her specific experience. Specifically, we consider a pool gathering losses of a community of n participants, numbered $i = 1, 2, \ldots, n$. The pool starts operating at time 0. Let $N_{i,t}$ be the number of claims recorded by participant i over time interval $(0, t)$, with $N_{i,0} = 0$. Throughout this paper, the claim number process $\{N_{i,t},\ t \geq 0\}$ is assumed to be Poisson with constant rate $\lambda_i > 0$. The size of the kth claim is denoted as $Y_{i,k}$. The random variables $Y_{i,1}, Y_{i,2}, \ldots$ are assumed to be non-negative, mutually independent, with common distribution function F_i. Claim sizes $Y_{i,k}$ are furthermore assumed to be independent of $\{N_{i,t},\ t \geq 0\}$. To ease exposition, we assume that individual claim sizes are absolutely continuous with respective probability density function f_i.

The total claim amount filed by participant i up to time t is given by

$$S_{i,t} = \sum_{k=1}^{N_{i,t}} Y_{i,k}, \quad t \geq 0,$$

starting from $S_{i,0} = 0$. The process $\{S_{i,t},\ t \geq 0\}$ is compound Poisson. Participants are required to contribute at constant rate c_i and to pay an initial deposit $\kappa_i \geq 0$. The amount κ_i can be interpreted as membership fees, for instance. The individual account for participant i is then given by

$$V_{i,t} = c_i t - S_{i,t} + \kappa_i, \quad t \geq 0,$$

where

$$c_i = (1 + \eta)\mathrm{E}[S_{i,1}] = (1 + \eta)\lambda_i \mathrm{E}[Y_{i,1}]$$

denotes the contribution rate for participant i, corresponding to expected claims over one period of time supplemented with safety loading proportional to pure premium, with safety coefficient $\eta > 0$.

Throughout the paper, we assume that individual claim experiences are mutually independent, that is, all the random variables associated to a participant are independent of the random variables associated to any other participant.

Remark 2.1. Notice that claim sizes $Y_{i,k}$ include loss adjustment expenses. The pool can contract with professional loss adjusters to settle the claims, including their fees within claim sizes, or appoint participants to act as claim assessors (through a consensus mechanism as with Nexus Mutual, for instance). When a partnering insurer is involved, claim sizes $Y_{i,k}$ may correspond to incurred losses to deal with long-tailed business developing beyond the maturity of the pool.

2.2. Pooled fund

Participants agree to join the pool in order to diversify claims experience. Contributions, claims and initial deposits are aggregated into

$$c = \sum_{i=1}^{n} c_i,\ \ S_t = \sum_{i=1}^{n} S_{i,t} \text{ and } \kappa = \sum_{i=1}^{n} \kappa_i.$$

The fund participants collectively own at time t is given by

$$V_t = \sum_{i=1}^{n} V_{i,t} = c\,t - S_t + \kappa, \quad t \geq 0. \tag{2.1}$$

We recognize in (2.1) the dynamics of the classical insurance surplus, or risk process.

The pool is impacted by claims recorded by the n participants. The number of claims filed during time interval $(0, t)$ is $N_t = \sum_{i=1}^{n} N_{i,t}$. The claim occurrence process at pool level $\{N_t,\ t \geq 0\}$ is Poisson with parameter $\lambda = \sum_{i=1}^{n} \lambda_i$. Let

$$T_k = \inf\{t \geq 0 | N_t = k\}$$

be the time at which the kth claim is filed. The total claim amount up to time t can then be represented as

$$S_t = \sum_{k=1}^{N_t} Y_k, \quad t \geq 0,$$

where the claim size Y_k is given by

$$Y_k = \sum_{i=1}^{n} \mathrm{I}[N_{i,T_k} - N_{i,T_k -} = 1] Y_{i,N_{i,T_k}}$$

where $\mathrm{I}[\cdot]$ is the indicator function, equal to 1 if the event appearing within brackets is realized and to 0 otherwise. The random variables $Y_1, Y_2, \ldots$ are independent with common distribution function $F = \frac{1}{\lambda}\sum_{i=1}^{n} \lambda_i F_i$. We denote as $f = \frac{1}{\lambda}\sum_{i=1}^{n} \lambda_i f_i$ the corresponding probability density function. The total claim process $\{S_t,\ t \geq 0\}$ is compound Poisson with claim frequency λ and severity distribution F.

2.3. Conditional mean risk-sharing rule at occurrence time

Let us now explain how individual losses can be shared among participants at occurrence time. The mixture representation linking individual accounts and pooled fund is used to allocate losses among participants. In the collective model, losses Y_k impacting the community are not attributed to a given participant but are allocated among them according to the respective chances they would have produced Y_k. In this way, heterogeneity is accounted for. It turns out that this intuitive approach corresponds to the allocation according to the conditional mean risk-sharing rule proposed by Denuit and Dhaene (2012). Formally, consider the kth loss Y_k impacting the pool at time T_k. Define $Z_{k,1}, Z_{k,2}, \ldots, Z_{k,n}$ as

$$Z_{k,i} = \mathrm{I}[N_{i,T_k} - N_{i,T_k-} = 1]Y_{i,N_{i,T_k}}$$

for $i = 1, 2, \ldots, n$ with $\sum_{i=1}^{n} Z_{k,i} = Y_k$.

The conditional mean risk sharing of Y_k among participants is defined as

$$h_i^{\mathrm{cmrs}}(y) = \mathrm{E}[Z_{k,i}|Y_k = y],\ i = 1, 2, \ldots, n.$$

The next result shows that this allocation rule indeed distributes losses among participants according to the respective chances they would have produced them.

Proposition 2.2. *The amount contributed by participant i to the kth loss $Y_k = y$ according to the conditional mean risk-sharing rule is given by*

$$h_i^{\mathrm{cmrs}}(y) = y\frac{\lambda_i f_i(y)}{\sum_{j=1}^{n} \lambda_j f_j(y)} \text{ for } i = 1, 2, \ldots, n.$$

Proof. Let I_k denote the participant who filed the kth claim Y_k. From the mixture representation, we have that

$$\mathrm{P}[I_k = i] = \frac{\lambda_i}{\lambda} \text{ for } i = 1, 2, \ldots, n.$$

We then get

$$\mathrm{P}[I_k = i|Y_k = y] = \frac{\lambda_i f_i(y)}{\sum_{j=1}^{n} \lambda_j f_j(y)} \text{ for } i = 1, 2, \ldots, n \text{ and } y \geq 0.$$

The random variables $Z_{k,1}, Z_{k,2}, \ldots, Z_{k,n}$ are mutually exclusive in the sense that

$$\mathrm{P}[Z_{k,i} > 0, Z_{k,j} > 0] = 0 \text{ for all } i \neq j \in \{1, 2, \ldots, n\}.$$

According to Section 4.4.3 in Denuit and Dhaene (2012), the conditional mean risk-sharing rule applied to mutually exclusive risks is based on

$$p_j(y) = \frac{\mathrm{d}}{\mathrm{d}y}\mathrm{P}[I_k = j, Y_k \leq y] = f(y)\mathrm{P}[I_k = j|Y_k = y] = \frac{\lambda_j}{\lambda} f_j(y).$$

This results in the allocation

$$h_i^{\mathrm{cmrs}}(y) = \mathrm{E}[Z_{k,i}|Y_k = y] = y\frac{p_i(y)}{\sum_{j=1}^{n} p_j(y)} = y\frac{\lambda_i f_i(y)}{\sum_{j=1}^{n} \lambda_j f_j(y)}$$

as an application of the conditional mean risk-sharing rule. □

With this risk sharing rule, the pooled individual risk account for participant i at time t is now given by

$$V_{i,t}^{\mathrm{pool}} = c_i t - S_{i,t}^{\mathrm{pool}} + \kappa_i \text{ where } S_{i,t}^{\mathrm{pool}} = \sum_{k=1}^{N_t} h_i^{\mathrm{cmrs}}(Y_k), \quad t \geq 0.$$

Whereas $S_{i,t}$ aggregates all losses specific to participant i over $(0,t)$, $S_{i,t}^{\mathrm{pool}}$ gives all losses allocated to participant i over $(0,t)$ after pooling has taken place at every occurrence time $T_k \in (0,t)$.

