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Asymptotics for a time-dependent by-claim model with dependent subexponential claims

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ABSTRACT

Consider a by-claim risk model with a constant force of interest, where each main claim may induce a by-claim after a random time. We propose a time-claim-dependent framework, that incorporates dependence between not only the waiting time and the claim but also the main claim and the corresponding by-claim. Based on this framework, we derive some asymptotic estimates for the finite-time ruin probabilities in the case of subexponential claims. We also provide examples and verify the assumptions on dependence. Numerical studies are conducted to examine the performance of these asymptotic formulas.

1. Introduction

In the insurance practice, disasters such as earthquakes, storms, and severe accidents can lead to not only direct property losses and medical compensations but also consequential losses that are causally linked. For example, a major earthquake may result in an outbreak of fire or even strong aftershocks several years later. Continued high temperatures can lead to fatalities, while also resulting in loss of agricultural yield. As such, catastrophic events may trigger two types of claims with different distributions. The first type is called the main claim, which involves immediate settlement of losses such as direct property damages and immediate death compensation. The second type is the by-claim, which may occur gradually in an uncertain period of time after the settlement of the main claim and includes claims related to subsequent treatment or secondary perils. It is natural to consider that dependence may exist between the main claim, the by-claim, and the corresponding waiting time. Some researchers believe that ignoring potential dependence may have serious repercussions for practical use (see, for example, Garrido et al. (2016)). Therefore, we use a by-claim model that considers dependence when modeling an insurer against catastrophic events.

Concretely speaking, let $\{(X_i, Y_i); i \in \mathbb{N}_+\}$ be a sequence of independent and identically distributed (i.i.d.) non-negative random pairs. For each $i \in \mathbb{N}_+$, X_i describes the i-th main claim arriving at time τ_i, and Y_i describes the corresponding by-claim at time $\tau_i + D_i$, where D_i denotes an uncertain delay time. Assume that the arrival times $\tau_i, i \in \mathbb{N}$ constitute a renewal sequence such that the inter-arrival times $\theta_i = \tau_i - \tau_{i-1}, i \in \mathbb{N}_+$ are non-negative i.i.d. random variables, non-degenerate at zero. The corresponding counting process $\{N_t; t \geq 0\}$ is a renewal process with a finite mean function $\lambda_t = \mathrm{E}N_t = \sum_{i=1}^{\infty} \mathrm{P}(\tau_i \leq t)$. The delay time $\{D_i; i \in \mathbb{N}\}$ is a sequence of non-negative (possibly degenerate at 0) i.i.d. random variables. Moreover, denote by x the initial reserve, by $r \geq 0$ the constant force of interest, and by $C(t)$ the premium accumulation process. Assume that the process $C(t)$ is non-negative and non-decreasing, satisfying $C(0) = 0$ and $C(t) < \infty$

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almost surely (a.s.) for $0 < t < \infty$. Besides, $C(t)$ is independent of the other sources of randomness. Accordingly, the surplus process of the insurer can be described as:

$$U_t = xe^{rt} + \int_0^t e^{r(t-s)} C(ds) - \sum_{i=1}^{N_t} X_i e^{r(t-\tau_i)} - \sum_{i=1}^{N_t} Y_i e^{r(t-\tau_i-D_i)} I_{\{\tau_i+D_i \le t\}} \qquad t \ge 0, \tag{1.1}$$

where I_A denotes the indicator function for an event A. For such a model, the finite-time ruin probability with a finite-horizon $t > 0$ can be formulated as

$$\psi(x,t) = \mathrm{P}\left(\inf_{0\le s\le t} U(s) < 0 \middle| U(0) = x\right). \tag{1.2}$$

The region of the variable t needs to be restricted to the range of $\Lambda = \{t : 0 < \lambda_t < \infty\}$ with $\underline{t} = \inf\{t : \mathrm{P}(\theta_1 \le t) > 0\}$. Clearly, $\Lambda = [\underline{t}, \infty)$ if $\mathrm{P}\left(\theta_1 = \underline{t}\right) > 0$ while $\Lambda = (\underline{t}, \infty)$ if $\mathrm{P}\left(\theta_1 = \underline{t}\right) = 0$. Denote $\Lambda_T = [0, T] \cap \Lambda$ for every fixed $T \in \Lambda$.

The study of the above by-claim model can be traced back to the work of Waters and Papatriandafylou (1985) where the authors considered a discrete-time risk model allowing for the delay in claims settlements. Since then, this topic has been studied extensively in insurance mathematics and applied probability. See Yuen and Guo (2001), Xiao and Guo (2007), and Li and Wu (2015) for discrete-time models; see Yuen et al. (2005), Xie and Zou (2010, 2011), Meng and Wang (2012), Zou and Xie (2013) for continuous-time versions, among many others. The references mentioned above focused on the light-tailed case, which is suitable for small claims. However, it is worth noting that small claims are no longer sufficient to meet the current demands of the insurance practice. For catastrophe insurance, heavy-tailed claims might be more appropriate for better modeling. Additionally, various extensions to the renewal risk model have been proposed to relax assumptions on independence. See some relevant studies including Li (2013), Fu and Li (2016), Gao et al. (2019), Zhang et al. (2021), and Lu and Yuan (2022) for ruin estimation of dependent main claims and by-claims under heavy-tailed conditions.

On the other hand, originating from Albrecher and Teugels (2006), a clear trend of the mainstream study focuses on time-dependent problems of an extended renewal risk model, in which $(X_i, \theta_i), i \in \mathbb{N}_+$ are assumed to be i.i.d. copies of a generic pair (X, θ) with dependent components X and θ. Asimit and Badescu (2010) introduced a general dependence structure for (X, θ), via the conditional tail probability of X given θ. They studied the tail behavior of discounted aggregate claims in the compound Poisson risk model in the presence of a constant force of interest and heavy-tailed claim sizes. Later, Li et al. (2010) used the same dependence structure in Asimit and Badescu (2010), that is,

$$\mathrm{P}(X > x \mid \theta = t) \sim \mathrm{P}(X > x)h(t), \quad x \to \infty, \tag{1.3}$$

uniformly for t, where the function $h(\cdot) : [0, \infty) \to (0, \infty)$ is measurable. Under the time-dependent structure, they derived the asymptotic estimates for discounted aggregate claims with subexponential tails and verified the assumptions on the dependence structure through copulas. For the study on this so-called time-dependent risk model, we refer readers to Li (2012), Fu and Ng (2014), Jiang et al. (2015), and Li (2016) among many others. Especially, Jiang et al. (2015) and Li (2016) considered certain general dependence between the claim vector and inter-arrival time as well as dependence among the claim sizes from different lines of businesses. Recently, Liu et al. (2021) applied the time-dependent assumption mentioned above to a by-claim model. They introduced (1.3) to the main claim and its waiting time, and proposed a similar time-dependent structure for the by-claim Y, via the conditional tail probability of Y given $\theta + D$.

Most of the references cited above focus on either time-claim-dependence or inter-claim-dependence while ruling out the other one. However, dependence may exist among different types of accident frequency, claim frequency, and claim severity due to various factors such as weather conditions, seasonal effects, business cycles, and queuing bottlenecks in claims processing. In the context of insurance and finance, risk assessment calls on the priority topic in the interplay of different factors shaping claim events in reinsurance or catastrophe insurance. Therefore, pricing models need to be flexible enough to incorporate dependence to enhance their capabilities when calibrated against historical data and economic factors. Our goal in this paper is to extend the existing work to a more general framework, in which dependence allows between not only the claim and its waiting time but also the main claim and the corresponding by-claim. We introduce the time-dependent assumption as (1.3), to the pair of the main claim and the inter-arrival time (X, θ). Accordingly, a similar time-dependent structure for the by-claim Y, via the conditional tail probability of Y given (θ, D), is proposed. Under the impact of dependence between the main claim and the corresponding by-claim, an asymptotic study of the ruin probability (1.2) for the case of subexponential claims is conducted.

The rest of this paper is organized as follows: Section 2 introduces some necessary preliminaries and states the main results. Section 3 proposes some examples and verification of the assumptions on dependence. Section 4 implements numerical studies on the accuracy of the obtained asymptotic results. The proofs of lemmas and the main results are relegated to the Appendix.

2. Preliminaries and main results

For convenience, we introduce the following notation that is used throughout this paper. For two positive functions $f(\cdot)$ and $g(\cdot)$, we write $f(x) \lesssim g(x)$ or $g(x) \gtrsim f(x)$ if $\limsup_{x\to\infty} f(x)/g(x) \le 1$, write $f(x) \sim g(x)$ if both $f(x) \lesssim g(x)$ and $f(x) \gtrsim g(x)$, and write $f(x) = o(g(x))$ if $\lim_{x\to\infty} f(x)/g(x) = 0$ and write $f(x) \asymp g(x)$ if $0 < \liminf_{x\to\infty} f(x)/g(x) \le \limsup_{x\to\infty} f(x)/g(x) < \infty$. Hereafter, unless otherwise stated, all limit relationships hold as $x \to \infty$.

Now we give a brief review of some heavy-tailed distribution classes that will appear later. Let $\overline{V} = 1 - V$ be a distribution on $[0, \infty)$ with an infinite upper endpoint, i.e., $\overline{V}(x) > 0$ for all $x \ge 0$. The distribution V on $[0, \infty)$ is said to belong to the long-tailed class, denoted by $V \in \mathcal{L}$, if for all $x \ge 0$ and the relation

$$\lim_{x\to\infty} \frac{\overline{V}(x+y)}{\overline{V}(x)} = 1, \quad y \in (-\infty, \infty)$$

holds. An elementary property of the class $\mathcal{L}$ is that, if $V \in \mathcal{L}$, there exists a function $l(x) \in \mathcal{H}(V)$ such that $\overline{V}(x - Kl(x)) \sim \overline{V}(x)$ for every $K > 0$, where

$$\mathcal{H}(V) = \left\{ l \text{on } [0, \infty) : l(x) \uparrow \infty, \frac{l(x)}{x} \downarrow 0 \text{and } \overline{V}(x - l(x)) \sim \overline{V}(x) \right\}.$$

Furthermore, for any fixed $n \geq 1$, if $V_i \in \mathcal{L}$ and $l_i(x) \in \mathcal{H}(V_i)$ for $1 \leq i \leq n$, we have $\overline{V}_i(x - l(x)) \sim \overline{V}_i(x)$ for all $1 \leq i \leq n$, where $l(x) = \wedge_{i=1}^{n} l_i(x)$, and write $l(x) \in \mathcal{H}(V_1, \ldots, V_n)$.

A natural and tractable subclass of $\mathcal{L}$ is the subexponential class $\mathcal{S}$. By definition, $V \in \mathcal{S}$ if the relation

$$\lim_{x \to \infty} \frac{\overline{V^{n*}}(x)}{\overline{V}(x)} = n$$

holds for all (or, equivalently, for some) $n = 2, 3, \ldots$, where V^{n*} is the n-fold convolution of V with itself.

One of the useful subclasses of $\mathcal{S}$ is the class $\mathcal{C}$ of distributions with consistently varying tails, and we write $V \in \mathcal{C}$ if

$$\lim_{y \downarrow 1} \liminf_{x \to \infty} \frac{\overline{V}(xy)}{\overline{V}(x)} = 1.$$

In particular, the class $\mathcal{C}$ covers the famous class $\mathcal{R}$ of regular variation. The distribution V is said to have a regularly varying tail with an index $\alpha \geq 0$, denoted by $V \in \mathcal{R}_{-\alpha}$ if

$$\lim_{x \to \infty} \frac{\overline{V}(xy)}{\overline{V}(x)} = y^{-\alpha}$$

holds for every $y > 0$. In addition, we need to mention the class $\mathcal{D}$ of dominated variation, which is not contained in the class $\mathcal{L}$. The distribution V is said to have a dominatedly varying tail, denoted by $V \in \mathcal{D}$ if the relation

$$\limsup_{x \to \infty} \frac{\overline{V}(xy)}{\overline{V}(x)} < \infty$$

holds for every (or, equivalently, for some) $0 < y < 1$. The intersection $\mathcal{L} \cap \mathcal{D}$ also forms a useful subclass of the class $\mathcal{S}$. The aforementioned classes satisfy the following proper inclusion relations:

$$\mathcal{R} \subset \mathcal{C} \subset \mathcal{L} \cap \mathcal{D} \subset \mathcal{S} \subset \mathcal{L}.$$

2.1. Main results

Recall the by-claim risk model constructed in (1.1). From now on, assume that $\{(X_i, Y_i, \theta_i, D_i); i \in \mathbb{N}_+\}$ is a sequence of i.i.d. copies of a generic vector (X, Y, θ, D) with generic marginal distributions F, G, H_θ, H_D, respectively. Moreover, let D be independent of (X, θ).

In what follows, we introduce a time-dependent structure through the following assumption, in which relation (2.2) was initiated by Asimit and Badescu (2010) and revisited by many researchers afterwards.

Assumption 2.1. *There exist a univariate measurable function $h(\cdot) : [0, \infty) \to (0, \infty)$ and a bivariate measurable function $\varphi(\cdot, \cdot) : [0, \infty)^2 \to (0, \infty)$ satisfying*

$$0 < \inf_{s \in \Lambda_T} h(s) \leq \sup_{s \in \Lambda_T} h(s) < \infty, \qquad 0 < \inf_{s,u \in \Lambda_T} \varphi(s, u) \leq \sup_{s,u \in \Lambda_T} \varphi(s, u) < \infty, \tag{2.1}$$

such that the random vector (X, Y, θ, D) fulfills the relations:

$$\mathrm{P}(X > x \mid \theta = s) \sim \mathrm{P}(X > x) h(s) \tag{2.2}$$

and

$$\mathrm{P}(Y > x \mid \theta = s, D = u) \sim \mathrm{P}(Y > x) \varphi(s, u), \tag{2.3}$$

uniformly for $s, u \in \Lambda_T$.

Remark 2.1. When s, u are not a possible value of θ and D, the conditional probabilities shown in (2.2) and (2.3) are understood as unconditional. Therefore, $h(s) = 1$ and $\varphi(s, u) = 1$. The time-dependent structure adapts to some commonly-used copulas such as the Farlie–Gumbel–Morgenstern copula (FGM copula), the Ali-Mikhail-Haq copula and the Frank copula. See Asimit and Badescu (2010) and Li et al. (2010) for a detailed discussion on (2.2). For discussion on (2.3), see examples and verification in Section 3.

Now we are ready to state our main results. In what follows, we introduce a general framework for (X, Y, θ, D) shown in relation (2.4) to capture the dependence among the main claim, the by-claim and their corresponding waiting time.

