Insurance: Mathematics and Economics 111 (2023) 23–40

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Comparing utility derivative premia under additive and multiplicative risks

Christoph Heinzel

INRAE, Institut Agro, UMR SMART, 35000, Rennes, France

ARTICLE INFO

Article history:
Received October 2021
Received in revised form February 2023
Accepted 22 February 2023
Available online 2 March 2023

JEL classification:
D81
E21

Keywords:
Utility premium
Multiplicative risk
Higher-order risk
Comparative risk aversion
Precautionary premium

ABSTRACT

This paper develops the risk comparative statics of utility derivatives with a focus on utility premia. I compare extensions of two kinds of normalized premia: the "rate of substitution between n^{th}- and m^{th}-degree risk increases" (Liu and Meyer, 2013) and the "normalized utility premium" (Li and Liu, 2014). Under additive risk, those premia provide separate, but equivalent characterizations. Multiplicative risk, on the other hand, provides for distinct characterizations for the two premia. The comparative reasoning is illustrated at interpersonal precautionary saving comparisons and the intrapersonal conditions for decreasing Ross absolute and relative risk aversion.

1. Introduction

Risk comparative statics often hinge on comparing direct risk effects on utility derivatives. For example, the risk-induced variation of a decision maker's optimal choice depends on how risk impacts marginal utility in the first-order condition (Rothschild and Stiglitz, 1971). A case in point is precautionary saving: saving increases with future risk in a well-behaved problem if and only if risk raises future marginal expected utility (Kimball, 1990; Eeckhoudt and Schlesinger, 2008).[1] The risk impact on the second utility derivative, in turn, shapes properties of the optimal choice, such as the willingness to change that choice in reaction to a risk increase (see Section 5). In a similar vein, a comparison of the risk impacts on consecutive utility derivatives determines the effect that background risk has on the aversion to an independent foreground risk (Gollier and Pratt, 1996; Wang and Li, 2014).

Despite the economic importance of direct risk effects on utility derivatives, the comparative analysis of the preferences sustaining those effects has so far been limited to the level of marginal utility and to preference measures that focus on tradeoffs with first-degree risk. For example, Liu's (2014) precautionary premium measures the strength of the precautionary saving motive as the safe reduction of risky future income that has the same effect on saving as an n^{th}-degree increase of future income risk. As a result, that premium is based on the tradeoff between an n^{th}- and a first-degree risk increase at the level of marginal utility.

By contrast, at the level of expected utility (EU), comparative risk aversion has long been formulated for tradeoffs between n^{th}- and m^{th}-degree risk increases, for any n and m with $n > m \geq 1$. In a crucial contribution, Liu and Meyer (2013) extend characterizations of comparative n^{th}-degree Ross risk aversion (Jindapon and Neilson, 2007; Li, 2009; Denuit and Eeckhoudt, 2010) to tradeoffs with m^{th}-degree risk. Their $(n/m)^{th}$-degree Ross more risk aversion order covers Ross's (1981) original order for $(n, m) = (2, 1)$. Liu and Meyer introduce the "rate of substitution between n^{th}- and m^{th}-degree risk increases" as a new preference measure. That rate measures the willingness to substitute an m^{th}-degree risk increase for a given n^{th}-degree risk increase. Using such rates, two decision makers' attitudes toward a

E-mail address: Christoph.Heinzel@inrae.fr.

[1] Analogous observations hold for static choice. For instance, the background risk impact on partial insurance or deductible choices similarly plays out via the influence of the background risk on marginal utility in the first-order condition (Eeckhoudt and Kimball, 1992; Schlesinger, 2013).

https://doi.org/10.1016/j.insmatheco.2023.02.006
0167-6687/

downside-risk increase ($n = 3$), for example, can be compared regarding their willingness to substitute it for a first-degree risk increase ($m = 1$) or a mean-preserving spread ($m = 2$).

A key element for such preference comparisons at EU level is the utility premium. As introduced by Friedman and Savage (1948), the utility premium is an intuitive, non-monetary measure of a decision maker's risk-induced pain. It is usually defined as the EU reduction due to a risk increase. For an additive risk to wealth, the utility premium of a k^{th}-degree risk increase is positive if and only if the decision maker is k^{th}-degree risk averse (Ekern, 1980). Importantly, Liu and Meyer's risk substitution rate is the ratio of two utility premia, that of the n^{th}-degree risk increase is divided by that of an m^{th}-degree risk increase.[2] By definition, this measure is unit-free. Li and Liu (2014) introduce the "normalized utility premium" as an alternative. It consists of the ratio of the utility premium for the n^{th}-degree risk increase and the m^{th} utility derivative with a sign adjustment. Both measures provide characterizations of $(n/m)^{th}$-degree Ross more risk aversion.

Interestingly, comparative utility premium analysis has been concentrating on attitudes toward additive risk, regardless of the importance that multiplicative risk has in economic decisions.[3] A reason might be that, for additive and multiplicative risk alike, signing the utility premium merely involves inspecting the sign of the appropriate utility derivative (Loubergé et al., 2020). However, this homogeneity does not extend to risk impacts on marginal utility. Consider Bostian and Heinzel's (2018) marginal EU premium. This premium captures the risk-induced variation of marginal EU as a function of the attitudes that determine precautionary saving in the discounted EU model. The sign of the marginal EU premium follows by inspecting the appropriate utility derivative only in the case of an additive risk to exogenous future income. In the face of the (multiplicative) risk on the interest rate on saving, its sign rather hinges on the interplay of a (positive) precautionary and a (negative) substitution effect, and it is positive or negative depending on which effect is stronger. Bostian and Heinzel (2022) show that for the same risk averse and prudent decision maker, who has a positive marginal EU premium under income risk, the substitution effect typically prevails, providing for a negative premium under return risk. Similarly, studies on how the utility premium evolves with wealth and its riskiness find different conditions per risk type (Eeckhoudt and Schlesinger, 2009; Loubergé et al., 2020; Loubergé and Rey, 2022).

Motivated by the importance of direct risk effects on utility derivatives and the analytical differences for additive versus multiplicative risk, this paper extends comparative utility premium analysis to the level of arbitrary utility derivatives and multiplicative risk. The aim is to develop a general framework to explore and compare the economic implications of the impacts of increases of additive versus multiplicative risks on the derivatives of a decision maker's utility. I apply this framework to comparative precautionary saving analysis and the intrapersonal conditions that determine the shape of Ross risk aversion and background risk effects. The new preference measures developed in this paper help to quantify decision makers' preferences and to compare them by inspecting the stated preference criteria.

My starting point is the definition of the j^{th} utility derivative premium for additive risk and for multiplicative risk. For arbitrary $n \geq 2$ and $j \geq 0$, these premia quantify the impact of an n^{th}-degree risk increase on the j^{th} derivative of a decision maker's EU.[4] In view of comparing such premia, I extend Liu and Meyer's $(n/m)^{th}$-degree Ross more risk aversion order to the $((n + j)/(m + j))^{th}$ degree. Then, I provide characterizations in terms of adaptations of the two mentioned normalized utility premium measures.

The main result is twofold. Whereas the extension of the comparative analysis from EU level is straightforward under additive risk, the analysis for multiplicative risk differs markedly as soon as it is situated at the level of a utility derivative. The reason is that, in the standard case of a decision maker with utility derivatives with alternating sign, the impact of multiplicative risk on any utility derivative depends on the relative strength of counteracting effects. The Ross conditions entailing the characterizations in terms of the two new normalized measures must account for that sign ambiguity according to the places where the measures involve utility derivative premia. As a consequence, different Ross conditions are associated with the two measures, and the two characterizations are not equivalent.

Two applications illustrate the general results. The first is situated at marginal utility level and generalizes characterizations of comparative precautionary saving to comparisons of the willingness to save in response to an n^{th}- versus an m^{th}-degree increase in future risk. Conventional characterizations – notably, in terms of precautionary premia (Liu, 2014; Bostian and Heinzel, 2018) – focus on tradeoffs between n^{th}- and first-degree risk increases. Under multiplicative risk, only the comparison of two decision-makers' normalized marginal utility premia for $m = 1$ is equivalent to comparing their multiplicative precautionary premia; their comparison in terms of risk substitution rates is distinct.

The second set of applications studies the intrapersonal conditions for an independent background risk to raise foreground risk aversion of second order and higher. Those conditions bear on comparing direct risk effects on two successive utility derivatives up to the $(m + 1)^{th}$ for $m \geq 1$. Wang and Li's (2014) decreasing $(n/m)^{th}$-degree Ross risk aversion is crucial to capture such background risk effects under additive risk. In order to consider the direct risk effects on utility derivatives, I extend their definition to the $((n + j)/(m + j))^{th}$ degree. Special cases when $m = 1$ are the often-used traits of decreasing Ross absolute risk aversion (Ross DARA) for $j = 0$ and decreasing Ross absolute prudence (Ross DAP) for $j = 1$.[5] For multiplicative risk, I introduce decreasing $((n + j)/(m + j))^{th}$-degree Ross relative risk aversion, which covers, when $m = 1$, Ross decreasing relative risk aversion (Ross DRRA) for $j = 0$ and Ross decreasing relative prudence (Ross DRP) for $j = 1$.[6]

---

[2] The values of simple utility premia are unique only up to positive linear transformations and cannot be compared interpersonally. The remedy is appropriate normalizations. See Crainich and Eeckhoudt (2008); Huang and Stapleton (2015) on the normalization and comparison of utility premia for risk added to certainty and Wong (2018) for an extension to higher-order risk aversion and prudence. Fleurbaey et al. (2021) discuss utility normalizations and apply normalized utility premia in a social-choice context.

[3] Examples of economic decisions under multiplicative risk include partial insurance (Schlesinger, 2013), portfolio choice with a risky asset (Chiu et al., 2012), saving with risky return (Eeckhoudt and Schlesinger, 2008), production with risky prices or output (Broll and Wong, 2013), labor/leisure decisions with risky wage rate (Chiu and Eeckhoudt, 2010), abatement policy with risky damage rate (Barro, 2015; Bramoullé and Treich, 2009), and the effect of economic convergence on the social discount rate (Gollier, 2015).

[4] The $j = 0$ special case covers the analysis at EU level. In parallel work, Loubergé et al. (2020) implicitly use these premia in a proof and provide conditions for their uniform signing in relation to risk apportionment theory. The focus in this paper is on the economic importance and comparison of those premia.

[5] Eeckhoudt et al. (1996) show in a basic result that a second-order increase of background risk raises foreground risk aversion if the decision maker exhibits Arrow-Pratt DAP and Ross DARA. In the same vein, Ross DARA and Ross DAP jointly guarantee the preservation of Kimball's (1993) "standard risk aversion," which combines Arrow-Pratt DARA and DAP (Keenan et al., 2008). That same joint condition warrants that adding a fair (zero-mean) or unfair (negative-mean) background risk to safe wealth raises Ross aversion to risk increases, also called Ross risk vulnerability (Keenan and Snow, 2012).

[6] Only little research has concerned the effects of multiplicative background risk on foreground risk aversion, see especially Franke et al. (2006, 2011) and Jokung (2013).

Under additive risk, characterizing decreasing $((n+j)/(m+j))^{th}$-degree Ross ARA in terms of the new normalized measures is straightforward for any $j \geq 0$. Under multiplicative risk, a general result yields Ross conditions to order each of the measures for any pair of successive utility derivatives, and the conditions determining the decreasing (increasing) Ross RRA shape involve these orderings. Due to the ambiguous risk impacts, those conditions are only sufficient (necessary) and, in view of their complexity, only stated for $j = 0, 1$.

Two specific examples apply these results for each risk type. The first provides three characterizations of a decreasing precautionary saving motive, including one in terms of temperance and precautionary premia. The temperance premium measures here the decision maker's willingness to pay to avoid changing the optimal saving choice due to the risk increase. The second example shows that $(n-m)^{th}$-degree increases of an additive background risk raise $(m+1)^{th}$-degree Arrow-Pratt ARA if and only if, for either of the two normalized measures, that at marginal utility level exceeds that at EU level. As regards multiplicative background risk, I provide the conditions on preferences such that the normalized $(m+1)^{th}$ utility derivative premium exceeding the m^{th} is necessary and sufficient for an n^{th}-degree increase to raise $(m+1)^{th}$-degree Arrow-Pratt RRA. Because of the focus on Arrow-Pratt risk aversion, only normalized j^{th} utility derivative premia apply here; a corresponding result in terms of risk substitution rates is not available.

Section 2 introduces the analytical framework and analyzes the risk comparative statics of utility derivatives in terms of $(n/m)^{th}$-degree risk substitution rates and normalized utility derivative premia for additive risk. Section 3 develops the corresponding analysis for multiplicative risk. Section 4 treats comparative precautionary saving as an illustration. Section 5 extends the existing definitions, characterizations, and applications of decreasing Ross absolute and relative risk aversion. Section 6 concludes.

2. Utility derivative premia for additive risk

This paper focuses on attitudes toward n^{th}-degree increases in risk as in Ekern (1980). An expected utility maximizer u is k^{th}-degree risk averse if $(-1)^{k+1} u^{(k)}(x) > 0$ for all x, where utility function u is defined on $[a, b] \in \mathbb{R}_0^+$ and $u^{(k)}$ denotes its k^{th} derivative.

To define Ekern risk increases, consider the cumulative distribution functions (CDFs) $F(z) = F^{[1]}(z)$ and $G(z) = G^{[1]}(z)$ of two random variables, with finite support in $[z_a, z_b]$ and equal start and end points, $F(z_a) = G(z_a) = 0$ and $F(z_b) = G(z_b) = 1$. Denote higher-order CDFs $F^{[k]}(z) = \int_{z_a}^{z} F^{[k-1]}(t)\,dt$ for $k = 2, 3, \ldots$, and similarly for $G(z)$.

Definition 1 *(n^{th}-Degree Risk Increase [Ekern, 1980]).* $G(z)$ has more n^{th}-degree risk than $F(z)$ if

$$G^{[k]}(z_b) = F^{[k]}(z_b) \quad \text{for } k = 1, \ldots, n$$
$$G^{[n]}(z) \geq F^{[n]}(z) \quad \text{for all } z \in [z_a, z_b] \text{ and ``}>\text{'' holding for some } z \in (z_a, z_b).$$

Such a risk increase implies an n^{th}-degree stochastic dominance shift with unchanged first $n-1$ moments. Well-known special cases are mean-preserving spreads for $n = 2$ (Rothschild and Stiglitz, 1970); increases in downside risk for $n = 3$ (Menezes et al., 1980); and increases in outer risk for $n = 4$ (Menezes and Wang, 2005).

Below, I will denote risk in its low state, associated with F, $\tilde{z}_l$ and risk in its high state, associated with G, $\tilde{z}_h$. I will further consider CDFs H_m and H_{n-m}, defined like F and G, which have, respectively, more m^{th}- and more $(n-m)^{th}$-degree risk than F. These intermediate risks will also be denoted $\tilde{z}_m$ and $\tilde{z}_{n-m}$. Throughout, I assume $n > m \geq 1$.

For an additive risk $\tilde{\varepsilon}$ to wealth x, I use the shorthand $\tilde{x} \equiv x + \tilde{\varepsilon}$. Ekern shows that $G(x)$ having more n^{th}-degree risk than $F(x)$ is equivalent to every n^{th}-degree risk averter preferring $F(x)$ to $G(x)$. Ekern derives this result by proving that utility premium

$$Eu(\tilde{x}_l) - Eu(\tilde{x}_h) \tag{1}$$

is positive whenever a decision maker is n^{th}-degree risk averse.[7] Hence, in the face of a first- (second-; third-) degree risk increase, the positive sign holds if and only if the decision maker exhibits non-satiation (risk aversion; downside risk aversion) or $u' > 0$ ($u'' < 0$; $u''' > 0$).