Proposition 2.2 shows that participant i contributes to the kth loss $Y_k = y_k$ covered by the pool the amount

$$h_i^{\mathrm{cmrs}}(y_k) = \beta_i(y_k)y_k$$

where $\beta_i(y_k) = \mathrm{P}[I_k = i|Y_k = y_k] = \frac{\lambda_i f_i(y_k)}{\sum_{j=1}^{n} \lambda_j f_j(y_k)}$.

Clearly, $\sum_{i=1}^{n} h_i^{\mathrm{cmrs}}(y_k) = y_k$ and hence the loss is totally paid by the participants in the pool. This means that

$$V_t = \sum_{i=1}^{n} V_{i,t} = \sum_{i=1}^{n} V_{i,t}^{\mathrm{pool}}. \tag{2.2}$$

Since the conditional mean risk-sharing rule is known to be actuarially fair, joining the pool does not modify average payments compared to the situation where participants stand alone. This is formally established next.

Proposition 2.3. *For all $t \geq 0$, $\mathrm{E}[S_{i,t}] = \mathrm{E}[S_{i,t}^{\mathrm{pool}}]$.*

Proof. We have

$$\mathrm{E}[S_{i,t}^{\mathrm{pool}}] = \mathrm{E}\left[\sum_{k=1}^{N_t} \beta_i(Y_k)Y_k\right] = \sum_{l=1}^{\infty} \mathrm{P}[N_t = l] \sum_{k=1}^{l} \mathrm{E}[\beta_i(Y_k)Y_k]$$

with

$$\mathrm{E}[\beta_i(Y_k)Y_k] = \frac{1}{\lambda}\int_0^{\infty} \lambda_i f_i(y) y \mathrm{d}y = \frac{1}{\lambda}\lambda_i \mathrm{E}[Y_{i,1}].$$

This results in

$$\mathrm{E}[S_{i,t}^{\mathrm{pool}}] = \lambda_i t \mathrm{E}[Y_{i,1}] = \mathrm{E}[N_{i,t}]\mathrm{E}[Y_{i,1}] = \mathrm{E}[S_{i,t}]$$

and ends the proof. □

Remark 2.4. Denuit and Robert (2022) demonstrated that participants can be grouped in teams according to some meaningful criterion (family, friends, profession, etc.), resulting in a hierarchical decomposition of the community. Such a team partitioning is possible in the compound Poisson model, by first allocating losses to the different teams and then within these teams according to the formulas given in this section.

3. Impact of pooling

3.1. Default probabilities

Infinite-time default probabilities for participant i are defined as

$$\psi_i(\kappa_i) = \mathrm{P}[V_{i,t} < 0 \text{ for some } t \geq 0] \text{ and}$$
$$\psi_i^{\mathrm{pool}}(\kappa_i) = \mathrm{P}[V_{i,t}^{\mathrm{pool}} < 0 \text{ for some } t \geq 0]$$

if he or she stands alone or joins the pool, respectively. The following lemmas are useful to prove our main result comparing infinite-time default probabilities. The first result uses the idea of thinning the Poisson process $\{N_t,\ t \geq 0\}$, which refers to classifying each loss Y_k, independently, into one of a finite number of different types. Then, the losses of a given type also form a Poisson process. This construction allows us to represent $\{S_{i,t},\ t \geq 0\}$ as a compound Poisson process with losses occurring according to $\{N_t,\ t \geq 0\}$ by discarding losses produced by other participants.

Lemma 3.1. *Let $Y'_{i,1}, Y'_{i,2}, \ldots$ be independent and identically distributed random variables, with*

$$\mathrm{P}[Y'_{i,1}=0]=1-\frac{\lambda_i}{\lambda} \text{ and } \mathrm{P}[Y'_{i,1}>y]=\frac{\lambda_i}{\lambda}\big(1-F_i(y)\big).$$

Define

$$S'_{i,t}=\sum_{k=1}^{N_t} Y'_{i,k}, \quad t\geq 0.$$

Then, the process $\{S_{i,t},\ t\geq 0\}$ is distributed as $\{S'_{i,t},\ t\geq 0\}$.

The next lemma uses the convex order $\preceq_{\text{CX}}$. Recall that given two random variables W_1 and W_2, $W_1 \preceq_{\text{CX}} W_2$ holds when $\mathrm{E}[W_1]=\mathrm{E}[W_2]$ and $\mathrm{E}[(W_1-t)_+]\leq \mathrm{E}[(W_2-t)_+]$ for all t. This ensures that $\mathrm{E}[g(W_1)]\leq \mathrm{E}[g(W_2)]$ holds true for any convex function g for which the expectations exist. The relation $\preceq_{\text{CX}}$ expresses the common preferences of all risk-averse economic agents in the expected utility paradigm for choice under risk.

Lemma 3.2. *For any $k=1,2,\ldots$, we have $\sum_{j=1}^k h_i^{\text{cmrs}}(Y_j)\preceq_{\text{CX}}\sum_{j=1}^k Y'_{i,j}$.*

Proof. First,

$$\begin{aligned}\mathrm{E}[h_i^{\text{cmrs}}(Y_k)]&=\int_0^\infty y\beta_i(y)\left(\sum_{j=1}^n\frac{\lambda_j}{\lambda}f_j(y)\right)\mathrm{d}y\\&=\frac{\lambda_i}{\lambda}\mathrm{E}[Y_{i,k}]=\mathrm{E}[Y'_{i,k}].\end{aligned}$$

Let g be a convex function. We have

$$\mathrm{E}[g(Y'_{i,k})]=\left(1-\frac{\lambda_i}{\lambda}\right)g(0)+\frac{\lambda_i}{\lambda}\mathrm{E}[g(Y_{i,k})]$$

and

$$\begin{aligned}\mathrm{E}[g(h_i^{\text{cmrs}}(Y_k))]&=\mathrm{E}\big[g\big(Y_k\beta_i(Y_k)\big)\big]\\&=\mathrm{E}\big[g\big((1-\beta_i(Y_k))\times 0+\beta_i(Y_k)Y_k\big)\big]\\&\leq \mathrm{E}\big[(1-\beta_i(Y_k))\,g(0)+\beta_i(Y_k)g(Y_k)\big]\\&=\mathrm{E}[1-\beta_i(Y_k)]\,g(0)+\mathrm{E}[\beta_i(Y_k)g(Y_k)]\\&=\left(1-\frac{\lambda_i}{\lambda}\right)g(0)+\frac{\lambda_i}{\lambda}\mathrm{E}[g(Y_{i,k})]\end{aligned}$$

since

$$\mathrm{E}[1-\beta_i(Y_k)]=1-\int_0^\infty \beta_i(y)\left(\sum_{j=1}^n\frac{\lambda_j}{\lambda}f_j(y)\right)\mathrm{d}y=1-\frac{\lambda_i}{\lambda}$$

and

$$\begin{aligned}\mathrm{E}[\beta_i(Y_k)g(Y_k)]&=\int_0^\infty \beta_i(y)g(y)\left(\sum_{j=1}^n\frac{\lambda_j}{\lambda}f_j(y)\right)\mathrm{d}y\\&=\frac{\lambda_i}{\lambda}\int_0^\infty g(y)f_i(y)\mathrm{d}y\\&=\frac{\lambda_i}{\lambda}\mathrm{E}[g(Y_{i,k})].\end{aligned}$$

This shows that $h_i^{\text{cmrs}}(Y_k)\preceq_{\text{CX}} Y'_{i,k}$ holds for any k. The announced result then follows from the stability of the convex order under convolution. This ends the proof. □

We are now ready to state the main result of this section, showing that pooling reduces infinite-time default probabilities.

