Insurance: Mathematics and Economics 112 (2023) 80–96

Contents lists available at ScienceDirect

Insurance: Mathematics and Economics

journal homepage: www.elsevier.com/locate/ime

Annuitizing at a bounded, absolutely continuous rate to minimize the probability of lifetime ruin

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ARTICLE INFO

Article history:
Received November 2022
Received in revised form June 2023
Accepted 19 June 2023
Available online 26 June 2023

JEL classification:
C730
G220

Keywords:
Probability of lifetime ruin
Optimal annuitization
Absolutely continuous annuitization rate
Barrier strategy
Age-dependent force of mortality

ABSTRACT

We minimize the probability of lifetime ruin in a deterministic financial and insurance model, although the investor's time of death is random, with an age-dependent force of mortality. By contrast with the traditional anything-anytime annuitization model (that is, individuals can annuitize any fraction of their wealth at anytime), the individual only purchases life annuity income gradually, using a bounded, absolutely continuous rate. As in the anything-anytime annuitization case, we find that it is optimal for the individual not to purchase additional annuity income when her wealth is less than a specific linear function of her existing annuity income, which we call the *buy boundary*. Interestingly, we find the buy boundary in our model is identical to the one in the anything-anytime annuitization model. However, there is a separate threshold, which we call the *safe level*. (This threshold degenerates to the buy boundary in the anything-anytime annuitization model.) When wealth is greater than the safe level, the minimum probability of lifetime ruin is zero; when wealth lies between the buy boundary and the safe level, the individual's best choice is to purchase annuity income at the maximum allowable rate.

1. Introduction

The probability of lifetime ruin is an important criterion in measuring the risk that individuals outlive their wealth; it was first proposed by Milevsky and Robinson (2000). Young (2004) extended their model by allowing individuals to optimally invest in a Black-Scholes financial market to minimize the probability of lifetime ruin. Bayraktar and Young (2007) further explored the problem by adding borrowing constraints. Bayraktar and Zhang (2015a) considered model ambiguity in the drift of the risky asset. Bayraktar et al. (2011) minimized the probability of lifetime ruin under stochastic volatility. Bayraktar and Zhang (2015b) applied Stochastic Perron's Method to analyze the lifetime ruin problem under transaction costs. Liang and Young (2019) simplified the model in Bayraktar and Zhang (2015b) by considering two riskless assets and found explicit solutions for an age-dependent force of mortality. Liang and Young (2020) extended the original model of Young (2004) by replacing *ruin* with *exponential Parisian ruin*.

In recent decades, pensions have changed from defined-benefit to defined-contribution schemes or to a hybrid blend of them. Thus, individuals are more responsible for the management of their pension funds, in particular, to converting those funds into lifetime income when they retire. Also, the quickly growing proportion of older people in many countries means that governments face unprecedented financial difficulties to provide enough government pension income; therefore, many governments encourage their citizens to purchase individual pensions to complement government pensions. In view of these changes, we expect more people to purchase life annuities to provide retirement income. However, voluntary annuity purchasing is less than predicted by many models. Some research proposed that the reason for this phenomenon might be the absence of bequest motives, market imperfections, and frictions; see, for example, Mitchell et al. (1999), Brown (2001), Bütler and Teppa (2007), Brown et al. (2008), and Lockwood (2012). To further understand this so-called *annuity puzzle*, a number of researchers minimized the individuals' probability of lifetime ruin by purchasing annuities and investing in a

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[1] X. Liang thanks the National Natural Science Foundation of China (No. 12271274) for financial support of her research.
[2] V. R. Young thanks the Nesbitt Professorship for financial support of her research.

https://doi.org/10.1016/j.insmatheco.2023.06.003
0167-6687/

risky financial market. For example, Milevsky et al. (2006) found the optimal asset allocation and annuity-purchase strategies to minimize the probability of lifetime ruin. Bayraktar and Young (2009) further explored the minimum probability of lifetime ruin by allowing the individual to purchase deferred life annuities. Wang and Young (2012) investigated commutable annuities to minimize the probability of lifetime ruin.

In this paper, we consider an optimal annuity-purchasing problem. Unlike the "all or nothing" or "anything anytime" annuitization framework of Milevsky and Young (2007), we assume the individual purchases annuity income gradually with an absolutely continuous rate, in which the per-unit-time annuity purchasing rate is restricted to a bounded interval. We propose three advantages of purchasing annuity income in this manner. First, because annuity purchasing is an irreversible transaction, an individual might be reluctant to purchase life annuity income with a lump sum payment, or she might delay that purchase. Instead, an individual might find it more acceptable to purchase annuity income gradually to reduce the perceived risk from this commitment, as individuals in the U.S. do when depositing money into their 401(k)s, 403(b)s, and IRAs. Horneff et al. (2008a,b) considered optimal gradually annuitization and asset allocation in a discrete-time framework. Horneff et al. (2015) extended their work by further investigating the optimal investment and consumption strategies with guaranteed minimum withdrawal benefit variable annuities. Second, buying annuity income gradually gives an individual more chances to evaluate how much annuity income she needs. As time passes, if she realizes she has enough annuity income for her lifestyle, she can stop altogether or wait before continuing. Third, purchasing annuity income gradually also addresses concerns about interest rates by making it less likely to invest money as a lump sum at today's low rate because future purchases might be made at a higher rate. Quoting from Warshawsky (2011), "Timing risk... may deter investors from the purchase of life annuities unless it is... managed through, for example, gradual purchases, that is ladders, of smaller immediate life annuities."

In our model, the individual aims to minimize her probability of lifetime ruin (that is, the probability that the individual runs out of money before she dies, which consuming at a pre-fixed level), and we assume that her future lifetime is governed by an age-dependent mortality intensity. First, we observe that when the individual's wealth is greater than or equal to $(c - A)/r$, and if the individual does not purchase additional annuity income, then consumption can be covered by interest income and existing annuity income.[3] Thus, the minimum probability of lifetime ruin is zero. By further analyzing the dynamics of the wealth and cumulative annuity processes between 0 and $(c - A)/r$, we find two additional boundaries which we call the *buy boundary* and the *safe level*. We show that the annuity's *buy boundary* is identical to that of the "anything anytime" annuitization model in Milevsky et al. (2006), which can be seen as the limiting case when the maximum allowable rate of buying annuity income goes to infinity. Our results show that it is optimal for the individual not to purchase additional annuity income when wealth is less than the buy boundary. On the other hand, we find when wealth becomes greater than the *safe level*, then the minimum probability of lifetime ruin is zero. When wealth lies between the buy boundary and the safe level, it is optimal for the individual to purchase annuity income at the maximum allowable rate till wealth decreases to the buy boundary.

The rest of the paper is organized as follows. In Section 2.1, we formulate the model, introduce the annuity structure in which the cumulative annuitization process accrues according to an absolutely continuous rate, and define admissible strategies. In Section 2.2, we summarize the main results and contributions of this paper. In Section 3, first, we prove two lemmas to analyze the trajectories of the wealth and the accumulated annuity processes and, then, provide a verification theorem. After the verification theorem, we hypothesize that the optimal annuitization strategy is a barrier strategy, in which the barrier equals the buy boundary in the anything-anytime annuitization model in Milevsky et al. (2006). Based on our hypothesis, we construct a candidate strategy and the associated probability of lifetime ruin. We find there is another barrier for our problem, the safe level, and when wealth is greater than the safe level, the minimum probability of lifetime ruin is zero. By using the verification theorem, we show that the constructed candidate probability of lifetime ruin is, indeed, equal to the *minimum* probability of lifetime ruin. Thus, our hypothesized annuitization strategy is optimal. We also provide several numerical examples to illustrate our results. Section 4 concludes the paper.

2. Problem formulation and main contributions

2.1. Model formulation

In this section, we present the financial ingredients that affect the individual's wealth, namely, consumption, income, a riskless asset, and a life annuity. We assume that the individual buys annuities in order to minimize the probability that her wealth reaches zero before she dies. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$. Let τ_d be the random future lifetime random variable of the individual, with deterministic force of mortality $\lambda(t)$. τ_d is defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

We assume the individual invests her wealth in a riskless asset that earns a constant, positive, continuous rate of interest r. The individual consumes at a continuous rate c. As an aside, if the individual receives income other than from her life annuity, then we interpret c as the rate of consumption *net* of that non-annuity income.

The lump-sum price of an immediate life annuity that pays \$1 per year continuously until the insured's death equals

$$\bar{a}^o(t) = \tilde{\mathbb{E}}\left[\int_t^{\tau_d} e^{-r(s-t)} ds \Big| \tau_d > t \right] = \int_t^{\infty} e^{-\int_t^s (r+\lambda^o(u))du} ds, \tag{2.1}$$

in which $\tilde{\mathbb{E}}$ is an expectation defined on a probability space $(\Omega, \mathcal{F}, \tilde{\mathbb{P}})$. In this probability space, τ_d has a deterministic force of mortality $\lambda^o(t)$. We also call it the *objective* force of mortality that the insurance company uses to price annuities.

We assume that $\lambda^o(t)$ is a continuous and non-decreasing function of t; the latter is a reasonable assumption because the force of mortality is usually increasing for people over age 30 or so, and younger people are generally not interested in purchasing annuity income. Note that we do *not* assume that the function λ^o and λ are equal. In other words, at time t, in return for each $\bar{a}^o(t)$ the individual

[3] c, A, and r are the rates of consumption, existing annuity income, and return on the riskless investment, respectively.

pays for an immediate life annuity, she receives \$1 per year of continuous annuity income until she dies. However, we assume that the individual buys annuity income at an absolutely continuous rate, so she accrues annuity income gradually.