Theorem 2.1. *Consider the by-claim model given by (1.1) with $F, G \in \mathcal{S}$. Assume that Assumption 2.1 is fulfilled,* $\mathrm{P}(\tau_1 \le t) > 0$ *for any* $t \in \Lambda_T$, *and* $\mathrm{P}\left(\int_0^t e^{-rs} C(ds) > x\right) = o\left(\overline{F}(x/a)\right)$ *for any fixed* $t, r > 0$ *and some* $a > 0$. *For any positive bounded measurable functions* $g_1(x) : (0,\infty) \to (0,\infty)$, $g_2(x,y) : (0,\infty)^2 \to (0,\infty)$ *satisfying* $\inf_{x\in\Lambda_T} g_1(x) > 0$ *and* $\inf_{x,y\in\Lambda_T} g_2(x,y) > 0$, *if the relation*

$$\mathrm{P}(g_1(\theta)X + g_2(\theta, D)Y > x \mid \theta = s, D = u) \sim \mathrm{P}(g_1(\theta)X > x)h(s) + \mathrm{P}(g_2(\theta, D)Y > x)\varphi(s,u) \tag{2.4}$$

holds uniformly for $s, u \in \Lambda_T$, *then*
(i) for $\overline{F}(x) \asymp \overline{G}(x)$, *it holds uniformly for* $t \in \Lambda_T$ *that*

$$\psi(x,t) \sim \int_{0-}^{t} \overline{F}\left(xe^{rs}\right) \mathrm{d}\tilde{\lambda}_s + \int_{0-}^{t}\int_{0-}^{t-u} \overline{G}\left(xe^{r(s+u)}\right) \mathrm{d}\hat{\lambda}_{s,u} H_D(\mathrm{d}u), \tag{2.5}$$

where $\tilde{\lambda}_s = \int_{0-}^{s} (1+\lambda_{s-s^*})\, h(s^*)\, H_\theta(\mathrm{d}s^*)$ *and* $\hat{\lambda}_{s,u} = \int_{0-}^{s} (1+\lambda_{s-s^*})\, \varphi(s^*, u)\, H_\theta(\mathrm{d}s^*)$;
(ii) for $\overline{G}(x) = o\left(\overline{F}(x)\right)$, *it holds uniformly for* $t \in \Lambda_T$ *that*

$$\psi(x,t) \sim \int_{0-}^{t} \overline{F}\left(xe^{rs}\right) \mathrm{d}\tilde{\lambda}_s. \tag{2.6}$$

Remark 2.2. We should note that relation (2.4) is insensitive to the specific form of functions g_1 and g_2. Provided that $g_1, g_2 \in [a,b]$ for some $0 < a < b < \infty$, relation (2.4) holds uniformly for g_1 and g_2.

Remark 2.3. The restriction that D should be independent of (X, θ) is not an essential condition when seeking asymptotic results of $\psi(x,t)$. The methodology used to prove Theorem 2.1 can also be applied to obtain asymptotic estimates without this restriction. Nevertheless, the asymptotic estimates may be more intricate than that in Theorem 2.1 due to the unknown joint distribution of θ and D.

Now we aim to provide further insight into relation (2.4). In the context of the time-dependent structure in Assumption 2.1, relation (2.4) incorporates dependence among claims as well as the waiting time. Nevertheless, (2.4) may appear to contain some uncertain requirements of X and Y. We recommend first characterizing the tail behavior of $g_1(\theta)X + g_2(\theta, D)Y$. For example, when X, Y, θ and D are mutually independent, it follows from Theorem 1 of Tang and Yuan (2014) that (2.4) holds. Then our results can then be used to derive asymptotic estimates of the ruin probability. Specially, we propose an appropriate dependence structure satisfying (2.4):

$$\lim_{x\wedge y\to\infty} \sup_{s,u\in\Lambda_T} \mathrm{P}(Y > y \mid X > x, \theta = s, D = u) = \lim_{x\wedge y\to\infty} \sup_{s,u\in\Lambda_T} \mathrm{P}(X > x \mid Y > y, \theta = s, D = u) = 0. \tag{2.7}$$

This structure can be seen as a conditional and non-negative version of the so-called pairwise strongly quasi-asymptotic independence (PSQAI), shown as

$$\lim_{x\wedge y\to\infty} \mathrm{P}(|\xi| > x \mid \eta > y) = \lim_{x\wedge y\to\infty} \mathrm{P}(|\eta| > y \mid \xi > x) = 0$$

for real-valued random variables ξ and η. See Geluk and Tang (2009) and Li (2013) for more details of the PSQAI case which covers a wide range of dependence structures. A detailed discussion and verification of (2.7) are postponed to Section 3.

On the other hand, we should point out that (2.4) largely applies to cases of asymptotic independence. Further investigation into cases of asymptotic dependence is expected for future projects. Existing research, such as Fougeres and Mercadier (2012), Chen and Yuan (2017), and Chen and Yang (2019) may shed some light on this topic using the multivariate regular variation framework. With more restrictive conditions that $\overline{G}(x) = o\left(\overline{F}(x)\right)$ and $F \in \mathcal{C}$, the corollary below considers the arbitrary dependence structure between the main claim and the corresponding by-claim. The asymptotic formula is fully consistent with (2.6) in Theorem 2.1.

Corollary 2.1. *Consider the by-claim risk model given by (1.1). Assume that* (X, Y, θ, D) *fulfills Assumption 2.1 where* X *and* Y *are arbitrarily dependent. If* $F \in \mathcal{C}$ *and* $\overline{G}(x) = o\left(\overline{F}(x)\right)$, *then (2.6) holds uniformly for* $t \in \Lambda_T$.

3. Examples and verification of the assumptions on dependence

This section aims to verify the assumptions on the dependence structure of (X, Y, θ, D) and provide examples to illustrate our results. In view of Assumption 2.1, we use the dependence structure shown in (2.7) to illustrate Theorem 2.1, and the scale mixture for X and Y to illustrate Corollary 2.1. Moreover, we verify Assumption 2.1 and relation (2.7) using some well-known copulas.

3.1. Examples

Example 3.1. Consider the by-claim risk model given by (1.1). Assume that (X, Y, θ, D) fulfills Assumption 2.1 and relation (2.7). If $F, G \in \mathcal{L} \cap \mathcal{D}$ and $\overline{F}(x) \asymp \overline{G}(x)$, we can verify that relation (2.4) is satisfied in the above setup with the help of Lemma A.5 in Appendix. Hence, Theorem 2.1 is applicable.

Example 3.2. Consider that (X, θ) fulfills relation (2.2) with X distributed as $F \in \mathcal{R}_{-\alpha}$ for some $0 < \alpha < \infty$, and $Y = W\sqrt{X}$ where W is a non-negative random variable independent of (X, θ) with $\mathrm{E}W^p < \infty$ for some $p > 2\alpha$. Assume that there exists some univariate function $\tilde{h} : [0, \infty) \to (0, \infty)$ satisfying

$$0 < \inf_{u \in \Lambda_T} \tilde{h}(u) \le \sup_{u \in \Lambda_T} \tilde{h}(u) < \infty \tag{3.1}$$

such that the relation

$$\mathrm{P}(W > x \mid D = u) \sim \mathrm{P}(W > x)\tilde{h}(u) \tag{3.2}$$

holds uniformly for $u \in \Lambda_T$. By Lemma 3.7 of Tang and Tsitsiashvili (2003), there is some positive function $l(x)$ satisfying $l(x) \to \infty$ and $l(x) = o(x)$ such that $\mathrm{P}(W > l(x)) = o(\mathrm{P}(\sqrt{X} > x))$. Let $\tilde{F}$ denote the distribution of $\sqrt{X}$ and note that $\tilde{F} \in \mathcal{R}_{-2\alpha}$. By the well-known Breiman's theorem (see Breiman (1965)),

$$\mathrm{P}(Y > x) \sim \mathrm{E}W^{2\alpha}\mathrm{P}(\sqrt{X} > x) = o(\overline{F}(x)).$$

Referring to the proof given in Example 3.1 in Li (2016), we have

$$\mathrm{P}(Y > x \mid \theta = s, D = u) \sim h(s)\tilde{h}(u)\mathrm{P}(Y > x).$$

Denote $\varphi(s, u) = h(s)\tilde{h}(u)$, then Assumption 2.1 is fulfilled. Recall that $\mathcal{R}_{-\alpha} \subset \mathcal{C}$. By Corollary 2.1, we can derive (2.6).

3.2. Verification of Assumption 2.1 and relation (2.7)

In the following parts, we consider the FGM family and the Frank family for the verification of Assumption 2.1 and relation (2.7). What needs to be pointed out is that some parameters of the FGM copula need to be set to 0 to ensure the condition that D should be independent of (X, θ). The one-parameter Frank copula cannot guarantee this independence restriction by adjusting its parameter. Nevertheless, recalling Remark 2.3 which mentions that independence can be relaxed, we provide the verification of the Frank copula as an extension. For a detailed discussion of some concrete copulas satisfying the dependence assumptions, see Asimit and Badescu (2010), Li et al. (2010), and Jiang et al. (2015).

According to Sklar's theorem, if all of the marginal distributions $F_1, \ldots, F_n$ of $(X_1, \ldots, X_n)$ are continuous, then there is a unique n-copula C such that, for all $(x_1, \ldots, x_n) \in [-\infty, \infty]^n$,

$$\mathrm{P}(X_1 \le x_1, \ldots, X_n \le x_n) = C(F_1(x_1), \ldots, F_n(x_n)).$$

The corresponding survival copula is defined as

$$\mathrm{P}(X_1 > x_1, \ldots, X_n > x_n) = \hat{C}\left(\overline{F}_1(x_1), \ldots, \overline{F}_n(x_n)\right).$$

Let $\hat{C}$ be the survival copula of (X, Y, θ, D). The functions $h(\cdot)$ and $\varphi(\cdot, \cdot)$ mentioned in Assumption 2.1, if it exists, can be calculated through the following equations:

$$h(s) = \lim_{u_1 \to 0+} \left. \frac{\partial \hat{C}(u_1, 1, u_3, 1)/\partial u_3}{u_1} \right|_{u_3 = \overline{H}_\theta(s)}$$

and

$$\varphi(s, u) = \lim_{u_2 \to 0+} \left. \frac{\partial \hat{C}(1, u_2, u_3, u_4)/\partial u_3 \partial u_4}{u_2 \partial \hat{C}(1, 1, u_3, u_4)/\partial u_3 \partial u_4} \right|_{\substack{u_3 = \overline{H}_\theta(s) \\ u_4 = \overline{H}_D(u)}}.$$

Then the uniformity of (2.2) and (2.3) in Assumption 2.1 can be restated as

$$\lim_{u_1 \to 0+} \sup_{u_3 \in [\delta, 1]} \left| \frac{\partial \hat{C}(u_1, 1, u_3, 1)/\partial u_3}{u_1 h(s)} - 1 \right| = 0 \tag{3.3}$$

and

$$\lim_{u_2 \to 0+} \sup_{u_3, u_4 \in [\delta, 1]} \left| \frac{\partial \hat{C}(1, u_2, u_3, u_4)/\partial u_3 \partial u_4}{\partial \hat{C}(1, 1, u_3, u_4)/\partial u_3 \partial u_4} \cdot \frac{1}{u_2 \varphi(s, u)} - 1 \right| = 0, \tag{3.4}$$

for $\delta \in (0, 1)$, respectively. In terms of the survival copula $\hat{C}$, the verification of relation (2.7) can be restated as

$$\begin{aligned} &\lim_{x \wedge y \to \infty} \sup_{s, u \in \Lambda_T} \mathrm{P}(X > x \mid Y > y, \theta = s, D = u) \\ &= \limsup_{u_1, u_2 \to 0+} \sup_{u_3, u_4 \in [\delta, 1]} \left. \frac{\partial \hat{C}(u_1, u_2, u_3, u_4)/\partial u_3 \partial u_4}{\partial \hat{C}(1, u_2, u_3, u_4)/\partial u_3 \partial u_4} \right|_{\substack{u_3 = \overline{H}_\theta(s) \\ u_4 = \overline{H}_D(u)}} = 0 \end{aligned} \tag{3.5}$$

and

$$\lim_{x\wedge y\to\infty}\sup_{s,u\in\Lambda_T} \mathrm{P}(Y>y \mid X>x,\theta=s,D=u) \\ = \lim_{u_1,u_2\to 0+}\sup\sup_{u_3,u_4\in[\delta,1]} \left.\frac{\partial\hat{C}(u_1,u_2,u_3,u_4)/\partial u_3\partial u_4}{\partial\hat{C}(u_1,1,u_3,u_4)/\partial u_3\partial u_4}\right|_{\substack{u_3=\overline{H}_\theta(s)\\ u_4=\overline{H}_D(u)}} = 0. \tag{3.6}$$

3.2.1. FGM copula

Let the survival copula $\hat{C}$ belong to the FGM family, written as

$$\hat{C}\,(u_1,\ldots,u_4) = u_1\cdots u_4\left[1+\sum_{l=2}^{4}\sum_{1\le j_1<\cdots<j_l\le 4}\vartheta_{j_1\cdots j_l}\prod_{k=1}^{l}\left(1-u_{j_k}\right)\right], \tag{3.7}$$

where the parameters satisfy the following constraints:

$$1+\sum_{l=2}^{4}\sum_{1\le j_1<\cdots<j_l\le 4}\varepsilon_{j_1}\cdots\varepsilon_{j_l}\vartheta_{j_1\cdots j_l}>0, \quad \varepsilon_1=\pm1,\ldots,\varepsilon_4=\pm1. \tag{3.8}$$

In addition, we need to set parameters ϑ_{14}, ϑ_{34} and ϑ_{134} to 0 to ensure the condition that D should be independent of (X,θ). It is easy to verify that relation (3.3) holds when the parameters satisfy the constraints shown in (3.8), which is consistent with the one obtained by Li et al. (2010). For verification of φ, it follows from the constraints shown in (3.8) that

$$\varphi(s,u)=\frac{1+\vartheta_{23}(1-2u_3)+\vartheta_{24}(1-2u_4)+\vartheta_{34}(1-2u_3)(1-2u_4)+\vartheta_{234}(1-2u_3)(1-2u_4)}{1+\vartheta_{34}(1-2u_3)(1-2u_4)}>0$$

and

$$\lim_{u_2\to 0+}\sup_{u_3,u_4\in[\delta,1]}\left|\frac{\partial\hat{C}(1,u_2,u_3,u_4)/\partial u_3\partial u_4}{\partial\hat{C}(1,1,u_3,u_4)/\partial u_3\partial u_4}\cdot\frac{1}{u_2\varphi(s,u)}-1\right| \\ =\lim_{u_2\to 0+}\sup_{u_3,u_4\in[\delta,1]}\left|\frac{\frac{\partial\hat{C}(1,u_2,u_3,u_4)/\partial u_3\partial u_4}{u_2\partial\hat{C}(1,1,u_3,u_4)/\partial u_3\partial u_4}-\varphi(s,u)}{\varphi(s,u)}\right| \\ =\lim_{u_2\to 0+}\sup_{u_3,u_4\in[\delta,1]}\left|\frac{\vartheta_{23}(1-2u_3)+\vartheta_{24}(1-2u_4)+\vartheta_{234}(1-2u_3)(1-2u_4)}{(1+\vartheta_{34}(1-2u_3)(1-2u_4))\,\varphi(s,u)}\right||u_2|=0,$$

which implies that relation (3.4) holds.

In terms of the FGM copula, we carry on the verification of relation (2.7). Note that

$$\frac{\partial\hat{C}(u_1,u_2,u_3,u_4)}{\partial u_3\partial u_4} \\ =u_1u_2\Bigg(1+\vartheta_{12}\prod_{i=1}^{2}(1-u_i)+\vartheta_{34}\prod_{j=3}^{4}(1-2u_j)+\sum_{i=1}^{2}\vartheta_{i3}(1-2u_3)(1-u_i) \\ +\sum_{i=1}^{2}\vartheta_{i4}(1-2u_4)(1-u_i)+\vartheta_{123}(1-2u_3)\prod_{i=1}^{2}(1-u_i)+\vartheta_{124}(1-2u_4)\prod_{i=1}^{2}(1-u_i) \\ +\sum_{i=1}^{2}\vartheta_{i34}(1-u_i)\prod_{j=3}^{4}(1-2u_j)+\vartheta_{1234}\prod_{i=1}^{2}(1-u_i)\prod_{j=3}^{4}(1-2u_j)\Bigg).$$

A direct calculation shows that (3.5) and (3.6) hold, which implies (2.7).