Proposition 3.3. *For all $\kappa_i\geq 0$, we have $\psi_i^{\text{pool}}(\kappa_i)\leq\psi_i(\kappa_i)$.*

Proof. Lemma 3.1 allows us to write

$$\psi_i(\kappa_i)=\mathrm{P}[V'_{i,t}<0 \text{ for some } t\geq 0] \text{ where } V'_{i,t}=c_i t-S'_{i,t}+\kappa_i.$$

Infinite-time default probabilities are ordered for the processes $V_{i,t}$ and $V'_{i,t}$ if the corresponding severities are ordered in the convex order $\preceq_{\text{CX}}$. The result then follows from Lemma 3.2 for $k=1$. □

In practice, pools often operate over a finite time horizon, exactly as insurance covers run over successive one-year periods. Let us now consider a pool operating over a finite time horizon $(0,m)$ for some maturity $0<m<\infty$.

When $\kappa_i=0$ for all i, Beekman formula shows that $\psi_i^{\text{pool}}(0)$ and $\psi_i(0)$ only depend on η, so that we have $\psi_i^{\text{pool}}(0)=\psi_i(0)$. Hence, pooling has no effect on infinite-time default probabilities. Finite-time default probabilities for participant i are defined as

$$\psi_i(\kappa_i,m)=\mathrm{P}[V_{i,t}<0 \text{ for some } 0\leq t\leq m]$$

if he or she stands alone and as

$$\psi_i^{\text{pool}}(\kappa_i,m)=\mathrm{P}[V_{i,t}^{\text{pool}}<0 \text{ for some } 0\leq t\leq m]$$

if he or she joins the pool.

The next result questions the benefits of risk pooling when starting with zero deposits.

Proposition 3.4. *For all $m\geq 0$, we have $\psi_i(0,m)\leq\psi_i^{\text{pool}}(0,m)$.*

Proof. Takacs-type formula provides us with the following convenient representation of the finite-time default probability when $\kappa_i=0$:

$$\begin{aligned}\psi_i(0,m)&=1-\frac{1}{c_i m}\mathrm{E}[(c_i m-S_{i,m})_+]\\ \psi_i^{\text{pool}}(0,m)&=1-\frac{1}{c_i m}\mathrm{E}[(c_i m-S_{i,m}^{\text{pool}})_+].\end{aligned}$$

See Proposition 3.2 in Lefèvre and Loisel (2008). Since the convex order is stable under mixture, we deduce from Lemma 3.2 that $S_{i,m}^{\text{pool}}\preceq_{\text{CX}} S_{i,m}$ which implies

$$\mathrm{E}[(s-S_{i,m}^{\text{pool}})_+]\leq\mathrm{E}[(s-S_{i,m})_+] \text{ for all } s,$$

so that we end up with the announced result. □

Proposition 3.4 shows that over a finite time interval, default probability favors larger risks in the convex order, when initial deposit is 0. This can be understood as follows. Without initial deposit, default typically occurs when a claim is filed rapidly, at a time when the surplus is still small. What matters is thus the left-part of the graph of the excess function, for relatively small values of the argument, and risks dominating in the convex order tend to have a heavier left tail. Proposition 3.4 shows that a positive initial deposit is required, to avoid that pooling becomes detrimental in terms of default probabilities.

Numerical illustrations in Cheung et al. (2023) show that this may also happen for initial capital close to 0. See Examples 4.2, 4.4 and 4.5 in that paper where the finite-time ruin probability decreases when the variance of claim severities increases for small initial capital. Considering for instance Example 4.2 in that paper, let J, W_1 and W_2 be independent random variables with $\mathrm{P}[J=1]=1-\mathrm{P}[J=0]=\frac{1}{3}$, W_1 obeying the Negative Exponential distribution with mean 2 and W_2 obeying the Negative Exponential distribution with mean 0.5. Claim severities are then distributed as $\mathrm{E}[J]W_1+(1-\mathrm{E}[J])W_2$, that is, as a sum of independent, Negative Exponentially distributed random variables, or as a mixture $JW_1+(1-J)W_2$ of Negative Exponentially distributed random variables. The variance is higher in the latter case, which also corresponds to larger severities in the convex order since

$$\mathrm{E}[J]W_1+(1-\mathrm{E}[J])W_2=\mathrm{E}\big[JW_1+(1-J)W_2\big|W_1,W_2\big].$$

Tables 9–10 in Cheung et al. (2023) show that for an initial capital equal to 1, ruin probabilities over time horizons 2, 4, 6, 8, 10, 20, and 40 are larger with the sum compared to the mixture.

In general, $\psi_i(\kappa_i,m)$ and $\psi_i^{\text{pool}}(\kappa_i,m)$ cannot be compared for $\kappa_i>0$. But even if finite-time default probabilities do not necessarily recognize the beneficial aspect of pooling, their integrals do so, as shown in the next section.

3.2. Largest excess of claims over contributions

Pooling reduces timing risk, that is, the risk that accumulated surplus is too low to face losses occurring rapidly. Define the non-negative random variables

$$L_i^{\text{pool}}=\sup_{t\geq 0}(S_{i,t}^{\text{pool}}-c_it)\text{ and }L_i=\sup_{t\geq 0}(S_{i,t}-c_it)$$

as well as

$$L_i^{\text{pool}}(m)=\sup_{0\leq t\leq m}(S_{i,t}^{\text{pool}}-c_it)\text{ and }L_i(m)=\sup_{0\leq t\leq m}(S_{i,t}-c_it)$$

corresponding to the largest excesses of claims over collected contributions over an infinite or finite time horizon, respectively. The next result shows that pooling is effective in improving these excesses.

Proposition 3.5. *We have*

$$\mathrm{P}[L_i^{\text{pool}}>s]\leq \mathrm{P}[L_i>s]\text{ for all }s\geq 0$$

and

$$\mathrm{E}\left[\left(L_i^{\text{pool}}(m)-s\right)_+\right]\leq \mathrm{E}\left[(L_i(m)-s)_+\right]\text{ for all }s\geq 0.$$

Proof. We have

$$\mathrm{P}[L_i^{\text{pool}}>s]=\psi_i^{\text{pool}}(s)\text{ and }\mathrm{P}[L_i>s]=\psi_i(s).$$

The announced result then follows from Proposition 3.3. Considering a finite time horizon, Proposition 2 in Lefèvre et al. (2017) shows that for all $s\geq 0$ and $m>0$,

$$\int_s^\infty \psi_i^{\text{pool}}(z,m)\,\mathrm{d}z\leq\int_s^\infty \psi_i(z,m)\,\mathrm{d}z.$$

Since

$$\psi_i^{\text{pool}}(z,m)=\mathrm{P}\left[L_i^{\text{pool}}(m)>z\right]\text{ and }\psi_i(z,m)=\mathrm{P}[L_i(m)>z],$$

we obtain the announced inequality between stop-loss premiums. This ends the proof. □

Over an infinite time horizon, Proposition 3.5 shows that pooling succeeds in reducing the largest excess of claims over contributions in the usual stochastic order. This is considered as being beneficial by all economic agents. Over a finite time horizon $(0,m)$, Proposition 3.5 shows that the stop-loss premiums for $L_i^{\text{pool}}(m)$ are smaller than the corresponding stop-loss premiums for $L_i(m)$. This means that pooling succeeds in reducing the largest excess of claims over contributions until maturity in the stop-loss order.