Let $W(t)$ denote the individual's wealth at time t, and let $A(t)$ denote the cumulative amount of (immediate) life annuity income purchased at or before time t. Let $q(t) \geq 0$ denote the *rate* at which the individual purchases annuity income at time t. In this paper, we require $q(t) \in [0, \bar{q}]$ for all $t \geq 0$, in which $\bar{q} > 0$ is the maximum allowable annuitization rate.[4] This maximum rate could arise because the individual might only have $\bar{a}^o(t)\bar{q}$ of disposable income for buying annuities.[5] The individual's wealth evolves according to the following dynamics:

$$\begin{cases} dW(s) = \big(rW(s) - c + A(s)\big)ds - \bar{a}^o(s)q(s)ds, \quad s \geq t, \\ dA(s) = q(s)ds, \quad s \geq t, \\ W(t) = w \geq 0, \qquad A(t) = A \geq 0. \end{cases} \tag{2.2}$$

Define a hitting time τ_0 associated with the wealth process by $\tau_0 = \inf\{t \geq 0 : W(t) < 0\}$. This hitting time is the time of ruin. By *probability of lifetime ruin*, we mean the probability that wealth reaches 0 before the individual dies at time τ_d. We minimize this probability over the set of admissible annuity purchasing strategies. We call a strategy $Q = \{q(s)\}_{t \leq s < \tau_d} \in \mathcal{A}(t)$ *admissible* if the annuitization rate process Q is non-negative, right-continuous, adapted to the filtration $\{\mathcal{F}_s\}_{s \geq t}$, and satisfies $q(s) \in [0, \bar{q}]$ for all $t \leq s < \tau_d$.

Note that, if $w \geq (c - A)/r$, and if the individual does not buy additional annuity income, then consumption is covered by interest income and existing annuity income, which implies that the minimum probability of lifetime ruin is zero. Therefore, it remains for us to study the minimum probability of lifetime ruin on the domain

$$\mathcal{D} \triangleq \left\{ (t, w, A) \in \mathbb{R}^3_+ : 0 \leq w < \frac{c - A}{r}, \ 0 \leq A \leq c \right\}.$$

Denote the minimum probability of lifetime ruin by ψ; then, for $(t, w, A) \in \mathcal{D}$,

$$\begin{aligned} \psi(t, w, A) &= \inf_{Q \in \mathcal{A}(t)} \mathbb{P}\left[\tau_0 < \tau_d \,\middle|\, W(t) = w, A(t) = A, \min(\tau_d, \tau_0) > t\right] \\ &= \inf_{Q \in \mathcal{A}(t)} e^{-\int_t^{\tau_0} \lambda(s)ds}. \end{aligned} \tag{2.3}$$

2.2. Main results and contributions

We summarize the main results and associated optimal annuitization strategy for our model in this section. Please refer to Fig. 1 for graphs of the important curves in (w, A)-space.

1. When the individual's wealth is greater than or equal to $(c - A)/r$, and if the individual does not purchase additional annuity income, then consumption can be covered by interest income and existing annuity income. Thus, the minimum probability of lifetime ruin is zero.
2. When the individual's wealth lies between 0 and the *buy boundary* $(c - A)\bar{a}^o(t)$, it is optimal for the individual not to purchase any additional annuity income (Theorem 3.2 and Proposition 3.2).
3. When wealth lies between the buy boundary $(c - A)\bar{a}^o(t)$ and the *safe level* $\bar{w}(t, A)$ (to be defined), it is optimal for the individual to purchase annuities at the maximum allowable rate till wealth decreases to the buy boundary (Theorem 3.2 and Proposition 3.2).
4. When wealth lies between the safe level $\bar{w}(t, A)$ and $(c - A)/r$, then it is optimal for the individual to purchase annuity income at the maximum allowable rate till annuity income plus interest income reaches c, after which the individual does not purchase additional annuity income. Via this strategy, ruin does not occur and the minimum probability of lifetime ruin is zero (Proposition 3.1).
5. As $\bar{q} \to \infty$, the model degenerates to the anything-anytime annuitization model. For this model, it is optimal for individual not to purchase any annuity income when wealth $0 < w < (c - A)\bar{a}^o(t)$ and to purchase an amount of $c - A$ annuity income if wealth is greater than $(c - A)\bar{a}^o(t)$. That is, the safe level approaches the buy boundary as $\bar{q} \to \infty$ (Proposition 3.4).

In analyzing the model, we find two "crossing-only-once" phenomena. First, if wealth plus the wealth equivalence of the annuity income $\bar{a}^o(t)A$ is less than $c\bar{a}^o(t)$, then subsequent wealth plus the wealth equivalence of the annuity income is less than $c\bar{a}^o(s)$, for all $s \geq t$, regardless of whether $q(s)$ is constrained (Lemma 3.2). The other similar phenomenon is if wealth lies below the safe level $\bar{w}(t, A)$, then future wealth will also be less than $\bar{w}(s, A(s))$, for all $s \geq t$ (Lemma 3.3). Moreover, if wealth lies between the buy boundary $(c - A)\bar{a}^o(t)$ and the safe level $\bar{w}(t, A)$, then the individual's wealth subject to the maximum allowable annuity rate $W^{\bar{q}}$ will eventually cross the buy boundary into the no-buy region. Otherwise, if the initial wealth w is greater than the safe level $\bar{w}(t, A)$, then $W^{\bar{q}}$ will never cross the buy boundary (Lemma 3.4), so ruin cannot occur.

We find it surprising that the individual does not purchase annuity income until her wealth reaches $(c - A)\bar{a}^o(t)$, the same level that triggers annuity buying when she is allowed to purchase annuity income via lump-sum payments. Before we analyzed the problem, we expected the individual to start buying annuity income at some wealth level *less than* $(c - A)\bar{a}^o(t)$ because of the constraint on how much she can spend on annuities during any time period. That said, we have seen this myopia in other problems related to goal-seeking problems. For example, Bayraktar and Young (2007) found the optimal investment strategy to minimize the probability of lifetime ruin under constant consumption and under a no-borrowing constraint on investment. Under the no-borrowing constraint, when the constraint

[4] The limiting result when $\bar{q}$ goes to infinity is given in Proposition 3.4, which is identical to the anything-anytime result in Milevsky et al. (2006) assuming no stochastic investment return.

[5] In the discrete-time case of Horneff et al. (2008a,b), individuals are allowed to purchase annuities with a certain part of their wealth several times during retirement.

did not bind (that is, at greater wealth levels), then the individual invested as if the constraint did not exist. Also, Bayraktar et al. (2016) and Bayraktar and Young (2016) determined the optimal investment strategy to maximize the probability of reaching a bequest goal with and without life insurance, respectively. In the wealth regions for which it is optimal not to buy life insurance (that is, at lower wealth levels), then the individual invested as if life insurance were not available. Finally, Angoshtari et al. (2015) found the optimal investment strategy to minimize the time spent in drawdown; they found that, when not in drawdown, individuals invest identically to those who seek to minimize the probability of drawdown. We conjecture that this myopia concerning constraints and opportunities is the rule, rather than the exception, in goal-seeking problems.

3. Minimum probability of lifetime ruin

In this section, we first prove three lemmas to analyze the behavior of the wealth process, then we prove a verification theorem that we use to find the optimal annuitization strategy.

Lemma 3.1. *$\bar{a}^o$ is a non-increasing function on $\mathbb{R}^+$ with*

$$\frac{d\bar{a}^o(t)}{dt} = -1 + (r + \lambda^o(t))\bar{a}^o(t). \tag{3.1}$$

Proof. By differentiating $\bar{a}^o(t)$ in (2.1) with respect to t, we obtain

$$\begin{aligned}\frac{d\bar{a}^o(t)}{dt} &= -1 + (r + \lambda^o(t)) \int_t^\infty e^{-\int_t^s (r+\lambda^o(u))du}\, ds \\ &\le -1 + (r + \lambda^o(t)) \int_t^\infty e^{-(r+\lambda^o(t))(s-t)}\, ds \\ &= 0,\end{aligned}$$

in which the inequality follows from the fact that $\lambda^o(t)$ is non-decreasing with respect to t. Also, (3.1) follows immediately from the first expression for $\frac{d\bar{a}^o(t)}{dt}$. □

Lemma 3.2. *Suppose $Q = \{q(s)\}_{t \le s < \tau_d}$ is an admissible strategy in $\mathcal{A}(t)$. Then, the corresponding wealth and annuity processes are such that $W(s) - (c - A(s))\bar{a}^o(s)$ decreases until time s_r defined by*

$$s_r = \inf\left\{ s \ge t : W(s) \ge \frac{c - A(s)}{r} \right\}, \tag{3.2}$$

with $\inf \emptyset = \infty$. Note that, after time s_r, the individual can avoid ruin by setting $q(s) = 0$ for all $s_r \le s < \tau_d$. Thus, if $(t, W(t), A(t)) = (t, w, A) \in \mathcal{D}$ satisfies the inequality

$$w < (c - A)\bar{a}^o(t),$$

then

$$W(s) < (c - A(s))\bar{a}^o(s),$$

for all $t \le s < s_r \wedge \tau_d$.

Proof. Define the function h by

$$h(s) = W(s) - (c - A(s))\bar{a}^o(s), \tag{3.3}$$

for all $t \le s < s_r$. By using (2.2) and (3.1), we compute h's derivative as follows:

$$\begin{aligned}h'(s) &= W'(s) + A'(s)\bar{a}^o(s) - (c - A(s))\frac{d\bar{a}^o(s)}{ds} \\ &= \big(rW(s) - c + A(s)\big) - \bar{a}^o(s)q(s) + q(s)\bar{a}^o(s) - (c - A(s))\frac{d\bar{a}^o(s)}{ds} \\ &= rW(s) - c + A(s) - (c - A(s))\big(-1 + (r + \lambda^o(s))\bar{a}^o(s)\big) \\ &= rW(s) - (c - A(s))(r + \lambda^o(s))\bar{a}^o(s) \\ &= rh(s) - (c - A(s))\lambda^o(s)\bar{a}^o(s) < rh(s),\end{aligned} \tag{3.4}$$

for all $t \le s < s_r$. Hence, if $h(s) < 0$, then $h'(s) < 0$.