3.2.2. Frank copula

Let the survival copula $\hat{C}$ belong to the Frank family, written as

$$\hat{C}\,(u_1,\ldots,u_n)=-\frac{1}{\vartheta}\log\left(1+\frac{\prod_{i=1}^{n}\left(\mathrm{e}^{-\vartheta u_i}-1\right)}{\left(\mathrm{e}^{-\vartheta}-1\right)^{n-1}}\right), \quad \vartheta>0.$$

It is easy to verify that relation (3.3) holds, which is consistent with the one obtained by Li et al. (2010). For verification of φ, a direct calculation shows

$$\varphi(s,u)=\frac{\vartheta\left(\mathrm{e}^{-\vartheta}-1+\left(\mathrm{e}^{-\vartheta u_3}-1\right)\left(\mathrm{e}^{-\vartheta u_4}-1\right)\right)^2}{\left(1-\mathrm{e}^{-\vartheta}\right)^3}>0$$

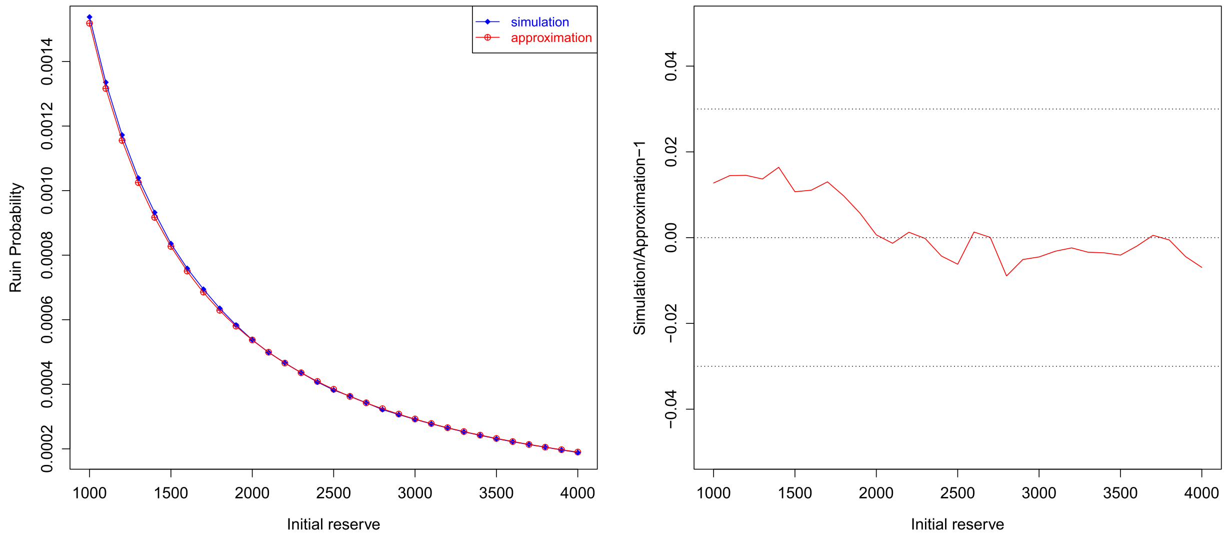

Fig. 1. Comparision of Simulation and Approximation of Ruin Probabilities when $\overline{F}(x) \asymp \overline{G}(x)$

and

$$\lim_{u_2\to 0+}\sup_{u_3,u_4\in[\delta,1]}\left|\frac{\partial\hat{C}(1,u_2,u_3,u_4)/\partial u_3\partial u_4}{\partial\hat{C}(1,1,u_3,u_4)/\partial u_3\partial u_4}\cdot\frac{1}{u_2\varphi(s,u)}-1\right|$$

$$=\lim_{u_2\to 0+}\sup_{u_3,u_4\in[\delta,1]}\left|\frac{\left(e^{-\vartheta}-1\right)^4\left(1-e^{-\vartheta u_2}\right)}{\vartheta u_2\left(\left(e^{-\vartheta}-1\right)^2+\prod_{i=2}^4\left(e^{-\vartheta u_i}-1\right)\right)^2}-1\right|=0,$$

which implies that (3.4) holds. In terms of the Frank copula, we carry on the verification of relation shown in (2.7). Direct calculation shows that

$$\frac{\partial\hat{C}(u_1,u_2,u_3,u_4)}{\partial u_3\partial u_4}=\frac{-\vartheta e^{-\vartheta u_3}e^{-\vartheta u_4}\left(e^{-\vartheta}-1\right)^3\prod_{i=1}^2\left(e^{-\vartheta u_i}-1\right)}{\left(\left(e^{-\vartheta}-1\right)^3+\prod_{i=1}^4\left(e^{-\vartheta u_i}-1\right)\right)^2}.$$

It follows that

$$\begin{aligned}&\limsup_{u_1,u_2\to 0+}\sup_{u_3,u_4\in[\delta,1]}\left|\frac{\partial\hat{C}(u_1,u_2,u_3,u_4)/\partial u_3\partial u_4}{\partial\hat{C}(1,u_2,u_3,u_4)/\partial u_3\partial u_4}\right|_{\substack{u_3=\overline{H}_\theta(s)\\ u_4=\overline{H}_D(u)}}\\ &=\limsup_{u_1,u_2\to 0+}\sup_{u_3,u_4\in[\delta,1]}\left|\frac{\left(e^{-\vartheta}-1\right)\left(e^{-\vartheta u_1}-1\right)\left(\left(e^{-\vartheta}-1\right)^2+\prod_{i=2}^4\left(e^{-\vartheta u_i}-1\right)\right)^2}{\left(\left(e^{-\vartheta}-1\right)^3+\prod_{i=2}^4\left(e^{-\vartheta u_i}-1\right)\right)^2}\right|_{\substack{u_3=\overline{H}_\theta(s)\\ u_4=\overline{H}_D(u)}}=0,\end{aligned}\tag{3.9}$$

which implies that (3.5) holds. Similarly, we have (3.6). Thus, we have verified (2.7).

4. Numerical studies

In this section, we examine the accuracy of asymptotic estimates obtained in Theorem 2.1 and Corollary 2.1. To this end, the crude Monte Carlo (CMC) method is used to compare the simulated ruin probabilities with the asymptotic estimates.

We consider the by-claim given in (1.1). For the simulated estimation $\hat{\psi}$, given the finite-horizon $[0,T]$, we divide the time interval $[0,T]$ into n equally-spaced partitions. Let $t_k = kT/n, k = 1,\ldots,n$. The ruin probability $\psi(x,T)$ can be estimated by

$$\hat{\psi}(x;T)=\frac{1}{N}\sum_{j=1}^{N}I_{\left\{\min\limits_{k=1,\ldots,n}U_{t_k}^{(j)}<0\right\}}$$

where $U_{t_k}^{(j)}$ is the surplus at time t_k for path j. Throughout this section, the simulation is conducted with $n = 200$.

As for Theorem 2.1(i), let the main claim X follow a Pareto distribution

$$F(x)=1-\left(\frac{\gamma}{x+\gamma}\right)^{\alpha},\quad x>0,$$

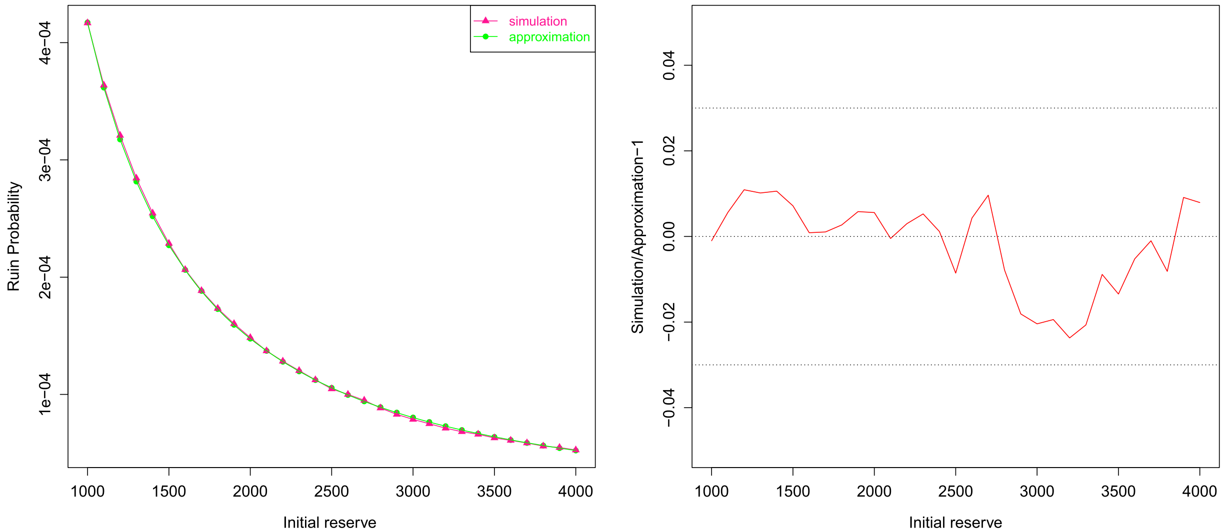

Fig. 2. Comparision of Simulation and Approximation of Ruin Probabilities when $\overline{G}(x) = o(\overline{F}(x))$

where the shape parameter is specified to $\alpha = 1.5$ and the scale parameter is specified to $\gamma = 1$, written as $X \sim Pareto(1.5, 1)$. Let $Y \sim Pareto(1.5, 2)$. Assume that both the inter-arrival time θ and the delay time D follow an exponential distribution with rate 1 such that the corresponding counting process N_t is a Poisson process with rate $\lambda = 1$. The dependence of (X, Y, θ, D) is characterized via the FGM copula given in (3.7), in which all parameters are specified to 0.1 except that $\vartheta_{14} = \vartheta_{34} = \vartheta_{134} = 0$. The premium accumulation process $C(t) = ct$ where the constant premium rate is specified to $c = 10 > \mathrm{E}X + \mathrm{E}Y$. The remaining parameters are set to be $T = 100$ and $r = 0.05$. In Fig. 1, the asymptotic estimation obtained from (2.5) and the simulation based on $N = 10^7$ are compared on the left, and the ratios are shown on the right.

As for Corollary 2.1, we employ the framework mentioned in Example 3.2. Assume that X still follows the Pareto distribution $Pareto(1.5, 1)$. The inter-arrival time θ and the delay time D still follow an exponential distribution with rate 1. The dependence of (X, θ) is characterized via the FGM copula with parameter $\vartheta = 0.25$. Let W follow the Uniform distribution $U(0, 1.2)$. The dependence of (W, D) is also characterized via the FGM copula with parameter $\vartheta = 0.25$. The constant premium rate is specified as $c = 5$ and other parameters remain unchanged. In Fig. 2, we compare the asymptotic estimation obtained from Corollary 2.1 with the simulation based on $N = 10^7$, and the ratios are shown on the right.

Figs. 1 and 2 show that the approximation fits well with the simulation, indicating the accuracy of the asymptotic estimation. The ratios stay around 0 and the errors are less than 3%. We should point out that fluctuation is due to the poor performance of the CMC method and a larger sample size is required in order to offset the negative effect of the ruin probability $\psi(x, T)$ being extremely small.

Declaration of competing interest

The authors declare that they have no known competing financial interests that could have appeared to influence the work reported in this paper.

Data availability

Data will be made available on request.

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Appendix A. Proofs

A.1. Lemmas

We start with several lemmas in the following. The proof of Theorem 2.1 leans heavily on Lemmas A.1-A.4. Particularly, Lemma A.5 is applied in Example 3.1.

Lemma A.1. *Assume that non-negative random vectors (X, θ) and (Y^*, θ^*, D^*) satisfy relations (2.2) and (2.3), respectively. Besides, (X, θ) is independent of (Y^*, θ^*, D^*) where θ^* is independent of D^*. Let the distributions of X and Y^* be F and G, respectively. If $F, G \in \mathcal{S}$ and $\overline{F}(x) \asymp \overline{G}(x)$, for positive bounded measurable functions $g_1(x)$, $g_2(x, y) \in [a, b]$ where $0 < a < b < \infty$, we have*

$$\begin{aligned} &\mathrm{P}\left(Xg_1(\theta) + Y^* g_2(\theta^*, D^*) > x \mid \theta = s, \theta^* = s^*, D^* = u^*\right) \\ \sim &\mathrm{P}\left(Xg_1(\theta) > x \mid \theta = s\right) + \mathrm{P}\left(Y^* g_2(\theta^*, D^*) > x \mid \theta^* = s^*, D^* = u^*\right) \end{aligned} \tag{A.1}$$

holds uniformly for $s, s^*, u^* \in \Lambda_T$.

Proof. Relation (A.1) amounts to the conjunction of

$$\begin{aligned} &\mathrm{P}\left(Xg_1(\theta)+Y^*g_2(\theta^*,D^*)>x \mid \theta=s,\theta^*=s^*,D^*=u^*\right) \\ \lesssim &\mathrm{P}\left(Xg_1(\theta)>x \mid \theta=s\right)+\mathrm{P}\left(Y^*g_2(\theta^*,D^*)>x \mid \theta^*=s^*,D^*=u^*\right) \end{aligned} \tag{A.2}$$

and

$$\begin{aligned} &\mathrm{P}\left(Xg_1(\theta)+Y^*g_2(\theta^*,D^*)>x \mid \theta=s,\theta^*=s^*,D^*=u^*\right) \\ \gtrsim &\mathrm{P}\left(Xg_1(\theta)>x \mid \theta=s\right)+\mathrm{P}\left(Y^*g_2(\theta^*,D^*)>x \mid \theta^*=s^*,D^*=u^*\right) \end{aligned} \tag{A.3}$$

uniformly for $s, s^*, u^* \in \Lambda_T$. Firstly, we shall verify (A.2). According to the value of $Xg_1(\theta)$ belonging to $(0, l(x)]$, $(x-l(x), \infty)$ and $(l(x), x-l(x)]$ for some function $l(x) \in \mathcal{H}(F,G)$, we split the following probability into three parts as

$$\begin{aligned} &\mathrm{P}\left(Xg_1(\theta)+Y^*g_2(\theta^*,D^*)>x \mid \theta=s,\theta^*=s^*,D^*=u^*\right) \\ =&\mathrm{P}\left(Xg_1(\theta)+Y^*g_2(\theta^*,D^*)>x, 0<Xg_1(\theta)\le l(x) \mid \theta=s,\theta^*=s^*,D^*=u^*\right) \\ &+\mathrm{P}\left(Xg_1(\theta)+Y^*g_2(\theta^*,D^*)>x, Xg_1(\theta)>x-l(x) \mid \theta=s,\theta^*=s^*,D^*=u^*\right) \\ &+\mathrm{P}\left(Xg_1(\theta)+Y^*g_2(\theta^*,D^*)>x, l(x)<Xg_1(\theta)\le x-l(x) \mid \theta=s,\theta^*=s^*,D^*=u^*\right) \\ :=&I_1+I_2+I_3. \end{aligned} \tag{A.4}$$