3.3. Risk-reducing effect of pooling

The following result shows that pooling improves the total amount of losses and the amounts on the individual insurance accounts, at all time points.

Proposition 3.6. *We have* $V_{i,t}^{\text{pool}}\preceq_{\text{CX}}V_{i,t}$ *for all* $t\geq 0$.

Proof. Since the convex order is stable under mixture, we know from Lemma 3.2 that $S_{i,m}^{\text{pool}}\preceq_{\text{CX}}S_{i,m}$ holds true, so that

$$-S_{i,t}^{\text{pool}}\preceq_{\text{CX}}-S_{i,t}\text{ for all }t\geq 0\Rightarrow V_{i,t}^{\text{pool}}\preceq_{\text{CX}}V_{i,t}\text{ for all }t\geq 0,$$

where the implication follows from the stability of the convex order under constant shifts. This ends the proof. □

Proposition 3.6 shows that every risk-averse economic agent prefers holding an insurance account with amount $V_{i,t}^{\text{pool}}$ over an insurance account with amount $V_{i,t}$, for all $t\geq 0$.

3.4. Numerical illustration

In this section, we illustrate the results derived earlier with the help of a simple example. Let us simulate Poisson rates λ_i as realizations of independent random variables with a common Gamma distribution with mean equal to 0.1 and variance equal to 0.001 (shape of 10 and scale of 0.01). The severity distribution is identical for all participants and given by the Negative Exponential distribution with parameter ν. Here, we set $\nu=1$, without loss of generality. Then, $S_{i,t}^{\text{pool}}$ follows a compound Poisson process with intensity $\lambda=\sum_{j=1}^n\lambda_j$ and Negative Exponential severity distribution with parameter equal to $\frac{\nu}{\lambda_i}\lambda$.

This particular setting has been selected because analytic expressions are available for the ruin probabilities in both individual and pooled surplus models. Notice that the pooling of losses is proportional when severity distributions are identical for all participants, as assumed here: if $f_1=f_2,\ldots=f_n$ then Proposition 2.2 shows that

$$h_i^{\text{cmrs}}(y)=\frac{\lambda_i}{\lambda}y.$$

Losses are thus shared proportionally, according to the respective expected claim frequencies. With pooling, contributions must be paid more often, at rate λ, but the amounts are scaled downwards by the factor λ_i/λ to adapt to participants' propensities to report losses. This has no impact on the average payment over time but reduces its variability in a way considered as being beneficial by all risk-averse agents (whose common preferences generate $\preceq_{\text{CX}}$). Any other proportional rule would modify expected payments and be detrimental for at least one participant.

Henceforth, we indicate the relevant parameters entering the calculation of ruin probabilities as additional arguments compared to the notation adopted before (appearing after a semi-colon). The infinite-time default probabilities for participant i are given by

$$\psi_i(\kappa_i;\nu) = \frac{1}{1+\eta}\exp\left(-\frac{\eta}{1+\eta}\nu\kappa_i\right)$$
$$\psi_i^{\text{pool}}(\kappa_i;\lambda_i,\nu,\lambda) = \psi_i\left(\kappa_i;\frac{\nu}{\lambda_i}\lambda\right) = \frac{1}{1+\eta}\exp\left(-\frac{\eta}{1+\eta}\frac{\nu}{\lambda_i}\lambda\kappa_i\right).$$

Theorem 5.6.3 in Rolski et al. (1999) shows that the finite-time default probabilities for the same participant are given by

$$\psi_i(\kappa_i,m;\lambda_i,\nu) = 1 - \mathrm{e}^{-\nu\kappa_i-(2+\eta)\lambda_i m} g\big(\nu\kappa_i + (1+\eta)\lambda_i m, \lambda_i m\big)$$

where the function $g(\cdot)$ is defined as

$$g(z,\theta) = J(\theta z) + \theta J^{(1)}(\theta z) + \int_0^z \mathrm{e}^{z-v} J(\theta v)\,\mathrm{d}v - \frac{1}{1+\eta}\int_0^{(1+\eta)\theta} \mathrm{e}^{(1+\eta)\theta - v} J\left(\frac{zv}{1+\eta}\right)\mathrm{d}v,$$

with $J(x) = I_0\left(2\sqrt{x}\right) = \sum_{n=0}^{\infty} x^n/(n!)^2$ and

$$\psi_i^{\text{pool}}(\kappa_i,m;\lambda_i,\nu,\lambda) = \psi_i\left(\kappa_i,m;\lambda,\frac{\nu}{\lambda_i}\lambda\right).$$

Let us now consider a pool with $n = 4$ participants. The respective Poisson rates are 0.1190, 0.0980, 0.1103, and 0.1139, and we set $\eta = 0.4$. The results are given for participant 1. Calculations are performed with the software R, using the package Bessel.

Fig. 1 displays the default probabilities as functions of the initial deposit κ_1, for selected time horizons (1 year, 50 years and ultimate). Over infinite time horizon ($m = \infty$), we can see there that the default probabilities without pooling ψ_i printed in black are larger than default probabilities with pooling ψ_i^{pool} printed in red, whatever the amount of initial deposit. Over finite time horizons however (when $m = 1$ and $m = 50$), a crossing of default probabilities ψ_i and ψ_i^{pool} is visible. For small amounts of deposit, ψ_i^{pool} dominates ψ_i when $m < \infty$, as expected from Proposition 3.4 and the comments made earlier. Pooling implies that participants contribute more often compared to the situation where they stay alone and pay for their own losses. With zero initial deposits, ruin typically occurs early, at a time when accumulated surplus is not high enough to cover first losses. Since contributions tend to be due earlier with pooling, this tends to increase ruin probabilities with zero deposit and provides an explanation to the detrimental effect of pooling in that case. This also applies to low levels of initial surplus but the phenomenon disappears when deposits get large enough.

4. Negative balance and surplus at maturity

Consider a pool with n participants operating as explained in Sections 2–3. We assume that when $V_t < 0$, the pool can borrow money until it recovers. Notice that only the aggregate V_t matters here and we do not consider individual accounts $V_{i,t}^{\text{pool}}$ summing to V_t according to (2.2). The cost of the loan covering $V_t < 0$ must be allocated to participants, each of them paying an extra contribution so that the pool can pay interests. This question is addressed in Section 4.1. If the pool operates over a finite time horizon, it may end up with positive surplus to be shared among participants. Likewise, if the pool terminates in deficit then the latter must also be shared among participants, who are charged an additional contribution at maturity. The way to allocate terminal surplus is studied in Section 4.2.