We claim that $h(s) < 0$ for all $t \le s < s_r$. On the contrary, suppose $h(s) \ge 0$ for some $t \le s < s_r$, and let $\bar{s} = \inf\{t < s < s_r : h(s) \ge 0\}$. Because $h(t) < 0$, it follows from (3.4) that $h'(t) < 0$. Moreover, because h' is continuous and $h(t) < h(\bar{s})$, there exists a point $\hat{s} \in (t, \bar{s})$ with $h'(\hat{s}) \ge 0$. Again, from (3.4), we observe that, if $h'(s) \ge 0$, then $h(s) > 0$; therefore, we deduce $h(\hat{s}) > 0$, a contradiction to the definition of $\bar{s}$ and $\hat{s} < \bar{s}$. □

Remark 3.1. $\bar{a}^o(s)A(s)$ is the so-called *wealth equivalence* of the annuity income at time s because $\bar{a}^o(s)A(s)$ is the amount of wealth required to purchase annuity income of $A(s)$. Lemma 3.2 tells us that if wealth at time t plus wealth equivalence of the annuity income at that time, namely, $w + \bar{a}^o(t)A$, is less than $c\,\bar{a}^o(t)$, then all subsequent wealth plus wealth equivalence of the annuity income is less than $c\bar{a}^o(s)$, for $s \ge t$ regardless of the constraint on $q(s)$. Intuitively, because there are no risky investments—investments with possibly large returns—if the individual's existing wealth is less than the threshold $(c - A)\bar{a}^o(t)$, then future wealth of the individual remains below that threshold. Lemma 3.2 also tells us that if wealth plus wealth equivalence of the annuity income begins above $c\,\bar{a}^o(t)$, it crosses that dynamic boundary at most once. □

We also have the following lemma that further describes the trajectory of $(s, W(s), A(s))$ for $s \ge t$.

Lemma 3.3. *Suppose $Q = \{q(s)\}_{t \le s < \tau_d}$ is an admissible strategy in $\mathcal{A}(t)$ with $(t, W(t), A(t)) = (t, w, A) \in \mathcal{D}$. If $w < \bar{w}(t, A)$, in which*

$$\bar{w}(t, A) = \frac{c - A}{r} - \frac{\bar{q}}{r^2}\left(1 - e^{-\frac{r}{\bar{q}}(c-A)}\right) + \bar{q}\int_t^{t+\frac{c-A}{\bar{q}}} \bar{a}^o(u)e^{-r(u-t)}du\,, \tag{3.5}$$

then $W(s) < \bar{w}(s, A(s))$ for all $t \le s < \tau_d$. Moreover, $\bar{w}(t, A) \in \big((c - A)\bar{a}^o(t), (c - A)/r\big)$ for $0 \le A < c$.

Proof. Suppose $w < \bar{w}(t, A)$, and define the function j by

$$j(s) = \bar{w}(s, A(s)) - W(s), \tag{3.6}$$

for all $t \le s < \tau_d$. Use (2.2) to compute j's derivative:

$$\begin{aligned}
j'(s) &= \frac{\partial \bar{w}(s, A(s))}{\partial s} + \frac{\partial \bar{w}(s, A(s))}{\partial A}A'(s) - W'(s) \\
&= \bar{q}\left\{\bar{a}^o\left(s + \frac{c - A(s)}{\bar{q}}\right)e^{-\frac{r}{\bar{q}}(c-A(s))} - \bar{a}^o(s)\right\} + r\bar{q}\int_s^{s+\frac{c-A(s)}{\bar{q}}} \bar{a}^o(u)e^{-r(u-s)}du \\
&\quad + \frac{q(s)}{r}\left\{-1 + \left(1 - r\bar{a}^o\left(s + \frac{c - A(s)}{\bar{q}}\right)\right)e^{-\frac{r}{\bar{q}}(c-A(s))}\right\} \\
&\quad - \big(rW(s) - (c - A(s))\big) + \bar{a}^o(s)q(s) \\
&= \bar{q}\left\{\bar{a}^o\left(s + \frac{c - A(s)}{\bar{q}}\right)e^{-\frac{r}{\bar{q}}(c-A(s))} - \bar{a}^o(s)\right\} + r\bar{q}\int_s^{s+\frac{c-A(s)}{\bar{q}}} \bar{a}^o(u)e^{-r(u-s)}du \\
&\quad - \frac{q(s)}{r}\left\{1 - r\bar{a}^o(s) - \left(1 - r\bar{a}^o\left(s + \frac{c - A(s)}{\bar{q}}\right)\right)e^{-\frac{r}{\bar{q}}(c-A(s))}\right\} \\
&\quad - \big(rW(s) - (c - A(s))\big).
\end{aligned}$$

The coefficient of $q(s)$ is positive. Indeed, define $\zeta(t) = (1 - r\bar{a}^o(t))e^{-rt}$, then

$$\begin{aligned}
\zeta'(t) &= -r\frac{d\bar{a}^o(t)}{dt}\cdot e^{-rt} - (1 - r\bar{a}^o(t))re^{-rt} \\
&\propto -\frac{d\bar{a}^o(t)}{dt} - (1 - r\bar{a}^o(t)) \\
&= -\lambda^o(t)\bar{a}^o(t) < 0,
\end{aligned}$$

in which the last equality follows from (3.1) in Lemma 3.1. Thus, $\zeta(s) - \zeta\left(s + \frac{c-A(s)}{\bar{q}}\right) > 0$ for all $s > 0$; note that $e^{rs}\left(\zeta(s) - \zeta\left(s + \frac{c-A(s)}{\bar{q}}\right)\right)$ equals the coefficient of $q(s)$.

Also, $q(s) \le \bar{q}$, so we have

$$\begin{aligned}
j'(s) &\ge \bar{q}\left\{\bar{a}^o\left(s + \frac{c - A(s)}{\bar{q}}\right)e^{-\frac{r}{\bar{q}}(c-A(s))} - \bar{a}^o(s)\right\} + r\bar{q}\int_s^{s+\frac{c-A(s)}{\bar{q}}} \bar{a}^o(u)e^{-r(u-s)}du \\
&\quad - \frac{\bar{q}}{r}\left\{1 - r\bar{a}^o(s) - \left(1 - r\bar{a}^o\left(s + \frac{c - A(s)}{\bar{q}}\right)\right)e^{-\frac{r}{\bar{q}}(c-A(s))}\right\} \\
&\quad - \big(rW(s) - (c - A(s))\big)
\end{aligned}$$

$$= -\frac{\bar{q}}{r}\left(1 - e^{-\frac{r}{\bar{q}}(c - A(s))}\right) + r\bar{q}\int_s^{s+\frac{c-A(s)}{\bar{q}}} \bar{a}^o(u)e^{-r(u-s)}du + \big(c - A(s) - rW(s)\big)$$
$$= r\big(\bar{w}(s, A(s)) - W(s)\big) = rj(s).$$

Hence, if $j(s) > 0$, then $j'(s) > 0$.

We assert that $j(s) > 0$ for all $t \le s < \tau_d$. Indeed, suppose $j(s) > 0$, for all $t \le s < \tau_d$, is not true, and let $s_1 = \inf\{s > t : j(s) \le 0\}$. Because $j(t) > 0$ and $j'(s) = rj(s)$, it follows that $j'(t) > 0$. Moreover, because j' is continuous, there exists a point s_c, such that $t < s_c < s_1$, with $j'(s_c) \le 0$. Recall that $j(s) > 0$ implies $j'(s) > 0$; by the contrapositive, we deduce $j(s_c) \le 0$, a contradiction to the definition of s_1 and $s_c < s_1$.

To show $\bar{w}(t, A) \in \big((c - A)\bar{a}^o(t), (c - A)/r\big)$, we compute

$$\bar{w}(t, A) > (c - A)\bar{a}^o(t)$$
$$\iff \frac{c - A}{r} - \frac{\bar{q}}{r^2}\left(1 - e^{-\frac{r}{\bar{q}}(c-A)}\right) + \bar{q}\int_t^{t+\frac{c-A}{\bar{q}}} \bar{a}^o(u)e^{-r(u-t)}du > (c - A)\bar{a}^o(t)$$
$$\iff \frac{c - A}{r}\big(1 - r\bar{a}^o(t)\big) - \frac{\bar{q}}{r^2}\left(1 - e^{-\frac{r}{\bar{q}}(c-A)}\right) + \bar{q}\int_t^{t+\frac{c-A}{\bar{q}}} \bar{a}^o(u)e^{-r(u-t)}du > 0$$
$$\iff \frac{1 - r\bar{a}^o(t)}{r}\cdot\frac{c - A}{\bar{q}} - \frac{1}{r^2}\left(1 - e^{-\frac{r}{\bar{q}}(c-A)}\right) + \int_t^{t+\frac{c-A}{\bar{q}}} \bar{a}^o(u)e^{-r(u-t)}du > 0. \tag{3.7}$$

Define a function β on $\mathbb{R}_+$ by

$$\beta(x) = \frac{1 - \bar{a}^o(t)r}{r}x - \frac{1}{r^2}\left(1 - e^{-rx}\right) + \int_t^{t+x} \bar{a}^o(u)e^{-r(u-t)}du.$$

We compute $\beta(0) = 0$,

$$\beta'(x) = \frac{1 - r\bar{a}^o(t)}{r} - \frac{1}{r}e^{-rx} + \bar{a}^o(t + x)e^{-rx} = \frac{1}{r}\big(1 - r\bar{a}^o(t)\big) - \frac{1}{r}e^{-rx}\big(1 - r\bar{a}^o(t + x)\big),$$

$\beta'(0) = 0$, and

$$\beta''(x) = e^{-rx}\big(1 - r\bar{a}^o(t + x)\big) + e^{-rx}\frac{d\bar{a}^o(t + x)}{dx} = e^{-rt}\left(1 - r\bar{a}^o(t + x) + \frac{d\bar{a}^o(t + x)}{dx}\right) > 0,$$

in which we used (3.1) in Lemma 3.1 to deduce $\beta''(x) > 0$ for $x \ge 0$. Thus, β is an increasing, convex function on $\mathbb{R}_+$, which implies (3.7) holds for $0 \le A < c$.