Risk comparative statics often hinge on direct risk effects on utility derivatives, as noted in Section 1. To measure the impact of a risk increase on the j^{th} utility derivative, I define the j^{th} utility derivative premium as

$$(-1)^j \left\{ Eu^{(j)}(\tilde{x}_l) - Eu^{(j)}(\tilde{x}_h) \right\} \tag{2}$$

For $j = 0$, this premium covers (1). For $j = 1$, (2) measures the risk impact on marginal utility. The marginal EU premium in Section 4.1 is a particular case defined in a saving problem with income risk. For $j = 2$, (2) captures the risk impact on the slope of marginal utility. That premium can represent attitudes toward the willingness to change an optimal choice (see Section 5.2). For arbitrary $j \geq 0$, (2) is positive if and only if the decision maker is $(n+j)^{th}$-degree risk averse, $(-1)^{n+j-1} u^{(n+j)}(x) > 0$ (Loubergé et al., 2020).

Other applications have used utility derivative premia to study the shape of utility premium (1) when wealth changes. By establishing (2)'s positive sign for $j = 0, 1, 2$, Eeckhoudt and Schlesinger (2006) show that the utility premium of a zero-mean risk added to safe wealth is positive, decreasing, and convex in wealth if and only if the decision maker is, respectively, risk averse ($u'' < 0$), downside risk averse ($u''' > 0$), and temperant ($u'''' < 0$).[8] Loubergé and Rey (2022) generalize that result to general n^{th}-degree Ekern risk increases.

Elaborating on that line of reasoning, Loubergé et al. provide a general result on risk apportionment. Under risk apportionment, a decision maker prefers to disaggregate risks, which may be degenerate, in different states of the world, instead of cumulating them in one state of the world. Starting from an m^{th}-degree deterioration of risky initial wealth, Loubergé et al. determine the preference condition for apportioning the n^{th}-degree increase of an independent background risk in the state of the world with undeteriorated initial wealth. The

[7] The proof uses (1)'s n-fold integration by parts, Definition 1, and n^{th}-degree risk aversion.

[8] For the purposes of their discussion, Eeckhoudt and Schlesinger (2006) consider the negative of (2) and omit sign correction $(-1)^j$.

derivation of that condition in the proof of their Theorem 1 is equivalent to signing (2) for $j = m$. This observation implies the following remark.

Remark 1 *(Relation to Additive Risk Apportionment).* Consider an n^{th}-degree background risk increase and an m^{th}-degree foreground risk increase, with $m = j$. Any decision maker with a positive j^{th} utility derivative premium prefers a lottery that disaggregates the risk increases in different states of the world, $\left[\tilde{x}_l + \tilde{x}_h, \tilde{x}_m + \tilde{x}_l\right]$, to a lottery that cumulates the risk increases in one state of the world, $\left[\tilde{x}_m + \tilde{x}_h, \tilde{x}_l + \tilde{x}_l\right]$.

This paper uses risk apportionment only for illustration purposes. The main focus is on comparing the attitudes responsible for the risk impact on the j^{th} utility derivative.

To that end, I first extend Liu and Meyer's (2013) $(n/m)^{th}$-degree Ross more risk aversion.

Definition 2 *($((n+j)/(m+j))^{th}$-Degree Ross More Risk Aversion, Additive Risk).* Suppose $j \geq 0$, and let u and v be $(n+j)^{th}$- and $(m+j)^{th}$-degree risk averse. Then, u is $((n+j)/(m+j))^{th}$-degree Ross more risk averse than v on $[a,b] \subseteq \mathbb{R}_0^+$ if there exists a scalar $\lambda > 0$ such that

$$\frac{u^{(n+j)}(x_a)}{v^{(n+j)}(x_a)} \geq \lambda \geq \frac{u^{(m+j)}(x_b)}{v^{(m+j)}(x_b)} \quad \text{for all } x_a, x_b \in [a,b] \tag{3}$$

When $j = 0$, this definition covers Ross's (1981) more risk aversion order for $(n,m) = (2,1)$, Jindapon and Neilson's (2007) n^{th}-degree extension for $(n,m) = (n,1)$, and Liu and Meyer's $(n/m)^{th}$-degree Ross more risk aversion for arbitrary n and m.

The preference comparison in (3) equivalently applies to $((n+j)/(m+j))^{th}$-degree risk tradeoffs at EU level and $(n/m)^{th}$-degree risk tradeoffs at j^{th} utility derivative level.[9]

Lemma 1 *(Relation to $(n/m)^{th}$-Degree Ross More Risk Aversion, Additive Risk). The $((n+j)/(m+j))^{th}$-degree Ross more risk averse order between u and v is equivalent to $(-1)^j u^{(j)}$ being $(n/m)^{th}$-degree Ross more risk averse than $(-1)^j v^{(j)}$.*

The lemma derives from applying Liu and Meyer's $(n/m)^{th}$-degree order at the level of the j^{th} utility derivative and comparing with (3).

To characterize $((n+j)/(m+j))^{th}$-degree Ross more risk aversion in terms of preferences measures, I compare two kinds of normalizations of utility derivative premia. Definition 3 extends Liu and Meyer's risk substitution rate to the level of utility derivatives.

Definition 3 *(Risk Substitution Rate for the j^{th} Utility Derivative, Additive Risk).* Let u be $(n+j)^{th}$- and $(m+j)^{th}$-degree risk averse. The $(n/m)^{th}$-degree risk substitution rate for the j^{th} utility derivative is

$$T_{j,u}^{(n/m)}\left(\tilde{x}_l, \tilde{x}_h, \tilde{x}_m\right) = \frac{Eu^{(j)}\left(\tilde{x}_l\right) - Eu^{(j)}\left(\tilde{x}_h\right)}{Eu^{(j)}\left(\tilde{x}_l\right) - Eu^{(j)}\left(\tilde{x}_m\right)} \tag{4a}$$

Alternatively, $T_{j,u}^{(n/m)}$ is the scalar solving

$$Eu^{(j)}\left(\tilde{x}_h\right) = \left(1 - T_{j,u}^{(n/m)}\right) Eu^{(j)}\left(\tilde{x}_l\right) + T_{j,u}^{(n/m)} Eu^{(j)}\left(\tilde{x}_m\right) \tag{4b}$$

In (4a), premium (2) is normalized by its analog for an m^{th}-degree risk increase. For an $(n+j)^{th}$- and $(m+j)^{th}$-degree risk averse decision maker, both of those premia are positive, and so is the risk substitution rate. Because both premia are measured in the same unit, the ratio is unit-free. In light of Lemma 1, $T_j^{(n/m)}$ gauges equivalently the willingness to substitute an m^{th}- for an n^{th}-degree risk increase at the level of the j^{th} utility derivative and the willingness to substitute an $(m+j)^{th}$- for an $(n+j)^{th}$-degree risk increase at EU level.

Li and Liu's normalized utility premium involves dividing premium (1) by $(-1)^{m-1} Eu^{(m)}\left(\tilde{x}_l\right)$. To adapt their approach, I use the normalizer $(-1)^{j+m-1} Eu^{(j+m)}\left(\tilde{x}_l\right)$.

Definition 4 *(Normalized j^{th} Utility Derivative Premium, Additive Risk).* Let u be $(m+j)^{th}$-degree risk averse. The normalized j^{th} utility derivative premium for a risk increase from $\tilde{x}_l$ to $\tilde{x}_h$ is defined as

$$N_jUP_u^{(n/m)}\left(\tilde{x}_l, \tilde{x}_h\right) = \frac{Eu^{(j)}\left(\tilde{x}_l\right) - Eu^{(j)}\left(\tilde{x}_h\right)}{(-1)^{m-1} Eu^{(j+m)}\left(\tilde{x}_l\right)} \tag{5}$$

Similar to Li and Liu's normalized utility premium, the utility derivative in (5)'s denominator is m times higher than in the numerator. As for the Li and Liu premium, this normalization converts (2) into the unit of the m^{th} moment of random wealth and indicates the cost of the impact of the risk increase on the j^{th} utility derivative in terms of a decrease (increase) in $\tilde{x}_l$'s m^{th} moment, for m odd (even). For an $(m+j)^{th}$-degree risk averse decision maker, $N_jUP_u^{(n/m)}$ has the same sign as (2).

The two kinds of normalized utility derivative premia admit equivalent characterizations of $((n+j)/(m+j))^{th}$-degree Ross more risk aversion.

[9] Liu (2014) points to this relation in the context of saving for marginal utility ($j = 1$) and $m = 1$.

Theorem 1 *(Comparative $((n+j)/(m+j))^{th}$-Degree Ross Risk Aversion, Additive Risk). Conditions (i)–(iv) are equivalent:*

(i) $(-1)^j u^{(j)}$ *is* $(n/m)^{th}$*-degree Ross more risk averse than* $(-1)^j v^{(j)}$.
(ii) *There exist* $\lambda > 0$ *and* $\phi(x)$ *with* $\phi^{(j)}(x)$ *such that* $u^{(j)}(x) = \lambda v^{(j)}(x) + \phi^{(j)}(x)$*, where* $(-1)^{j+m-1}\phi^{(j+m)}(x) \le 0$ *and* $(-1)^{j+n-1}\phi^{(j+n)}(x) \ge 0$ *for all x.*
(iii) $T_{j,u}^{(n/m)} \ge T_{j,v}^{(n/m)}$ *for all* $\tilde{x}_l, \tilde{x}_h, \tilde{x}_m$.
(iv) $N_jUP_u^{(n/m)} \ge N_jUP_v^{(n/m)}$ *for all* $\tilde{x}_l, \tilde{x}_h$.

Given Lemma 1, the equivalence of (i), (ii), and (iii) is immediate from Liu and Meyer (2013, Theorem 1) when substituting $\left((-1)^j u^{(j)}, (-1)^j v^{(j)}\right)$ for (u, v), and the equivalence of (i), (ii), and (iv) arises similarly from Li and Liu (2014, Theorem 2). Hence, decision maker u is $((n+j)/(m+j))^{th}$-degree Ross more risk averse than decision maker v if and only if u's willingness to substitute an $(m+j)^{th}$- for an $(n+j)^{th}$-degree risk increase is uniformly larger than v's or, alternatively, u's normalized j^{th} utility derivative premium for an n^{th}-degree risk increase uniformly exceeds v's.

Although the two normalized utility derivative premium measures are closely related on formal grounds, there is mutually no implication between the two characterizations.

Remark 2 *(Relation of* $T_j^{(n/m)}$ *and* $N_jUP^{(n/m)}$ *in Theorem 1).* A positive conversion factor links $T_j^{(n/m)}$ from (4a) and $N_jUP^{(n/m)}$ from (5):

$$N_jUP_u^{(n/m)}\left(\tilde{x}_l, \tilde{x}_h\right) = T_{j,u}^{(n/m)}\left(\tilde{x}_l, \tilde{x}_h, \tilde{x}_m\right) \cdot \frac{Eu^{(j)}\left(\tilde{x}_l\right) - Eu^{(j)}\left(\tilde{x}_m\right)}{(-1)^{m-1} Eu^{(j+m)}\left(\tilde{x}_l\right)}$$

When applied for two decision makers u and v, the order of their conversion factors is ambiguous and need not be aligned with the order of their $T_j^{(n/m)}$ and $N_jUP^{(n/m)}$ measures.

The conversion factor is the ratio of the two normalizers. If the order of two decision makers' conversion factors were aligned with the order of one type of their normalized measures, then, knowing the order of that type would imply knowing the order of the other type. Remark 2 implies that such a hierarchy between the orders of the two measures is not given (see Supplemental Appendix B for a proof). This underlines that the two equivalent characterizations in Theorem 1 are associated with separate quantifications. The conceptual difference between the two normalized measures is accentuated under multiplicative risk.

3. Comparative aversion to multiplicative risk

This section turns to comparing the attitudes that determine the impact that the increase of a multiplicative risk $\tilde{\rho}$ to wealth x has on decision makers' utility and utility derivatives. More complex tradeoffs emerge as compared to additive risk as soon as the risk effects concern utility derivatives. The source of the difference is the random variable's multiplicative nature.

To measure the impact of multiplicative risk on a decision maker's utility and utility derivatives, consider the j^{th} utility derivative premium for an increase in return risk,

$$(-1)^j \left\{E\left[u^{(j)}\left(x\tilde{\rho}_l\right)\tilde{\rho}_l^j\right] - E\left[u^{(j)}\left(x\tilde{\rho}_h\right)\tilde{\rho}_h^j\right]\right\} \tag{6}$$

For $j=0$, utility premium $Eu\left(x\tilde{\rho}_l\right) - Eu\left(x\tilde{\rho}_h\right)$ is the analog for multiplicative risk of (1). For $j=1$, (6) measures the impact of the return risk increase on marginal utility. Bostian and Heinzel's marginal EU premium is an application in a saving problem with interest rate risk (see Section 4.2). For $j=2$, (6) captures the impact of the return risk increase on the slope of marginal utility, and the premium can represent attitudes toward the willingness to change an optimal choice in the face of the risk increase (see Section 5.4).

Signing (6) depends on the product of $(-1)^{n+j-1}$ and the n^{th} derivative of j^{th} utility derivative

$$h_{[j],u}(\rho) = \rho^j u^{(j)}(x\rho) \tag{7a}$$

(Loubergé et al., 2020). Premium (6) is positive (negative) if and only if[10]

$$(-1)^{n+j-1} h_{[j],u}^{(n)}(\rho) = (-1)^{n+j-1} \sum_{k=0}^{n} \binom{n}{k} \prod_{i=1}^{k} (j+1-i)\, u^{(j+n-k)}(x\rho)\, x^{n-k} \rho^{j-k} > (<)\ 0 \tag{7b}$$

The formula for $h_{[j],u}^{(n)}(\rho)$ in (7b) is equivalent to the one in Loubergé et al.'s Theorem 2. That theorem shows how premia (6) apply to multiplicative risk apportionment.

Remark 3 *(Relation to Multiplicative Risk Apportionment).* Consider an m^{th}-degree increase of the risk on wealth, with $m=j$, and an n^{th}-degree return risk increase. Any decision maker with a positive j^{th} utility derivative premium (6) prefers a lottery that disaggregates the risk increases in different states of the world, $\left[\tilde{x}_m\tilde{\rho}_l, \tilde{x}_l\tilde{\rho}_h\right]$, to a lottery that cumulates the risk increases in one state of the world, $\left[\tilde{x}_m\tilde{\rho}_h, \tilde{x}_l\tilde{\rho}_l\right]$.

[10] By convention, $\prod_{i=1}^{0}(j+1-i) = 1$ for any j. The recursive formula is due to straighforward calculations.

The preference conditions controlling the risk impact on the j^{th} utility derivative are more involved than under additive risk. Only for $j=0$, (7b) collapses to $(-1)^{n-1} u^{(n)}(x\rho) > (<)\ 0$, so that the utility premium for a return risk increase is positive for any n^{th}-degree risk-averse decision maker, similar to (1).

For $j=1$, the sign of the marginal utility premium depends on two effects associated with the right-hand terms in

$$(-1)^n h^{(n)}_{[1],u}(\rho) = (-1)^n \left\{ u^{(n+1)}(x\rho)\, x^n \rho + n x^{n-1} u^{(n)}(x\rho) \right\} \tag{8a}$$

If $x, \rho > 0$, the first term is positive and the second negative for any $(n+1)^{th}$- and n^{th}-degree risk averse decision maker. Chiu et al. (2012) motivate those counteracting effects using risk apportionment theory. Consider two 50-50 lotteries, one with high wealth $\bar{x}$ associated with high risk $\tilde{\rho}_h$ and low wealth $\underline{x}$ associated with low risk $\tilde{\rho}_l$ and another one with reversed associations. The decision maker's $(n+1)^{th}$-degree risk aversion is the basis of the preference to disaggregate – or apportion – low wealth and high risk in different states of the world, making the first lottery more attractive. At the same time, the decision maker's n^{th}-degree risk aversion makes the second lottery more attractive: the associations of high risk $\tilde{\rho}_h$ with low wealth $\underline{x}$ and of low risk $\tilde{\rho}_l$ with high wealth $\bar{x}$ both imply lower risk exposure. The relative strength of these "apportionment" and "(n^{th}-degree) risk aversion" effects determines the overall sign of the return risk impact on marginal utility. In saving models, these counteracting forces have the well-known interpretation as precautionary effect and substitution effect (see Section 4.2).