Note that $G \in \mathcal{S} \subset \mathcal{L}$. Applying relation (2.3) leads to

$$I_1 \le \mathrm{P}\left(Y^*g_2(\theta^*,D^*)>x-l(x) \mid \theta^*=s^*,D^*=u^*\right) \sim \mathrm{P}\left(Y^*g_2(\theta^*,D^*)>x \mid \theta^*=s^*,D^*=u^*\right) \tag{A.5}$$

uniformly for $s^*, u^* \in \Lambda_T$. In a similar way, it holds uniformly for $s \in \Lambda_T$ that

$$I_2 \lesssim \mathrm{P}\left(Xg_1(\theta)>x \mid \theta=s\right). \tag{A.6}$$

Now we deal with I_3. Recall that (X,θ) is independent of (Y^*,θ^*,D^*). Applying Assumption 2.1 in the third step, we have, for sufficiently large x and every $\varepsilon=\varepsilon(a,b)>0$,

$$\begin{aligned} I_3 =&\mathrm{P}\left(Xg_1(\theta)+Y^*g_2(\theta^*,D^*)>x, l(x)<Xg_1(\theta)\le x-l(x) \mid \theta=s,\theta^*=s^*,D^*=u^*\right) \\ =&\int_{l(x)}^{x-l(x)} \mathrm{P}\left(Y^*g_2(\theta^*,D^*)>x-t \mid \theta^*=s^*,D^*=u^*\right)\mathrm{P}\left(Xg_1(\theta)\in \mathrm{d}t \mid \theta=s\right) \\ \le&(1+\varepsilon)h(s)\varphi(s^*,u^*)\int_{l(x)}^{x-l(x)} \mathrm{P}\left(Y^*g_2(s^*,u^*)>x-t\right)\mathrm{P}\left(Xg_1(s)\in \mathrm{d}t\right) \end{aligned} \tag{A.7}$$

holds uniformly for $s, s^*, u^* \in \Lambda_T$. Without loss of generality we can assume that $g_1(s) \ge g_2(s^*,u^*)$ uniformly for $s, s^*, u^* \in \Lambda_T$ because otherwise we may use the value of $Y^*g_2(\theta^*,D^*)$ to split the probability on the left side in (A.4). Recall that $\overline{F}(x) \asymp \overline{G}(x)$ and $g_1, g_2 \in [a,b]$. Then there exists a constant M, irrespective of s that

$$\begin{aligned} &\int_{l(x)}^{x-l(x)} \mathrm{P}\left(Y^*g_2(s^*,u^*)>x-t\right)\mathrm{P}\left(Xg_1(s)\in \mathrm{d}t\right) \\ \le&\int_{l(x)}^{x-l(x)} \mathrm{P}\left(Y^*g_1(s)>x-t\right)\mathrm{P}\left(Xg_1(s)\in \mathrm{d}t\right) \\ \le& M\int_{l(x)}^{x-l(x)} \mathrm{P}\left(X^*g_1(s)>x-t\right)\mathrm{P}\left(Xg_1(s)\in \mathrm{d}t\right) \\ =&M\mathrm{P}\left(Xg_1(s)+X^*g_1(s)>x, l(x)<X^*g_1(s)\le x-l(x)\right), \end{aligned}$$

where X^* is an i.i.d. copy of X. Note that, for sufficiently large x and every $\varepsilon^*=\varepsilon^*(a,b)>0$ (irrespective of s),

$$\begin{aligned} &\mathrm{P}\left(Xg_1(s)+X^*g_1(s)>x, l(x)<X^*g_1(s)\le x-l(x)\right) \\ \le&\mathrm{P}\left(Xg_1(s)+X^*g_1(s)>x\right)-\mathrm{P}\left(Xg_1(s)>x, X^*g_1(s)\le l(x)\right)-\mathrm{P}\left(X^*g_1(s)>x, Xg_1(s)\le l(x)\right) \\ \le&(1+\varepsilon^*)\left(\mathrm{P}\left(Xg_1(s)>x\right)+\mathrm{P}\left(X^*g_1(s)>x\right)\right)-\mathrm{P}\left(Xg_1(s)>x\right)\mathrm{P}\left(X^*g_1(s)\le l(x)\right) \\ &-\mathrm{P}\left(X^*g_1(s)>x\right)\mathrm{P}\left(Xg_1(s)\le l(x)\right) \\ \le&4\varepsilon^*\mathrm{P}\left(Xg_1(s)>x\right), \end{aligned}$$

where the second step is due to Proposition 5.1 of Tang and Tsitsiashvili (2003). Then we have

$$\int_{l(x)}^{x-l(x)} \mathrm{P}\left(Y^* g_2(s^*, u^*) > x - t\right) \mathrm{P}\left(X g_1(s) \in \mathrm{d}t\right) \le \varepsilon^* 4M \mathrm{P}\left(X g_1(s) > x\right). \tag{A.8}$$

Substituting (A.8) into (A.7) and using Assumption 2.1, we have

$$I_3 = o\left(\mathrm{P}\left(X g_1(s) > x\right)\right) = o\left(\mathrm{P}\left(X g_1(\theta) > x \mid \theta = s\right)\right). \tag{A.9}$$

Thus, combining relations (A.5), (A.6) and (A.9) leads to (A.2).

Next we turn to verify the relation (A.3). Recalling that (X, θ) is independent of (Y^*, θ^*, D^*), we have, uniformly for $s, s^*, u^* \in \Lambda_T$,

$$\begin{aligned}
&\mathrm{P}\left(X g_1(\theta) + Y^* g_2(\theta^*, D^*) > x \mid \theta = s, \theta^* = s^*, D^* = u^*\right)\\
\ge& \mathrm{P}\left(\{X g_1(\theta) > x\} \cup \{Y^* g_2(\theta^*, D^*) > x\} \mid \theta = s, \theta^* = s^*, D^* = u^*\right)\\
=& \mathrm{P}\left(X g_1(\theta) > x \mid \theta = s\right) + \mathrm{P}\left(Y^* g_2(\theta^*, D^*) > x \mid \theta^* = s^*, D^* = u^*\right)\\
&- \mathrm{P}\left(X g_1(\theta) > x \mid \theta = s\right) \mathrm{P}\left(Y^* g_2(\theta^*, D^*) > x \mid \theta^* = s^*, D^* = u^*\right)\\
\gtrsim& \mathrm{P}\left(X g_1(\theta) > x \mid \theta = s\right) + \mathrm{P}\left(Y^* g_2(\theta^*, D^*) > x \mid \theta^* = s^*, D^* = u^*\right),
\end{aligned}$$

which implies (A.3). This completes the proof. □

In addition, going along the same lines as the proof of Lemma A.1 with obvious modifications, we obtain

$$\mathrm{P}\left(X g_1(\theta) + \tilde{X} g_1(\tilde{\theta}) > x \mid \theta = s, \tilde{\theta} = \tilde{s}\right) \sim \mathrm{P}\left(X g_1(\theta) > x \mid \theta = s\right) + \mathrm{P}\left(\tilde{X} g_1(\tilde{\theta}) > x \mid \tilde{\theta} = \tilde{s}\right) \tag{A.10}$$

uniformly for $s, \tilde{s} \in \Lambda_T$ when $(\tilde{X}, \tilde{\theta})$ is an i.i.d. copy of (X, θ), and

$$\begin{aligned}
&\mathrm{P}\left(Y g_2(\theta, D) + \tilde{Y} g_2(\tilde{\theta}, \tilde{D}) > x \mid \theta = s, \tilde{\theta} = \tilde{s}, D = u, \tilde{D} = \tilde{u}\right)\\
\sim& \mathrm{P}\left(Y g_2(\theta, D) > x \mid \theta = s, D = u\right) + \mathrm{P}\left(\tilde{Y} g_2(\tilde{\theta}, \tilde{D}) > x \mid \tilde{\theta} = \tilde{s}, \tilde{D} = \tilde{u}\right)
\end{aligned} \tag{A.11}$$

uniformly for $s, \tilde{s}, u, \tilde{u} \in \Lambda_T$ when $(\tilde{Y}, \tilde{\theta}, \tilde{D})$ is an i.i.d. copy of (Y, θ, D). The above two relations (A.10) and (A.11) will be used in the proof of Lemma A.2 below. The following lemma deals with the tail probability of the aggregate claims conditioning on the waiting time, which forms the basis for Lemma A.4. For notational convenience, we write $t_n = \sum_{i=1}^{n} s_i$ and $\Delta_n = \{1, \ldots, n\}$ for every $n \in \mathbb{N}_+$ hereafter.

Lemma A.2. *Under the conditions of Theorem 2.1, for every $n \in \mathbb{N}_+$, it holds uniformly for $s_i, u_i \in \Lambda_T$, $1 \le i \le n$ and $t \in \Lambda_T$ that*
(i) for $\overline{F}(x) \asymp \overline{G}(x)$,

$$\begin{aligned}
&\mathrm{P}\left(\sum_{i=1}^{n} X_i e^{-r\tau_i} + \sum_{i=1}^{n} Y_i e^{-r(\tau_i + D_i)} I_{\{\tau_i + D_i \le t\}} > x \mid \theta_i = s_i, D_i = u_i, i \in \Delta_n\right)\\
\sim& \sum_{i=1}^{n} \mathrm{P}\left(X_i e^{-rt_i} > x \mid \theta_i = s_i\right) + \sum_{i=1}^{n} \mathrm{P}\left(Y_i e^{-r(t_i + u_i)} I_{\{t_i + u_i < t\}} > x \mid \theta_i = s_i, D_i = u_i\right);
\end{aligned} \tag{A.12}$$

(ii) for $\overline{G}(x) = o\left(\overline{F}(x)\right)$,

$$\begin{aligned}
&\mathrm{P}\left(\sum_{i=1}^{n} X_i e^{-r\tau_i} + \sum_{i=1}^{n} Y_i e^{-r(\tau_i + D_i)} I_{\{\tau_i + D_i \le t\}} > x \mid \theta_i = s_i, D_i = u_i, i \in \Delta_n\right)\\
\sim& \sum_{i=1}^{n} \mathrm{P}\left(X_i e^{-rt_i} > x \mid \theta_i = s_i\right).
\end{aligned} \tag{A.13}$$

Proof. We proceed by the induction method to prove (A.12). Firstly, we shall prove that the assertion holds for $n = 1$. Note that $e^{-r\theta_1}, e^{-r(\theta_1 + D_1)} \in [e^{-rT}, 1]$. By (2.4) in the last step, it holds uniformly for $s_1, u_1, t \in \Lambda_T$ that

$$\begin{aligned}
&\mathrm{P}\left(X_1 e^{-r\theta_1} + Y_1 e^{-r(\theta_1 + D_1)} I_{\{\theta_1 + D_1 \le t\}} > x \mid \theta_1 = s_1, D_1 = u_1\right)\\
=& \mathrm{P}\left(X_1 e^{-rs_1} > x \mid \theta_1 = s_1\right) I_{\{s_1 + u_1 > t\}} + \mathrm{P}\left(X_1 e^{-rs_1} + Y_1 e^{-r(s_1 + u_1)} > x \mid \theta_1 = s_1, D_1 = u_1\right) I_{\{s_1 + u_1 \le t\}}\\
\sim& \mathrm{P}\left(X_1 e^{-rs_1} > x \mid \theta_1 = s_1\right) + \mathrm{P}\left(Y_1 e^{-r(s_1 + u_1)} I_{\{s_1 + u_1 \le t\}} > x \mid \theta_1 = s_1, D_1 = u_1\right),
\end{aligned}$$

which implies that (A.12) holds for $n = 1$. Next, assume that assertion (A.12) holds for some positive integer $n = m - 1$. We aim to show that the assertion holds for $n = m$, that is, it holds uniformly for $s_i, u_i \in \Lambda_T, i \in \Delta_m$ and $t \in \Lambda_T$ that

$$\begin{aligned}&\mathrm{P}\left(\sum_{i=1}^{m}X_ie^{-r\tau_i}+\sum_{i=1}^{m}Y_ie^{-r(\tau_i+D_i)}I_{\{\tau_i+D_i\le t\}}>x\mid\theta_i=s_i,D_i=u_i,i\in\Delta_m\right)\\&\sim\sum_{i=1}^{m}\mathrm{P}\left(X_ie^{-rt_i}>x\mid\theta_i=s_i\right)+\sum_{i=1}^{m}\mathrm{P}\left(Y_ie^{-r(t_i+u_i)}I_{\{t_i+u_i\le t\}}>x\mid\theta_i=s_i,D_i=u_i\right).\end{aligned}\tag{A.14}$$

According to the value of $\sum_{i=1}^{m-1}X_ie^{-r\tau_i}+\sum_{i=1}^{m-1}Y_ie^{-r(\tau_i+D_i)}I_{\{\tau_i+D_i\le t\}}$ belonging to $(0,l(x)]$, $(x-l(x),\infty)$ and $(l(x),x-l(x)]$ for some function $l(x)\in\mathcal{H}(F,G)$, we split the probability on the left side in (A.14) into three parts as

$$\mathrm{P}\left(\sum_{i=1}^{m}X_ie^{-r\tau_i}+\sum_{i=1}^{m}Y_ie^{-r(\tau_i+D_i)}I_{\{\tau_i+D_i\le t\}}>x\mid\theta_i=s_i,D_i=u_i,i\in\Delta_m\right):=I_1+I_2+I_3.$$

For I_1,

$$\begin{aligned}I_1=&\mathrm{P}\left(\sum_{i=1}^{m-1}\left(X_ie^{-rt_i}+Y_ie^{-r(t_i+u_i)}I_{\{t_i+u_i\le t\}}\right)\le l(x),\right.\\&\left.\sum_{i=1}^{m}\left(X_ie^{-rt_i}+Y_ie^{-r(t_i+u_i)}I_{\{t_i+u_i\le t\}}\right)>x\mid\theta_i=s_i,D_i=u_i,i\in\Delta_m\right)\\\le&\mathrm{P}\left(X_me^{-rt_m}+Y_me^{-r(t_m+u_m)}I_{\{t_m+u_m\le t\}}>x-l(x)\mid\theta_m=s_m,D_m=u_m\right).\end{aligned}$$

Referring to the proof for $n=1$, by the fact that $F,G\in\mathcal{L}$, it holds uniformly for $s_i,u_i\in\Lambda_T,i\in\Delta_m$ and $t\in\Lambda_T$ that

$$I_1\lesssim\mathrm{P}\left(X_me^{-rt_m}>x\mid\theta_m=s_m\right)+\mathrm{P}\left(Y_me^{-r(t_m+u_m)}I_{\{t_m+u_m\le t\}}>x\mid\theta_m=s_m,D_m=u_m\right).\tag{A.15}$$

For I_2, by the induction assumption and the fact that $F,G\in\mathcal{L}$, we have, uniformly for $s_i,u_i\in\Lambda_T,i\in\Delta_{m-1}$ and $t\in\Lambda_T$,

$$\begin{aligned}I_2\le&\mathrm{P}\left(\sum_{i=1}^{m-1}X_ie^{-r\tau_i}+\sum_{i=1}^{m-1}Y_ie^{-r(\tau_i+D_i)}I_{\{\tau_i+D_i\le t\}}>x-l(x)\mid\theta_i=s_i,D_i=u_i,i\in\Delta_{m-1}\right)\\\lesssim&\sum_{i=1}^{m-1}\mathrm{P}\left(X_ie^{-rt_i}>x\mid\theta_i=s_i\right)+\sum_{i=1}^{m-1}\mathrm{P}\left(Y_ie^{-r(t_i+u_i)}I_{\{t_i+u_i\le t\}}>x\mid\theta_i=s_i,D_i=u_i\right).\end{aligned}\tag{A.16}$$