Moreover,

$$\bar{w}(t, A) < \frac{c - A}{r}$$
$$\iff \frac{c - A}{r} - \frac{\bar{q}}{r^2}\left(1 - e^{-\frac{r}{\bar{q}}(c-A)}\right) + \bar{q}\int_t^{t+\frac{c-A}{\bar{q}}} \bar{a}^o(u)e^{-r(u-t)}du < \frac{c - A}{r}$$
$$\iff -\frac{1}{r^2}\left(1 - e^{-\frac{r}{\bar{q}}(c-A)}\right) + \int_t^{t+\frac{c-A}{\bar{q}}} \bar{a}^o(u)e^{-r(u-t)}du < 0.$$

Because $\bar{a}^o(t)$ is non-increasing with respect to t, we have

$$\int_t^{t+\frac{c-A}{\bar{q}}} \bar{a}^o(u)e^{-r(u-t)}du \le \int_t^{t+\frac{c-A}{\bar{q}}} \bar{a}^o(t)e^{-r(u-t)}du$$
$$= r\bar{a}^o(t)\cdot\frac{1}{r^2}\left(1 - e^{-\frac{r}{\bar{q}}(c-A)}\right);$$

thus, $\bar{w}(t, A) < \frac{c-A}{r}$ holds if the following stronger inequality is true:

$$- \frac{1}{r^2}\left(1 - e^{-\frac{r}{q}(c-A)}\right)\left(1 - r\bar{a}^o(t)\right) < 0,$$

which holds because $\frac{r}{q}(c - A) > 0$. □

Theorem 3.1. *Let $f = f(t, w, A)$ be a non-negative function on $\mathcal{D}$, in which*

$$\mathcal{D} = \left\{(t, w, A) \in \mathbb{R}^3_+ : 0 \le w < \frac{c-A}{r},\ 0 \le A \le c\right\},$$

that satisfies the following properties:

1. *f is continuous on $\mathcal{D}$, except possibly along the surfaces*

$$L = \left\{(t, w, A) \in \mathcal{D} : w = (c - A)\bar{a}^o(t)\right\}$$

and

$$C = \left\{(t, w, A) \in \mathcal{D} : w = \bar{w}(t, A)\right\}.$$

2. *f is continuously differentiable with respect to t on the domain $\mathcal{D}$ and is continuously differentiable with respect to w and A in the interior of $\mathcal{D}$, except possibly on L and C. On L and C, f has right- and left-partial derivatives with w and A.*
3. *f satisfies the following Hamilton-Jacobi (HJ) equation in the interior of $\mathcal{D} \setminus (L \cup C)$:*

$$\min_{q\in[0,\bar{q}]} \left\{f_t + (rw - c + A)f_w - \lambda(t)f + (f_A - \bar{a}^o(t)f_w)q\right\} = 0, \tag{3.8}$$

with boundary conditions

$$f(t, 0, A) = 1, \qquad \text{if } 0 \le A < c, \tag{3.9}$$

and

$$f\left(t, \frac{c-A}{r}, A\right) = 0, \qquad \text{if } 0 \le A \le c. \tag{3.10}$$

Then,

$$f \le \psi,$$

on $\mathcal{D}$. □

Proof. Suppose $\{q(s)\}_{s\ge t} \in \mathcal{A}(t)$ is an admissible strategy for purchasing life annuities for some $t \ge 0$, and suppose f is a function that satisfies the conditions of this theorem. Then, associated with this strategy, define $\tau_b = \inf\left\{s \ge t : W(s) < (c - A(s))\bar{a}^o(s)\right\}$ and $\tau = \inf\left\{s \ge t : W(s) < \bar{w}(s, A(s))\right\}$, either of which might equal infinity. By choosing a constant n sufficiently large, we can write

$$\begin{aligned}
&e^{-\int_t^{\tau_0\wedge n}\lambda(u)du} f\left(\tau_0 \wedge n, W(\tau_0 \wedge n), A(\tau_0 \wedge n)\right) \\
&= f(t, w, A) - \int_t^{\tau_0\wedge n} \lambda(s)e^{-\int_t^s \lambda(u)du} f\left(s, W(s), A(s)\right) ds \\
&\quad + \int_t^{\tau_0\wedge n} e^{-\int_t^s \lambda(u)du} \left\{\left(rW(s) - c + A(s)\right) f_w\left(s, W(s), A(s)\right) + f_t\left(s, W(s), A(s)\right)\right\} dt \\
&\quad + \int_t^{\tau_0\wedge n} e^{-\int_t^s \lambda(u)du} \left(f_A\left(s, W(s), A(s)\right) - \bar{a}^o(s) f_w\left(s, W(s), A(s)\right)\right) q(s)\, ds,
\end{aligned} \tag{3.11}$$

in which we interpret the integrals as being split into two or three integrals if $\tau < \tau_b < \tau_0 \wedge n$ or if $\tau_b < \tau_0 \wedge n$. From Lemmas 3.2 and 3.3, we know that the processes $\left\{W(s) - (c - A(s))\bar{a}^o(s)\right\}$ and $\left\{W(s) - \bar{w}(s, A(s))\right\}$ cross 0 at most once and only from above; therefore, (1) $\tau = \tau_b = \infty$ if $w < (c - A)\bar{a}^o(t)$, (2) $\tau_b \le \tau = \infty$ if $(c - A)\bar{a}^o(t) \le w < \bar{w}(t, A)$, and (3) $\tau < \tau_b$ if $w \ge \bar{w}(t, A)$. Note that the projection of surface C onto (w, A)-space lies to the northeast of the projection of surface L; see Fig. 1 for an illustration.

By using the HJ equation in (3.8), we obtain

$$e^{-\int_t^{\tau_0\wedge n}\lambda(u)du} f\left(\tau_0 \wedge n, W(\tau_0 \wedge n), A(\tau_0 \wedge n)\right) \ge f(t, w, A). \tag{3.12}$$

By letting n go to infinity in (3.12) and noting that $f\left(\tau_0, W(\tau_0), A(\tau_0)\right) = 1$, we obtain

$$e^{-\int_t^{\tau_0}\lambda(u)du} \ge f(t, w, A).$$

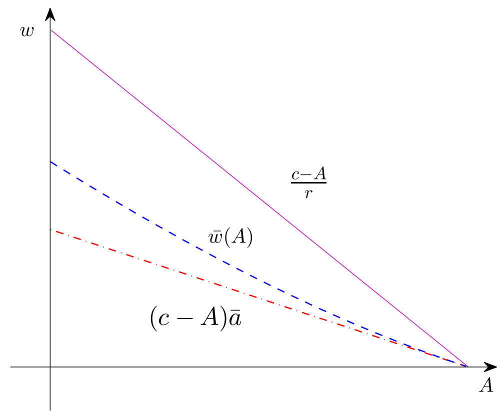

Fig. 1. Projections of surfaces L and C in (w, A)-space at a fixed value of time, along with the line $w = \frac{c-A}{r}$.

By taking the infimum of the left side over admissible strategies and by referring to the expression for ψ in (2.3), we obtain

$$\psi \geq f$$

on $\mathcal{D}$. $\square$

We state a corollary that follows immediately from Theorem 3.1.

Corollary 3.1. *Suppose a function f satisfies the conditions of Theorem 3.1. Moreover, if f equals the probability of lifetime ruin associated with an admissible strategy for purchasing life annuity income, then the minimum probability of lifetime ruin ψ equals f on $\mathcal{D}$.* $\square$

In the remainder of this section, we construct a function ϕ that satisfies the conditions of Theorem 3.1 and that equals the probability of lifetime ruin associated with an admissible strategy for purchasing life annuity income; then, Corollary 3.1 allows us to deduce that ϕ equals the minimum probability of lifetime ruin.

Recall that when an individual can invest in a Black-Scholes financial market, Milevsky et al. (2006) show that the individual will not buy additional annuity income until her wealth is large enough to cover all her consumption. One can show that the same result holds when an individual can invest only in a riskless asset, as in this paper, but is allowed to purchase annuity income with a lump-sum payment. Specifically, if $w < (c-A)\bar{a}^o(t)$, then it is optimal for the individual not to buy any life annuity income. However, if $w \geq (c-A)\bar{a}^o(t)$, then it is optimal for her to spend the lump-sum amount $(c-A)\bar{a}^o(t)$ to buy an annuity that will pay at the continuous rate $c - A$ for the rest of her life. This additional annuity income, together with the prior annuity income of A, will cover her desired consumption rate c. Based on this finding, we propose the following hypothesis for purchasing life annuity income at an absolutely continuous rate. Recall that wealth and annuity income at time t equal $(W(t), A(t)) = (w, A) \in \mathcal{D}$.

Hypothesis 3.1.