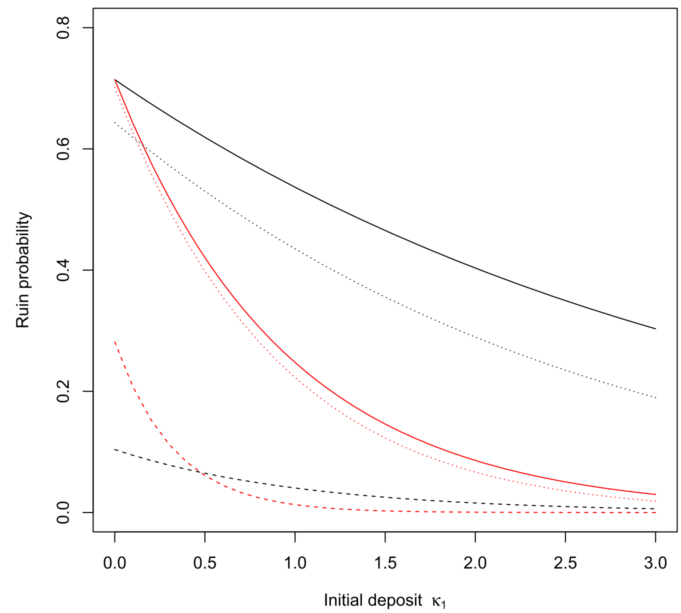

Fig. 1. Default probabilities as functions of the initial deposit κ_1 for participant 1, over time horizons $m = \infty$ (solid line), $m = 1$ (dashed line), and $m = 50$ (dotted line). Default probabilities without pooling printed in black and default probabilities with pooling printed in red. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

4.1. Sharing interest on negative balance

The surplus V_t may become negative at each occurrence time T_k, and remain negative for some time before it recovers (this period is referred to as time in red in the literature). In this case, participants borrow amounts $|V_t|$ as long as $V_t < 0$, paying interest rate until the fund recovers, in addition to individual contributions. This is similar to a finite risk program where participants' contributions are accumulated in an "experience account" out of which losses up to a specified amount are paid when they occur. The partnering insurer fills up the experience account if it has a negative balance and charges the participant a specified interest on the funds added to the account. Funds added to an experience account with a negative balance are thus debited at an agreed rate. This approach is standard when dealing with insurance captives. See e.g. Maeda et al. (2011).

Let δ be the force of interest to be paid by participants when $V_t < 0$. Here, we assume that the functions $v \mapsto \mathrm{E}[V_{i,t}|V_t = v]$ are continuously increasing (and thus one-to-one) for every $i \in \{1,2,\ldots,n\}$ and $t \geq 0$. This imposes a kind of solidarity among participants as no individual account may increase on average given that the pooled fund decreases. This condition is similar to the no-sabotage requirement in risk sharing, meaning that allocations must be comonotonic. It is equivalent to requiring that the functions $s \mapsto \mathrm{E}[S_{i,t}|S_t = s]$ are continuously increasing for every $i \in \{1,2,\ldots,n\}$ and $t \geq 0$. This condition is generally fulfilled provided the number n of participants is large enough, as demonstrated in Denuit and Robert (2021).

Participants must pay interest when V_t is negative, at rate

$$\delta|V_t|\mathrm{I}[V_t < 0] = \delta(S_t - \kappa - ct)_+.$$

Under the no-sabotage requirement, this can be decomposed into

$$\delta|V_t|\mathrm{I}[V_t < 0] = \sum_{i=1}^n \delta\Big(w_{i,t} - \mathrm{E}[V_{i,t}|V_t]\Big)_+$$

with

$w_{i,t} = \mathrm{E}[V_{i,t}|V_t = 0] = \kappa_i + c_i t - \mathrm{E}[S_{i,t}|S_t = \kappa + ct]$.

Alternatively,

$$\delta|V_t|\mathrm{I}[V_t < 0] = \delta\sum_{i=1}^{n}\Big(\mathrm{E}[S_{i,t}|S_t] - w'_{i,t}\Big)_+$$

where $w'_{i,t} = \mathrm{E}[S_{i,t}|S_t = \kappa + ct]$. The contribution rate for participant i then becomes

$$c_i + c_i^+(t)$$

where $c_i^+(t) = \delta\Big(w_{i,t} - \mathrm{E}[V_{i,t}|V_t]\Big)_+ = \delta\Big(\mathrm{E}[S_{i,t}|S_t] - w'_{i,t}\Big)_+$ as long as $V_t < 0$.

Notice that the dynamics of the pooled fund V_t and individual accounts $V_{i,t}$ and $V_{i,t}^{\text{pool}}$ is not modified. This is because interests at rate $c_i^+(t)$ are paid in addition to regular contributions c_i, in favor of the partnering institution financing the debt $V_t < 0$.

Denuit and Robert (2020) established representations for conditional mean risk sharing of compound Poisson losses, which can be used in the present context. With $v_t < 0$, we have

$$\begin{aligned}&\mathrm{E}\left[V_{i,t}|V_t = v_t\right]\\ &= \kappa_i + c_i t - \mathrm{E}\left[S_{i,t}|S_t = \kappa + ct - v_t\right]\\ &= \kappa_i + c_i t - \frac{\lambda_i \mathrm{E}\left[Y_{i,1}\right] f_{S_t+\widetilde{Y}_{i,1}}(\kappa + ct - v_t)}{\sum_{j=1}^{n}\lambda_j \mathrm{E}\left[Y_{j,1}\right] f_{S_t+\widetilde{Y}_{j,1}}(\kappa + ct - v_t)}(\kappa + ct - v_t)\end{aligned}$$

where the random variables $\widetilde{Y}_{j,1}$ are independent of S_t, with respective probability density function

$$\widetilde{f}_j(y) = \frac{y}{\mathrm{E}\left[Y_{j,1}\right]} f_j(y), \quad y > 0, \quad j = 1, 2, \dots, n,$$

corresponding to the size-biased distribution associated to F_i. This formula allows participants to compute $c_i^+(t)$ once V_t is known to be equal to $v_t < 0$.

Remark 4.1. Notice that in the proposed system, we do not penalize participants with $V_{i,t}^{\text{pool}} < 0$ as long as $V_t > 0$. Interest is charged to all participants when $V_t < 0$, each one contributing $c_i^+(t)$ in addition to the initial c_i as long as V_t remains negative. This provides an added benefit of joining the pool. Notice also that $c_i^+(t)$ accounts for each participant's risk profile.

As an alternative, we might consider another system where participants are penalized as soon as their individual account becomes negative. Thus, participant i would be forced to pay extra contribution over c_i as soon as $V_{i,t}^{\text{pool}} < 0$, even when $V_t > 0$. This possibility is not considered further in this paper and left for future research.

4.2. Sharing accumulated surplus at maturity

Participants share losses occurring over $(0, m)$ according to conditional mean risk-sharing rule and cover deficit by paying interest as long as $V_t < 0$. When the pool operates over a finite time horizon, participants also share the surplus at maturity, if positive. Under the no-sabotage condition, the allocation of $V_m > 0$ at time m can be based on the decomposition

$$V_m\mathrm{I}[V_m > 0] = \sum_{i=1}^{n}\Big(\mathrm{E}[V_{i,m}|V_m] - w_{i,m}\Big)_+$$

where $w_{i,m} = \mathrm{E}[V_{i,m}|V_m = 0]$,

or

$$(\kappa + cm - S_m)_+ = \sum_{i=1}^{n}(w'_{i,m} - \mathrm{E}[S_{i,m}|S_m])_+$$

where $w'_{i,m} = \mathrm{E}[S_{i,m}|S_m = \kappa + cm]$.