For $j=2$, the risk impact on the second utility derivative hinges on three effects associated with the right-hand terms in

$$(-1)^{n+1} h^{(n)}_{[2],u}(\rho) = (-1)^{n+1} \left\{ u^{(n+2)}(x\rho)\, x^n \rho^2 + 2n\, u^{(n+1)}(x\rho)\, x^{n-1} \rho + n(n-1)\, u^{(n)}(x\rho)\, x^{n-2} \right\} \tag{8b}$$

Those effects are, respectively, positive, negative, and positive for any $(n+2)^{th}$- , $(n+1)^{th}$-, and n^{th}-degree risk averse decision maker. The overall impact sign reflects the net effect.[11]

The cases $j=0,1,2$ illustrate a number of more general observations that signing condition (7b) entails (see Supplemental Appendix C). First, for any $j \geq 0$, the number of different utility derivatives that jointly determine the return risk impact on the j^{th} utility derivative is equal to the number of times product $\prod_{i=1}^{k}(j+1-i)$ is positive for $k=0,\ldots,n$. Second, the maximum number of utility derivatives involved is $j+1$, and that number grows with j.[12] Third, the minimum number of involved utility derivatives across all $j \geq 0$ is one, and it is attained if and only if $j=0$. Finally, the risk effects always involve the $(n+j)^{th}$ as the highest utility derivative and depend, in addition, on the consecutive lower ones if two or more are activated. Remark 4 summarizes the main insights from these observations.

Remark 4 *(Return Risk Effects on Utility and Utility Derivatives).* Only at the level of EU ($j=0$), the sign of the impact of a return risk increase is completely determined by the sign of one single utility derivative, namely, the n^{th}. For $j \geq 1$, the sign of the impact on the j^{th} utility derivative always reflects the net of several effects, which are counteracting for a decision maker with utility derivatives which alternate in sign.

The upshot is that any comparison of utility derivative premia needs to control for the net sign of the overall risk impact. Merely controlling for the sign of one single utility derivative, as it is convenient for $j=0$, is not sufficient.

As regards the normalized preference measures I define below, the latter observation applies to all places where utility derivative premia are involved. Because the two measures differ in that regard, the characterizations hinge on two different notions of $((n+j)/(m+j))^{th}$-degree Ross more aversion to return risk increases.

Definition 5 *($((n+j)/(m+j))^{th}$-Degree Ross More Risk Aversion, Return Risk).* Suppose $j \geq 0$. Let u and v be such that $(-1)^{n+j-1} h^{(n)}_{[j],f}(\rho) > [<]\ 0$ for $f \in \{u, v\}$ and let them have identical levels of x under reference return $\tilde{\rho}_l$. If u and v are, in addition,

(a) such that $(-1)^{m+j-1} h^{(m)}_{[j],f}(\rho) > [<]\ 0$ for $f \in \{u, v\}$, then u is $((n+j)/(m+j))^{th}$-degree Ross more risk averse than v in the first sense with respect to a return risk increase from $\tilde{\rho}_l$ to $\tilde{\rho}_h$, if there exists a scalar $\lambda > 0$ such that

$$\frac{h^{(n)}_{[j],u}(\rho_a)}{h^{(n)}_{[j],v}(\rho_a)} \geq \lambda \geq \frac{h^{(m)}_{[j],u}(\rho_b)}{h^{(m)}_{[j],u}(\rho_b)} \quad \text{for all } \rho_a, \rho_b \text{ such that } x\rho_a, x\rho_b \in [a,b] \tag{9a}$$

(b) $(m+j)^{th}$-degree risk averse, then u is $((n+j)/(m+j))^{th}$-degree Ross more risk averse than v in the second sense with respect to a return risk increase from $\tilde{\rho}_l$ to $\tilde{\rho}_h$, if there exists a scalar $\lambda > 0$ such that

$$\frac{h^{(n)}_{[j],u}(\rho_a)}{h^{(n)}_{[j],v}(\rho_a)} \geq \lambda \geq \frac{u^{(j+m)}(x\rho_b)}{v^{(j+m)}(x\rho_b)} \quad \text{for all } \rho_a, \rho_b \text{ such that } x\rho_a, x\rho_b \in [a,b] \tag{9b}$$

Both Ross conditions (9) involve terms that account for (6)'s sign ambivalence, namely, in the form of $h^{(k)}_{[j]}(\rho)$ from (7b) for $k=n,m$. For $j=0$, the two conditions are identical and coincide with (3) applied to return risk. However, whereas the first inequalities in (9) are always identical, it is easy to see that, as soon as $j \geq 1$, there is no necessary or sufficient condition that relates the second inequalities.

Because (9a) homogeneously involves controls for the impact signs of n^{th}- and m^{th}-degree risk increases, Definition 5(a) admits a relation analogous to Lemma 1.

[11] Interpreting these effects depends on the decision context and is left for future research.

[12] There is an interaction between the number of consecutive utility derivatives determining the risk impact and the degree n of the return risk increase for $j \geq 3$ and low n. For example, if $n=2$, signing condition $h^{(2)}_{[3],u}(\rho)$ only contains the terms with $u^{(5)}$, $u^{(4)}$, and u''', instead of four terms.

Lemma 2 *(Relation to $(n/m)^{th}$-Degree Ross More Risk Aversion, Return Risk). The $((n+j)/(m+j))^{th}$-degree Ross more risk averse order between u and v in the first sense with respect to a return risk increase is equivalent to $(-1)^j h_{[j],u}$ being $(n/m)^{th}$-degree Ross more risk averse than $(-1)^j h_{[j],v}$.*

The lemma derives by applying Liu and Meyer's $(n/m)^{th}$-degree Ross risk aversion analogously at the level of $u(x\rho)$'s j^{th} derivative with respect to x and comparing with (9a). By lack of a similar homogeneity of controls for (9b), there is no such relation for Definition 5(b).

The next two definitions adapt the normalized utility derivative premium measures from Section 2 to multiplicative risk. I start with the risk substitution rate.

Definition 6 *(Risk Substitution Rate for j^{th} Utility Derivative, Return Risk).* Let u be such that $(-1)^{k+j-1} h^{(k)}_{[j],u}(\rho) > [<]\ 0$ for $k = n, m$. The $(n/m)^{th}$-degree risk substitution rate for return risk $\tilde{\rho}$ and the j^{th} utility derivative is given by

$$\widehat{T}_j^{(n/m)}(\tilde{\rho}_l, \tilde{\rho}_h, \tilde{\rho}_m) = \frac{E\left[u^{(j)}(x\tilde{\rho}_l)\,\tilde{\rho}_l^j\right] - E\left[u^{(j)}(x\tilde{\rho}_h)\,\tilde{\rho}_h^j\right]}{E\left[u^{(j)}(x\tilde{\rho}_l)\,\tilde{\rho}_l^j\right] - E\left[u^{(j)}(x\tilde{\rho}_m)\,\tilde{\rho}_m^j\right]} \tag{10a}$$

or, alternatively, $\widehat{T}_j^{(n/m)}$ is the scalar solving

$$E\left[u^{(j)}(x\tilde{\rho}_h)\,\tilde{\rho}_h^j\right] = \left(1 - \widehat{T}_j^{(n/m)}\right) E\left[u^{(j)}(x\tilde{\rho}_l)\,\tilde{\rho}_l^j\right] + \widehat{T}_j^{(n/m)}\, E\left[u^{(j)}(x\tilde{\rho}_m)\,\tilde{\rho}_m^j\right] \tag{10b}$$

In (10a), (6) is normalized by the j^{th} utility derivative premium for an m^{th}-degree return-risk increase. Rate $\widehat{T}_j^{(n/m)}$ is positive and unit-free because the premia in its numerator and denominator have the same sign and are measured in the same unit. The cases with split signs are ruled out because a negative risk substitution rate would be hard to interpret. In light of Lemma 2, $\widehat{T}_j^{(n/m)}$ is, equivalently, a measure of the willingness to substitute an $(m+j)^{th}$- for an $(n+j)^{th}$-degree return risk increase and a measure of the willingness to substitute an m^{th}- for an n^{th}-degree return risk increase at j^{th} utility derivative level.

Dividing (6) by normalizer $(-1)^{j+m-1} x^m E\left[u^{(j+m)}(x\tilde{\rho}_l)\,\tilde{\rho}_l^j\right]$ yields the normalized j^{th} utility derivative premium for return risk.

Definition 7 *(Normalized j^{th} Utility Derivative Premium, Return Risk).* Let u be $(m+j)^{th}$-degree risk averse. The normalized j^{th} utility derivative premium for a risk increase from $\tilde{\rho}_l$ to $\tilde{\rho}_h$ is defined as

$$\widehat{N_jUP}_u^{(n/m)}(\tilde{\rho}_l, \tilde{\rho}_h) = \frac{E\left[u^{(j)}(x\tilde{\rho}_l)\,\tilde{\rho}_l^j\right] - E\left[u^{(j)}(x\tilde{\rho}_h)\,\tilde{\rho}_h^j\right]}{(-1)^{m-1} x^m E\left[u^{(j+m)}(x\tilde{\rho}_l)\,\tilde{\rho}_l^j\right]} \tag{11}$$

Division by $(-1)^{j+m-1} E\left[u^{(j+m)}(x\tilde{\rho}_l)\,\tilde{\rho}_l^j\right]$ converts (6) into the unit of the m^{th} moment of random wealth, like (5). Further division by x^m makes $\widehat{N_jUP}_u^{(n/m)}$ unit-free. Thus, (11) indicates the cost of the risk-induced pain in terms of a decrease (increase) in $\tilde{\rho}_l$'s m^{th} moment, for m odd (even), *per unit* of x^m. Premia (11) and (6) have the same sign.

Comparisons of two decision makers in terms of premia (10a) and (11) are each associated with one of the two kinds of Ross more risk aversion in Definition 5. For convenience, I consider decision makers whose successive utility derivatives alternate in sign until the $(n+j)^{th}$ starting with a positive first.[13]

Theorem 2 *(Comparative $((n+j)/(m+j))^{th}$-Degree Ross Risk Aversion, Return Risk). Given Definition 5(a) and $h^{(k)}_{[j],\phi}(\rho)$ as in (7b) for $k = m, n$, (a.i)–(a.iii) are equivalent:*

(a.i) *u is $((n+j)/(m+j))^{th}$-degree Ross more risk averse than v in the sense of (9a).*
(a.ii) *There exist $\lambda > 0$ and $\phi(x\rho)$ with $\phi^{(j)}(x\rho)$ such that $u^{(j)}(x\rho) = \lambda v^{(j)}(x\rho) + \phi^{(j)}(x\rho)$, where $(-1)^{m+j-1} h^{(m)}_{[j],\phi}(\rho) \le [\ge]\ 0$ and $(-1)^{n+j-1} h^{(n)}_{[j],\phi}(\rho) \ge [\le]\ 0$ for all $x\rho$.*
(a.iii) *$\widehat{T}_{j,u}^{(n/m)} \ge [\le]\ \widehat{T}_{j,v}^{(n/m)}$ for all $\tilde{\rho}_l, \tilde{\rho}_h, \tilde{\rho}_m$.*

Given Definition 5(b) and $h^{(n)}_{[j],\phi}(\rho)$ as in (7b), (b.i)–(b.iii) are equivalent:

(b.i) *u is $((n+j)/(m+j))^{th}$-degree Ross more risk averse than v in the sense of (9b).*
(b.ii) *There exist $\lambda > 0$ and $\phi(x\rho)$ with $\phi^{(j)}(x\rho)$ such that $u^{(j)}(x\rho) = \lambda v^{(j)}(x\rho) + \phi^{(j)}(x\rho)$, where $(-1)^{m+j-1} \phi^{(j+m)}(x\rho) \le 0$ and $(-1)^{n+j-1} h^{(n)}_{[j],\phi}(\rho) \ge [\le]\ 0$ for all $x\rho$.*
(b.iii) *$\widehat{N_jUP}_u^{(n/m)} \ge \widehat{N_jUP}_v^{(n/m)} \ge 0 \left[\widehat{N_jUP}_u^{(n/m)} \le \widehat{N_jUP}_v^{(n/m)} \le 0\right]$ for all $\tilde{\rho}_l, \tilde{\rho}_h$.*

[13] Under this assumption, the return risk impact on any utility derivative is marked by counteracting effects with a specific sign (see Remark 4). That assumption is often made in related literature (e.g., Courbage et al., 2018; Loubergé et al., 2020), but stronger than needed when dealing with Ekern risk increases.

See Appendix A for a proof.

The two characterizations in Theorem 2 reflect the different properties of the two normalized premia. The differences are rooted in the different involvement of utility derivative premia in their definitions in conjunction with the ambiguity of return risk impact. Comprising the premia for n^{th}- and m^{th}-degree risk increases, the risk substitution rate requires a common uniform sign for those premia, but it does not rely on the specific sign of a utility derivative. Normalized utility derivative premia, by contrast, have no sign requirement for utility derivative premia, but decision makers need to be $(m+j)^{th}$-degree risk averse. From a theoretical point of view, risk substitution rates may seem more coherent because of their equivalent representation of the willingness to face $(n/m)^{th}$-degree risk tradeoffs at j^{th} utility derivative level and $((n+j)/(m+j))^{th}$-degree risk tradeoffs at EU level (Lemma 2). However, the different requirements for the two measures provide for mutual advantages and disadvantages depending on the application context. Notably, normalized utility derivative premia imply specific equivalences and applications that risk substitution rates do not admit, as we will see below.

4. Illustration: comparative precautionary saving

This section applies the above reasoning to a two-period consumption/saving model. The new preference measures extend the characterizations of comparative precautionary saving to $(n/m)^{th}$-degree risk tradeoffs with generic $m \geq 1$. Characterizations in terms of the precautionary premium require a redefinition of the latter concept to cover $m > 1$. The two normalized utility premium measures differ regarding their equivalence with conventional representations under return risk.

Let the decision maker choose saving s out of first-period income y_1 such as to maximize

$$u\left(y_1 - s\right) + \beta E u\left(\tilde{y}_2 + s\tilde{R}\right) \tag{12}$$

In this intertemporal objective, felicity u jointly captures consumption-smoothing preference and risk attitudes. The utility discount factor β reflects pure time preference. Risk may enter either through second-period income $\tilde{y}_2$ or gross return $\tilde{R}$. Notation containing both $\tilde{y}_2$ and $\tilde{R}$ applies to each of the two risk types. By assumption, $y_2 \geq 0$ and $R > 0$ in any state of the world. If unambiguous, I abbreviate second-period consumption $\tilde{c}_2$ depending on the risk type as $\tilde{c}_{2,i}^{y_2} = \tilde{y}_{2,i} + sR$ and $\tilde{c}_{2,i}^{R} = y_2 + s\tilde{R}_i$, for $i \in \{l, h\}$.

According to the first-order optimality condition

$$u'\left(y_1 - s\right) = \beta E\left[u'\left(\tilde{y}_2 + s\tilde{R}\right)\tilde{R}\right] \tag{13}$$

the decision maker saves until the marginal utility from foregoing consumption in period 1 (i.e., saving a marginal amount) is equal to the discounted marginal utility from consuming in period 2 instead. Assuming the second-order condition to hold, (13)'s unique solution maximizes objective (12). For consistency, I assume that incentives are such that $s > 0$ under return risk.

Precautionary saving is the saving reaction to future risk and depends on the risk effect on future marginal utility. When applied to model (12), marginal utility premia (2) and (6) for $j = 1$ entail a simple rule regardless of the risk type: a positive (negative) premium predicts that the risk increase induces higher (lower) saving. The criteria for signing the marginal utility premium depend on the risk type. Theorems 1 and 2 applied to model (12) provide characterizations of comparative precautionary saving for each type.

4.1. Comparative precautionary saving under income risk

Marginal utility premium $Eu'\left(\tilde{c}_{2,h}^{y_2}\right) - Eu'\left(\tilde{c}_{2,l}^{y_2}\right)$ is positive if and only if $(-1)^n u^{(n+1)}(c_2^{y_2}) > 0$ (Eeckhoudt and Schlesinger, 2008). A conventional comparison of precautionary saving motives can use precautionary premium θ^{y_2}, from

$$Eu'\left(\tilde{y}_{2,l} + sR - \theta^{y_2}\right) = Eu'\left(\tilde{y}_{2,h} + sR\right) \tag{14}$$

(Liu, 2014). This premium measures the strength of the precautionary saving motive in terms of the safe reduction in $\tilde{y}_{2,l}$ that has the same effect on saving as the n^{th}-degree increase in y_2 risk. Premium θ^{y_2} is positive under the same condition as (2) for $j = 1$.