1. If $w < (c-A)\bar{a}^o(t)$, then it is optimal for the individual not to purchase any additional annuity income. We call the surface $w = (c-A)\bar{a}^o(t)$ the *buy boundary*, and we call the region in $\mathcal{D}$ such that $w < (c-A)\bar{a}^o(t)$ the *no-buy region*.
2. If $w \geq (c-A)\bar{a}^o(t)$, then it is optimal for the individual to purchase annuity income at the maximum allowable rate $\bar{q}$, unless $W(s)$ equals $(c - A(s))/r$, after which time the individual will not purchase additional annuity income and will not ruin. We call the region in $\mathcal{D}$ such that $w \geq (c-A)\bar{a}^o(t)$ the *buy region*. $\square$

Let ϕ denote the probability of lifetime ruin under the strategy given in Hypothesis 3.1. Via a verification theorem similar to Theorem 3.1, if we find a suitably continuous and differentiable solution of

$$f_t + (rw - c + A)f_w - \lambda(t)f + (f_A - \bar{a}^o(t)f_w)\bar{q}\,\mathbb{1}_{\{w \geq (c-A)\bar{a}^o(t)\}} = 0 \tag{3.13}$$

in the interior of $\mathcal{D}$ that satisfies the boundary conditions in Item 3 of Theorem 3.1, then that solution equals ϕ.

Lemma 3.2 shows that if wealth $W(t)$ at time t lies below the surface $w = (c-A)\bar{a}^o(t)$, then it remains below that surface thereafter. Thus, for $w \in \big(0, (c-A)\bar{a}^o(t)\big)$, under Item 1 of Hypothesis 3.1, ϕ solves the equation

$$\phi_t + (rw - c + A)\phi_w - \lambda(t)\phi = 0, \tag{3.14}$$

with $\phi(t, 0, A) = 1$ if $0 \leq A < c$. By solving this boundary-value problem, we get

$$\phi(t, w, A) = e^{-\int_t^{s_0} \lambda(s)ds}, \tag{3.15}$$

in which

$$s_0 = t - \frac{1}{r}\ln\left(1 - \frac{rw}{c-A}\right) \tag{3.16}$$

equals the time when wealth reaches 0 under the strategy in Hypothesis 3.1, assuming $w \in \big(0, (c-A)\bar{a}^o(t)\big)$.

Before computing ϕ for $w \ge (c-A)\bar{a}^o(t)$, we first analyze the behavior of $(s, W(s), A(s))$ under Item 2 of Hypothesis 3.1. Let $\big(W^{\bar{q}}(s), A^{\bar{q}}(s)\big)$ denote the wealth and annuity income processes when $q(s) \equiv \bar{q}$ for all $s \ge t$. Clearly,

$$A^{\bar{q}}(s) = A + \bar{q}(s-t), \tag{3.17}$$

from which it follows

$$W^{\bar{q}}(s) = e^{r(s-t)}\left(w - \frac{c-A}{r} + \frac{\bar{q}}{r^2} - \bar{q}\int_t^s \bar{a}^o(u)e^{-r(u-t)}du\right) + \frac{c-A}{r} - \frac{\bar{q}}{r}(s-t) - \frac{\bar{q}}{r^2}. \tag{3.18}$$

We present the following lemma that describes the trajectory of $W^{\bar{q}}$. Let s_b denote the minimum time when $W^{\bar{q}}$ (for $w \ge (c-A)\bar{a}^o(t)$) crosses the buy boundary, that is,

$$s_b = \inf\left\{s \ge t : W^{\bar{q}}(s) < \big(c - A^{\bar{q}}(s)\big)\bar{a}^o(s)\right\}, \tag{3.19}$$

in which we use the subscript b for **buy**.

Lemma 3.4. *Let $\bar{w}(t, A)$ denote the function of A defined in (3.5), and assume that $0 \le A < c$. If $(c-A)\bar{a}^o(t) \le w < \bar{w}(t, A)$, then $W^{\bar{q}}$ in (3.18) eventually crosses the buy boundary into the no-buy region. Moreover, s_b increases with w, and*

$$\lim_{w \to \bar{w}(t,A)^-} s_b = t_e, \tag{3.20}$$

in which $t_e = t + (c-A)/\bar{q}$. Otherwise, if $w \ge \bar{w}(t, A)$, then $W^{\bar{q}}$ never crosses the buy boundary and $s_b = \infty$.

Proof. For $t \le s < \tau_d$, define the function g by

$$\begin{aligned} g(s) &= W^{\bar{q}}(s) - \big(c - A^{\bar{q}}(s)\big)\bar{a}^o(s) \\ &= e^{r(s-t)}\left(w - \frac{c-A}{r} + \frac{\bar{q}}{r^2} - \bar{q}\int_t^s \bar{a}^o(u)e^{-r(u-t)}du\right) + \frac{c-A^{\bar{q}}(s)}{r}\big(1 - r\bar{a}^o(s)\big) - \frac{\bar{q}}{r^2}, \end{aligned} \tag{3.21}$$

that is, g measures the (signed) distance from $W^{\bar{q}}$ to the buy boundary. Equivalently, g equals the function h in (3.3) when $q(s) \equiv \bar{q}$ for all $t \le s < \tau_d$.

If g becomes negative, then $W^{\bar{q}}$ has crossed the buy boundary into the no-buy region, which it will never leave; see Lemma 3.2. Moreover, by direct calculation, we obtain

$$g(t_e) = e^{\frac{r}{\bar{q}}(c-A)}\big(w - \bar{w}(t, A)\big),$$

which implies

$$g(t_e) < 0 \iff w < \bar{w}(t, A). \tag{3.22}$$

Therefore, if $(c-A)\bar{a}^o(t) \le w < \bar{w}(t, A)$, then s_b is finite and less than t_e. In words, we have shown that $W^{\bar{q}}$ in (3.18) eventually crosses the buy boundary into the no-buy region.

Furthermore, if $w < \bar{w}(t, A)$, then from the definitions of s_b and g in (3.19) and (3.21), respectively, we see that $g(s_b) = 0$. By fully differentiating this equation with respect to w (and by using the second expression for g in (3.21)), we obtain

$$g'(s_b)\frac{\partial s_b}{\partial w} + e^{r(s_b - t)} = 0. \tag{3.23}$$

Moreover, by further careful calculation and using (3.1), one can show that

$$\begin{aligned} g'(s_b) &= -\big(c - A^{\bar{q}}(s_b)\big)\left(\big(1 - r\bar{a}^o(s_b)\big) + \frac{d\bar{a}^o(s)}{ds}\bigg|_{s=s_b}\right) \\ &= -\big(c - A^{\bar{q}}(s_b)\big)\lambda^o(s_b)\bar{a}^o(s_b) < 0. \end{aligned} \tag{3.24}$$

$g'(s_b) < 0$ and (3.23) imply $\frac{\partial s_b}{\partial w}$ is strictly positive, that is, s_b increases with respect to w.

To show the limit in (3.20), note that (because $s_b < t_e$ when $(c-A)\bar{a}^o(t) \le w < \bar{w}(t, A)$)

$$\tilde{s} := \lim_{w \to \bar{w}(t,A)^-} s_b \le t_e.$$

If the above inequality is strict, that is, if $\tilde{s} < t_e$, then (as shown in Lemma 3.3) once wealth drops below the buy boundary, the future wealth of the individual will always lie below the buy boundary, which implies $\lim_{w \to \bar{w}(t,A)^-} g(t_e) < 0$. However, from (3.22), we observe that $\lim_{w \to \bar{w}(t,A)^-} g(t_e) = 0$, a contradiction. Hence, $\tilde{s} = t_e$, and we have proved (3.20).

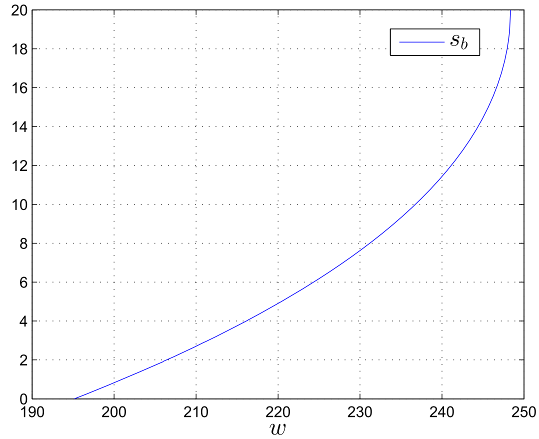

Fig. 2. s_b as a function of w.

We claim that if $w \geq \bar{w}(t, A)$, then $W^{\bar{q}}$ never crosses the buy boundary. Otherwise, there must exist a point $\hat{s} > t$, such that $g(\hat{s}) < 0$, which is equivalent to

$$w < \frac{c-A}{r} - \frac{\bar{q}}{r^2} + \bar{q} \int_t^{\hat{s}} \bar{a}^o(u) e^{-r(u-t)} du - e^{-r(\hat{s}-t)} \left\{ \frac{c - A^{\bar{q}}(\hat{s})}{r} \left(1 - r\bar{a}^o(\hat{s})\right) - \frac{\bar{q}}{r^2} \right\}. \tag{3.25}$$

Define a function ℓ via the right side of (3.25), that is, for $s \geq t$,

$$\ell(s) = \frac{c-A}{r} - \frac{\bar{q}}{r^2} + \bar{q} \int_t^{s} \bar{a}^o(u) e^{-r(u-t)} du - e^{-r(s-t)} \left\{ \frac{c - A^{\bar{q}}(s)}{r} \left(1 - r\bar{a}^o(s)\right) - \frac{\bar{q}}{r^2} \right\},$$

then, by using (3.1), we compute

$$\begin{aligned} \ell'(s) &= e^{-r(s-t)} \left(c - A^{\bar{q}}(s)\right) \left(\left(1 - r\bar{a}^o(s)\right) + \frac{d\bar{a}^o(s)}{ds} \right) \\ &= e^{-r(s-t)} \left(c - A^{\bar{q}}(s)\right) \lambda^o(s) \bar{a}^o(s). \end{aligned}$$

Thus, $\ell'(s) > 0$ for all $t < s < t_e$, and $\ell'(s) \leq 0$ for all $s \geq t_e$. Hence, $\ell(t_e) = \bar{w}(t, A)$ is a global maximum, and for all $\hat{s} \geq t$, we have $\ell(\hat{s}) \leq \bar{w}(t, A)$. Inequality (3.25), then, implies $w < \bar{w}(t, A)$, which contradicts our original assumption $w \geq \bar{w}(t, A)$. Therefore, if $w \geq \bar{w}(t, A)$, then $s_b = \infty$. □