Any deficit at maturity must also be shared among participants. This can be done according to the decomposition

$$(V_m)_- = |V_m|\mathrm{I}[V_m < 0] = \sum_{i=1}^{n}\Big(w_{i,m} - \mathrm{E}[V_{i,m}|V_m]\Big)_+$$

or

$$(S_m - \kappa - cm)_+ = \sum_{i=1}^{n}(\mathrm{E}[S_{i,m}|S_m] - w'_{i,m})_+.$$

5. Large pools

5.1. Motivation

Diversification typically increases when the number of independent risks comprised in the pool gets larger. This is also the case here: within large pools, that is, letting the number of participants n increase, individual risk disappears at the limit under mild conditions. This can be seen as follows. Assume for instance that $\mathrm{E}[Y_{i,1}^2] < \infty$ for all i and that, for any y such that $f_i(y) > 0$, $\sum_{j=1}^{n}\lambda_j f_j(y) \to \infty$ as $n \to \infty$. Then, the variance of $S_{i,t}^{\text{pool}}$ given by

$$\begin{aligned}&\lambda t\int_0^\infty\left(y\frac{\lambda_i f_i(y)}{\sum_{j=1}^{n}\lambda_j f_j(y)}\right)^2\left(\sum_{j=1}^{n}\frac{\lambda_j}{\lambda}f_j(y)\right)\mathrm{d}y\\ &= \lambda_i t\int_0^\infty y^2\frac{\lambda_i f_i(y)}{\sum_{j=1}^{n}\lambda_j f_j(y)}f_i(y)\mathrm{d}y\end{aligned}$$

tends to 0 by the dominated convergence theorem. It follows that

$$V_{i,t}^{\text{pool}} \xrightarrow{\mathrm{d}} \eta\lambda_i\mathrm{E}[Y_{i,1}]t + \kappa_i, \quad t \geq 0,$$

and that $\psi_i^{\text{pool}}(\kappa_i) \to 0$ as $n \to \infty$. Diversification increases with the number of participants and this section studies how infinite-time and finite-time default probabilities decrease towards 0.

5.2. Assumptions

We need some additional assumptions to study the asymptotic behavior of default probabilities in the pool. Precisely, we will use the following set of assumptions.

Assumption 5.1. We assume that there exist two positive constants λ_∞ and c_∞ and a limit distribution function F_∞ such that $\lambda/n \to \lambda_\infty$, $c/n \to c_\infty$ and $F \xrightarrow{\mathrm{d}} F_\infty$ as $n \to \infty$. Moreover, we assume that the moment generating function $\mathrm{L}_\infty(\nu) = \int_0^\infty \mathrm{e}^{\nu y}\mathrm{d}F_\infty(y)$ of F_∞ is such that either

$$\mathrm{L}_\infty(\nu) < \infty \text{ for all } \nu < \infty,$$

or there exists $\nu_\infty < \infty$ such that

$$\mathrm{L}_\infty(\nu) < \infty \text{ for all } \nu < \nu_\infty \text{ and } \mathrm{L}_\infty(\nu) = \infty \text{ for all } \nu \geq \nu_\infty.$$

Let us briefly comment on Assumption 5.1. The existence of a moment generating function for the limit implies that this is also the case for all individual distributions with probability density functions f_i. The convergence in distribution of F guarantees that the moment generating functions converge pointwise, but the other conditions appearing in Assumption 5.1 may not be fulfilled.

Under Assumption 5.1, there exists n_1, such that, for any $n \geq n_1$, an adjustment coefficient $R > 0$ (also known as Lundberg's exponent) satisfying

$$1 + \frac{c}{\lambda} R = \mathrm{L}(R) = \int_0^\infty \mathrm{e}^{Ry} \mathrm{d}F(y)$$

exists and is unique. Moreover $R \to R_\infty$ as $n \to \infty$, where R_∞ is defined as the solution of

$$1 + \frac{c_\infty}{\lambda_\infty} R_\infty = \mathrm{L}_\infty(R_\infty).$$

For the asymptotic behavior of individual default probabilities, we will use the following set of assumptions.

Assumption 5.2. We assume that there exist a positive constant λ_∞ and a probability density function f_∞ such that $\lambda/n \to \lambda_\infty$, $f \to f_\infty$ uniformly as $n \to \infty$. Let

$$Z_i = Y \frac{\lambda_i f_i(Y)}{\lambda_\infty f_\infty(Y)}$$

where Y denotes a random variable with probability density function f_∞. We assume that the moment generating function $\mathrm{L}_{Z_i}(\nu) = \int_0^\infty \mathrm{e}^{\nu y} \mathrm{d}F_{Z_i}(y)$ of F_{Z_i} is such that either

$$\mathrm{L}_{Z_i}(\nu) < \infty \text{ for all } \nu < \infty,$$

or there exists $\nu_i < \infty$ such that

$$\mathrm{L}_{Z_i}(\nu) < \infty \text{ for all } \nu < \nu_i \text{ and } \mathrm{L}_{Z_i}(\nu) = \infty \text{ for all } \nu \geq \nu_i.$$

Considering Assumption 5.2, it is indeed possible that one of the probability density functions f_i is not light-tailed and that f_∞ exists. But in such a case, the moment generating function of Z_i would not exist. It is therefore necessary that each density f_i be light-tailed (and hence has all moments).

Under Assumption 5.2, there exists $n_{1,i}$, such that for any $n \geq n_{1,i}$, an adjustment coefficient $R_i > 0$ satisfying

$$1 + \frac{c_i}{(\lambda/n)} R_i = \int_0^\infty \exp\left(R_i y \frac{\lambda_i f_i(y)}{(\lambda/n) \sum_{j=1}^n (\lambda_j/\lambda) f_j(y)} \right) \mathrm{d}F(y)$$

exists and is unique. Moreover $R_i \to R_{i,\infty}$ as $n \to \infty$, where $R_{i,\infty}$ is defined as the solution of

$$1 + \frac{c_i}{\lambda_\infty} R_{i,\infty} = \mathrm{L}_{Z_i}\left(R_{i,\infty}\right).$$

5.3. Infinite time horizon

Define

$$\breve{V}_t = (c/n)t - \breve{S}_t + \kappa \text{ with } \breve{S}_t = \sum_{k=1}^{\breve{N}_t} Y_k$$

where $\{\breve{N}_t,\ t \geq 0\}$ is a Poisson process with intensity λ/n. The infinite-time default probability of the pool $\psi(\kappa) = \mathrm{P}[V_t < 0$ for some $t \geq 0]$ is then also equal to the probability

$$\breve{\psi}(\kappa) = \mathrm{P}[\breve{V}_t < 0 \text{ for some } t \geq 0].$$

Define

$$C = \frac{\eta/(1+\eta)}{(\lambda/c)\mathrm{L}'(R) - 1}$$

and notice that $C \to C_\infty$ as $n \to \infty$, where

$$C_\infty = \frac{\eta/(1+\eta)}{(\lambda_\infty/c_\infty)\mathrm{L}'(R_\infty) - 1}.$$

Proposition 5.3. *If Assumption 5.1 holds and $\kappa \to \infty$ as $n \to \infty$ then*

$$\psi(\kappa) \sim C \exp(-R\kappa) \text{ as } n \to \infty.$$

Proof. By Theorem IV.5.3 in Asmussen and Albrecher (2010), for any (fixed) n, as $z \to \infty$,

$$\breve{\psi}(z) \sim C \exp(-Rz).$$

Let

$$\breve{\psi}_\infty(\kappa) = \mathrm{P}[\breve{V}_t^{(\infty)} < 0 \text{ for some } t \geq 0]$$

where $\{\breve{V}_t^{(\infty)},\ t \geq 0\}$ is the surplus process for which c/n and λ/n have been replaced by c_∞ and λ_∞, respectively. Let $\varepsilon > 0$. For each $n \geq n_1$, there exists $z_n < \infty$ such that, for $z > z_n$,

$$\left| \frac{\breve{\psi}(z)}{C\exp(-Rz)} - 1 \right| < \varepsilon.$$

Moreover by Assumption 5.1, there exists $z_\infty < \infty$ such that for $z > z_\infty$

$$\left| \frac{\breve{\psi}_\infty(z)}{C_\infty \exp(-R_\infty z)} - 1 \right| < \varepsilon.$$