A precautionary premium measure that captures the tradeoff between n^{th}- and m^{th}-degree risk increases can be defined elaborating on Liu and Neilson (2019) (see Heinzel, 2021). The *path-dependent* m^{th}-degree precautionary premium $\overline{\theta}^{y_2}$ arises from

$$Eu'\left(\tilde{y}_2\left(\overline{\theta}^{y_2}\right) + sR\right) = Eu'\left(\tilde{y}_{2,h} + sR\right) \tag{15a}$$

where $\left\{\tilde{y}_2\left(\overline{\theta}^{y_2}\right)\right\}_{\overline{\theta}^{y_2} \in [0,1)}$ is the continuous path of m^{th}-degree increasing risk from $\tilde{y}_{2,l}$, with $\tilde{y}_2(0) = \tilde{y}_{2,l}$ and such that $\tilde{y}_2\left(\overline{\theta}^{y_2\prime}\right)$ has more m^{th}-degree risk than $\tilde{y}_2\left(\overline{\theta}^{y_2}\right)$ for every $\overline{\theta}^{y_2\prime} > \overline{\theta}^{y_2} \geq 0$. Thus, $\overline{\theta}^{y_2}$ reflects the tradeoff between the impacts on marginal utility of an n^{th}-degree and an m^{th}-degree risk increase from $\tilde{y}_{2,l}$.

Another interpretation for $\overline{\theta}^{y_2}$ arises when rewriting defining equation (15a) with CDFs F, G, H_m from Section 2 applied to $\tilde{y}_2$. Thus, $\tilde{y}_2\left(\overline{\theta}^{y_2}\right)$'s CDF can be stated as $\left(1 - \overline{\theta}^{y_2}\right)F + \overline{\theta}^{y_2}H_m$ for all $\overline{\theta}^{y_2}$, which leads to

$$\left(1 - \overline{\theta}^{y_2}\right)Eu'\left(\tilde{y}_{2,l} + sR\right) + \overline{\theta}^{y_2}Eu'\left(\tilde{y}_{2,m} + sR\right) = Eu'\left(\tilde{y}_{2,h} + sR\right) \tag{15b}$$

When $\overline{\theta}^{y_2}, T_1^{(n/m)} \in (0,1)$, premium $\overline{\theta}^{y_2}$ in (15b) coincides with $T_1^{(n/m)}$ in (4b) applied to model (12). As a result, the interpretation and analysis for $T_1^{(n/m)}$ holds for $\overline{\theta}^{y_2}$ analogously.

The latter insight together with Theorem 1 for $j = 1$ imply that u's stronger precautionary motive can alternatively be expressed in three ways.

Proposition 1 *(Comparative Precautionary Saving, Income Risk). Comparative precautionary saving for a $\tilde{y}_{2,l}$ to $\tilde{y}_{2,h}$ increase is equivalently expressed by*

(i) $\overline{\theta}_u^{y_2} \geq \overline{\theta}_v^{y_2}$ *for all* $\tilde{y}_{2,l}$, $\tilde{y}_{2,h}$, $\tilde{y}_{2,m}$.
(ii) $T_{1,u}^{(n/m)} \geq T_{1,v}^{(n/m)}$ *for all* $\tilde{y}_{2,l}$, $\tilde{y}_{2,h}$, $\tilde{y}_{2,m}$.
(iii) $N_1UP_u^{(n/m)} \geq N_1UP_v^{(n/m)}$ *for all* $\tilde{y}_{2,l}$, $\tilde{y}_{2,h}$.

The proposition covers characterizations equivalent to $\theta_u^{y_2} \geq \theta_v^{y_2}$ from Liu (2014, Theorem 3) for $m=1$.[14] By (i), (ii), and Lemma 1, those express alternatively the willingness to replace the y_2 risk increase by a first-degree increase (e.g., in the form of $-\theta^{y_2}$) at marginal utility level and the willingness to replace an $(n+1)^{th}$- by a second-degree $\tilde{y}_2$ increase at EU level. Similar to θ^{y_2}, $N_1UP^{(n/1)}$ provides a monetary quantification of the precautionary motive. Obviously, Proposition 1 is more general in that it admits comparisons for any $m \geq 1$.

4.2. Comparative precautionary saving under return risk

Marginal utility premium $E\left[u'\left(\tilde{c}_{2,h}^R\right)\tilde{R}_h\right] - E\left[u'\left(\tilde{c}_{2,l}^R\right)\tilde{R}_l\right]$ is positive (negative) if and only if the product of $(-1)^n$ and the n^{th} derivative of marginal utility $h_{[1],u}(R) = Ru'\left(c_2^R\right)$,

$$h_{[1],u}^{(n)}(R) = u^{(n+1)}\left(c_2^R\right)s^nR + ns^{n-1}u^{(n)}\left(c_2^R\right) \tag{16}$$

is positive (negative) (Eeckhoudt and Schlesinger, 2008). The sign of that product depends on two counteracting effects: for an n^{th}- and $(n+1)^{th}$-degree risk averse decision maker, the first term expresses the positive precautionary effect and the second expresses the negative substitution effect (Bostian and Heinzel, 2022). A conventional comparison of the strength of precautionary motives can refer to multiplicative precautionary premium θ^R, from

$$E\left[u'\left(y_2 + s\left(\tilde{R}_l - \theta_u^R\right)\right)\tilde{R}_l\right] = E\left[u'\left(y_2 + s\tilde{R}_h\right)\tilde{R}_h\right] \tag{17}$$

(Bostian and Heinzel, 2018). This premium indicates the *proportion* of saving such that the safe change $s\theta_u^R$ in $\tilde{c}_2^R$ has the same effect on saving as the risk increase. Similar to (11), θ^R shares the sign of (6) for $j=1$ and is unit-free. The safe change $s\theta_u^R$ captures the strength of the precautionary motive.

The path-dependent m^{th}-degree multiplicative precautionary premium $\overline{\theta}^R$ arises from

$$E\left[u'\left(y_2 + s\tilde{R}\left(\overline{\theta}^R\right)\right)\tilde{R}\left(\overline{\theta}^R\right)\right] = E\left[u'\left(y_2 + s\tilde{R}_h\right)\tilde{R}_h\right] \tag{18}$$

where path $\left\{\tilde{R}\left(\overline{\theta}^R\right)\right\}_{\overline{\theta}^R \in [0,1)}$ is defined in analogy to $\left\{\tilde{y}_2\left(\overline{\theta}^{y_2}\right)\right\}_{\overline{\theta}^{y_2} \in [0,1)}$ in the previous section. Premium $\overline{\theta}^R$ reflects the tradeoff between the impacts on marginal utility of an n^{th}-degree and an m^{th}-degree risk increase from $\tilde{R}_l$. With arguments analogous to above, $\overline{\theta}^R$ coincides with $\widehat{T}_1^{(n/m)}$ in (10b) applied to model (12) when $\overline{\theta}^R, \widehat{T}_1^{(n/m)} \in (0,1)$.

Theorem 2 for $j=1$ implies further comparative precautionary saving characterizations. For convenience, I assume the decision makers $(n+1)^{th}$- and $(m+1)^{th}$-degree risk averse.

Proposition 2 *(Comparative Precautionary Saving, Return Risk). Among decision makers u, v with identical incentives and saving amounts under $\tilde{R}_l$ and increasing [decreasing] responses to any n^{th}-degree return risk increase, u shows the stronger precautionary reaction if and only if $\widehat{N_1UP}_u^{(n/m)} \geq [\leq]\ \widehat{N_1UP}_v^{(n/m)}$ for all $\tilde{R}_l$, $\tilde{R}_h$.*

If, in addition, u, v increase [decrease] saving in response to any m^{th}-degree increase, u shows the stronger precautionary reaction if and only if either condition holds:

(i) $\overline{\theta}_u^R \geq [\leq]\ \overline{\theta}_v^R$ *for all* $\tilde{R}_l$, $\tilde{R}_h$, $\tilde{R}_m$.
(ii) $\widehat{T}_{1,u}^{(n/m)} \geq [\leq]\ \widehat{T}_{1,v}^{(n/m)}$ *for all* $\tilde{R}_l$, $\tilde{R}_h$, $\tilde{R}_m$.

In the saving context, Theorem 2's sign conditions on $(-1)^k h_{[1]}^{(k)}(\rho)$ for $k=n,m$ translate into conditions on u's and v's common direction of the saving reaction to n^{th}- and m^{th}-degree return risk increases. Thus, expressing u's stronger precautionary motive in terms of u's higher (lower) normalized marginal utility premium only requires the common positive (negative) saving reaction to n^{th}-degree risk increases. But, expressing u's stronger motive in terms of u's higher (lower) path-dependent multiplicative precautionary premium or higher (lower) risk substitution rate at marginal utility level requires the decision makers, in addition, to react in the analogous way to m^{th}-degree increases.

Interestingly, for $m=1$, only the characterization in terms of $\widehat{N_1UP}^{(n/1)}$ is equivalent to $\theta_u^R \geq [\leq]\theta_v^R$ from Bostian and Heinzel (2018, Lemma 2); those in terms of $\overline{\theta}^R$ and $\widehat{T}_1^{(n/1)}$ are not.[15] For $m=1$, $\overline{\theta}^R$ captures the willingness to substitute a first- for an n^{th}-degree return risk increase at marginal utility level, whereas θ^R quantifies a saving fraction. Those two amounts would coincide only by accident.

[14] The equivalences for (ii), (iii), and $\theta_u^{y_2} \geq \theta_v^{y_2}$ when $m=1$ arise in analogy to Proposition 3 below.

[15] The equivalence for $\widehat{N_1UP}^{(n/1)}$ arises in analogy to Proposition 5 below.

Comparisons in terms of $\widehat{N_1UP}^{(n/m)}$, $\widehat{T}_1^{(n/m)}$, and $\overline{\theta}^R$ are more general as they apply for any $m \geq 1$. However, neither $\widehat{T}_1^{(n/m)}$ nor $\overline{\theta}^R$ reveal the sign of the decision makers' saving reaction.

5. Wealth dependence of Ross risk aversion

The normalized premia from Sections 2 and 3 help to generalize characterizations of the intrapersonal conditions for how risk aversion evolves with the utility argument, and yield new interpretations. Those intrapersonal conditions are instrumental to predict whether risk aversion of second order and higher rises with background risk.

5.1. Increases in additive risk

Sections 5.1 and 5.2 focus on decreasing shapes of absolute risk aversion.[16]

Definition 8 first extends Wang and Li's (2014) decreasing $(n/m)^{th}$-degree Ross risk aversion.

Definition 8 *(Decreasing $((n+j)/(m+j))^{th}$-Degree Ross Absolute Risk Aversion).* Utility function $u(.)$, with $(-1)^{k+1}u^{(k)}(.) \geq 0$ for $k = m+j$, $m+1+j$, $n+j$, $n+1+j$, exhibits decreasing $((n+j)/(m+j))^{th}$-degree Ross absolute risk aversion if, for all x_a, x_b, $x_a + x_c$, $x_b + x_c \in [a, b]$ with $x_c > 0$,

$$(-1)^{n-m} \frac{u^{(n+j)}(x_a)}{u^{(m+j)}(x_b)} \geq (-1)^{n-m} \frac{u^{(n+j)}(x_a + x_c)}{u^{(m+j)}(x_b + x_c)} \tag{19a}$$

In words, a decision maker is decreasingly $((n+j)/(m+j))^{th}$-degree Ross absolute risk averse if the coefficient on the left decreases with the utility argument. When $m = 1$, this definition covers Ross DARA for $j = 0$ and Ross DAP for $j = 1$.

Intrapersonal condition (19a) has a close link to $((n+j)/(m+j))^{th}$-degree Ross more risk aversion in Definition 2. Namely, for some scalar $\lambda > 0$, this condition is equivalent to

$$-\frac{u^{(n+1+j)}(x_a)}{u^{(n+j)}(x_a)} \geq \lambda \geq -\frac{u^{(m+1+j)}(x_b)}{u^{(m+j)}(x_b)} \quad \text{for all } x_a, x_b \in [a, b] \tag{19b}$$

When replacing $\left(-u', u\right)$ by (u, v), (19b) is identical to (3). Thus, with Lemma 1, Ross DARA is equivalent to $-u'$ being n^{th}-degree Ross more risk averse than u, and Ross DAP is equivalent to u'' being n^{th}-degree Ross more risk averse than $-u'$ (see also Footnote 9).

Substituting $\left(-u', u\right)$ for (u, v) in Theorem 1 yields characterizations in terms of risk substitution rates and normalized utility premia.

Theorem 3 *(Decreasing $((n+j)/(m+j))^{th}$-Degree Ross Absolute Risk Aversion). Suppose that u is k^{th}-degree risk averse for $k = 1+j$, $2+j$, $m+j$, $m+1+j$, $n+j$, $n+1+j$. Then, (i)–(iv) are equivalent:*

(i) *$(-1)^{1+j}u^{(1+j)}$ is $(n/m)^{th}$-degree Ross more risk averse than $(-1)^j u^{(j)}$.*
(ii) *There exist $\lambda > 0$ and $\phi^{(j)}(x)$ such that $-u^{(1+j)}(x) = \lambda u^{(j)}(x) + \phi^{(j)}(x)$, where $(-1)^{j+m-1}\phi^{(j+m)}(x) \leq 0$ and $(-1)^{j+n-1}\phi^{(j+n)}(x) \geq 0$ for all x.*
(iii) *$T_{j+1}^{(n/m)} \geq T_j^{(n/m)}$ for all $\tilde{x}_l, \tilde{x}_h, \tilde{x}_m$.*
(iv) *$N_{j+1}UP_u^{(n/m)} \geq N_jUP_u^{(n/m)}$ for all $\tilde{x}_l, \tilde{x}_h$.*

The theorem provides two new characterizations of decreasing $((n+j)/(m+j))^{th}$-degree Ross absolute risk aversion. That preference trait is both equivalent to the decision maker's willingness to substitute an m^{th}- for an n^{th}-degree risk increase at $(j+1)^{th}$ utility derivative level uniformly exceeding that at j^{th} derivative level and to u's $(j+1)^{th}$ utility derivative premium being uniformly larger than the j^{th}. As in Theorem 1, there is mutually no direct implication between (iii) and (iv).

The two criteria have an intuitive link to decreasing Ross absolute risk aversion. Noting that $T_j^{(n/m)}$ and $N_jUP^{(n/m)}$ are measures of $((n+j)/(m+j))^{th}$-degree risk aversion, the two conditions arise equivalently from $T_j^{(n/m)}$'s and $N_jUP^{(n/m)}$'s nonpositive derivatives with respect to safe wealth $x = E\tilde{x}$. Hence, those conditions are equivalent, respectively, to a risk substitution rate for the j^{th} derivative and a normalized j^{th} utility derivative premium that decreases with the level of safe wealth.

5.2. Illustrative examples for additive risk

Two examples illustrate Theorem 3. The first elaborates on saving model (12) and provides alternative criteria for a decreasing precautionary saving motive. When applied to that model, Ross DAP captures the conditions on preferences for the precautionary motive to decrease with the level of expected second-period income $E\tilde{y}_2$. The conditions compare attitudes toward the risk impacts on the second and first utility derivatives.

A first criterion, namely $\zeta^{y_2} \geq \theta^{y_2}$, compares precautionary motive θ^{y_2} from (14) and Keenan and Snow's (2012) temperance premium ζ^{y_2}, generalized to n^{th}-degree risk, from[17]

[16] The corresponding results for increasing shapes merely require reversing inequality signs, starting with defining inequalities (19). Theorem 3 (for $\lambda > 0$) and Propositions 3 and 4 then hold analogously.