In the following proposition, we use Lemma 3.4 to show that, if $w \geq \bar{w}(t, A)$, then the probability of lifetime ruin under Item 2 of Hypothesis 3.1 is zero. We, therefore, call $\bar{w}(t, A)$ the *safe level.*

Proposition 3.1. *If $\bar{w}(t, A) \leq w \leq (c - A)/r$ and $0 \leq A \leq c$, then the probability of lifetime ruin ϕ equals zero.*

Proof. Lemma 3.4 shows that, if $w \geq \bar{w}(t, A)$, then the wealth process $W^{\bar{q}}$ never drops below the buy boundary into the no-buy region; thus, $s_b = \infty$. In this case, wealth $W^{\bar{q}}(s)$ reaches $(c - A^{\bar{q}}(s))/r$ (at time s_r) before time $s_b = \infty$, after which the individual does not purchase additional annuity income (according to Item 2 of Hypothesis 3.1) and does not ruin. □

Remark 3.2. Recall that s_r, defined in (3.2), is the time wealth and annuity income reach $W(s) = (c - A(s))/r$, and $t_e = t + (c - A)/\bar{q}$ is the time annuity income reaches c when annuities are purchased at the maximum possible rate. To see the relationship between s_r and t_e under the strategy given in Hypothesis 3.1, note that if wealth lies between the buy boundary $(c - A)\bar{a}^o(t)$ and the safe level $\bar{w}(t, A)$, and if the individual purchases annuity income at rate $\bar{q}$, then wealth decreases to the buy boundary, s_r is infinite, and annuity income never reaches c. However, if wealth is greater than the safe level $\bar{w}(t, A)$, then at time t_e annuity income equals c, and s_r occurs before time t_e. Also, Lemma 3.4 and Proposition 3.1 imply that once wealth w is greater than the safe level $\bar{w}(t, A)$, then subsequent wealth will never down cross the safe level. □

Example 3.1. Suppose the individual's force of mortality and the objective force of mortality used to price annuities are the same, both of which follow Makeham's law, that is, $\lambda(t) = \lambda^o(t) = a + Be^{bt}$, in which $a = 0.03$, $B = 0.001$, and $b = 0.01$. Moreover, we set $t = 0$, $r = 0.02$, $\lambda^o = 0.04$, $c = 10$, $\bar{q} = 0.5$, and $A(0) = 0$. It follows that the per-dollar price of the immediate life annuity at time 0 equals $\bar{a}^o(0) = 19.52$, and

$$\bar{a}^o(t) = \int_0^\infty e^{-0.05u - 0.1e^{0.01t}(e^{0.01u-1})} du.$$

At time 0, the buy boundary $(c - A(0))\bar{a}^o(0) = 195.16$, and the safe level $\bar{w}(0, A(0)) = 248.37$. In Fig. 2, we plot the graph of s_b as a function of $w \in [195.16, 248.37]$. As expected from Lemma 3.4, we see that s_b increases from 0 to $t_e = 20$ as w increases from the buy boundary to the safe level. □

Based on our analysis of $W^{\bar{q}}$, we present the following proposition that gives a (semi-) explicit expression for ϕ in $\mathcal{D}$.

Proposition 3.2. *The probability of lifetime ruin associated with the annuitization strategy in Hypothesis 3.1 equals*

$$\phi(t, w, A) = \begin{cases} e^{-\int_t^{s_0} \lambda(s)ds}, & 0 \le w < (c - A)\bar{a}^o(t), \\ e^{-\int_t^{\varpi_b} \lambda(s)ds}, & (c - A)\bar{a}^o(t) \le w < \bar{w}(t, A), \\ 0, & \bar{w}(t, A) \le w \le \dfrac{c - A}{r}, \end{cases} \tag{3.26}$$

in which $\varpi_b = s_b - \frac{1}{r}\ln\big(1 - r\bar{a}^o(s_b)\big)$, and s_0 and s_b are defined in (3.16) and (3.19), respectively. Moreover, ϕ is continuous and continuously differentiable in the interior of $\mathcal{D}\backslash C$.

Proof. Note that, if $w = (c - A)\bar{a}^o(t)$ and $0 \le A < c$, then $s_b = 0$ because $W^{\bar{q}}$ will immediately cross the buy boundary into the no-buy region. Thus, ϕ is continuous across the buy boundary in the interior of $\mathcal{D}$. However, as $w \to \bar{w}(t, A)^-$, the hitting time s_b approaches t_e; thus, ϕ is not continuous across $w = \bar{w}(t, A)$.

By construction, ϕ in (3.26) satisfies the differential equation in (3.13) when $0 < w < (c - A)\bar{a}^o(t)$.

We wish to show that ϕ satisfies the differential equation in (3.13) when $(c - A)\bar{a}^o(t) \le w < \bar{w}(t, A)$. To that end, when $(c - A)\bar{a}^o(t) \le w < \bar{w}(t, A)$, s_b is finite and solves the equation $g(s_b) = 0$, or equivalently,

$$e^{r(s_b - t)}\left(w - \frac{c - A}{r} + \frac{\bar{q}}{r^2} - \bar{q}\int_t^{s_b} \bar{a}^o(u)e^{-r(u-t)}du\right) + \frac{c - A^{\bar{q}}(s_b)}{r}\big(1 - r\bar{a}^o(s_b)\big) - \frac{\bar{q}}{r^2} = 0. \tag{3.27}$$

By differentiating $g(s_b) = 0$ with respect to t (refer to (3.27) and recall that $A^{\bar{q}}(s)$ depends on t), we obtain

$$g'(s_b)\frac{\partial s_b}{\partial t} - re^{r(s_b - t)}\left(w - \frac{c - A}{r} + \frac{\bar{q}}{r^2}\big(1 - r\bar{a}^o(t)\big)\right) + \frac{\bar{q}}{r}\big(1 - r\bar{a}^o(s_b)\big) = 0.$$

Again, differentiate (3.27) with respect to w and A, respectively, to obtain

$$g'(s_b)\,\frac{\partial s_b}{\partial w} + e^{r(s_b - t)} = 0,$$

as in (3.23), and

$$g'(s_b)\,\frac{\partial s_b}{\partial A} + \frac{1}{r}\,e^{r(s_b - t)} - \frac{1}{r}\big(1 - r\bar{a}^o(s_b)\big) = 0.$$

By differentiating ϕ in (3.26), we get

$$\begin{aligned} \phi_t &= \lambda(t)\phi - \lambda(\varpi_b)\phi\,\frac{\partial \varpi_b}{\partial t} = \lambda(t)\phi - \lambda(\varpi_b)\phi\,\frac{\partial s_b}{\partial t}\left(1 + \frac{1}{1 - r\bar{a}^o(s_b)}\frac{d\bar{a}^o(s)}{ds}\bigg|_{s=s_b}\right) \\ &= \lambda(t)\phi - \lambda(\varpi_b)\phi\,\frac{\partial s_b}{\partial t}\cdot\frac{\lambda^o(s_b)\bar{a}^o(s_b)}{1 - r\bar{a}^o(s_b)}, \end{aligned} \tag{3.28}$$

$$\phi_w = -\lambda(\varpi_b)\phi\,\frac{\partial s_b}{\partial w}\cdot\frac{\lambda^o(s_b)\bar{a}^o(s_b)}{1 - r\bar{a}^o(s_b)}, \tag{3.29}$$

and

$$\phi_A = -\lambda(\varpi_b)\phi\,\frac{\partial s_b}{\partial A}\cdot\frac{\lambda^o(s_b)\bar{a}^o(s_b)}{1 - r\bar{a}^o(s_b)}. \tag{3.30}$$

By substituting ϕ in (3.26) into the left side of (3.13), and by substituting the derivatives of ϕ in (3.28)-(3.30), we obtain

$$\begin{aligned} &\phi_t + (rw - c + A)\phi_w - \lambda(t)\phi + (\phi_A - \bar{a}^o(t)\phi_w)\bar{q} \\ &= -\lambda(\varpi_b)\phi\,\frac{\lambda^o(s_b)\bar{a}^o(s_b)}{1 - r\bar{a}^o(s_b)}\left(\frac{\partial s_b}{\partial t} + \big(rw - c + A - \bar{q}\,\bar{a}^o(t)\big)\frac{\partial s_b}{\partial w} + \bar{q}\,\frac{\partial s_b}{\partial A}\right) \\ &= 0. \end{aligned}$$

Thus, we have shown that ϕ satisfies the differential equation in (3.13) when $(c - A)\bar{a}^o(t) \le w < \bar{w}(t, A)$.

When $\bar{w}(t, A) \le w \le \frac{c-A}{r}$, clearly $\phi = 0$ satisfies the differential equation in (3.13) and satisfies the boundary condition in (3.10). Finally, it is straightforward to show (by using some of the above computations) that

$$\lim_{w \to (c-A)\bar{a}^o(t)^-} \phi_t(t, w, A) = \lim_{w \to (c-A)\bar{a}^o(t)^+} \phi_t(t, w, A) = (\lambda(t) - \lambda(\bar{s}_0)) e^{-\int_t^{\bar{s}_0} \lambda(u) du},$$

$$\lim_{w \to (c-A)\bar{a}^o(t)^-} \phi_w(t, w, A) = \lim_{w \to (c-A)\bar{a}^o(t)^+} \phi_w(t, w, A) = -\frac{\lambda(\bar{s}_0)}{(c-A)(1 - r\bar{a}^o(t))} e^{-\int_t^{\bar{s}_0} \lambda(u) du},$$

and

$$\lim_{w \to (c-A)\bar{a}^o(t)^-} \phi_A(t, w, A) = \lim_{w \to (c-A)\bar{a}^o(t)^+} \phi_A(t, w, A) = -\frac{\lambda(\bar{s}_0)\bar{a}^o(t)}{(c-A)(1 - r\bar{a}^o(t))} e^{-\int_t^{\bar{s}_0} \lambda(u) du},$$

in which $\bar{s}_0 = t - \frac{1}{r} \ln\big(1 - r\bar{a}^o(t)\big)$. Thus, ϕ is continuous and continuously differentiable with respect to t, w, and A in the interior of $\mathcal{D} \backslash C$, and this proposition follows from a verification theorem similar to Theorem 3.1. □

In the following theorem, we show that ϕ equals the *minimum* probability of lifetime ruin ψ.