The sequence (z_n) is bounded and converges to z_∞. Therefore there exists $n_2 \geq n_1$ such that, for $n \geq n_2$, $\kappa > \max_{n \geq n_1} z_n$. We deduce that, for $n \geq n_2$,

$$\left| \frac{\breve{\psi}(\kappa)}{C\exp(-R\kappa)} - 1 \right| < \varepsilon,$$

and the announced result follows. □

Remark 5.4. Notice that, since $C \to C_\infty$, we also have

$$\psi(\kappa) \sim C_\infty \exp(-R\kappa) \quad \text{as } n \to \infty.$$

It is worth to stress that R can not be replaced by R_∞ in general. Silvestrov (2014) presents several asymptotic results for perturbed risk processes that can be used to replace R by a power series expansion in $1/n$ (see Theorem 5.4 there). However stronger assumptions are needed, in particular on the rate of convergences of the sequences $(\lambda/n)_{n\geq 1}$, $(c/n)_{n\geq 1}$ and $(\mathrm{L}(R_\infty))_{n\geq 1}$.

Let us now consider individual infinite-time default probability when joining the pool. As we did at the pool level, define

$$\breve{V}_{i,t}^{\text{pool}} = c_i t - \breve{S}_{i,t}^{\text{pool}} + n\kappa_i \text{ with } \breve{S}_{i,t}^{\text{pool}} = \sum_{k=1}^{\breve{N}_t} n h_i^{\text{cmrs}}(Y_k).$$

Default probability $\psi_i^{\text{pool}}(\kappa_i) = \mathrm{P}[V_{i,t}^{\text{pool}} < 0$ for some $t \geq 0]$ is then also equal to the probability

$$\breve{\psi}_i^{\text{pool}}(\kappa_i) = \mathrm{P}[\breve{V}_{i,t}^{\text{pool}} < 0 \text{ for some } t \geq 0].$$

Let

$$C_i = \frac{\eta/(1+\eta)}{(\lambda/(nc_i))\mathrm{L}'_{nh_i^{\text{cmrs}}(Y_k)}(R_i) - 1}$$

and notice that $C_i \to C_{i,\infty}$ as $n \to \infty$, where

$$C_{i,\infty} = \frac{\eta/(1+\eta)}{(\lambda_\infty/c_i)\mathrm{L}'_{Z_i}\left(R_{i,\infty}\right) - 1}.$$

Proposition 5.5. *If Assumption 5.2 holds then*

$$\psi_i^{\text{pool}}(\kappa_i) \sim C_i \exp(-R_i n \kappa_i) \text{ as } n \to \infty.$$

Proof. By Assumption 5.2, we have for ν such that $\mathrm{L}_{Z_i}(\nu) < \infty$ that

$$\mathrm{L}_{nh_i^{\text{cmrs}}(Y_k)}(\nu) \to \mathrm{L}_{Z_i}(\nu) \text{ as } n \to \infty.$$

We can then use the same type of arguments as in the proof of Proposition 5.3 to conclude. □

Remark 5.6. We also have that

$$\psi_i^{\text{pool}}(\kappa_i) \sim C_{i,\infty} \exp(-R_i n \kappa_i) \text{ as } n \to \infty.$$

5.4. Finite time horizon

The finite-time default probability for the pool $\psi(\kappa, m) = \mathrm{P}[V_t < 0 \text{ for some } 0 \le t \le m]$ is also equal to

$$\breve{\psi}(\kappa, nm) = \mathrm{P}[\breve{V}_t < 0 \text{ for some } 0 \le t \le nm].$$

Let us assume that there exists $\kappa_\infty > 0$ such that $\kappa/n \to \kappa_\infty$ as $n \to \infty$ and we write

$$\breve{\psi}(\kappa, nm) = \breve{\psi}(\kappa, \kappa y)$$

where $y = m/(\kappa/n) \to y_\infty = m/\kappa_\infty$ as $n \to \infty$. Define

$$\beta_F(r) = \frac{\lambda}{c}\left(\int_0^\infty e^{ry}\mathrm{d}F(y) - 1\right) - r \quad \text{and}$$

$$\beta_{F_\infty}(r) = \frac{\lambda_\infty}{c_\infty}\left(\int_0^\infty e^{ry}\mathrm{d}F_\infty(y) - 1\right) - r.$$

Under Assumption 5.1, for r such that $\beta_{F_\infty}(r) < \infty$, then $\beta_F(r) \to \beta_{F_\infty}(r)$ as $n \to \infty$.

Let us now define (α_y, R_y) by

$$\beta'_F(\alpha_y) = \frac{1}{y}, \quad R_y = \alpha_y - y\beta_F(\alpha_y)$$

and $(\alpha_{y_\infty}, R_{y_\infty})$ by

$$\beta'_{F_\infty}(\alpha_{y_\infty}) = \frac{1}{y_\infty}, \quad R_{y_\infty} = \alpha_{y_\infty} - y_\infty\beta_{F_\infty}(\alpha_{y_\infty}).$$

Proposition 5.7. *If Assumption 5.1 holds and $\kappa/n \to \kappa_\infty$ as $n \to \infty$ then*

(i) *If $y_\infty < 1/\beta'_{F_\infty}(R_\infty)$ then there exists n_2 such that for $n \ge n_2$ the solution $\tilde{\alpha}_y < \alpha_y$ of $\beta_F(\tilde{\alpha}_y) = \beta_F(\alpha_y)$ is negative, and as $n \to \infty$*

$$\psi(\kappa, m) \sim \frac{\alpha_y - \tilde{\alpha}_y}{\alpha_y|\tilde{\alpha}_y|\sqrt{2\pi(\lambda/c)y\mathrm{L}''(\alpha_y)}} \frac{e^{-R_y\kappa}}{\sqrt{\kappa}}.$$

(ii) *If $y_\infty > 1/\beta'_{F_\infty}(R_\infty)$, then there exists n_3 such that for $n \ge n_3$ the solution $\tilde{\alpha}_y$ is positive, and as $n \to \infty$*

$$\psi(\kappa) - \psi(\kappa, m) \sim \frac{\alpha_y - \tilde{\alpha}_y}{\alpha_y|\tilde{\alpha}_y|\sqrt{2\pi(\lambda/c)y\mathrm{L}''(\alpha_y)}} \frac{e^{-R_y\kappa}}{\sqrt{\kappa}}.$$

Proof. Under Assumption 5.1, $\beta'_F(R) \to \beta'_{F_\infty}(R_\infty)$ as $n \to \infty$, and since $\kappa/n \to \kappa_\infty$ as $n \to \infty$, we also have $y \to y_\infty$.