[17] Like all premia above, the temperance premium from (20) refers, in general, to a generic risk increase and is thus a *partial* premium (Denuit and Eeckhoudt, 2010). Its name applies literally for $n = 2$. Higher-order risk increases involve conditions on preferences of order higher than four.

$$Eu''\left(\tilde{y}_{2,l}+sR-\zeta^{y_2}\right)=Eu''\left(\tilde{y}_{2,h}+sR\right) \tag{20}$$

Generally, ζ^{y_2} measures the maximum willingness-to-pay to avoid the risk increase impact on the slope of marginal utility. In the saving context, ζ^{y_2} expresses the aversion to changing the saving decision due to the risk increase. It is positive if and only if $(-1)^{n+1}\,u^{(n+2)}>0$, like $N_2UP^{(n/1)}$. The criterion derives from Liu (2014, Theorem 3) by analogy given the definitions of Ross DAP, θ^{y_2}, and ζ^{y_2}.

Theorem 3 above for $j=m=1$ yields two further criteria.

Proposition 3 *(Decreasing Precautionary Saving Motive, Income Risk). Let $m=1$. Decision maker u's precautionary saving motive in the face of any risk increase from $\tilde{y}_{2,l}$ to $\tilde{y}_{2,h}$ decreases with $E\tilde{y}_2$ if and only if either condition holds:*

(i) $T_2^{(n/1)}\geq T_1^{(n/1)}$ *for all* $\tilde{y}_{2,l},\tilde{y}_{2,h},\tilde{y}_{2,m}$.
(ii) $N_2UP^{(n/1)}\geq N_1UP^{(n/1)}$ *for all* $\tilde{y}_{2,l},\tilde{y}_{2,h}$.
(iii) $\zeta^{y_2}\geq\theta^{y_2}$ *for all* $\tilde{y}_{2,l},\tilde{y}_{2,h}$.

Supplemental Appendix D contains an explicit proof. Hence, the precautionary saving motive declines with the level of future labor income $E\tilde{y}_2$ if and only if, alternatively, the willingness to substitute a first- for the n^{th}-degree risk increase at second utility derivative level exceeds that willingness at marginal utility level; the normalized second utility derivative premium is uniformly larger than the normalized marginal utility premium; and the willingness-to-pay to avoid a risk-induced saving change is uniformly higher than the precautionary motive. Establishing any of these relationships confirms an EU decision maker's Ross DAP.

The second example concerns the effects of background risk increases on risk aversion of second order and higher. According to Wang and Li, decreasing $(n/m)^{th}$-degree Ross absolute risk aversion is necessary and sufficient for any $(n-m)^{th}$-degree background risk increase, from $\tilde{\varepsilon}_l$ to $\tilde{\varepsilon}_{n-m}$, to raise $(m+1)^{th}$-degree Arrow-Pratt risk aversion (see Supplemental Appendix E for an explicit proof). The conditions on preferences here involve comparative risk impacts on marginal utility and plain EU.

Two criteria ensuring that relationship derive from Theorem 3 for $j=0$.

Proposition 4 *(Background Risk Effect on $(m+1)^{th}$-Degree ARA). An $(n-m)^{th}$-degree background risk increase from $\tilde{\varepsilon}_l$ to $\tilde{\varepsilon}_{n-m}$ raises $(m+1)^{th}$-degree Arrow-Pratt absolute risk aversion if and only if either condition holds:*

(i) $T_1^{(n/m)}\geq T_0^{(n/m)}$ *for all* $\tilde{x}_l,\tilde{x}_h,\tilde{x}_m$.
(ii) $N_1UP^{(n/m)}\geq NUP^{(n/m)}$ *for all* $\tilde{x}_l,\tilde{x}_h$.

Thus, a decision maker's $(m+1)^{th}$-degree absolute foreground risk aversion increases due to more $(n-m)^{th}$-degree background risk if and only if the willingness to substitute m^{th}- for n^{th}-degree risk at marginal utility level uniformly exceeds that at EU level or, equivalently, the normalized marginal utility premium is uniformly larger than Li and Liu's normalized utility premium. Those characterizations hold for any $m\geq 1$.

5.3. Increases in return risk

The results under return risk are less clear-cut due to the sign ambivalence of risk impacts on utility derivatives (Remark 4). Relative risk aversion (RRA) is the appropriate measure of aversion to return risk.[18] Both decreasing and increasing RRA shapes can be motivated empirically (Gollier, 2001; Meyer and Meyer, 2005). Yet, the representations for the two shapes are not analogous. In coherence with Section 5.1, this section focuses on decreasing shapes.[19]

I define decreasing Ross RRA in analogy to Definition 8 as follows.[20]

Definition 9 *(Decreasing $((n+j)/(m+j))^{th}$-Degree Ross Relative Risk Aversion).* Utility function u, with $(-1)^{k+1}u^{(k)}\geq 0$ for $k=m+j,\ m+1+j,\ n+j,\ n+1+j$, exhibits decreasing $((n+j)/(m+j))^{th}$-degree Ross relative risk aversion if, for all $w_a,\ w_b,\ w_a+w_c,\ w_b+w_c\in[a,b]$ with $w_c>0$,

$$(-1)^{n-m}\,\frac{u^{(n+j)}\left(w_a\right)}{u^{(m+j)}\left(w_b\right)}\,w_a\geq(-1)^{n-m}\,\frac{u^{(n+j)}\left(w_a+w_c\right)}{u^{(m+j)}\left(w_b+w_c\right)}\,\left(w_a+w_c\right) \tag{21a}$$

In words, a decision maker is decreasingly $((n+j)/(m+j))^{th}$-degree Ross relative risk averse if the coefficient on the left decreases with the utility argument. When $m=1$, this definition covers Ross DRRA for $j=0$ and Ross DRP for $j=1$.

Given some $\lambda>0$, intrapersonal condition (21a) is equivalent to Ross condition

$$-\frac{u^{(n+1+j)}\left(\omega_a\right)}{u^{(n+j)}\left(\omega_a\right)}\,\omega_a-1\geq\lambda\geq-\frac{u^{(m+1+j)}\left(\omega_b\right)}{u^{(m+j)}\left(\omega_b\right)}\,\omega_b\quad\text{for all }\omega_a,\omega_b\in[a,b] \tag{21b}$$

[18] For example, Pratt (1964) shows that the RRA coefficient is proportional to the multiplicative risk premium $\hat{\pi}^{\rho}$, from $u\left(x\left(E\tilde{\rho}-\hat{\pi}^{\rho}\right)\right)=Eu\left(x\tilde{\rho}\right)$, for a small zero-mean risk added to the safe return.

[19] Supplemental Appendix H states the analysis for the case with increasing shapes. Defining the IRRA case in analogs of (21) and Lemma 3's analog merely require reversing the inequality signs. But, in Theorem 4's analog, the implications reverse: the premium orderings now follow from a given preference characteristic.

[20] The closest definition I am aware of, in Jokung (2013), stipulates Ross DRRA based on (21b) for $(m,j)=(1,0)$. See Supplemental Appendix F for a proof of the equivalence of (21a) and (21b).

which links the $(n+j)^{th}$- and $(m+j)^{th}$-degree RRA coefficients. When $\omega_a = \omega_b$, $j = 0$, and $(n,m) = (2,1)$, (21b) expresses the well-known condition for DRRA, namely that relative prudence be uniformly larger than RRA plus one. The conditions for DRP and decreasing relative temperance are similarly covered for $j = 1$ and $j = 2$.

In the following, I will use inequalities (21) applied to the model with return risk from Section 3. Thus, w is replaced by $x\rho$, for some x and all $\rho_a, \rho_b, \rho_c > 0$ such that $x\rho_a$, $x\rho_b$, $x(\rho_a + \rho_c)$, $x(\rho_b + \rho_c) \in [a, b]$, and similarly for ω.

Lemma 3 provides Ross conditions that order, respectively, successive risk substitution rates and normalized utility derivative premia for arbitrary j, n, and m.

Lemma 3 *(Ross Conditions to Order Successive $\widehat{T}_j^{(n/m)}$s and $\widehat{N_jUP}^{(n/m)}$s). Consider $h_{[j],f}^{(k)}(\rho)$ from (7b) for $f \in \{u, \phi\}$ such that $(-1)^{k+j-1}\, h_{[j],u}^{(k)}(\rho) > [<]\ 0$ and $(-1)^{k+j}\, h_{[j+1],u}^{(k)}(\rho) \geq [\leq]\ 0$ for $k = m, n$. Then, (a.i)–(a.iii) are equivalent:*

(a.i) *$-u'$ is Ross more risk averse than u in the sense that*

$$-\frac{xh_{[j+1],u}^{(n)}(\rho_a)}{h_{[j],u}^{(n)}(\rho_a)} \geq \lambda \geq -\frac{xh_{[j+1],u}^{(m)}(\rho_b)}{h_{[j],u}^{(m)}(\rho_b)} \quad \text{for some } x \text{ and all } \rho_a, \rho_b \tag{22a}$$

(a.ii) *There exist $\lambda > 0$ and $\phi(x\rho)$ such that $-xh_{[j+1],u}(\rho) = \lambda h_{[j],u}(\rho) + h_{[j],\phi}(\rho)$, where $(-1)^{m+j-1}\, h_{[j],\phi}^{(m)}(\rho) \leq [\geq]\ 0$ and $(-1)^{n+j-1}\, h_{[j],\phi}^{(n)}(\rho) \geq [\leq]\ 0$ for some x and all ρ.*

(a.iii) *$\widehat{T}_{j+1}^{(n/m)} \geq [\leq]\ \widehat{T}_j^{(n/m)}$ for all $\tilde{\rho}_l$, $\tilde{\rho}_h$, $\tilde{\rho}_m$.*

Suppose that u is k^{th}-degree risk averse for $k = 1+j, 2+j, m+j, m+1+j, n+\iota$ with $\iota \in \{0, \ldots, j+1\}$, and consider $h_{[j],f}^{(n)}(\rho)$ from (7b) for $f \in \{u, \phi\}$ such that $(-1)^{n+j-1}\, h_{[j],u}^{(n)}(\rho) > [<]\ 0$ and $(-1)^{n+j}\, h_{[j+1],u}^{(n)}(\rho) \geq [\leq]\ 0$. Then, (b.i)–(b.iii) are equivalent:

(b.i) *$-u'$ is Ross more risk averse than u in the sense that*

$$-\frac{xh_{[j+1],u}^{(n)}(\rho_a)}{h_{[j],u}^{(n)}(\rho_a)} \geq \lambda \geq -\frac{u^{(m+1+j)}(x\rho_b)}{u^{(m+j)}(x\rho_b)}\, x\rho_b \quad \text{for some } x \text{ and all } \rho_a, \rho_b \tag{22b}$$

(b.ii) *There exist $\lambda > 0$ and $\phi(x\rho)$ such that $-xh_{[j+1],u}(\rho) = \lambda h_{[j],u}(\rho) + h_{[j],\phi}(\rho)$, where $(-1)^{m+j-1}\, h_{[j],\phi}^{(m)}(\rho) \leq [\leq]\ (-1)^{m+j} \Big\{xh_{[j+1],u}^{(m)}(\rho) + \lambda h_{[j],u}^{(m)}(\rho) - x^m \rho^j \left[u^{(m+1+j)}(x\rho)\, x\rho\ + \lambda u^{(m+j)}(x\rho)\right]\Big\}$ and $(-1)^{n+j-1}\, h_{[j],\phi}^{(n)}(\rho) \geq [\leq]\ 0$ for some x and all ρ.*

(b.iii) *$\widehat{N_{j+1}UP}_u^{(n/m)} \geq \widehat{N_jUP}_u^{(n/m)} \geq 0 \left[\widehat{N_{j+1}UP}_u^{(n/m)} \leq \widehat{N_jUP}_u^{(n/m)} \leq 0\right]$ for all $\tilde{\rho}_l$, $\tilde{\rho}_h$.*

The equivalences of (a.i)-(a.iii) on the one hand and (b.i)-(b.iii) on the other hand arise similarly as for Theorem 2 above.

Relating the premium orderings in Lemma 3 to risk attitude characteristics is more intricate than under additive risk (see Theorem 3). The conditions now need to account for the counteracting effects at the levels of both the $(j+1)^{th}$ and the j^{th} utility derivative. In (22), the $h_{[j+1],u}^{(k)}$ and $h_{[j],u}^{(k)}$ expressions for $k = n, m$ capture those effects. In view of the increasing complexity with j, the next theorem only treats the two prominent cases, $j = 0, 1$. For $j = 0$, Lemma 3, and so, the premium orderings in (a.iii) and (b.iii), imply decreasing $(n/m)^{th}$-degree Ross RRA in (21). The implication follows because (21b)'s left side compares to that of (a rewriting of) (22a) as $-\frac{u^{(n+1)}(x\rho_a)}{u^{(n)}(x\rho_a)}\, x\rho_a - 1 \geq -\frac{u^{(n+1)}(x\rho_a)}{u^{(n)}(x\rho_a)}\, x\rho_a - (n-m)$ and to (22b)'s as $-\frac{u^{(n+1)}(x\rho_a)}{u^{(n)}(x\rho_a)}\, x\rho_a - 1 > -\frac{u^{(n+1)}(x\rho_a)}{u^{(n)}(x\rho_a)}\, x\rho_a - n$. For $j = 1$, decreasing $((n+1)/(m+1))^{th}$-degree Ross RRA follows from $\widehat{N_2UP}^{(n/m)} \geq [\leq]\ \widehat{N_1UP}^{(n/m)}$, if the decision maker has increasing [decreasing] $((n+1)/n)^{th}$-degree Ross RRA; and $\widehat{T}_2^{(n/m)} \geq [\leq]\ \widehat{T}_1^{(n/m)}$ follows if the decision maker's $(m+1)^{th}$-degree Ross RRA is, in addition, sufficiently stronger [weaker] than the m^{th}-degree one (see Appendix B). For consistency, the implications require that $n - m > 1$.

Theorem 4 summarizes these conditions for $j = 0, 1$.

Theorem 4 *(Decreasing $(n/m)^{th}$- and $((n+1)/(m+1))^{th}$-degree Ross RRA). Lemma 3 for $j = 0$ implies decreasing $(n/m)^{th}$-degree Ross RRA. For $j = 1$ and $n - m > 1$, given the conditions of Lemma 3(a), $\widehat{T}_2^{(n/m)} \geq [\leq]\ \widehat{T}_1^{(n/m)}$ implies decreasing $((n+1)/(m+1))^{th}$-degree Ross RRA, if (i) the decision maker has increasing [decreasing] $((n+1)/n)$-degree Ross RRA and (ii) $(m+1)^{th}$-degree Ross RRA is sufficiently stronger [weaker] than m^{th}-degree Ross RRA, whereas, given the conditions of Lemma 3(b), the implication from $\widehat{N_2UP}^{(n/m)} \geq [\leq]\ \widehat{N_1UP}^{(n/m)}$ only requires (i).*

When $m = 1$, this theorem covers Ross DRRA for $j = 0$ and Ross DRP for $j = 1$, like Definition 9. For some intuition for $j = 1$, consider the simplest feasible case, namely, $(n,m) = (3,1)$. In that case, $\widehat{N_2UP}^{(3/1)} \geq [\leq]\ \widehat{N_1UP}^{(3/1)}$ implies decreasing $(4/2)^{th}$-degree Ross RRA, if the decision maker has increasing [decreasing] $(4/3)^{th}$-degree Ross RRA. If, in addition, second-degree Ross RRA is sufficiently stronger [weaker] than first-degree Ross RRA, then decreasing $(4/2)^{th}$-degree Ross RRA holds if the decision maker's willingness to substitute a third- for a fifth-degree return risk increase is uniformly higher [lower] than that to substitute a second- for a fourth-degree one.

5.4. Illustrative examples for return risk

Two examples similar to Section 5.2 illustrate those findings. The first one provides criteria for the precautionary saving motive to decrease with the level of expected return $E\tilde{R}$. Ross DRP – as in (21) for $j = m = 1$ – applied to model (12) captures the related conditions. Those compare risk impacts on the second and first utility derivatives.