Theorem 3.2. *The minimum probability of lifetime ruin ψ in (2.3) equals ϕ in (3.26). Moreover, the corresponding rate of annuitization is given in feedback form by*

$$q^*(s) = \begin{cases} \bar{q}, & (c - A^*(s))\bar{a}^o(s) \le W^*(s) < \dfrac{c - A^*(s)}{r}, \\ 0, & \text{otherwise}, \end{cases} \tag{3.31}$$

in which $(W^*(s), A^*(s))_{s \ge t}$ denotes optimally controlled wealth and annuity income after time t.

Proof. From Propositions 3.1 and 3.2, to prove this theorem, all that remains is to show

$$\arg\min_{q \in [0, \bar{q}]} (\phi_A - \bar{a}^o(t)\phi_w) q = \begin{cases} 0, & 0 < w < (c-A)\bar{a}^o(t), \\ \bar{q}, & (c-A)\bar{a}^o(t) \le w < \bar{w}(t, A), \end{cases} \tag{3.32}$$

or equivalently,

$$\phi_A - \bar{a}^o(t)\phi_w > 0 \iff w < (c-A)\bar{a}^o(t). \tag{3.33}$$

If $w < (c-A)\bar{a}^o(t)$, then

$$\phi_w = -\frac{\lambda(s_0)}{c-A} \left(1 - \frac{rw}{c-A}\right)^{-1} \phi,$$

and

$$\phi_A = -\frac{\lambda(s_0) w}{(c-A)^2} \left(1 - \frac{rw}{c-A}\right)^{-1} \phi,$$

which implies

$$\phi_A - \bar{a}^o(t)\phi_w = \big((c-A)\bar{a}^o(t) - w\big) \frac{\lambda(s_0)}{(c-A)^2} \left(1 - \frac{rw}{c-A}\right)^{-1} \phi > 0.$$

If $(c-A)\bar{a}^o(t) \le w < \bar{w}(t, A)$, then from (3.29) and (3.30),

$$\begin{aligned} \phi_A - \bar{a}^o(t)\phi_w &= \lambda(\varpi_b)\phi \left(\bar{a}^o(t) \frac{\partial s_b}{\partial w} - \frac{\partial s_b}{\partial A}\right) \frac{\lambda^o(s_b)\bar{a}^o(s_b)}{1 - r\bar{a}^o(s_b)} \\ &= -\frac{\lambda(\varpi_b)\phi}{g'(s_b)} \cdot \frac{\lambda^o(s_b)\bar{a}^o(s_b)}{1 - r\bar{a}^o(s_b)} \cdot \frac{1}{r} \Big(\big(1 - r\bar{a}^o(s_b)\big) - \big(1 - r\bar{a}^o(t)\big) e^{r(s_b - t)}\Big) \le 0, \end{aligned}$$

in which the inequality follows from $g'(s_b) < 0$ in (3.24) and from

$$\big(1 - r\bar{a}^o(s_b)\big) e^{-rs_b} \le \big(1 - r\bar{a}^o(t)\big) e^{-rt}. \tag{3.34}$$

Indeed, to prove (3.34), define a function ϱ on $\mathbb{R}_+$ by $\varrho(t) = \big(1 - r\bar{a}^o(t)\big)e^{-rt}$, then

$$\varrho'(t) = -re^{-rt} \left(1 - r\bar{a}^o(t) + \frac{d\bar{a}^o(t)}{dt}\right) = -re^{-rt} \lambda^o(t)\bar{a}^o(t) < 0.$$

Thus, $\varrho(t)$ decreases with respect to t, and (3.34) holds because $s_b \ge t$.

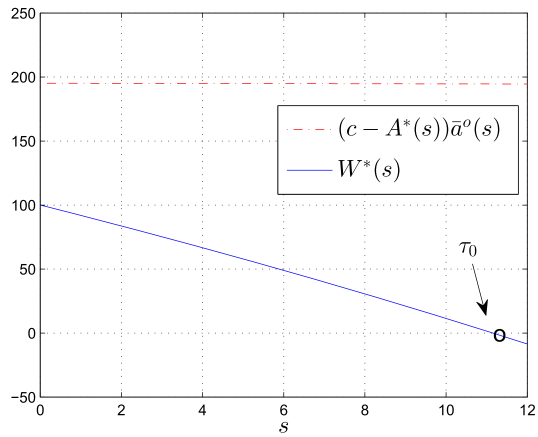

Fig. 3. $W^*(s)$ as a function of time when $w = 100$.

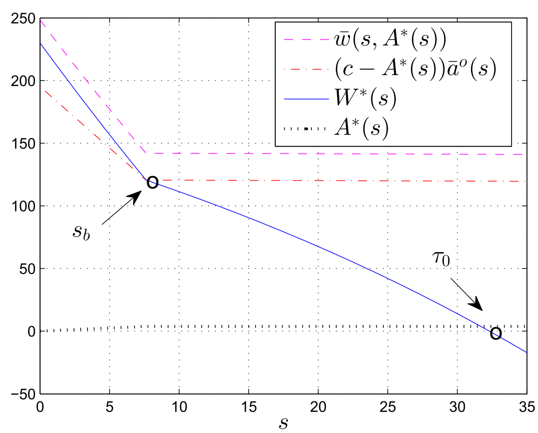

Fig. 4. $W^*(s)$ as a function of time when $w = 230$.

Therefore, we have proved (3.33), and from Theorem 3.1 and Corollary 3.1, we deduce $\psi = \phi$ on $\mathcal{D}$. □

Note that the annuitization strategy when wealth lies above the safe level is not unique. As long as the individual keeps her wealth above the safe level (and there are uncountably many admissible annuitization strategies that will do so), then she will not ruin. However, the minimum probability of lifetime ruin itself is unique, due to the verification theorem and its corollary, namely, Theorem 3.1 and Corollary 3.1, respectively.

Example 3.2. For this example, we use the same parameters as in Example 3.1. In Figs. 3–5, we graph optimal wealth $W^*(s)$ as a function of s when wealth at time $t = 0$ lies in one of three intervals: $0 \le w < (c - A(0))\bar{a}^o(0) = 195.16$, $(c - A(0))\bar{a}^o(0) \le w < \bar{w}(0, A(0)) = 248.37$, and $\bar{w}(0, A(0)) \le w \le (c - A(0))/r = 500$, respectively. In the scenario reflected in Fig. 3, it is optimal for the individual not to purchase any additional annuity income because $w = 100$ lies below the buy boundary. In this case, ruin occurs at $\tau_0 = 11.1572$, and the probability of lifetime ruin equals 0.7071.

In Fig. 4, initial wealth $w = 230$ lies between $(c - A)\bar{a}^o(0) = 195.16$ and $\bar{w}(0, A(0)) = 248.37$, so it is optimal for the individual to first purchase annuity income at the maximum allowable rate $\bar{q}$ till the wealth reaches the buy boundary at time $s_b = 7.6221$, after which the individual does not purchase additional annuity income. Ruin occurs at $\tau_0 = 32.3034$, and the probability of lifetime ruin equals 0.3652.

In Fig. 5, initial wealth $w = 280$ is greater than the safe level, and we see that optimally controlled wealth $W^*(s)$ never drops below $\bar{w}(s, A^*(s))$, as we expect from Lemma 3.4. Hence, lifetime ruin will not occur. In this case, it is optimal for the individual to purchase annuity income at the maximum allowable rate $\bar{q}$ till time $s_r = 17.0032$, the time at which $W^*(s)$ equals $(c - A^*(s))/r$; see (3.2).

In Fig. 6, we plot the graph of the minimum probability of lifetime ruin as a function of w for $(t, A) = (0, 0)$. We assume the individual's mortality hazard rate follows Makeham's law and plot the graphs in three cases when $\lambda_1(t) = 0.01 + 0.0001e^{0.001t} < \lambda^o(t)$, $\lambda_2(t) = 0.03 + 0.001e^{0.01t} = \lambda^o(t)$, and $\lambda_3(t) = 0.06 + 0.01e^{0.02t} > \lambda^o(t)$. We see that $\psi_i(0, w, 0)$ for $i = 1, 2, 3$ decreases with respect to w, as expected, and appears to be first convex, then concave under these parameters.

Finally, in Fig. 7, we compare the minimum probability of lifetime ruin when the forces of mortality follow Makeham's law, that is, $\lambda^o(t) = 0.03 + 0.001e^{0.01t}$ and $\lambda(t) = 0.06 + 0.01e^{0.02t}$, with the minimum probability of lifetime ruin when the forces of mortality are constants (with equal expected values of τ_d), that is, $\lambda^o = 0.0314$ and $\lambda = 0.0734$. We use $\psi_c(0, w, 0)$ to denote the minimum probability of lifetime ruin with the constant mortality hazard rate, and we observe the values of ψ and ψ_c are quite close under these specific parameters. □

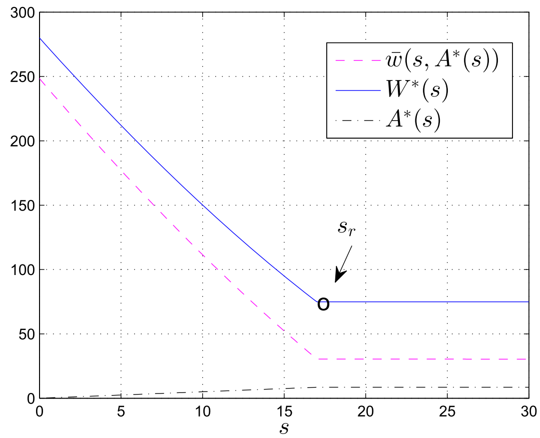

Fig. 5. $W^*(s)$ as a function of time when $w = 280$.