Consider statement (i). If $y_\infty < 1/\beta'_{F_\infty}(R_\infty)$, then there exists n_2 such that, for $n \ge n_2$, $y < 1/\beta'_F(R)$. By Theorem V.4.9 in Asmussen and Albrecher (2010), for any (fixed) $n \ge n_2$, as $z \to \infty$,

$$\breve{\psi}(z, zy) \sim \frac{\alpha_y - \tilde{\alpha}_y}{\alpha_y|\tilde{\alpha}_y|\sqrt{2\pi(\lambda/c)y\mathrm{L}''(\alpha_y)}} \frac{e^{-R_y z}}{\sqrt{z}}.$$

Using the same type of arguments as in the proof of Proposition 5.3, we can conclude that, as $n \to \infty$,

$$\begin{aligned}\psi(\kappa, m) &= \breve{\psi}(\kappa, nm)\\ &= \breve{\psi}(\kappa, \kappa y) \sim \frac{\alpha_y - \tilde{\alpha}_y}{\alpha_y|\tilde{\alpha}_y|\sqrt{2\pi(\lambda/c)y\mathrm{L}''(\alpha_y)}} \frac{e^{-R_y\kappa}}{\sqrt{\kappa}}.\end{aligned}$$

Turning to statement (ii), if $y_\infty > 1/\beta'_{F_\infty}(R_\infty)$, the same type of reasoning and Theorem V.4.9 in Asmussen and Albrecher (2010) lead to, as $n \to \infty$

$$\psi(\kappa) - \psi(\kappa, m) \sim \frac{\alpha_y - \tilde{\alpha}_y}{\alpha_y|\tilde{\alpha}_y|\sqrt{2\pi(\lambda/c)y\mathrm{L}''(\alpha_y)}} \frac{e^{-R_y\kappa}}{\sqrt{\kappa}}.$$

This ends the proof. □

Equivalent forms for individual finite-time default probabilities can also be derived under Assumption 5.2.

5.5. Numerical illustration

Let us continue with the example of Section 3.4. For any $n \ge 1$, let us rewrite the infinite-time ruin probability as

$$\psi_i^{\text{pool}}(\kappa_i; \lambda_i, \nu, \lambda) = \frac{1}{1+\eta} \exp(-R_i n \kappa_i)$$

with

$$R_i = \frac{\eta}{1+\eta} \frac{\nu\lambda}{n\lambda_i}.$$

For large pools, $\lambda/n \to \lambda_\infty = 0.1$ and Z_i obeys the Negative Exponential distribution with parameter $\frac{\lambda_\infty}{\lambda_i}\nu$. This is clearly in line with the result stated under Proposition 5.5.

Fig. 2 displays default probabilities as functions of the number n of participants. Again, we only consider participant 1. Results are represented for three levels of initial deposit: $\kappa_1 = 0.05$, 0.1, and 0.5. We consider different time horizons, those in Fig. 1 supplemented with an intermediate value; specifically, results for $m = \infty$, $m = 1$, $m = 10$, and $m = 50$ are represented in Fig. 2. We can see there that the default probabilities with pooling ψ_i^{pool} increase as m increases, as expected. However, default probabilities are not always decreasing functions of the number of participants except if $m = \infty$. They are nevertheless ultimately decreasing as n becomes large. Let us mention that the curves depicted in Fig. 2 are not perfectly smooth because Poisson rates λ_i have been simulated, as explained in Section 3.4.

6. Discussion

In this paper, we have proposed a new way of allocating insurance losses within a pool, where the conditional mean risk-sharing rule applies at each occurrence time. An explicit expression is derived for this rule, based on the results derived for mutually exclusive risks in Denuit and Dhaene (2012). The impact of pooling is assessed with the help of default probabilities and largest excesses

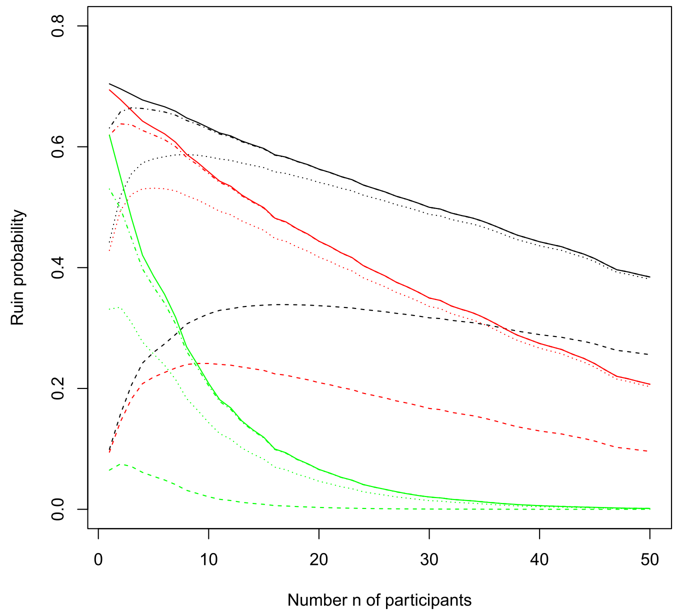

Fig. 2. Default probabilities for participant 1 as functions of the number n of participants, for initial deposit $\kappa_1 = 0.05$ (printed in black), 0.1 (printed in red), and 0.5 (printed in green) over time horizons $m = \infty$ (solid line), $m = 1$ (dashed line), $m = 10$ (dotted line), and $m = 50$ (dotdash line). (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

of accumulated claims over contributions. Thanks to the convex order, pooling is shown to be beneficial, except for finite-time default probability with zero initial deposit. Under the no-sabotage condition, formulas splitting the burden of interest on temporary deficits and the accumulated surplus at maturity have been derived. Large pools have been considered, showing that individual risk can be fully diversified when the number of participants tends to infinity, under mild technical conditions. The asymptotic behavior of finite-time and infinite-time default probabilities has also been studied.

The paper only considered pure pooling solutions so far. There is just a partnering financial institution helping participants to face timing risk by providing them with an access to loan when $V_t < 0$. If participants wish to benefit from some guarantees, they can contract with a partnering insurer. When an insurer is involved, we adopt the standard practice working with annual coverage periods so that we set $m = 1$ and we consider one period, $(0, 1)$ say. The partnering insurer offers credit on V_t when negative, with cost

$$\int_0^1 \delta |V_t| \mathrm{I}[V_t < 0] \mathrm{d}t = \delta \sum_{i=1}^n \int_0^1 \Big(w_{i,t} - \mathrm{E}[V_{i,t}|V_t] \Big)_+ \mathrm{d}t$$

or

$$\int_0^1 \delta (S_t - \kappa - ct)_+ \mathrm{d}t = \delta \sum_{i=1}^n \int_0^1 \Big(\mathrm{E}[S_{i,t}|S_t] - w'_{i,t} \Big)_+ \mathrm{d}t$$

under the no-sabotage condition, where δ is the force of interest. Notice that these integrals are referred to as area in red after Loisel (2005) who derived expressions for the corresponding expectations. The partnering insurer also covers the negative surplus at time 1 given by

$$|V_1| \mathrm{I}[V_1 < 0] = \sum_{i=1}^n \Big(w_{i,1} - \mathrm{E}[V_{i,1}|V_1] \Big)_+$$

or

$$(S_1 - \kappa - c)_+ = \sum_{i=1}^n \Big(\mathrm{E}[S_{i,1}|S_1] - w'_{i,1} \Big)_+$$

under the no-sabotage condition. These guarantees are priced according to some premium calculation principle, assumed to be comonotonic additive (so that the premium can be allocated to participants with the help of the decomposition given above). Participants may still share (part of) $V_1 > 0$ in case of favorable experience.

The approach developed in this paper also helps to formalize the guarantees comprised in an insurance contract. Considering that commercial insurers operate pooling within their portfolios, on behalf of the policyholders, risk sharing as described in this paper disentangles pooling effects from additional guarantees. The latter can then be priced accordingly.

Declaration of competing interest

There is no competing interest.

Data availability

No data was used for the research described in the article.

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