Now we deal with I_3.

$$\begin{aligned}I_3=&\mathrm{P}\left(l(x)<\sum_{i=1}^{m-1}\left(X_ie^{-r\tau_i}+Y_ie^{-r(\tau_i+D_i)}I_{\{\tau_i+D_i\le t\}}\right)\le x-l(x),\right.\\&\left.\sum_{i=1}^{m}\left(X_ie^{-r\tau_i}+Y_ie^{-r(\tau_i+D_i)}I_{\{\tau_i+D_i\le t\}}\right)>x\mid\theta_i=s_i,D_i=u_i,i\in\Delta_m\right)\\\le&\mathrm{P}\left(\sum_{i=1}^{m}\left(X_ie^{-rt_i}+Y_ie^{-r(t_i+u_i)}I_{\{t_i+u_i\le t\}}\right)>x,\sum_{i=1}^{m-1}\left(X_ie^{-rt_i}+Y_ie^{-r(t_i+u_i)}I_{\{t_i+u_i\le t\}}\right)>l(x),\right.\\&\left.X_me^{-rt_m}+Y_me^{-r(t_m+u_m)}I_{\{t_m+u_m\le t\}}>l(x)\mid\theta_i=s_i,D_i=u_i,i\in\Delta_m\right)\\=&\int_{l(x)}^{\infty}\mathrm{P}\left(\sum_{i=1}^{m-1}\left(X_ie^{-rt_i}+Y_ie^{-r(t_i+u_i)}I_{\{t_i+u_i\le t\}}\right)>(x-y)\vee l(x)\mid\theta_i=s_i,D_i=u_i,i\in\Delta_{m-1}\right)\\&\mathrm{P}\left(X_me^{-rt_m}+Y_me^{-r(t_m+u_m)}I_{\{t_m+u_m\le t\}}\in\mathrm{d}y\mid\theta_m=s_m,D_m=u_m\right).\end{aligned}$$

Then applying the induction assumption and relation (2.4), uniformly for $s_i,u_i\in\Lambda_T,i\in\Delta_m$ and $t\in\Lambda_T$,

$$\begin{aligned}I_3\lesssim&\int_{l(x)}^{\infty}\sum_{i=1}^{m-1}\left(\mathrm{P}\left(X_ie^{-rt_i}>(x-y)\vee l(x)\mid\theta_i=s_i\right)\right.\\&\left.+\mathrm{P}\left(Y_ie^{-r(t_i+u_i)}I_{\{t_i+u_i\le t\}}>(x-y)\vee l(x)\mid\theta_i=s_i,D_i=u_i\right)\right)\\&\left(h(s_m)\mathrm{P}\left(X_me^{-rt_m}\in\mathrm{d}y\right)+\varphi(s_m,u_m)\mathrm{P}\left(Y_me^{-r(t_m+u_m)}I_{\{t_m+u_m\le t\}}\in\mathrm{d}y\right)\right)\\:=&I_{31}+I_{32}+I_{33}+I_{34},\end{aligned}\tag{A.17}$$

where

$$I_{31}=\sum_{i=1}^{m-1}\int_{l(x)}^{\infty}\mathrm{P}\left(X_ie^{-rt_i}>(x-y)\vee l(x)\mid\theta_i=s_i\right)h(s_m)\mathrm{P}\left(X_me^{-rt_m}\in\mathrm{d}y\right),$$

$$I_{32}=\sum_{i=1}^{m-1}\int_{l(x)}^{\infty}\mathrm{P}\left(X_ie^{-rt_i}>(x-y)\vee l(x)\mid\theta_i=s_i\right)\varphi(s_m,u_m)\mathrm{P}\left(Y_me^{-r(t_m+u_m)}I_{\{t_m+u_m\le t\}}\in\mathrm{d}y\right),$$

$$I_{33}=\sum_{i=1}^{m-1}\int_{l(x)}^{\infty}\mathrm{P}\left(Y_ie^{-r(t_i+u_i)}I_{\{t_i+u_i\le t\}}>(x-y)\vee l(x)\mid\theta_i=s_i,D_i=u_i\right)h(s_m)\mathrm{P}\left(X_me^{-rt_m}\in\mathrm{d}y\right),$$

$$I_{34}=\sum_{i=1}^{m-1}\int_{l(x)}^{\infty}\mathrm{P}\left(Y_ie^{-r(t_i+u_i)}I_{\{t_i+u_i\le t\}}>(x-y)\vee l(x)\mid\theta_i=s_i,D_i=u_i\right)$$
$$\cdot\varphi(s_m,u_m)\mathrm{P}\left(Y_me^{-r(t_m+u_m)}I_{\{t_m+u_m\le t\}}\in\mathrm{d}y\right).$$

We deal with I_{32} first. By (2.3) in Assumption 2.1, it holds uniformly for $s_i,u_i\in\Lambda_T, i\in\Delta_m$ and $t\in\Lambda_T$ that

$$\begin{aligned}
I_{32}\sim&\sum_{i=1}^{m-1}\mathrm{P}\Big(X_ie^{-rt_i}+Y_me^{-r(t_m+u_m)}I_{\{t_m+u_m\le t\}}>x, X_ie^{-rt_i}>l(x),\\
&\qquad Y_me^{-r(t_m+u_m)}I_{\{t_m+u_m\le t\}}>l(x)\mid\theta_i=s_i,\theta_m=s_m,D_m=u_m\Big)\\
\le&\sum_{i=1}^{m-1}\Big(\mathrm{P}\Big(X_ie^{-rt_i}+Y_me^{-r(t_m+u_m)}I_{\{t_m+u_m\le t\}}>x\mid\theta_i=s_i,\theta_m=s_m,D_m=u_m\Big)\\
&-\mathrm{P}\Big(X_ie^{-rt_i}>x, Y_me^{-r(t_m+u_m)}I_{\{t_m+u_m\le t\}}\le l(x)\mid\theta_i=s_i,\theta_m=s_m,D_m=u_m\Big)\\
&-\mathrm{P}\Big(X_ie^{-rt_i}\le l(x), Y_me^{-r(t_m+u_m)}I_{\{t_m+u_m\le t\}}>x\mid\theta_i=s_i,\theta_m=s_m,D_m=u_m\Big)\Big)\\
\lesssim&\sum_{i=1}^{m-1}\Big(\mathrm{P}\left(X_ie^{-rt_i}>x\mid\theta_i=s_i\right)+\mathrm{P}\Big(Y_me^{-r(t_m+u_m)}I_{\{t_m+u_m\le t\}}>x\mid\theta_m=s_m,D_m=u_m\Big)\\
&-\mathrm{P}\left(X_ie^{-rt_i}>x\mid\theta_i=s_i\right)\mathrm{P}\Big(Y_me^{-r(t_m+u_m)}I_{\{t_m+u_m\le t\}}\le l(x)\mid\theta_m=s_m,D_m=u_m\Big)\\
&-\mathrm{P}\left(X_ie^{-rt_i}\le l(x)\mid\theta_i=s_i\right)\mathrm{P}\Big(Y_me^{-r(t_m+u_m)}I_{\{t_m+u_m\le t\}}>x\mid\theta_m=s_m,D_m=u_m\Big)\Big)\\
=&o\left(\sum_{i=1}^{m-1}\mathrm{P}\left(X_ie^{-rt_i}>x\mid\theta_i=s_i\right)+(m-1)\mathrm{P}\Big(Y_me^{-r(t_m+u_m)}I_{\{t_m+u_m\le t\}}>x\mid\theta_m=s_m,D_m=u_m\Big)\right),
\end{aligned}\tag{A.18}$$

where in the third step we used Lemma A.1 and the fact that (X_i,θ_i) is independent of (Y_m,θ_m,D_m) for $i\ne m$. Going along the similar lines of I_{32}, it holds uniformly for $s_i,u_i\in\Lambda_T, i\in\Delta_m$ and $t\in\Lambda_T$ that

$$I_{31}=o\left(\sum_{i=1}^{m-1}\mathrm{P}\left(X_ie^{-rt_i}>x\mid\theta_i=s_i\right)+(m-1)\mathrm{P}\left(X_me^{-rt_m}>x\mid\theta_m=s_m\right)\right),\tag{A.19}$$

$$I_{33}=o\left(\sum_{i=1}^{m-1}\mathrm{P}\left(Y_ie^{-r(t_i+u_i)}I_{\{t_i+u_i\le t\}}>x\mid\theta_i=s_i,D_i=u_i\right)+(m-1)\mathrm{P}\left(X_me^{-rt_m}>x\mid\theta_m=s_m\right)\right)\tag{A.20}$$

and

$$\begin{aligned}
I_{34}=&o\Bigg(\sum_{i=1}^{m-1}\mathrm{P}\left(Y_ie^{-r(t_i+u_i)}I_{\{t_i+u_i\le t\}}>x\mid\theta_i=s_i,D_i=u_i\right)\\
&+(m-1)\mathrm{P}\left(Y_me^{-r(t_m+u_m)}I_{\{t_m+u_m\le t\}}>x\mid\theta_m=s_m,D_m=u_m\right)\Bigg).
\end{aligned}\tag{A.21}$$

Substituting (A.18)-(A.21) back into (A.17), we have, uniformly for $s_i,u_i\in\Lambda_T, i\in\Delta_m$ and $t\in\Lambda_T$,

$$I_3=o\left(\sum_{i=1}^{m}\mathrm{P}\left(X_ie^{-rt_i}>x\mid\theta_i=s_i\right)+\sum_{i=1}^{m}\mathrm{P}\left(Y_ie^{-r(t_i+u_i)}I_{\{t_i+u_i\le t\}}>x\mid\theta_i=s_i,D_i=u_i\right)\right).\tag{A.22}$$

Then combining relations (A.15), (A.16) and (A.22) leads to the upper bound version of (A.14):

$$
\begin{aligned}
&\mathrm{P}\left(\sum_{i=1}^{m} X_i e^{-r\tau_i} + \sum_{i=1}^{m} Y_i e^{-r(\tau_i+D_i)} I_{\{\tau_i+D_i\le t\}} > x \mid \theta_i = s_i, D_i = u_i, i \in \Delta_m\right) \\
&\lesssim \sum_{i=1}^{m} \mathrm{P}\left(X_i e^{-rt_i} > x \mid \theta_i = s_i\right) + \sum_{i=1}^{m} \mathrm{P}\left(Y_i e^{-r(t_i+u_i)} I_{\{t_i+u_i<t\}} > x \mid \theta_i = s_i, D_i = u_i\right),
\end{aligned} \tag{A.23}
$$

uniformly for $s_i, u_i \in \Lambda_T, i \in \Delta_m$ and $t \in \Lambda_T$.

Now we turn to the lower bound version of (A.14). We have

$$
\begin{aligned}
&\mathrm{P}\left(\sum_{i=1}^{m} X_i e^{-r\tau_i} + \sum_{i=1}^{m} Y_i e^{-r(\tau_i+D_i)} I_{\{\tau_i+D_i\le t\}} > x \mid \theta_i = s_i, D_i = u_i, i \in \Delta_m\right) \\
\ge &\mathrm{P}\left(\sum_{i=1}^{m-1} X_i e^{-rt_i} + \sum_{i=1}^{m-1} Y_i e^{-r(t_i+u_i)} I_{\{t_i+u_i\le t\}} > x \mid \theta_i = s_i, D_i = u_i, i \in \Delta_{m-1}\right) \\
&+ \mathrm{P}\left(X_m e^{-rt_m} + Y_m e^{-r(t_m+u_m)} I_{\{t_m+u_m\le t\}} > x \mid \theta_m = s_m, D_m = u_m\right) - J_1,
\end{aligned} \tag{A.24}
$$

where

$$
\begin{aligned}
J_1 = &\mathrm{P}\left(\sum_{i=1}^{m-1} X_i e^{-rt_i} + \sum_{i=1}^{m-1} Y_i e^{-r(t_i+u_i)} I_{\{t_i+u_i\le t\}} > x \mid \theta_i = s_i, D_i = u_i, i \in \Delta_{m-1}\right) \\
&\cdot \mathrm{P}\left(X_m e^{-rt_m} + Y_m e^{-r(t_m+u_m)} I_{\{t_m+u_m\le t\}} > x \mid \theta_m = s_m, D_m = u_m\right).
\end{aligned}
$$

Applying the induction assumption yields that

$$
\begin{aligned}
J_1 \sim &\sum_{i=1}^{m-1} \left(\mathrm{P}\left(X_i e^{-rt_i} > x \mid \theta_i = s_i\right) + \mathrm{P}\left(Y_i e^{-r(t_i+u_i)} I_{\{t_i+u_i\le t\}} > x \mid \theta_i = s_i, D_i = u_i\right)\right) \\
&\cdot \mathrm{P}\left(X_m e^{-rt_m} + Y_m e^{-r(t_m+u_m)} I_{\{t_m+u_m\le t\}} > x \mid \theta_m = s_m, D_m = u_m\right) \\
= &o\left(\sum_{i=1}^{m-1} \left(\mathrm{P}\left(X_i e^{-rt_i} > x \mid \theta_i = s_i\right) + \mathrm{P}\left(Y_i e^{-r(t_i+u_i)} I_{\{t_i+u_i\le t\}} > x \mid \theta_i = s_i, D_i = u_i\right)\right)\right)
\end{aligned} \tag{A.25}
$$

holds uniformly for $s_i, u_i \in \Lambda_T, i \in \Delta_m$ and $t \in \Lambda_T$. Then, substituting (A.25) back into (A.24) yields that

$$
\begin{aligned}
&\mathrm{P}\left(\sum_{i=1}^{m} X_i e^{-r\tau_i} + \sum_{i=1}^{m} Y_i e^{-r(\tau_i+D_i)} I_{\{\tau_i+D_i\le t\}} > x \mid \theta_i = s_i, D_i = u_i, i \in \Delta_m\right) \\
&\gtrsim \sum_{i=1}^{m} \mathrm{P}\left(X_i e^{-rt_i} > x \mid \theta_i = s_i\right) + \sum_{i=1}^{m} \mathrm{P}\left(Y_i e^{-r(t_i+u_i)} I_{\{t_i+u_i<t\}} > x \mid \theta_i = s_i, D_i = u_i\right)
\end{aligned} \tag{A.26}
$$

holds uniformly for $s_i, u_i \in \Lambda_T, i \in \Delta_m$ and $t \in \Lambda_T$. Combining (A.26) and (A.23) leads to (A.12).