The first criterion, namely $\zeta^R \geq \theta^R \geq 0$ $\left[\zeta^R \leq \theta^R \leq 0\right]$, compares (17)'s multiplicative precautionary premium θ^R and the multiplicative temperance premium ζ^R, from

$$E\left[u''\left(s\left(\tilde{R}_l - \zeta^R\right)\right)\tilde{R}_l^2\right] = E\left[u''\left(s\tilde{R}_h\right)\tilde{R}_h^2\right] \tag{23}$$

As applied to saving, ζ^R indicates the *proportion* of s such that $s\zeta^R$ is the maximum amount of $\tilde{c}_2^R$ the decision maker is willing to forgo to avoid changing the saving decision due to the risk increase. Like $\widehat{N_2UP}^{(n/1)}$, ζ^R is positive [negative] if and only if $(-1)^{n+1}\, h_{[2],u}^{(n)} > [<]\, 0$; and it is unit-free, like θ^R and $\widehat{N_2UP}^{(n/1)}$.

The mentioned criterion is equivalent to $\widehat{N_2UP}^{(n/1)} \geq [\leq]\, \widehat{N_1UP}^{(n/1)}$ as implied, by analogy, by the equivalence of the comparative precautionary saving characterizations in terms of θ^R in Bostian and Heinzel (2018, Lemma 2) and $\widehat{N_1UP}^{(n/1)}$ in Proposition 2 (see Supplemental Appendix I for an explicit proof). By Theorem 4, thus, both criteria ensure Ross DRP for any decision maker with increasing [decreasing] $((n+1)/n)^{th}$-degree Ross RRA in the case of an increasing [a decreasing] saving response, provided that $n > 2$.

The $\zeta^R \geq [\leq]\, \theta^R$ comparison has an intuitive interpretation: the precautionary motive decreases with $E\tilde{R}$, if the willingness to avoid changing the saving decision due to the risk increase, $s\zeta^R$, uniformly exceeds [stays below] the precautionary motive, $s\theta^R$, for a decision maker with positive [negative] θ^R and ζ^R. The $\widehat{N_2UP}^{(n/1)} \geq [\leq]\, \widehat{N_1UP}^{(n/1)}$ criterion expresses that in that case equivalently the normalized second utility derivative premium exceeds [stays below] the normalized marginal utility premium.

Similarly based on Theorem 4, Proposition 5 adds to the latter equivalence the conditions under which an ordering of $(n/1)^{th}$-degree risk substitution rates implies Ross DRP.

Proposition 5 *(Decreasing Precautionary Saving Motive, Return Risk). Assume $n > 2$, and consider a decision maker with increasing [decreasing] $((n+1)/n)$-degree Ross RRA in the case of an increasing [a decreasing] saving response. Then, given the conditions of Lemma 3(b) for $j = m = 1$, either of the following equivalent conditions implies $(n/1)^{th}$-degree Ross DRP:*

(i) $\widehat{N_2UP}^{(n/1)} \geq [\leq]\, \widehat{N_1UP}^{(n/1)}$ *for all* $\tilde{R}_l$, $\tilde{R}_h$.
(ii) $\zeta^R \geq [\leq]\, \theta^R$ *for all* $\tilde{R}_l$, $\tilde{R}_h$.

If, in addition, the decision maker has a sufficiently stronger [weaker] second-degree Ross RRA than first-degree Ross RRA, then, given the conditions of Lemma 3(a) for $j = m = 1$, $\widehat{T}_2^{(n/1)} \geq [\leq]\, \widehat{T}_1^{(n/1)}$ for all $\tilde{R}_l$, $\tilde{R}_h$, $\tilde{R}_m$ implies $(n/1)^{th}$-degree Ross DRP.

Under the additional condition in the proposition, $(n/1)^{th}$-degree Ross DRP also follows if the decision maker's willingness to substitute a first- for the n^{th}-degree return risk increase at the level of the second utility derivative uniformly exceeds [stays below] that willingness to substitute at the level of marginal utility, given a positive [negative] risk impact on those two utility derivatives. Based on Theorem 4, it is furthermore clear that, for arbitrary m, the comparisons of the appropriate normalized utility derivative premia and risk substitution rates imply $(n/m)^{th}$-degree Ross DRP, provided that $n > m \geq 1$ and $n - m > 1$.

The final example provides the conditions for a multiplicative background risk increase to raise $(m+1)^{th}$-degree foreground RRA. Convenient conditions arise now for n^{th}-degree increases, from $\tilde{\rho}_l$ to $\tilde{\rho}_h$, and involve comparisons of the risk impacts on the $(m+1)^{th}$ and m^{th} utility derivatives. The result builds on Lemma 3(b) above and its analog for increasing Ross RRA (in Supplemental Appendix H): necessary and sufficient is that the normalized $(m+1)^{th}$ uniformly exceeds the normalized m^{th} utility derivative premium, given that $(2m+1)^{th}$- and $(m+1)^{th}$-degree Ross RRA compare in a certain way. It is interesting that the required premium orderings for positive and negative risk impacts are uniform and associated, respectively, with decision makers according to Lemma 3(b) and its increasing Ross RRA analog.

Proposition 6 *(Background Risk Effect on $(m+1)^{th}$-Degree RRA). For a decision maker fulfilling the conditions of Lemma 3(b) { Lemma 3(b)'s analog for increasing Ross RRA } for $j = m$, with decreasing $(2m/m)^{th}$-degree Ross RRA { $(2m+1)^{th}$-degree Ross RRA not exceeding $(m+1)^{th}$-degree Ross RRA } for $(-1)^{n+m-1}\, h_{[m]}^{(n)}(\rho) > \{<\}\, 0$, a background risk increase from $\tilde{\rho}_l$ to $\tilde{\rho}_h$ raises $(m+1)^{th}$-degree Arrow-Pratt RRA if and only if*

$$\widehat{N_{m+1}UP}^{(n/m)} \geq \{\geq\}\, \widehat{N_mUP}^{(n/m)} \text{ for all } \tilde{\rho}_l, \tilde{\rho}_h \tag{24}$$

The proof in Appendix C leans on Keenan and Snow (2012, Theorems 4-5).

Thus, in the simplest case $m = 1$, a decision maker with positive { negative } $\tilde{\rho}$ impact on marginal utility and Ross DRRA { third-degree Ross RRA not above second-degree Ross RRA } has a higher Arrow-Pratt foreground RRA if and only if the normalized second utility derivative premium is uniformly larger than the normalized marginal utility premium.

Proposition 6 extends Jokung (2013, Proposition 1), which gives for $m = 1$ an alternative criterion in the form of a Ross condition that involves (a rewriting of) (22b)'s left-hand term for $j = m = 1$ and second-degree Ross RRA.[21] Similar Ross conditions to Jokung's can readily be derived for the effects on higher-order RRA ($m > 1$), but do not lend themselves to characterizations in terms of premia.

[21] Jokung refers to n^{th}-degree stochastic dominance, instead of the n^{th}-degree risk special case here.

The comparison of (22b)'s left-hand term with $(m+1)^{th}$-degree RRA is due to the interest in the background risk effect on this latter attitude. Lemma 3(b) and its increasing Ross RRA analog, which both involve normalized utility derivative premia, are well suited for such comparisons, contrary to Lemma 3(a). Condition (24) for $m=1$ can alternatively be stated in terms conventional willingness-to-pay premia, namely, as $\zeta^{\rho} \geq \theta^{\rho}$. However, using normalized utility derivative premia enables comparisons for arbitrary $m \geq 1$.

6. Conclusion

This paper develops the risk comparative statics of utility derivatives for increases in additive and multiplicative risk. Direct risk impacts on utility derivatives play an important role in various economic contexts, including precaution and background risk effects. Such risk impacts depend on the decision makers' attitudes toward $(n/m)^{th}$-degree risk tradeoffs at the level of the j^{th} utility derivative.

To capture these attitudes, I extend two kinds of normalized utility premium measures. Under additive risk, the risk substitution rate for the j^{th} utility derivative and the normalized j^{th} utility derivative premium can equivalently characterize interpersonal comparisons of those attitudes, extending the EU level result. To multiplicative risk, this equivalence extends only at EU level. At utility derivative level, the two preference measures are associated with distinct characterizations with different properties.

Applications show that the new preference measures can characterize comparative precautionary saving in terms of attitudes toward $(n/m)^{th}$-degree tradeoffs with $m \geq 1$, whereas conventional characterizations have been restricted to $m=1$. In addition, simple comparison of the measures at the $(j+1)^{th}$ and j^{th} utility derivative levels determines whether a decision maker exhibits decreasing or increasing $((n+j)/(m+j))^{th}$-degree Ross ARA, for any $j \geq 0$. For $j=0$, this result implies a criterion for Arrow-Pratt ARA of second order or higher to rise with $(n-m)^{th}$-degree increases of additive background risk. Due to the nature of multiplicative risk, the conditions regarding Ross RRA are much more involved. A new proposition providing the conditions for Arrow-Pratt RRA of second order or higher to increase with multiplicative background risk involves as a necessary and sufficient condition a comparison of the normalized $(j+1)^{th}$ and j^{th} utility derivative premia.

Declaration of competing interest

There is no competing interest.

Data availability

No data was used for the research described in the article.

Acknowledgements

I am indebted to the editor and two anonymous referees for their help. Moreover, I thank Richard Peter for stimulating this research, and Liqun Liu, Normann Lorenz, Henri Loubergé, Béatrice Rey, as well as participants at MUSEES (Lyon) and the Journées Internationales du Risque (Niort) for valuable comments on earlier drafts. The usual disclaimer applies.

Appendix A. Proof of Theorem 2

Part (a). (*a.i*) $\Rightarrow$ (*a.ii*). From (9a), (*a.ii*)'s $\phi^{(j)}(x\rho)$, and (7b) applied for u, v, ϕ, we have for all ρ such that $x\rho \in [a,b]$ that $(-1)^{m+j-1} h^{(m)}_{[j],\phi}(\rho) = (-1)^{m+j-1}\left[h^{(m)}_{[j],u}(\rho) - \lambda h^{(m)}_{[j],v}(\rho)\right] \leq [\geq]\ 0$ and $(-1)^{n+j-1} h^{(n)}_{[j],\phi}(\rho) = (-1)^{n+j-1}\left[h^{(n)}_{[j],u}(\rho) - \lambda h^{(n)}_{[j],v}(\rho)\right] \geq [\leq]\ 0$ for all $x\rho$.

(*a.ii*) $\Rightarrow$ (*a.iii*). From Definition 6 and $u^{(j)}(x\rho) = \lambda v^{(j)}(x\rho) + \phi^{(j)}(x\rho)$ with (7a) and (7b), (a.iii) holds because

$$\begin{aligned}
\widehat{T}^{(n/m)}_{j,u} &= \frac{Eh_{[j],u}(\tilde{\rho}_l) - Eh_{[j],u}(\tilde{\rho}_h)}{Eh_{[j],u}(\tilde{\rho}_l) - Eh_{[j],u}(\tilde{\rho}_m)} = \frac{Eh_{[j],u}(\tilde{\rho}_l) - \left[Eh_{[j],u}(\tilde{\rho}_h)\right]}{\lambda\left[Eh_{[j],v}(\tilde{\rho}_l) - Eh_{[j],v}(\tilde{\rho}_m)\right] + Eh_{[j],\phi}(\tilde{\rho}_l) - Eh_{[j],\phi}(\tilde{\rho}_m)} \\
&\geq [\leq]\ \frac{Eh_{[j],u}(\tilde{\rho}_l) - \left[Eh_{[j],u}(\tilde{\rho}_h)\right]}{\lambda\left[Eh_{[j],v}(\tilde{\rho}_l) - Eh_{[j],v}(\tilde{\rho}_m)\right]} = \frac{\lambda\left[Eh_{[j],v}(\tilde{\rho}_l) - Eh_{[j],v}(\tilde{\rho}_h)\right] + Eh_{[j],\phi}(\tilde{\rho}_l) - Eh_{[j],\phi}(\tilde{\rho}_h)}{\lambda\left[Eh_{[j],v}(\tilde{\rho}_l) - Eh_{[j],v}(\tilde{\rho}_m)\right]} \\
&\geq [\leq]\ \frac{Eh_{[j],v}(\tilde{\rho}_l) - Eh_{[j],v}(\tilde{\rho}_h)}{Eh_{[j],v}(\tilde{\rho}_l) - Eh_{[j],v}(\tilde{\rho}_m)} = \widehat{T}^{(n/m)}_{j,v}
\end{aligned}$$

where the first inequality follows because $(-1)^j\left\{Eh_{[j],\phi}(\tilde{\rho}_l) - Eh_{[j],\phi}(\tilde{\rho}_m)\right\} \leq [\geq]\ 0$ due to $(-1)^{m+j-1} h^{(m)}_{[j],\phi}(\rho) \leq [\geq]\ 0$, and the second because $(-1)^j\left\{Eh_{[j],\phi}(\tilde{\rho}_l) - Eh_{[j],\phi}(\tilde{\rho}_h)\right\} \geq [\leq]\ 0$ due to $(-1)^{n+j-1} h^{(n)}_{[j],\phi}(\rho) \geq [\leq]\ 0$.

(*a.iii*) $\Rightarrow$ (*a.i*). Let F, G, H_m be the cumulative density functions (CDFs) of $\tilde{\rho}_l$, $\tilde{\rho}_h$, $\tilde{\rho}_m$, respectively, defined on $[\rho_{lb}, \rho_{ub}]$, with associated higher-order CDFs $F^{[k]}(\rho) = \int_{\rho_{lb}}^{\rho} F^{[k-1]}(q)\,dq$, $G^{[k]}(\rho) = \int_{\rho_{lb}}^{\rho} G^{[k-1]}(q)\,dq$, $H^{[k]}_m(\rho) = \int_{\rho_{lb}}^{\rho} H^{[k-1]}_m(q)\,dq$ and $F^{[k]}(\rho) = G^{[k]}(\rho) = H^{[k]}_m(\rho)$ for $\rho = \rho_{lb}, \rho_{ub}$ and all $k = 1, \ldots, n-1$.[22] Then, $\widehat{T}^{(n/m)}_{j,u} \geq [\leq]\ \widehat{T}^{(n/m)}_{j,v}$ can be rewritten by using an n-fold integration by parts in the numerators and an m-fold one in the denominators as

$$\frac{(-1)^{n+j-1}\int_{\rho_{lb}}^{\rho_{ub}} h^{(n)}_{[j],u}(\rho)\left[G^{[n]}(\rho) - F^{[n]}(\rho)\right]d\rho}{(-1)^{m+j-1}\int_{\rho_{lb}}^{\rho_{ub}} h^{(m)}_{[j],u}(\rho)\left[H^{[m]}_m(\rho) - F^{[m]}(\rho)\right]d\rho} \geq [\leq]\ \frac{(-1)^{n+j-1}\int_{\rho_{lb}}^{\rho_{ub}} h^{(n)}_{[j],v}(\rho)\left[G^{[n]}(\rho) - F^{[n]}(\rho)\right]d\rho}{(-1)^{m+j-1}\int_{\rho_{lb}}^{\rho_{ub}} h^{(m)}_{[j],v}(\rho)\left[H^{[m]}_m(\rho) - F^{[m]}(\rho)\right]d\rho} \tag{25a}$$

[22] The lb and ub subscripts to ρ stand for lower bound and upper bound.