Finally, we turn to prove (A.13). Note that $e^{-r(t_i+u_i)} I_{\{t_i+u_i\le t\}} \le e^{-rt_i}$. By Assumption 2.1 and the fact that $\overline{G}(x) = o(\overline{F}(x))$, for $i \in \Delta_m$ and every $\varepsilon_i > 0$,

$$
\begin{aligned}
&\mathrm{P}\left(Y_i e^{-r(t_i+u_i)} I_{\{t_i+u_i\le t\}} > x \mid \theta_i = s_i, D_i = u_i\right) \\
\lesssim &\varphi(s_i, u_i) \mathrm{P}\left(Y_i e^{-rt_i} > x\right) \\
\le &h(s_i)\varepsilon_i \mathrm{P}\left(X_i e^{-rt_i} > x\right) \sup_{s_i, u_i \in \Lambda_T} \frac{\varphi(s_i, u_i)}{h(s_i)} \\
= &o\left(\mathrm{P}\left(X_i e^{-rt_i} > x \mid \theta_i = s_i\right)\right)
\end{aligned}
$$

holds uniformly for $s_i, u_i \in \Lambda_T, i \in \Delta_m$ and $t \in \Lambda_T$. Note that

$$
\mathrm{P}\left(X_m e^{-rt_m} + Y_m e^{-r(t_m+u_m)} I_{\{t_m+u_m\le t\}} > x \mid \theta_m = s_m, D_m = u_m\right) \ge \mathrm{P}\left(X_m e^{-rt_m} > x \mid \theta_m = s_m\right).
$$

Then proceeding in a similar way as we did for the proof of (A.12), we derive (A.13). This completes the proof. □

In what follows we establish a Kesten-type upper bound, which is a useful means to deal with the tail probability of randomly weighted sums of infinitely many terms. See Lemma 1.3.5(c) of Embrechts et al. (1997) for more details on the well-known Kesten's inequality.

Lemma A.3. *Under the conditions of Theorem 2.1, for every $\varepsilon > 0$ and $T \in \Lambda$, there exists some constant $K = K_{r,\varepsilon,T} > 0$ such that, for all $n \in \mathbb{N}_+$, $x \ge 0$, and $t \in \Lambda_T$,* *(i) if* $\overline{F}(x) \asymp \overline{G}(x)$,

$$\begin{aligned}&\mathrm{P}\left(\sum_{i=1}^{n}\left(X_i e^{-r\theta_i}+Y_i e^{-r(\theta_i+D_i)} I_{\{\theta_i+D_i\le t\}}\right)>x,\tau_n\le t\right)\\ \le& K(1+\varepsilon)^n\left(\mathrm{P}\left(X_1 e^{-r\theta_1}>x,\tau_n\le t\right)+\mathrm{P}\left(Y_1 e^{-r(\theta_1+D_1)} I_{\{\theta_1+D_1\le t\}}>x,\tau_n\le t\right)\right);\end{aligned}$$

(ii) if $\overline{G}(x) = o\left(\overline{F}(x)\right)$,

$$\mathrm{P}\left(\sum_{i=1}^{n}\left(X_i e^{-r\theta_i}+Y_i e^{-r(\theta_i+D_i)} I_{\{\theta_i+D_i\le t\}}\right)>x,\tau_n\le t\right)\le K(1+\varepsilon)^n \mathrm{P}\left(X_1 e^{-r\theta_1}>x,\tau_n\le t\right).$$

Proof. For notational convenience, write $Z_i = X_i \mathrm{e}^{-r\theta_i} + Y_i e^{-r(\theta_i+D_i)} I_{\{\theta_i+D_i\le t\}}$ for $i \in \Delta_{n+1}$ and

$$\begin{aligned}\alpha_n &= \sup_{x\ge 0, t\in\Lambda_T}\frac{\mathrm{P}\left(\sum_{i=1}^{n} X_i \mathrm{e}^{-r\theta_i}+Y_i e^{-r(\theta_i+D_i)} I_{\{\theta_i+D_i\le t\}}>x,\tau_n\le t\right)}{\mathrm{P}\left(X_1\mathrm{e}^{-r\theta_1}>x,\tau_n\le t\right)+\mathrm{P}\left(Y_1 e^{-r(\theta_1+D_1)} I_{\{\theta_1+D_1\le t\}}>x,\tau_n\le t\right)}\\ &= \sup_{x\ge 0, t\in\Lambda_T}\frac{\mathrm{P}\left(\sum_{i=1}^{n} Z_i>x,\tau_n\le t\right)}{\mathrm{P}\left(X_1\mathrm{e}^{-r\theta_1}>x,\tau_n\le t\right)+\mathrm{P}\left(Y_1 e^{-r(\theta_1+D_1)} I_{\{\theta_1+D_1\le t\}}>x,\tau_n\le t\right)}.\end{aligned}$$

By (2.4), for every $\varepsilon > 0$ (irrespective of s_1, u_1), there exists some constant $x_0 > 0$ such that, for all $x > x_0$,

$$\begin{aligned}&\mathrm{P}\left(Z_1>x\mid\theta_1=s_1,D_1=u_1\right)\\ \le&(1+\varepsilon)\left(\mathrm{P}\left(X_1 e^{-r\theta_1}>x\mid\theta_1=s_1\right)+\mathrm{P}\left(Y_1 e^{-r(\theta_1+D_1)} I_{\{\theta_1+D_1\le t\}}>x\mid\theta_1=s_1,D_1=u_1\right)\right)\end{aligned}\tag{A.27}$$

holds uniformly for $s_1, u_1 \in \Lambda_T$. By (2.2) in Assumption 2.1, the constant x_0 above can be chosen so large that, for all $t \in \Lambda_T$,

$$\mathrm{P}\left(X_1\mathrm{e}^{-r\theta_1}>x_0\mid\theta_1=s_1\right)\ge\frac{1}{2}\overline{F}\left(x_0\mathrm{e}^{rT}\right)h(s_1).\tag{A.28}$$

Now we start by evaluating α_{n+1} and focus on the denominator in α_{n+1}:

$$\mathrm{P}\left(\sum_{i=1}^{n+1} Z_i>x,\tau_{n+1}\le t\right)=\mathrm{P}\left(\sum_{i=1}^{n+1} Z_i>x,Z_{n+1}\le x,\tau_{n+1}\le t\right)+\mathrm{P}\left(Z_{n+1}>x,\tau_{n+1}\le t\right).$$

For all $x > x_0$, conditioning on (θ_{n+1}, D_{n+1}) and using (A.27) yields

$$\begin{aligned}&\mathrm{P}\left(Z_{n+1}>x,\tau_{n+1}\le t\right)\\ =&\int_{0-}^{t}\int_{0-}^{t}\mathrm{P}\left(Z_{n+1}>x\mid\theta_{n+1}=s_{n+1},D_{n+1}=u_{n+1}\right)\mathrm{P}\left(\tau_n\le t-s_{n+1}\right)H_\theta(\mathrm{d}s_{n+1})H_D(\mathrm{d}u_{n+1})\\ =&\int_{0-}^{t}\int_{0-}^{t}\mathrm{P}\left(Z_1>x\mid\theta_1=s,D_1=u\right)\mathrm{P}\left(\tau_n\le t-s\right)H_\theta(\mathrm{d}s)H_D(\mathrm{d}u)\\ \le&(1+\varepsilon)\int_{0-}^{t}\int_{0-}^{t}\left(\mathrm{P}\left(X_1 e^{-r\theta_1}>x\mid\theta_1=s\right)+\mathrm{P}\left(Y_1 e^{-r(\theta_1+D_1)} I_{\{\theta_1+D_1\le t\}}>x\mid\theta_1=s,D_1=u\right)\right)\\ &\cdot\mathrm{P}\left(\tau_n\le t-s\right)H_\theta(\mathrm{d}s)H_D(\mathrm{d}u)\\ \le&(1+\varepsilon)\left(\mathrm{P}\left(X_1 e^{-r\theta_1}>x,\tau_{n+1}\le t\right)+\mathrm{P}\left(Y_1 e^{-r(\theta_1+D_1)} I_{\{\theta_1+D_1\le t\}}>x,\tau_{n+1}\le t\right)\right),\end{aligned}\tag{A.29}$$

where in the last step we used the fact that $\{\theta_i; i \in \mathbb{N}_+\}$ is a sequence of i.i.d. copies of θ. Conditioning on $(Z_{n+1}, \theta_{n+1}, D_{n+1})$, we have

$$\begin{aligned}
&\mathrm{P}\left(\sum_{i=1}^{n+1} Z_i > x, Z_{n+1} \le x, \tau_{n+1} \le t\right)\\
=&\int_{0-}^{t}\int_{0-}^{t}\int_{0-}^{x} \mathrm{P}\left(\sum_{i=1}^{n} Z_i > x-w, \tau_n \le t-s_{n+1}\right) \mathrm{P}\left(Z_{n+1}\in \mathrm{d}w, \theta_{n+1}\in \mathrm{d}s_{n+1}, D_{n+1}\in \mathrm{d}u_{n+1}\right)\\
=&\int_{0-}^{t}\int_{0-}^{t}\int_{0-}^{x} \frac{\mathrm{P}\left(\sum_{i=1}^{n} Z_i > x-w, \tau_n \le t-s_{n+1}\right)}{\mathrm{P}\left(X_1\mathrm{e}^{-r\theta_1} > x-w, \tau_n \le t-s_{n+1}\right)+\mathrm{P}\left(Y_1 e^{-r(\theta_1+D_1)} I_{\{\theta_1+D_1\le t\}} > x-w, \tau_n \le t-s_{n+1}\right)}\\
&\cdot\left(\mathrm{P}\left(X_1\mathrm{e}^{-r\theta_1} > x-w, \tau_n \le t-s_{n+1}\right)+\mathrm{P}\left(Y_1 e^{-r(\theta_1+D_1)} I_{\{\theta_1+D_1\le t\}} > x-w, \tau_n \le t-s_{n+1}\right)\right)\\
&\cdot \mathrm{P}\left(Z_{n+1}\in \mathrm{d}w, \theta_{n+1}\in \mathrm{d}s_{n+1}, D_{n+1}\in \mathrm{d}u_{n+1}\right)\\
\le&\alpha_n\left(\mathrm{P}\left(X_1\mathrm{e}^{-r\theta_1}+Z_{n+1} > x, Z_{n+1}\le x, \tau_{n+1}\le t\right)\right.\\
&\left.+\mathrm{P}\left(Y_1 e^{-r(\theta_1+D_1)} I_{\{\theta_1+D_1\le t\}}+Z_{n+1} > x, Z_{n+1}\le x, \tau_{n+1}\le t\right)\right)\\
:=&\alpha_n(I_1+I_2).
\end{aligned} \tag{A.30}$$

For all $x > x_0$ and $t \in \Lambda_T$,

$$\begin{aligned}
I_1 =&\mathrm{P}\left(X_1\mathrm{e}^{-r\theta_1}+Z_{n+1} > x, Z_{n+1}\le x, \tau_{n+1}\le t\right)\\
=&\mathrm{P}\left(X_1\mathrm{e}^{-r\theta_1}+Z_{n+1} > x, \tau_{n+1}\le t\right)-\mathrm{P}\left(Z_{n+1} > x, \tau_{n+1}\le t\right)\\
=&\int_{0-}^{t}\int_{0-}^{t}\int_{0-}^{t}\left(\mathrm{P}\left(X_1\mathrm{e}^{-r\theta_1}+Z_{n+1} > x \mid \theta_1=s_1, \theta_{n+1}=s_{n+1}, D_{n+1}=u_{n+1}\right)\right.\\
&\left.-\mathrm{P}\left(Z_{n+1} > x \mid \theta_{n+1}=s_{n+1}, D_{n+1}=u_{n+1}\right)\right)\\
&\cdot \mathrm{P}\left(\sum_{i=2}^{n}\theta_i \le t-s_1-s_{n+1}\right) H_\theta(\mathrm{d}s_1)\, H_\theta(\mathrm{d}s_{n+1})\, H_D(\mathrm{d}u_{n+1}).
\end{aligned} \tag{A.31}$$

Following the same lines as the proof of Lemma A.2 with some obvious modifications, we have, for every $\varepsilon_1 > 0$ (irrespective of s, s_{n+1}, u_{n+1}) and all $x > x_0$,

$$\begin{aligned}
&\mathrm{P}\left(X_1\mathrm{e}^{-r\theta_1}+Z_{n+1} > x \mid \theta_1=s_1, \theta_{n+1}=s_{n+1}, D_{n+1}=u_{n+1}\right)\\
\le&(1+\varepsilon_1)\left(\mathrm{P}\left(X_1\mathrm{e}^{-r\theta_1} > x \mid \theta_1=s_1\right)+\mathrm{P}\left(Z_{n+1} > x \mid \theta_{n+1}=s_{n+1}, D_{n+1}=u_{n+1}\right)\right)
\end{aligned} \tag{A.32}$$

holds uniformly for $s_1, s_{n+1}, u_{n+1} \in \Lambda_T$. Recall that $\{(X_i, Y_i, \theta_i, D_i); i \in \mathbb{N}_+\}$ is a sequence of i.i.d. copies of a generic vector (X, Y, θ, D). Thus, combining (A.27) and (A.32) yields

$$\begin{aligned}
&\mathrm{P}\left(X_1\mathrm{e}^{-r\theta_1}+Z_{n+1} > x \mid \theta_1=s_1, \theta_{n+1}=s_{n+1}, D_{n+1}=u_{n+1}\right)-\mathrm{P}\left(Z_{n+1} > x \mid \theta_{n+1}=s_{n+1}, D_{n+1}=u_{n+1}\right)\\
\le&(1+\varepsilon_1)\mathrm{P}\left(X_1\mathrm{e}^{-r\theta_1} > x \mid \theta_1=s_1\right)+(1+\varepsilon)\varepsilon_1\mathrm{P}\left(X_{n+1}\mathrm{e}^{-r\theta_{n+1}} > x \mid \theta_{n+1}=s_{+1}\right)\\
&+(1+\varepsilon)\varepsilon_1\mathrm{P}\left(Y_{n+1}e^{-r(\theta_{n+1}+D_{n+1})} I_{\{\theta_{n+1}+D_{n+1}\le t\}} > x \mid \theta_{n+1}=s_{n+1}, D_{n+1}=u_{n+1}\right).
\end{aligned}$$

Substituting the above relation back into (A.31) leads to

$$\begin{aligned}
I_1 \le&(1+\varepsilon_1)\mathrm{P}\left(X_1\mathrm{e}^{-r\theta_1} > x, \tau_{n+1}\le t\right)+(1+\varepsilon)\varepsilon_1\mathrm{P}\left(X_{n+1}\mathrm{e}^{-r\theta_{n+1}} > x, \tau_{n+1}\le t\right)\\
&+(1+\varepsilon)\varepsilon_1\mathrm{P}\left(Y_{n+1}e^{-r(\theta_{n+1}+D_{n+1})} I_{\{\theta_{n+1}+D_{n+1}\le t\}} > x, \tau_{n+1}\le t\right)\\
=&(1+2\varepsilon_1+\varepsilon_1\varepsilon)\mathrm{P}\left(X_1\mathrm{e}^{-r\theta_1} > x, \tau_{n+1}\le t\right)\\
&+(1+\varepsilon)\varepsilon_1\mathrm{P}\left(Y_{n+1}e^{-r(\theta_{n+1}+D_{n+1})} I_{\{\theta_{n+1}+D_{n+1}\le t\}} > x, \tau_{n+1}\le t\right),
\end{aligned}$$

which can be rewritten to the form: for every $\widetilde{\varepsilon}_1 > 0$, all $x > x_0$ and $t \in \Lambda_T$,

$$I_1 \le (1+\widetilde{\varepsilon}_1)\mathrm{P}\left(X_1\mathrm{e}^{-r\theta_1} > x, \tau_{n+1}\le t\right)+\widetilde{\varepsilon}_1\mathrm{P}\left(Y_1 e^{-r(\theta_1+D_1)} I_{\{\theta_1+D_1\le t\}} > x, \tau_{n+1}\le t\right). \tag{A.33}$$

Similarly, for every $\widetilde{\varepsilon}_2 > 0$, all $x > x_0$ and $t \in \Lambda_T$,

$$I_2 \le (1+\widetilde{\varepsilon}_2)\mathrm{P}\left(Y_1 e^{-r(\theta_1+D_1)} I_{\{\theta_1+D_1\le t\}} > x, \tau_{n+1}\le t\right)+\widetilde{\varepsilon}_2\mathrm{P}\left(X_1\mathrm{e}^{-r\theta_1} > x, \tau_{n+1}\le t\right). \tag{A.34}$$

Plugging (A.33) and (A.34) into (A.30), we have, for every $\widetilde{\varepsilon} > 0$, all $x > x_0$ and $t \in \Lambda_T$,

$$\begin{aligned}&\mathrm{P}\left(\sum_{i=1}^{n+1} Z_i > x, Z_{n+1} \leq x, \tau_{n+1} \leq t\right)\\ \leq&(1+\widetilde{\varepsilon})\alpha_n \left(\mathrm{P}\left(X_1 \mathrm{e}^{-r\theta_1} > x, \tau_{n+1} \leq t\right) + \mathrm{P}\left(Y_1 e^{-r(\theta_1+D_1)} I_{\{\theta_1+D_1\leq t\}} > x, \tau_{n+1} \leq t\right)\right).\end{aligned}$$

Thus, together with (A.29), we have

$$\sup_{x>x_0, t\in\Lambda_T} \frac{\mathrm{P}\left(\sum_{i=1}^{n+1} Z_i \mathrm{e}^{-r\theta_i} > x, \tau_{n+1} \leq t\right)}{\mathrm{P}\left(X_1 \mathrm{e}^{-r\theta_1} > x, \tau_{n+1} \leq t\right) + \mathrm{P}\left(Y_1 \mathrm{e}^{-r(\theta_1+D_1)} I_{\{\theta_1+D_1\leq t\}} > x, \tau_{n+1} \leq t\right)} \leq (1+\widetilde{\varepsilon})\alpha_n + 2. \tag{A.35}$$

By (A.28), it holds for all $t \in \Lambda_T$ and $x \leq x_0$ that

$$\frac{\mathrm{P}\left(\sum_{i=1}^{n+1} Z_i > x, \tau_{n+1} \leq t\right)}{\mathrm{P}\left(X_1 \mathrm{e}^{-r\theta_1} > x, \tau_{n+1} \leq t\right) + \mathrm{P}\left(Y_1 \mathrm{e}^{-r(\theta_1+D_1)} I_{\{\theta_1+D_1\leq t\}} > x, \tau_{n+1} \leq t\right)} \leq \frac{\mathrm{P}\left(\tau_{n+1} \leq t\right)}{\mathrm{P}\left(X_1 \mathrm{e}^{-r\theta_1} > x_0, \tau_{n+1} \leq t\right)}.$$

Going the same lines as the proof for (4.14) in Li et al. (2010), for some $0 < L < \infty$, we have

$$\sup_{x\leq x_0, t\in\Lambda_T} \frac{\mathrm{P}\left(\sum_{i=1}^{n+1} Z_i \mathrm{e}^{-r\theta_i} > x, \tau_{n+1} \leq t\right)}{\mathrm{P}\left(X_1 \mathrm{e}^{-r\theta_1} > x, \tau_{n+1} \leq t\right) + \mathrm{P}\left(Y_1 \mathrm{e}^{-r(\theta_1+D_1)} I_{\{\theta_1+D_1\leq t\}} > x, \tau_{n+1} \leq t\right)} \leq L.$$

Thus, together with (A.35), we obtain the recursive inequality:

$$\alpha_{n+1} \leq (1+\varepsilon)\alpha_n + 2 + L.$$

Note that we can obtain $\alpha_1 \leq 2 + L$ by proceeding in the same way as we did in above. We can deduce the Kesten-type upper bound with a suitably chosen constant $K_{r,\varepsilon,T}$. The proof under the condition $\overline{G}(x) = o\left(\overline{F}(x)\right)$ follows from the deduction above with some obvious modifications. This completes the proof. □

The next lemma plays a crucial role in the proof of Theorem 2.1. Considering the convenience and simplicity of the discussion, we merge the two assertions in Theorem 2.1 as one, namely, it holds that

$$\psi(x,t) \sim \int_{0-}^{t} \overline{F}\left(xe^{rs}\right) \mathrm{d}\tilde{\lambda}_s + \int_{0-}^{t}\int_{0-}^{t-u^*} \overline{G}\left(xe^{r\left(s+u^*\right)}\right) \mathrm{d}\hat{\lambda}_{s,u^*} H_D\left(\mathrm{d}u^*\right) \cdot I_{\{\beta>0\}} := \phi(x,t,\beta),$$

where $\beta = \limsup \dfrac{\overline{G}(x)}{\overline{F}(x)}$.

Lemma A.4. *Under the conditions of Theorem 2.1, it holds uniformly for $t \in \Lambda_T$ that*

$$\mathrm{P}\left(\sum_{i=1}^{N_t} X_i e^{-r\tau_i} + \sum_{i=1}^{N_t} Y_i e^{-r(\tau_i+D_i)} I_{\{\tau_i+D_i\leq t\}} > x\right) \sim \phi(x,t,\beta).$$

Proof. Arbitrarily choose some large integer M,

$$\begin{aligned}&\mathrm{P}\left(\sum_{i=1}^{N_t} X_i e^{-r\tau_i} + \sum_{i=1}^{N_t} Y_i e^{-r(\tau_i+D_i)} I_{\{\tau_i+D_i\leq t\}} > x\right)\\ =&\left(\sum_{n=1}^{M} + \sum_{n=M+1}^{\infty}\right) \mathrm{P}\left(\sum_{i=1}^{n} X_i e^{-r\tau_i} + \sum_{i=1}^{n} Y_i e^{-r(\tau_i+D_i)} I_{\{\tau_i+D_i\leq t\}} > x, N_t = n\right)\\ :=& I_1 + I_2.\end{aligned} \tag{A.36}$$

Recall that $\Delta_n = \{1,\ldots,n\}$ for every $n \in \mathbb{N}_+$ and $t_n = \sum_{i=1}^{n} s_i$. By Lemma A.2,

$$\begin{aligned}
I_1 =& \sum_{n=1}^{M} \int_{\Omega_{n,t}} \mathrm{P}\left(\sum_{i=1}^{n} X_i e^{-rt_i} + \sum_{i=1}^{n} Y_i e^{-r(t_i+u_i)} I_{\{t_i+u_i\le t\}} > x \mid \theta_i = s_i, D_i = u_i, i \in \Delta_n \right) \\
& \cdot \overline{H}_\theta (t-t_n) \prod_{i=1}^{n} H_\theta(\mathrm{d}s_i) H_D(\mathrm{d}u_i) \\
\sim & \sum_{n=1}^{M} \int_{\Omega_{n,t}} \left(\sum_{i=1}^{n} \mathrm{P}\left(X_i e^{-rt_i} > x \mid \theta_i = s_i\right) + \sum_{i=1}^{n} \mathrm{P}\left(Y_i e^{-r(t_i+u_i)} I_{\{t_i+u_i\le t\}} > x \mid \theta_i = s_i, D_i = u_i \right) \cdot I_{\{\beta>0\}} \right) \\
& \cdot \overline{H}_\theta (t-t_n) \prod_{i=1}^{n} H_\theta(\mathrm{d}s_i) H_D(\mathrm{d}u_i) \\
=& \sum_{n=1}^{M} \left(\sum_{i=1}^{n} \mathrm{P}\left(X_i e^{-r\tau_i} > x, N_t = n\right) + \sum_{i=1}^{n} \mathrm{P}\left(Y_i e^{-r(\tau_i+D_i)} I_{\{\tau_i+D_i\le t\}} > x, N_t = n \right) \cdot I_{\{\beta>0\}} \right) \\
=& \left(\sum_{n=1}^{\infty} - \sum_{n=M+1}^{\infty} \right) \left(\sum_{i=1}^{n} \mathrm{P}\left(X_i e^{-r\tau_i} > x, N_t = n\right) + \sum_{i=1}^{n} \mathrm{P}\left(Y_i e^{-r(\tau_i+D_i)} I_{\{\tau_i+D_i\le t\}} > x, N_t = n \right) \cdot I_{\{\beta>0\}} \right) \\
:=& I_{11} - I_{12},
\end{aligned} \tag{A.37}$$

where $\Omega_{n,t} = \{s_i \in [0,t], u_i \in [0,t], i \in \Delta_n : t_n \le t\}$. By interchanging the order of the sums and then conditioning on $(\tau_{i-1}, \theta_i, D_i)$,

$$\begin{aligned}
I_{11} =& \sum_{n=1}^{\infty} \left(\sum_{i=1}^{n} \mathrm{P}\left(X_i e^{-r\tau_i} > x, N_t = n\right) + \sum_{i=1}^{n} \mathrm{P}\left(Y_i e^{-r(\tau_i+D_i)} I_{\{\tau_i+D_i\le t\}} > x, N_t = n \right) \cdot I_{\{\beta>0\}} \right) \\
=& \sum_{i=1}^{\infty} \mathrm{P}\left(X_i e^{-r\tau_i} > x, \tau_i \le t\right) + \sum_{i=1}^{\infty} \mathrm{P}\left(Y_i e^{-r(\tau_i+D_i)} I_{\{\tau_i+D_i\le t\}} > x, \tau_i \le t \right) \cdot I_{\{\beta>0\}}
\end{aligned} \tag{A.38}$$

holds uniformly for $t \in \Lambda_T$. Then it follows from (4.18) in Li et al. (2010) that, uniformly for $t \in \Lambda_T$,

$$\sum_{i=1}^{\infty} \mathrm{P}\left(X_i e^{-r\tau_i} > x, \tau_i \le t\right) \sim \int_{0-}^{t} \overline{F}\left(xe^{rs}\right) \mathrm{d}\tilde{\lambda}_s, \tag{A.39}$$

where $\tilde{\lambda}_s = \int_{0-}^{s} (1+\lambda_{s-s^*})\, h(s^*)\, H_\theta(\mathrm{d}s^*)$. In a similar way, by (2.3) in Assumption 2.1 and integrating by parts with possible jumps (see Klebaner (2005)), uniformly for $t \in \Lambda_T$,

$$\begin{aligned}
& \sum_{i=1}^{\infty} \mathrm{P}\left(Y_i e^{-r(\tau_i+D_i)} I_{\{\tau_i+D_i\le t\}} > x, \tau_i \le t \right) \\
=& \sum_{i=1}^{\infty} \iiint_{\Gamma_t} \mathrm{P}\left(Y_i e^{-r(v+u+s)} > x \mid \theta_i = s, D_i = u \right) \mathrm{P}(\tau_{i-1} \in \mathrm{d}v)\, H_\theta(\mathrm{d}s)\, H_D(\mathrm{d}u) \\
\sim & \sum_{i=1}^{\infty} \iiint_{\Gamma_t} \overline{G}\left(xe^{r(v+s+u)}\right) \mathrm{P}(\tau_{i-1} \in \mathrm{d}v)\, \varphi(s,u)\, H_\theta(\mathrm{d}s)\, H_D(\mathrm{d}u) \\
=& \int_{0-}^{t} \int_{0-}^{t-u} \overline{G}\left(xe^{r(s+u)}\right) \varphi(s,u)\, H_\theta(\mathrm{d}s)\, H_D(\mathrm{d}u) \\
& + \iiint_{\Gamma_t} \overline{G}\left(xe^{r(v+s+u)}\right) \mathrm{P}(\tau_{i-1} \in \mathrm{d}v)\, \varphi(s,u)\, H_\theta(\mathrm{d}s)\, H_D(\mathrm{d}u) \\
=& \int_{0-}^{t} \int_{0-}^{t-u} \left(\overline{G}\left(xe^{r(s+u)}\right) + \int_{0-}^{t-u-s} \overline{G}\left(xe^{r(v+s+u)}\right) \lambda_v \right) \varphi(s,u)\, H_\theta(\mathrm{d}s)\, H_D(\mathrm{d}u) \\
:=& \int_{0-}^{t} \int_{0-}^{t-u} \overline{G}\left(xe^{r(s+u)}\right) \mathrm{d}\hat{\lambda}_{s,u} H_D(\mathrm{d}u),
\end{aligned} \tag{A.40}$$

where $\Gamma_t = \{v, u, s \in [0,t] : v+u+s \le t\}$ and $\hat{\lambda}_{s,u} = \int_{0-}^{s} (1+\lambda_{s-s^*})\, \varphi(s^*, u)\, H_\theta(\mathrm{d}s^*)$. Plugging (A.39) and (A.40) into (A.38), we have, uniformly for $t \in \Lambda_T$,

$$I_{11} \sim \phi(x,t,\beta). \tag{A.41}$$

For I_{12}, by Assumption 2.1 in the second step,

$$\begin{aligned}
I_{12} &= \sum_{n=M+1}^{\infty}\sum_{i=1}^{n}\left(\mathrm{P}\left(X_i e^{-r\tau_i} > x, N_t = n\right) + \mathrm{P}\left(Y_i e^{-r(\tau_i+D_i)} I_{\{\tau_i+D_i\le t\}} > x, N_t = n\right)\cdot I_{\{\beta>0\}}\right)\\
&\le \sum_{n=M+1}^{\infty}\sum_{i=1}^{n}\left(\int_{0-}^{t}\mathrm{P}\left(X_i e^{-r\theta_i} > x \mid \theta_i = s\right)\mathrm{P}\left(N_{t-s}=n-1\right)H_\theta(\mathrm{d}s)\right.\\
&\quad \left. + \int_{0-}^{t}\int_{0-}^{t-u}\mathrm{P}\left(Y_i e^{-r(\theta_i+D_i)} > x \mid \theta_i = s, D_i = u\right)\mathrm{P}\left(N_{t-s}=n-1\right)H_\theta(\mathrm{d}s)H_D(\mathrm{d}u)\cdot I_{\{\beta>0\}}\right)\\
&\sim \sum_{n=M+1}^{\infty} n\left(\int_{0-}^{t}\overline{F}\left(xe^{rs}\right)\mathrm{P}\left(N_{t-s}=n-1\right)h(s)H_\theta(\mathrm{d}s)\right.\\
&\quad \left. + \int_{0-}^{t}\int_{0-}^{t-u}\overline{G}\left(xe^{r(s+u)}\right)\mathrm{P}\left(N_{t-s}=n-1\right)\varphi(s,u)H_\theta(\mathrm{d}s)H_D(\mathrm{d}u)\cdot I_{\{\beta>0\}}\right)\\
&\le \mathrm{E}\left(N_T+1\right)I_{\{N_T\ge M\}}\left(\int_{0-}^{t}\overline{F}\left(xe^{rs}\right)h(s)H_\theta(\mathrm{d}s) + \int_{0-}^{t}\int_{0-}^{t-u}\overline{G}\left(xe^{r(s+u)}\right)\varphi(s,u)H_\theta(\mathrm{d}s)H_D(\mathrm{d}u)\cdot I_{\{\beta>0\}}\right).
\end{aligned}$$