Similar to Li and Liu (2014), the proof then proceeds by contradiction: assume that (9a) does not hold. That is, there exist some $[\overline{\rho}_{lb}, \overline{\rho}_{ub}], [\overline{\overline{\rho}}_{lb}, \overline{\overline{\rho}}_{ub}] \in (\rho_{lb}, \rho_{ub})$ such that, for all $\rho_a \in [\overline{\rho}_{lb}, \overline{\rho}_{ub}]$ and all $\rho_b \in [\overline{\overline{\rho}}_{lb}, \overline{\overline{\rho}}_{ub}]$,

$$\frac{h^{(n)}_{[j],u}(\rho_a)}{h^{(n)}_{[j],v}(\rho_a)} < \mu < \frac{h^{(m)}_{[j],u}(\rho_b)}{h^{(m)}_{[j],v}(\rho_b)} \tag{25b}$$

Choose $F(\rho)$, $G(\rho)$, $H_m(\rho)$, such that

$$\begin{cases} G^{[n]}(\rho) - F^{[n]}(\rho) > 0 \text{ for all } \rho \in (\overline{\rho}_{lb}, \overline{\rho}_{ub}) \\ G^{[n]}(\rho) - F^{[n]}(\rho) = 0 \text{ for all } \rho \notin (\overline{\rho}_{lb}, \overline{\rho}_{ub}) \end{cases} \tag{25c}$$

$$\begin{cases} H^{[m]}_m(\rho) - F^{[m]}(\rho) > 0 \text{ for all } \rho \in \left(\overline{\overline{\rho}}_{lb}, \overline{\overline{\rho}}_{ub}\right) \\ H^{[m]}_m(\rho) - F^{[m]}(\rho) = 0 \text{ for all } \rho \notin \left(\overline{\overline{\rho}}_{lb}, \overline{\overline{\rho}}_{ub}\right) \end{cases}$$

Then, we have from (25b) that

$$\begin{aligned} &(-1)^{n+j-1}\int_{\rho_{lb}}^{\rho_{ub}} h^{(n)}_{[j],u}(\rho)\left[G^{[n]}(\rho) - F^{[n]}(\rho)\right]d\rho < [>]\ \mu\ (-1)^{n+j-1}\int_{\rho_{lb}}^{\rho_{ub}} h^{(n)}_{[j],v}(\rho)\left[G^{[n]}(\rho) - F^{[n]}(\rho)\right]d\rho \\ &(-1)^{m+j-1}\int_{\rho_{lb}}^{\rho_{ub}} h^{(m)}_{[j],u}(\rho)\left[H^{[m]}_m(\rho) - F^{[m]}(\rho)\right]d\rho > [<]\ \mu\ (-1)^{m+j-1}\int_{\rho_{lb}}^{\rho_{ub}} h^{(m)}_{[j],v}(\rho)\left[H^{[m]}_m(\rho) - F^{[m]}(\rho)\right]d\rho \end{aligned} \tag{25d}$$

which, when combined, contradicts (25a). Therefore, (9a) must be true.

Part (b). $(b.i) \Rightarrow (b.ii)$. From (9b), $(b.ii)$'s $\phi^{(j)}(x\rho)$, and (7b), we have for all ρ such that $x\rho \in [a,b]$ that $(-1)^{j+m-1}\phi^{(j+m)}(x\rho) = (-1)^{j+m-1}\left[u^{(j+m)}(x\rho) - \lambda v^{(j+m)}(x\rho)\right] \le 0$ and $(-1)^{n+j-1}h^{(n)}_{[j],\phi}(\rho) = (-1)^{n+j-1}\left[h^{(n)}_{[j],u}(\rho) - \lambda h^{(n)}_{[j],v}(\rho)\right] \ge [\le]\ 0$ for all $x\rho$.

$(b.ii) \Rightarrow (b.iii)$. Note first that $(-1)^j\left\{E\left[u(x\tilde{\rho}_l)\tilde{\rho}_l^j\right] - E\left[u(x\tilde{\rho}_h)\tilde{\rho}_h^j\right]\right\} \ge [\le]\ 0$ if and only if $(-1)^{n+j-1}h^{(n)}_{[j],u}(\rho) \ge [\le]\ 0$. Then, from Definition 7 and $u^{(j)}(x\rho) = \lambda v^{(j)}(x\rho) + \phi^{(j)}(x\rho)$ with (7a) and (7b), (b.iii) holds because

$$\begin{aligned} \widehat{N_jUP}^{(n/m)}_u &= \frac{Eh_{[j],u}(\tilde{\rho}_l) - Eh_{[j],u}(\tilde{\rho}_h)}{(-1)^{m-1}x^m E\left[u^{(j+m)}(x\tilde{\rho}_l)\tilde{\rho}_l^j\right]} = \frac{(-1)^j\left\{Eh_{[j],u}(\tilde{\rho}_l) - Eh_{[j],u}(\tilde{\rho}_h)\right\}}{(-1)^{j+m-1}x^m\left\{\lambda E\left[v^{(j+m)}(x\tilde{\rho}_l)\tilde{\rho}_l^j\right] + E\left[\phi^{(j+m)}(x\tilde{\rho}_l)\tilde{\rho}_l^j\right]\right\}} \\ &\ge [\le]\ \frac{(-1)^j\left\{Eh_{[j]}(\tilde{\rho}_l) - Eh_{[j]}(\tilde{\rho}_h)\right\}}{\lambda(-1)^{j+m-1}x^m E\left[v^{(j+m)}(x\tilde{\rho}_l)\tilde{\rho}_l^j\right]} = \frac{(-1)^j\left\{\lambda\left[Eh_{[j],v}(\tilde{\rho}_l) - Eh_{[j],v}(\tilde{\rho}_h)\right] + \left[Eh_{[j],\phi}(\tilde{\rho}_l) - Eh_{[j],\phi}(\tilde{\rho}_h)\right]\right\}}{\lambda(-1)^{j+m-1}x^m E\left[v^{(j+m)}(x\tilde{\rho}_l)\tilde{\rho}_l^j\right]} \\ &\ge [\le]\ \frac{Eh_{[j],v}(\tilde{\rho}_l) - Eh_{[j],v}(\tilde{\rho}_h)}{(-1)^{m-1}x^m E\left[v^{(j+m)}(x\tilde{\rho}_l)\tilde{\rho}_l^j\right]} = \widehat{N_jUP}^{(n/m)}_v \end{aligned}$$

where the first inequality follows because $(-1)^{j+m-1}\phi^{(j+m)}(x\rho) \le 0$, and the second because $(-1)^j\left\{Eh_{[j],\phi}(\tilde{\rho}_l) - Eh_{[j],\phi}(\tilde{\rho}_h)\right\} \ge [\le]\ 0$ due to $(-1)^{n+j-1}h^{(n)}_{[j],\phi}(\rho) \ge [\le]\ 0$.

$(b.iii) \Rightarrow (b.i)$. The proof is analogous to that of (a.iii) $\Rightarrow$ (a.i), but focuses only on CDFs F and G. Using them and an n-fold integration by parts in the numerators, $\widehat{N_jUP}^{(n/m)}_u \ge [\le]\ \widehat{N_jUP}^{(n/m)}_v$ can be rewritten as

$$\frac{(-1)^{n+j-1}\int_{\rho_{lb}}^{\rho_{ub}} h^{(n)}_{[j],u}(\rho)\left[G^{[n]}(\rho) - F^{[n]}(\rho)\right]d\rho}{(-1)^{m-1}x^m\int_{\rho_{lb}}^{\rho_{ub}} h^{(m)}_{[j],u}(\rho)\,dF(\rho)} \ge [\le]\ \frac{(-1)^{n+j-1}\int_{\rho_{lb}}^{\rho_{ub}} h^{(n)}_{[j],v}(\rho_a)\left[G^{[n]}(\rho) - F^{[n]}(\rho)\right]d\rho}{(-1)^{m-1}x^m\int_{\rho_{lb}}^{\rho_{ub}} h^{(m)}_{[j],v}(\rho)\,dF(\rho)} \tag{26a}$$

For the proof by contradiction, assume first that (9b) does not hold. That is, there exist some $[\overline{\rho}_{lb}, \overline{\rho}_{ub}], [\overline{\overline{\rho}}_{lb}, \overline{\overline{\rho}}_{ub}] \in (\rho_{lb}, \rho_{ub})$ such that, for all $\rho_a \in [\overline{\rho}_{lb}, \overline{\rho}_{ub}]$ and all $\rho_b \in [\overline{\overline{\rho}}_{lb}, \overline{\overline{\rho}}_{ub}]$,

$$\frac{h^{(n)}_{[j],u}(\rho_a)}{h^{(n)}_{[j],v}(\rho_a)} < \mu < \frac{u^{(j+m)}(x\rho_b)}{v^{(j+m)}(x\rho_b)} \tag{26b}$$

The implication then follows as for (a.iii) $\Rightarrow$ (a.i), when replacing (25c)'s second part by

$$\begin{cases} dF(\rho) > 0 \text{ for all } \rho \in \left(\overline{\overline{\rho}}_{lb}, \overline{\overline{\rho}}_{ub}\right) \\ dF(\rho) = 0 \text{ for all } \rho \notin \left(\overline{\overline{\rho}}_{lb}, \overline{\overline{\rho}}_{ub}\right) \end{cases}$$

and (25d)'s second inequality by

$$(-1)^{m-1}x^m\int_{\rho_{lb}}^{\rho_{ub}} h^{(m)}_{[j],u}(\rho)\,dF(\rho) < [>]\ \mu\ (-1)^{m-1}x^m\int_{\rho_{lb}}^{\rho_{ub}} h^{(m)}_{[j],v}(\rho)\,dF(\rho) \quad \blacksquare$$

Appendix B. Theorem 4: proof of sufficiency for $j = 1$

For $j = 1$, sufficiency of Lemma 3 for conditions (21) follows in two steps. First, it is to be shown that (21b)'s and (22)'s expressions on the left compare as

$$-\frac{u^{(n+2)}(x\rho)}{u^{(n+1)}(x\rho)}x\rho - 1 \geq -\frac{xh^{(n)}_{[2],u}(\rho)}{h^{(n)}_{[1],u}(\rho)} \tag{27a}$$

By inserting from (7b) and given that $(-1)^n h^{(n)}_{[1],u}(\rho) > [<]\ 0$, (27a) is equivalent to[23]

$$-\frac{u^{(n+2)}(x\rho)}{u^{(n+1)}(x\rho)}x\rho - 1 \leq [\geq] - \frac{2n-1}{n}\cdot\frac{u^{(n+1)}(x\rho)}{u^{(n)}(x\rho)}x\rho - (n-1) \tag{27b}$$

Comparing (27b)'s right-hand side with that of increasing [decreasing] $((n+1)/n)^{th}$-degree Ross RRA, i.e., $-\frac{u^{(n+2)}(x\rho)}{u^{(n+1)}(x\rho)}x\rho - 1 \leq [\geq] - \frac{u^{(n+1)}(x\rho)}{u^{(n)}(x\rho)}x\rho$, yields a sufficient condition, namely,

$$-\frac{u^{(n+1)}(x\rho)}{u^{(n)}(x\rho)}x\rho \leq [\geq] - \frac{2n-1}{n}\cdot\frac{u^{(n+1)}(x\rho)}{u^{(n)}(x\rho)}x\rho - (n-1) \tag{27c}$$

for some x and all ρ, which is, in turn, equivalent to

$$-\frac{u^{(n+1)}(x\rho)}{u^{(n)}(x\rho)}x\rho \geq [\leq] n \tag{27d}$$

Note that $(-1)^n h^{(n)}_{[1],u}(\rho) > [<]\ 0 \Leftrightarrow -\frac{u^{(n+1)}(x\rho)}{u^{(n)}(x\rho)}x\rho > [<] n$ ensures (27d) to hold, if $n - m > 1$: for $(-1)^n h^{(n)}_{[1],u}(\rho) > 0$, the decreasing $((n+1)/n)^{th}$-degree Ross RRA claim from (21) (for $m = n-1$) and the increasing $((n+1)/n)^{th}$-degree Ross RRA claim – for the implication to hold in that case – mutually exclude each other; and for $(-1)^n h^{(n)}_{[1],u}(\rho) < 0$, decreasing $((n+1)/n)^{th}$-degree Ross RRA would otherwise be claim *and* implication. With this qualification, this proof part establishes for Theorem 4(a) the left-hand side comparison and Theorem 4(b) fully.

Second, the right-hand side comparison for Theorem 4(a) requires (21b)'s and (22a)'s expressions on the right to compare as

$$-\frac{u^{(m+2)}(x\rho)}{u^{(m+1)}(x\rho)}x\rho \leq -\frac{xh^{(m)}_{[2],u}(\rho)}{h^{(m)}_{[1],u}(\rho)} \tag{27e}$$

By inserting from (7b) and given that $(-1)^m h^{(m)}_{[1],u}(\rho) > [<]\ 0$, (27e) is equivalent to

$$-\frac{u^{(m+2)}(x\rho)}{u^{(m+1)}(x\rho)}x\rho - 1 \geq [\leq] - 2\cdot\frac{u^{(m+1)}(x\rho)}{u^{(m)}(x\rho)}x\rho - m \tag{27f}$$

Comparing the right-hand side of decreasing [increasing] $((m+1)/m)^{th}$-degree Ross RRA, i.e., $-\frac{u^{(m+2)}(x\rho)}{u^{(m+1)}(x\rho)}x\rho - 1 \geq [\leq] - \frac{u^{(m+1)}(x\rho)}{u^{(m)}(x\rho)}x\rho$, with (27f)'s shows that the former is only necessary; for, sufficiency requires $-2\cdot\frac{u^{(m+1)}(x\rho)}{u^{(m)}(x\rho)}x\rho - m \leq [\geq] - \frac{u^{(m+1)}(x\rho)}{u^{(m)}(x\rho)}x\rho \Leftrightarrow -\frac{u^{(m+1)}(x\rho)}{u^{(m)}(x\rho)}x\rho \leq [\geq]\ m$, which contradicts $(-1)^m h^{(m)}_{[1],u}(\rho) > [<]\ 0$. As a result, for (27e) to hold, $(m+1)^{th}$-degree Ross risk aversion must be sufficiently stronger [weaker] than m^{th}-degree Ross risk aversion. Then, jointly with the first proof part, Lemma 3(a) implies (21). ∎

Appendix C. Proof of Proposition 6

It is to be shown that, under the given conditions, $\widehat{N_{m+1}UP}^{(n/m)} \geq \{\geq\} \widehat{N_m UP}^{(n/m)}$ for $(-1)^{n+m-1} h^{(n)}_{[m]}(\rho) > \{<\}\ 0$ is necessary and sufficient for an n^{th}-degree risk increase from $\tilde{\rho}_l$ to $\tilde{\rho}_h$ to increase $(m+1)^{th}$-degree RRA in the sense that

$$-\frac{E\left[u^{(m+1)}\left(x\tilde{\rho}_h\right)\tilde{\rho}_h^{m+1}\right]}{E\left[u^{(m)}\left(x\tilde{\rho}_h\right)\tilde{\rho}_h^{m}\right]}x \geq -\frac{E\left[u^{(m+1)}\left(x\tilde{\rho}_l\right)\tilde{\rho}_l^{m+1}\right]}{E\left[u^{(m)}\left(x\tilde{\rho}_l\right)\tilde{\rho}_l^{m}\right]}x \quad \text{for all } x \tag{28}$$

Sufficiency. To relate $\widehat{N_{m+1}UP}^{(n/m)} \geq \{\geq\} \widehat{N_m UP}^{(n/m)}$ to (28), rewrite (28) first by multiplying by $(-1)^{m+1} E\left[u^{(m)}\left(x\tilde{\rho}_h\right)\tilde{\rho}_h^{m}\right]$ and adding $(-1)^{m+1} x E\left[u^{(m+1)}\left(x\tilde{\rho}_l\right)\tilde{\rho}_l^{m+1}\right]$,

[23] Note that (27a)'s right-hand side $-\frac{xh^{(n)}_{[2],u}(\rho)}{h^{(n)}_{[1],u}(\rho)} = -\frac{xh^{(n)}_{[2],u}(\rho) - h^{(n)}_{[1],u}(\rho)}{h^{(n)}_{[1],u}(\rho)} - 1$.

$$(-1)^m x \left\{ E\left[u^{(m+1)}\left(x\tilde{\rho}_h\right)\tilde{\rho}_h^{m+1}\right] - E\left[u^{(m+1)}\left(x\tilde{\rho}_l\right)\tilde{\rho}_l^{m+1}\right]\right\} \geq - \frac{E\left[u^{(m+1)}\left(x\tilde{\rho}_l\right)\tilde{\rho}_l^{m+1}\right]}{E\left[u^{(m)}\left(x\tilde{\rho}_l\right)\tilde{\rho}_l^{m}\right]} x(-1)^{m+1}\left\{E\left[u^{(m)}\left(x\tilde{\rho}_h\right)\tilde{\rho}_h^{m}\right] - E\left[u^{(m)}\left(x\tilde{\rho}_l\right)\tilde{\rho}_l^{m}\right]\right\} \tag{29a}$$