Note that $\mathrm{E}(N_T+1)I_{\{N_T\ge M\}} \to 0$ as $M\to\infty$ (see e.g. Stein (1946)). For every $\delta>0$, we can find some large positive integer M such that, uniformly for $t\in\Lambda_T$,

$$I_{12} \lesssim \delta\phi(x,t,\beta). \tag{A.42}$$

Next, we turn to I_2. By Lemma A.3 in the third step, for every $\varepsilon>0$, there exists some constant K such that, uniformly for $t\in\Lambda_T$,

$$\begin{aligned}
I_2 &= \sum_{n=M+1}^{\infty}\mathrm{P}\left(\sum_{i=1}^{n}\left(X_i e^{-r\tau_i} + Y_i e^{-r(\tau_i+D_i)} I_{\{\tau_i+D_i\le t\}}\right) > x, N_t = n\right)\\
&\le \sum_{n=M+1}^{\infty}\mathrm{P}\left(\sum_{i=1}^{n}\left(X_i e^{-r\theta_i} + Y_i e^{-r(\theta_i+D_i)} I_{\{\theta_i+D_i\le t\}}\right) > x, \tau_n \le t\right)\\
&\le \sum_{n=M+1}^{\infty} K(1+\varepsilon)^n\left(\mathrm{P}\left(X_1 e^{-r\theta_1} > x, \tau_n\le t\right) + \mathrm{P}\left(Y_1 e^{-r(\theta_1+D_1)} I_{\{\theta_1+D_1\le t\}} > x, \tau_n\le t\right)\cdot I_{\{\beta>0\}}\right)\\
&\sim \sum_{n=M+1}^{\infty} K(1+\varepsilon)^n\left(\int_{0-}^{t}\overline{F}(xe^{rs})\mathrm{P}\left(N_{t-s}\ge n-1\right)h(s)H_\theta(\mathrm{d}s)\right.\\
&\quad \left. + \int_{0-}^{t}\int_{0-}^{t-u}\overline{G}(xe^{r(s+u)})\mathrm{P}\left(N_{t-s}\ge n-1\right)\varphi(s,u)H_\theta(\mathrm{d}s)H_D(\mathrm{d}u)\right)\\
&\le \sum_{n=M}^{\infty} K(1+\varepsilon)^{n+1}\mathrm{P}\left(N_T\ge n\right)\left(\int_{0-}^{t}\overline{F}(xe^{rs})h(s)H_\theta(\mathrm{d}s) + \int_{0-}^{t}\int_{0-}^{t-u}\overline{G}(xe^{r(s+u)})\varphi(s,u)H_\theta(\mathrm{d}s)H_D(\mathrm{d}u)\right),
\end{aligned}$$

where in the fourth step we used Assumption 2.1. By Theorem 1 of Kočetova et al. (2009), $\sum_{n=M}^{\infty}(1+\varepsilon)^{n+1}\mathrm{P}(N_T\ge n)\to 0$ as $M\to\infty$. Thus, for every $\delta'>0$, we can find some large positive integer M such that, uniformly for $t\in\Lambda_T$,

$$I_2 \lesssim \delta'\phi(x,t,\beta). \tag{A.43}$$

A combination of (A.36), (A.37) and (A.41)-(A.43) gives that, uniformly for all $t\in\Lambda_T$,

$$(1-\delta)\phi(x,t,\beta) \lesssim \mathrm{P}\left(\sum_{i=1}^{N_t} X_i e^{-r\tau_i} + \sum_{i=1}^{N_t} Y_i e^{-r(\tau_i+D_i)} I_{\{\tau_i+D_i\le t\}} > x\right) \lesssim (1+\delta')\phi(x,t,\beta).$$

Noting the arbitrariness of δ and δ', we complete the proof. □

The following lemma is used to verify relation (2.4) such that Theorem 2.1 is applicable in Example 3.1.

Lemma A.5. *Assume that* (X, Y, θ, D) *fulfills Assumption 2.1 and relation (2.7). If* $F, G \in \mathcal{L} \cap \mathcal{D}$ *and* $\overline{F}(x) \asymp \overline{G}(x)$*, relation (2.4) is satisfied for positive bounded measurable functions* $g_1(x), g_2(x, y) \in [a, b]$ *for* $0 < a < b < \infty$.

Proof. According to the value of $Yg_2(\theta, D)$ belonging to $(0, l(x)]$, $(x - l(x), \infty)$ and $(l(x), x - l(x)]$ for some function $l(x) \in \mathcal{H}(F, G)$, we split the following probability into three parts as

$$P(Xg_1(\theta) + Yg_2(\theta, D) > x \mid \theta = s, D = u) := I_1 + I_2 + I_3, \tag{A.44}$$

where

$$\begin{aligned} I_1 =& P(Xg_1(\theta) + Yg_2(\theta, D) > x, 0 < Yg_2(\theta, D) \le l(x) \mid \theta = s, D = u), \\ I_2 =& P(Xg_1(\theta) + Yg_2(\theta, D) > x, Yg_2(\theta, D) > x - l(x) \mid \theta = s, D = u), \\ I_3 =& P(Xg_1(\theta) + Yg_2(\theta, D) > x, l(x) < Yg_2(\theta, D) \le x - l(x) \mid \theta = s, D = u). \end{aligned}$$

By Assumption 2.1 and the fact that $F, G \in \mathcal{L}$,

$$I_1 \le P(Xg_1(s) > x - l(x) \mid \theta = s) \sim h(s) P(Xg_1(s) > x) \sim P(Xg_1(s) > x \mid \theta = s) \tag{A.45}$$

and

$$I_2 \lesssim P(Yg_2(s, u) > x \mid \theta = s, D = u) \tag{A.46}$$

uniformly for $s, u \in \Lambda_T$. For I_3,

$$\begin{aligned} I_3 \le& P(Xg_1(s) > l(x), Yg_2(s, u) > l(x), Xg_1(s) + Yg_2(s, u) > x \mid \theta = s, D = u) \\ \le& P(Xg_1(s) > l(x), Yg_2(s, u) > x/2 \mid \theta = s, D = u) + P(Xg_1(s) > x/2, Yg_2(s, u) > l(x) \mid \theta = s, D = u) \\ \le& P(Xg_1(s) > l(x) \mid Yg_2(s, u) > x/2, \theta = s, D = u) P(Yg_2(s, u) > x/2 \mid \theta = s, D = u) \\ &+ P(Yg_2(s, u) > l(x) \mid Xg_1(s) > x/2, \theta = s, D = u) P(Xg_1(s) > x/2 \mid \theta = s). \end{aligned}$$

Applying relation (2.7), Assumption 2.1 and the fact that $F, G \in \mathcal{D}$ in the above inequality, we have

$$I_3 = o(P(Xg_1(s) > x \mid \theta = s) + P(Yg_2(s, u) > x \mid \theta = s, D = u)) \tag{A.47}$$

holds uniformly for $s, u \in \Lambda_T$. Plugging (A.45)–(A.47) into (A.44), we know that

$$\limsup_{x \to \infty} \sup_{s, u \in \Lambda_T} \frac{P(Xg_1(\theta) + Yg_2(\theta, D) > x \mid \theta = s, D = u)}{P(Xg_1(s) > x \mid \theta = s) + P(Yg_2(s, u) > x \mid \theta = s, D = u)} \le 1.$$

On the other hand,

$$\begin{aligned} & P(Xg_1(s) + Yg_2(s, u) > x \mid \theta = s, D = u) \\ =& P(Xg_1(s) > x \mid \theta = s) + P(Yg_2(s, u) > x \mid \theta = s, D = u) \\ & - P(Xg_1(s) > x, Yg_2(s, u) > x \mid \theta = s, D = u). \end{aligned}$$

By (2.7), we have

$$\begin{aligned} & P(Xg_1(s) > x, Yg_2(s, u) > x \mid \theta = s, D = u) \\ \le& P(Xg_1(s) > x \mid Yg_2(s, u) > x, \theta = s, D = u) P(Yg_2(s, u) > x \mid \theta = s, D = u) \\ =& o(P(Xg_1(s) > x \mid \theta = s) + P(Yg_2(s, u) > x \mid \theta = s, D = u)) \end{aligned}$$

holds uniformly for $s, u \in \Lambda_T$. Thus,

$$\liminf_{x \to \infty} \inf_{s, u \in \Lambda_T} \frac{P(Xg_1(\theta) + Yg_2(\theta, D) > x \mid \theta = s, D = u)}{P(Xg_1(s) > x \mid \theta = s) + P(Yg_2(s, u) > x \mid \theta = s, D = u)} \ge 1.$$

This completes the proof. □

A.2. Proof of main results

After the preliminaries above, we are now in the position to prove our main results.

Proof of Theorem 2.1. For the asymptotic upper bound, we have

$$\begin{aligned}\psi(x,t) =& \mathrm{P}\left(\inf_{0\le s\le t} U(s)<0 \middle| U(0)=x\right)\\ =& \mathrm{P}\left(\inf_{0\le s\le t} e^{-rs}U(s)<0 \middle| U(0)=x\right)\\ \le& \mathrm{P}\left(\sum_{i=1}^{N_t} X_i e^{-r\tau_i}+\sum_{i=1}^{N_t} Y_i e^{-r(\tau_i+D_i)} I_{\{\tau_i+D_i\le t\}}>x\right)\\ \sim& \int_{0-}^{t} \overline{F}\left(xe^{rs}\right)\mathrm{d}\tilde{\lambda}_s+\int_{0-}^{t}\int_{0-}^{t-u^*} \overline{G}\left(xe^{r(s+u^*)}\right)\mathrm{d}\hat{\lambda}_{s,u^*}H_D\left(\mathrm{d}u^*\right)\cdot I_{\{\beta>0\}},\end{aligned}$$

where in the last step we used Lemma A.4 and $\beta=\limsup\frac{\overline{G}(x)}{\overline{F}(x)}$. On the other hand, recall that $C(t)$ is non-negative and independent of the other sources of randomness. By conditioning on $\int_0^T e^{-rs}C(ds)$, the dominated convergence argument and Lemma A.4, it holds uniformly for $t\in\Lambda_T$ that

$$\begin{aligned}\psi(x,t) \ge& \mathrm{P}\left(\sum_{i=1}^{N_t} X_i e^{-r\tau_i}+\sum_{i=1}^{N_t} Y_i e^{-r(\tau_i+D_i)} I_{\{\tau_i+D_i\le t\}}>x+\int_0^T e^{-rs}C(ds)\right)\\ \sim& \int_{0-}^{t} \mathrm{P}\left(Xe^{-rs}-\int_0^T e^{-rz}C(dz)>x\right)\mathrm{d}\tilde{\lambda}_s\\ &+\int_{0-}^{t}\int_{0-}^{t-u^*} \mathrm{P}\left(Ye^{-r(s+u^*)}-\int_0^T e^{-rz}C(dz)>x\right)\mathrm{d}\hat{\lambda}_{s,u^*}H_D\left(\mathrm{d}u^*\right)\cdot I_{\{\beta>0\}}\\ \sim& \int_{0-}^{t} \overline{F}\left(xe^{rs}\right)\mathrm{d}\tilde{\lambda}_s+\int_{0-}^{t}\int_{0-}^{t-u^*} \overline{G}\left(xe^{r(s+u^*)}\right)\mathrm{d}\hat{\lambda}_{s,u^*}H_D\left(\mathrm{d}u^*\right)\cdot I_{\{\beta>0\}},\end{aligned}$$

where in the last step we used the assumption that $\mathrm{P}\left(\int_0^t e^{-rs}C(ds)>x\right)=o\left(\overline{F}(x/a)\right)$ for any fixed $t,r>0$ and some $a>0$, and Lemma 3.2 of Li (2017). Combining the above two estimates, we complete the proof. □

Proof of Corollary 2.1. We apply the idea of the deduction of Lemma 3.3 in Yang et al. (2018). Without loss of generality we can assume that $g_1,g_2\in[a,b]$ for some $0<a<b<\infty$. For any $0<v<1$, by the fact that $F\in\mathcal{C}\subset\mathcal{D}$,

$$\begin{aligned}&\frac{\mathrm{P}(g_1(\theta)X+g_2(\theta,D)Y>x\mid\theta=s,D=u)}{\mathrm{P}(g_1(\theta)X>x\mid\theta=s)}\\ =&\frac{\mathrm{P}(g_1(\theta)X+g_2(\theta,D)Y>x,\,g_2(\theta,D)Y>vx\mid\theta=s,D=u)}{\mathrm{P}(g_1(\theta)X>x\mid\theta=s)}\\ &+\frac{\mathrm{P}(g_1(\theta)X+g_2(\theta,D)Y>x,\,g_2(\theta,D)Y\le vx\mid\theta=s,D=u)}{\mathrm{P}(g_1(\theta)X>x\mid\theta=s)}\\ \le&\frac{\mathrm{P}(g_2(\theta,D)Y>vx\mid\theta=s,D=u)}{\mathrm{P}(g_1(\theta)X>x\mid\theta=s)}+\frac{\mathrm{P}(g_1(\theta)X>(1-v)x\mid\theta=s)}{\mathrm{P}(g_1(\theta)X>x\mid\theta=s)}\\ :=&I_1+I_2.\end{aligned}$$

By Assumption 2.1 and the fact that $\overline{G}(x)=o\left(\overline{F}(x)\right)$ and $F\in\mathcal{C}\subset\mathcal{D}$, for any $0<v<1$,

$$\begin{aligned}&\limsup_{x\to\infty}\sup_{s,u\in\Lambda_T} I_1\\ \le&\limsup_{x\to\infty}\sup_{s,u\in\Lambda_T}\frac{\mathrm{P}(bY>vx\mid\theta=s,D=u)}{\mathrm{P}(bX>vx\mid\theta=s)}\frac{\mathrm{P}(bX>vx\mid\theta=s)}{\mathrm{P}(aX>x\mid\theta=s)}\\ \le&\limsup_{x\to\infty}\sup_{s,u\in\Lambda_T}\frac{\varphi(s,u)\overline{G}(vx/b)}{h(s)\overline{F}(vx/b)}\frac{\overline{F}(vx/b)}{\overline{F}(x/a)}=0.\end{aligned}\tag{A.48}$$

By Assumption 2.1 and the fact that $F\in\mathcal{C}$,

$$\limsup_{v\downarrow 0}\limsup_{x\to\infty}\sup_{s,u\in\Lambda_T} I_2=1.\tag{A.49}$$

Combining (A.48) and (A.49) leads to

$$\limsup_{v\downarrow 0}\limsup_{x\to\infty}\sup_{s,u\in\Lambda_T}\frac{\mathrm{P}(g_1(\theta)X+g_2(\theta,D)Y>x\mid\theta=s,D=u)}{\mathrm{P}(g_1(\theta)X>x)h(s)}\le 1.$$

Note that

$$\mathrm{P}(g_1(\theta)X+g_2(\theta,D)Y>x\mid\theta=s,D=u)\ge\mathrm{P}(g_1(\theta)X>x\mid\theta=s)$$

holds uniformly for $s,u\in\Lambda_T$. We have

$$\mathrm{P}(g_1(\theta)X+g_2(\theta,D)Y>x\mid\theta=s,D=u)\sim\mathrm{P}(g_1(\theta)X>x\mid\theta=s) \tag{A.50}$$

holds uniformly for $s,u\in\Lambda_T$. Relation (A.50) can be seen as a special form of (2.4). Thus, it follows from (ii) of Theorem 2.1 that (2.6) holds uniformly for $t\in\Lambda_T$. This completes the proof. □

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