Using E's definition and formula (7a) for $j = m, m+1$, (29a) can be rewritten as

$$(-1)^m x \int h_{[m+1]}(\rho)\, d\left[G(\rho) - F(\rho)\right] \geq -\frac{E\left[u^{(m+1)}\left(x\tilde{\rho}_l\right)\tilde{\rho}_l^{m+1}\right]}{E\left[u^{(m)}\left(x\tilde{\rho}_l\right)\tilde{\rho}_l^{m}\right]} x(-1)^{m+1}\int h_{[m]}(\rho)\, d\left[G(\rho) - F(\rho)\right] \tag{29b}$$

Applying an n-fold integration by parts to the integrals yields

$$(-1)^{n+m} x \int h^{(n)}_{[m+1]}(\rho)\left[G^{[n]}(\rho) - F^{[n]}(\rho)\right] d\rho \geq -\frac{E\left[u^{(m+1)}\left(x\tilde{\rho}_l\right)\tilde{\rho}_l^{m+1}\right]}{E\left[u^{(m)}\left(x\tilde{\rho}_l\right)\tilde{\rho}_l^{m}\right]} x(-1)^{n+m+1}\int h^{(n)}_{[m]}(\rho)\left[G^{[n]}(\rho) - F^{[n]}(\rho)\right] d\rho$$

and, when combining,

$$(-1)^{n+m-1}\int\left[-\frac{x h^{(n)}_{[m+1]}(\rho)}{h^{(n)}_{[m]}(\rho)} + \frac{E\left[u^{(m+1)}\left(x\tilde{\rho}_l\right)\tilde{\rho}_l^{m+1}\right]}{E\left[u^{(m)}\left(x\tilde{\rho}_l\right)\tilde{\rho}_l^{m}\right]} x\right] h^{(n)}_{[m]}(\rho)\left[G^{[n]}(\rho) - F^{[n]}(\rho)\right] d\rho \geq 0 \tag{29c}$$

where the second term in square brackets is nonnegative because $G^{[n]}(\rho) - F^{[n]}(\rho)$ is an n^{th}-degree risk increase; moreover, $(-1)^{n+m-1} h^{(n)}_{[m]}(\rho) > \{<\}\ 0$. As a result, the equivalent inequalities (29) hold if (29c)'s first term in square brackets is nonnegative $\{$nonpositive$\}$ for all x and ρ, that is,

$$-\frac{x h^{(n)}_{[m+1]}(\rho)}{h^{(n)}_{[m]}(\rho)} \geq \{\leq\} - \frac{E\left[u^{(m+1)}\left(x\tilde{\rho}_l\right)\tilde{\rho}_l^{m+1}\right]}{E\left[u^{(m)}\left(x\tilde{\rho}_l\right)\tilde{\rho}_l^{m}\right]} x \tag{30}$$

Sufficient conditions for (30) derive in three steps. First, note that, from Lemma 3 $\{$ Lemma 3's analog for increasing Ross RRA $\}$ for $j = m$, $\widehat{N_{m+1}UP}^{(n/m)} \geq \{\geq\}\ \widehat{N_m UP}^{(n/m)}$ is equivalent to

$$-\frac{x h^{(n)}_{[m+1]}(\rho)}{h^{(n)}_{[m]}(\rho)} \geq \{\leq\} - \frac{u^{(2m+1)}(x\rho)}{u^{(2m)}(x\rho)} x\rho \tag{31a}$$

Second, for "$\geq$", given decreasing $(2m/m)^{th}$-degree Ross RRA, and, for "$\leq$", equivalently to u's $(2m+1)^{th}$-degree Ross RRA not exceeding u's $(m+1)^{th}$-degree Ross RRA,[24] we have that (31a)'s right side

$$-\frac{u^{(2m+1)}(x\rho_a)}{u^{(2m)}(x\rho_a)} x\rho_a \geq \{\leq\} - \frac{u^{(m+1)}(x\rho_b)}{u^{(m)}(x\rho_b)} x\rho_b \ \text{ for all } \ \rho_a, \rho_b \tag{31b}$$

Finally, compare $\widehat{M}_{max}(x) = max_\rho\left[-\frac{u^{(m+1)}(x\rho)}{u^{(m)}(x\rho)} x\rho\right]$ and $\widehat{M}_{min}(x) = min_\rho\left[-\frac{u^{(m+1)}(x\rho)}{u^{(m)}(x\rho)} x\rho\right]$ with $-\frac{E\left[u^{(m+1)}\left(x\tilde{\rho}_l\right)\tilde{\rho}_l^{m+1}\right]}{E\left[u^{(m)}\left(x\tilde{\rho}_l\right)\tilde{\rho}_l^{m}\right]} x = -x\int \rho\, \frac{u^{(m+1)}(x\rho)}{u^{(m)}(x\rho)} \cdot \frac{u^{(m)}(x\rho) f(\rho)}{\int u^{(m)}(x\rho) dF(\rho)} d\rho$, where the integrand's second ratio, with $f(\rho)$ being $F(\rho)$'s density, can be interpreted as a probability density, so that

$$\widehat{M}_{max}(x) \geq -\frac{E\left[u^{(m+1)}\left(x\tilde{\rho}_l\right)\tilde{\rho}_l^{m+1}\right]}{E\left[u^{(m)}\left(x\tilde{\rho}_l\right)\tilde{\rho}_l^{m}\right]} x \geq \widehat{M}_{min}(x) \tag{31c}$$

Sufficiency follows since conditions (31) jointly imply, for "$\geq$" {"$\leq$"}, that

$$-\frac{x h^{(n)}_{[m+1]}(\rho)}{h^{(n)}_{[m]}(\rho)} \geq \widehat{M}_{max}(x) \geq \left\{\leq \widehat{M}_{min}(x) \leq\right\} - \frac{E\left[u^{(m+1)}\left(x\tilde{\rho}_l\right)\tilde{\rho}_l^{m+1}\right]}{E\left[u^{(m)}\left(x\tilde{\rho}_l\right)\tilde{\rho}_l^{m}\right]} x$$

for all x and ρ, so that (30) and, hence, conditions (29) hold, implying (28).

Necessity. The proof proceeds by contradiction. Suppose (22b) from Lemma 3 $\{$ (22b)'s analog from Lemma 3's analog for increasing Ross RRA $\}$ for $j = m$ does not hold at some x^0, ρ_a^0, and ρ_b^0, that is,

[24] Note that decreasing [increasing] $(2m/m)^{th}$-degree Ross RRA is equivalent to

$$-\frac{u^{(2m+1)}(x\rho_a)}{u^{(2m)}(x\rho_a)} x\rho_a - 1 \geq [\leq] - \frac{u^{(m+1)}(x\rho_b)}{u^{(m)}(x\rho_b)} x\rho_b \ \text{ for all } \ \rho_a, \rho_b$$

The condition for "$\leq$" *implies* increasing $(2m/m)^{th}$-degree Ross RRA and is thus stricter.

$$-\frac{x^0 h^{(n)}_{[m+1]}\left(\rho^0_a\right)}{h^{(n)}_{[m]}\left(\rho^0_a\right)} \ \leq \{\geq\} \ -\frac{u^{(m+1)}\left(x^0\rho^0_b\right)}{u^{(m)}\left(x^0\rho^0_b\right)}x^0\rho^0_b \tag{32a}$$

Assume that $F(\rho)$ concentrates probability around $\rho=\rho^0_a$, so that $-\frac{E\left[u^{(m+1)}(x\tilde{\rho}_l)\tilde{\rho}_l^{m+1}\right]}{E\left[u^{(m)}(x\tilde{\rho}_l)\tilde{\rho}_l^{m}\right]}x$ can be made arbitrarily close to (32a)'s right-hand side. Now, observe that (29c) evaluated at x^0 and divided by $(-1)^{n+m+1}\int h^{(n)}_{[m]}(\rho)\left[G^{[n]}(\rho)-F^{[n]}(\rho)\right]d\rho\Big|_{x=x^0}>0$ yields, after rearranging,

$$-\int \frac{x^0 h^{(n)}_{[m+1]}(\rho)}{h^{(n)}_{[m]}(\rho)}\cdot\frac{h^{(n)}_{[m]}(\rho)\left[G^{[n]}(\rho)-F^{[n]}(\rho)\right]}{\int h^{(n)}_{[m]}(\rho)\left[G^{[n]}(\rho)-F^{[n]}(\rho)\right]d\rho}d\rho \ \geq\{\leq\} \ -\frac{E\left[u^{(m+1)}\left(x^0\tilde{\rho}_l\right)\tilde{\rho}_l^{m+1}\right]}{E\left[u^{(m)}\left(x^0\tilde{\rho}_l\right)\tilde{\rho}_l^{m}\right]}x^0 \tag{32b}$$

where $G^{[n]}(\rho)$ can be chosen so that (32b)'s left side approximates (32a)'s left side, while (32b)'s right already approximates (32a)'s right side. Since (32b) then contradicts (32a), the supposition that (22b) { (22b)'s analog } for $j=m$ need not hold must be false. ■

Appendix D. Supplementary material

Supplementary material related to this article can be found online at https://doi.org/10.1016/j.insmatheco.2023.02.006.

References

Barro, Robert J., 2015. Environmental protection, rare disasters and discount rates. Economica 82 (325), 1–23.
Bostian, AJ A., Heinzel, Christoph, 2018. Comparative precautionary saving under higher-order risk and recursive utility. Geneva Risk and Insurance Review 43 (1), 95–114.
Bostian, AJ A., Heinzel, Christoph, 2022. Precautionary Saving Under Recursive Preferences. Working paper.
Bramoullé, Yann, Treich, Nicolas, 2009. Can uncertainty alleviate the commons problem? Journal of the European Economic Association 7 (5), 1042–1067.
Broll, Udo, Wong, Kit P., 2013. The firm under uncertainty: real and financial decisions. Decisions in Economics and Finance 36 (2), 125–136.
Chiu, W. Henry, Eeckhoudt, Louis, 2010. The effects of stochastic wages and non-labor income on labor supply: update and extensions. Journal of Economics 100 (1), 69–83.
Chiu, W. Henry, Eeckhoudt, Louis, Rey, Béatrice, 2012. On relative and partial risk attitudes: theory and implications. Economic Theory 50 (1), 151–167.
Courbage, Christophe, Loubergé, Henri, Rey, Béatrice, 2018. On the properties of high-order non-monetary measures for risks. Geneva Risk and Insurance Review 43 (1), 77–94.
Crainich, David, Eeckhoudt, Louis, 2008. On the intensity of downside risk aversion. Journal of Risk and Uncertainty 36 (3), 267–276.
Denuit, Michel M., Eeckhoudt, Louis, 2010. Stronger measures of higher-order risk attitudes. Journal of Economic Theory 145 (5), 2027–2036.
Eeckhoudt, Louis, Kimball, Miles, 1992. Background risk, prudence, and the demand for insurance. In: Dionne, Georges (Ed.), Contributions to Insurance Economics. Kluwer Academic Publishers, pp. 239–254.
Eeckhoudt, Louis, Schlesinger, Harris, 2006. Putting risk in its proper place. American Economic Review 96 (1), 280–289.
Eeckhoudt, Louis, Schlesinger, Harris, 2008. Changes in risk and the demand for saving. Journal of Monetary Economics 55 (7), 1329–1336.
Eeckhoudt, Louis, Schlesinger, Harris, 2009. On the utility premium of Friedman and Savage. Economics Letters 105 (1), 46–48.
Eeckhoudt, Louis, Gollier, Christian, Schlesinger, Harris, 1996. Changes in background risk and risk taking behavior. Econometrica 64 (3), 683–689.
Ekern, Steinar, 1980. Increasing Nth degree risk. Economics Letters 6 (4), 329–333.
Fleurbaey, Marc, Zuber, Stéphane, 2021. Fair utilitarianism. American Economic Journal: Microeconomics 13 (2), 370–401.
Franke, Günter, Schlesinger, Harris, Stapleton, Richard C., 2006. Multiplicative background risk. Management Science 52 (1), 146–153.
Franke, Guenter, Schlesinger, Harris, Stapleton, Richard C., 2011. Risk taking with additive and multiplicative background risks. Journal of Economic Theory 146 (4), 1547–1568.
Friedman, Milton, Savage, L.J., 1948. The utility analysis of choices involving risk. Journal of Political Economy 56 (4), 279–304.
Gollier, Christian, 2001. The Economics of Risk and Time. MIT Press, Cambridge, MA.
Gollier, Christian, 2015. Discounting, inequality and economic convergence. Journal of Environmental Economics and Management 69, 53–61.
Gollier, Christian, Pratt, John W., 1996. Risk vulnerability and the tempering effect of background risk. Econometrica 64 (5), 1109–1123.
Heinzel, Christoph, 2021. The Risk Substitution Rate as a Fundamental Risk Aversion Measure. Working paper.
Huang, James, Stapleton, Richard, 2015. The utility premium of Friedman and Savage, comparative risk aversion, and comparative prudence. Economics Letters 134, 34–36.
Jindapon, Paan, Neilson, William S., 2007. Higher-order generalizations of Arrow-Pratt and Ross risk aversion: a comparative statics approach. Journal of Economic Theory 136 (1), 719–728.
Jokung, Octave, 2013. Changes in multiplicative background risk and risk-taking behavior. Theory and Decision 74 (1), 127–149.
Keenan, Donald C., Snow, Arthur, 2012. Ross risk vulnerability for introductions and changes in background risk. Journal of Mathematical Economics 48 (4), 197–206.
Keenan, Donald C., Rudow, Donald C., Snow, Arthur, 2008. Risk preferences and changes in background risk. Journal of Risk and Uncertainty 36 (2), 139–152.
Kimball, Miles S., 1990. Precautionary saving in the small and in the large. Econometrica 58 (1), 53–73.
Kimball, Miles S., 1993. Standard risk aversion. Econometrica 61 (3), 589–611.
Li, Jingyuan, 2009. Comparative higher-degree Ross risk aversion. Insurance. Mathematics & Economics 45 (3), 333–336.
Li, Jingyuan, Liu, Liqun, 2014. The monetary utility premium and interpersonal comparisons. Economics Letters 125 (2), 257–260.
Liu, Liqun, 2014. Precautionary saving in the large: nth degree deteriorations in future income. Journal of Mathematical Economics 52, 169–172.
Liu, Liqun, Meyer, Jack, 2013. Substituting one risk increase for another: a method for measuring risk aversion. Journal of Economic Theory 148 (6), 2706–2718.
Liu, Liqun, Neilson, William S., 2019. Alternative approaches to comparative nth-degree risk aversion. Management Science 65 (8), 3824–3834.
Loubergé, Henri, Rey, Béatrice, 2022. On the Shape of High Order Additive and Multiplicative Utility Premiums. Working paper.
Loubergé, Henri, Malevergne, Yannick, Rey, Béatrice, 2020. New results for additive and multiplicative risk apportionment. Journal of Mathematical Economics 90, 140–151.
Menezes, Carmen E., Geiss, Charles, Tressler, John, 1980. Increasing downside risk. American Economic Review 70 (5), 921–932.
Menezes, Carmen F., Wang, X. Henry, 2005. Increasing outer risk. Journal of Mathematical Economics 41 (7), 875–886.
Meyer, Donald J., Meyer, Jack, 2005. Relative risk aversion: what do we know? Journal of Risk and Uncertainty 31 (3), 243–262.
Pratt, John W., 1964. Risk aversion in the small and in the large. Econometrica 32 (1–2), 122–136.
Ross, Stephen A., 1981. Some stronger measures of risk aversion in the small and the large with applications. Econometrica 49 (3), 621–638.
Rothschild, Michael, Stiglitz, Joseph E., 1970. Increasing risk: I. A definition. Journal of Economic Theory 2 (3), 225–243.
Rothschild, Michael, Stiglitz, Joseph E., 1971. Increasing risk II: its economic consequences. Journal of Economic Theory 3 (1), 66–84.
Schlesinger, Harris, 2013. The theory of insurance demand. In: Dionne, Georges (Ed.), Handbook of Insurance. Springer, pp. 167–184. Chapter 7.
Wang, Jianli, Li, Jingyuan, 2014. Decreasing Ross risk aversion: higher-order generalizations and implications. Journal of Mathematical Economics 55, 136–142.
Wong, Kit P., 2018. Comparative higher-order risk aversion and higher-order prudence. Economics Letters 169, 38–42.