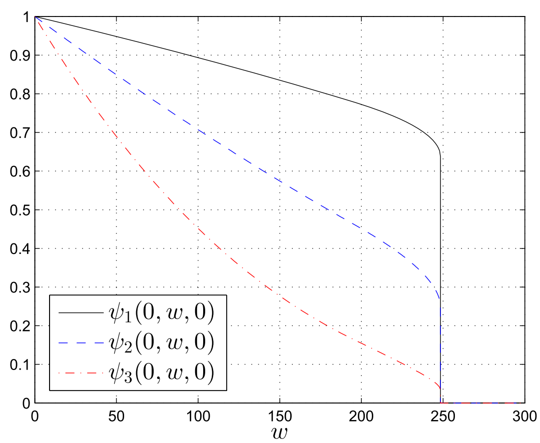

Fig. 6. $\psi(0, w, 0)$ as a function of w.

We end this section with an analysis of the minimum probability of lifetime ruin ψ as $\bar{q}$ changes. Note that ψ depends on $\bar{q}$ when $(c - A)\bar{a}^o(t) \le w < \bar{w}(t, A)$; also, the safe level $\bar{w}(t, A)$ depends on $\bar{q}$. We expect ψ to decrease as $\bar{q}$ increases because larger $\bar{q}$ more closely resembles the lump-sum purchase of annuity income $c - A$ when wealth reaches $(c - A)\bar{a}^o(t)$, the optimal strategy when annuity purchases are not constrained. This relationship is, indeed, the case, as we prove in the following proposition.

Proposition 3.3. *The minimum probability of lifetime ruin ψ on $\mathcal{D}$ is non-increasing with respect to $\bar{q}$.*

Proof. First, note that $\psi(t, w, A)$ is independent of $\bar{q}$ when $0 \le w < (c - A)\bar{a}^o(t)$. Second, differentiate $\bar{w}(t, A)$ in (3.5) with respect to $\bar{q}$:

$$\frac{\partial \bar{w}(t, A)}{\partial \bar{q}} = -\frac{1}{r^2}\left(1 - e^{-\frac{r}{\bar{q}}(c-A)}\right) + \frac{c - A}{r\bar{q}}\, e^{-\frac{r}{\bar{q}}(c-A)}\left(1 - r\bar{a}^o(t_e)\right) + \int_t^{t_e} \bar{a}^o(u) e^{-r(u-t)} du,$$

in which $t_e = t + (c - A)/\bar{q}$. Define a function ϑ on $\mathbb{R}_+$ as follows:

$$\vartheta(s) = -\frac{1}{r^2}\left(1 - e^{-rs}\right) + \frac{s}{r}\, e^{-rs}\left(1 - r\bar{a}^o(t + s)\right) + \int_t^{t+s} \bar{a}^o(u) e^{-r(u-t)} du.$$

Then, we see $\vartheta(0) = 0$ and

$$\vartheta'(s) = -se^{-rs}\left(\left(1 - r\bar{a}^o(t + s)\right) + \left.\frac{d\bar{a}^o(u)}{du}\right|_{u=t+s}\right) = -se^{-rs}\lambda^o(t + s)\bar{a}^o(t + s) < 0,$$

in which we use (3.1) to compute $\frac{d\bar{a}^o(u)}{du}$. Therefore, $\vartheta(s)$ decreases with s, which implies $\vartheta\left(\frac{c-A}{\bar{q}}\right) < \vartheta(0) = 0$. Also, note that

$$\frac{\partial \bar{w}(t, A)}{\partial \bar{q}} = \vartheta\left(\frac{c - A}{\bar{q}}\right) < 0.$$

Hence, the safe level $\bar{w}(t, A)$ decreases with respect to $\bar{q}$.

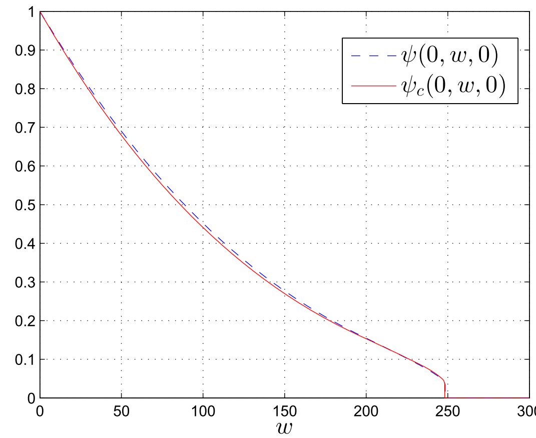

Fig. 7. $\psi(0,w,0)$ and $\psi_c(0,w,0)$ as functions of w.

Third, for $(c-A)\bar{a}^o(t) \le w < \bar{w}(t,A)$, differentiate $g(s_b)=0$ with respect to $\bar{q}$, in which g is defined in (3.21):

$$\frac{\partial g}{\partial \bar{q}} + g'(s_b)\,\frac{\partial s_b}{\partial \bar{q}} = 0,$$

or equivalently,

$$\frac{1}{r^2}\left(e^{r(s_b-t)}-1\right) - \frac{s_b-t}{r}\big(1-r\bar{a}^o(s_b)\big) - e^{r(s_b-t)}\int_t^{s_b} \bar{a}^o(u)e^{-r(u-t)}du + g'(s_b)\frac{\partial s_b}{\partial \bar{q}} = 0.$$

From (3.24), we know that $g'(s_b)<0$; thus, $\frac{\partial s_b}{\partial \bar{q}}$ is positively proportional to $\varsigma(t;s_b)$, in which

$$\varsigma(t;s) = \frac{1}{r^2}\left(e^{r(s-t)}-1\right) - \frac{s-t}{r}\big(1-r\bar{a}^o(s)\big) - e^{r(s-t)}\int_t^{s} \bar{a}^o(u)e^{-r(u-t)}du.$$

By further computation, one can show that $\varsigma(s_b;s_b)=0$, and

$$\frac{\partial \varsigma(t;s)}{\partial t} = \frac{e^{rs}}{r}\left(\big(1-r\bar{a}^o(s)\big)e^{-rs} - \big(1-r\bar{a}^o(t)\big)e^{-rt}\right) \le 0,$$

for $t \le s$, in which the last inequality can be derived similarly as (3.34). Thus, $\varsigma(t;s_b) \ge \varsigma(s_b;s_b)=0$ because $t \le s_b$. Hence, s_b is non-decreasing with respect to $\bar{q}$, which implies that $\psi(t,w,A)$ is non-increasing with respect to $\bar{q}$ when $(c-A)\bar{a}^o(t) \le w < \bar{w}(t,A)$. □

An alternative, shorter proof of Proposition 3.3 is to observe that, if we increase $\bar{q}$, then we expand the allowable set of rates for purchasing annuity income (namely, $[0,\bar{q}]$), which implies that the minimum probability of lifetime ruin will (weakly) decrease with increasing $\bar{q}$.

We also present the following limiting result.

Proposition 3.4. *As $\bar{q}$ approaches infinity, the minimum probability of lifetime ruin approaches*

$$\psi_\infty(t,w,A) = \begin{cases} e^{-\int_t^{s_0}\lambda(s)ds}, & 0 \le w < (c-A)\bar{a}^o(t), \\ 0, & (c-A)\bar{a}^o(t) \le w \le \dfrac{c-A}{r}, \end{cases} \tag{3.35}$$

in which s_0 is defined in (3.16). Furthermore, ψ_∞ is identical *to the probability of lifetime ruin when the individual is allowed to purchase annuity income via lump-sum payments.*

Proof. From (3.5), we write the safe level $\bar{w}(t,A)$ in powers of $\bar{q}$:

$$\begin{aligned}
\bar{w}(t,A) &= \frac{c-A}{r} - \frac{\bar{q}}{r^2}\left(1-e^{-\frac{r}{\bar{q}}(c-A)}\right) + \bar{q}\int_t^{t+\frac{c-A}{\bar{q}}} \bar{a}^o(u)e^{-r(u-t)}du \\
&= \frac{c-A}{r} - \frac{\bar{q}}{r^2}\left(1-e^{-\frac{r}{\bar{q}}(c-A)}\right) + \bar{q}\cdot\frac{c-A}{\bar{q}}\bar{a}^o(t)e^{-\frac{r}{\bar{q}}(c-A)} + \mathcal{O}(1/\bar{q}) \\
&= \frac{c-A}{r} - \frac{\bar{q}}{r^2}\left(1-\left\{1-\frac{r}{\bar{q}}(c-A)+\mathcal{O}\big(1/\bar{q}^2\big)\right\}\right) + (c-A)\bar{a}^o(t) + \mathcal{O}(1/\bar{q}) \\
&= (c-A)\bar{a}^o(t) + \mathcal{O}(1/\bar{q}),
\end{aligned}$$

which approaches $(c-A)\bar{a}^o(t)$ as $\bar{q} \to \infty$. Thus, ψ $(=\phi)$ in (3.26) approaches ψ_∞ in (3.35) as $\bar{q}$ goes to infinity.

To show that ψ_∞ equals the minimum probability of lifetime ruin when an individual is allowed to purchase annuity income via lump-sum payments, we rely on a verification theorem similar to Theorem 3.1. In place of Item 3 in Theorem 3.1, the minimum probability of lifetime ruin solves the following HJ variational inequality on the interior of $\mathcal{D}\backslash L$ (see, for example, Wang and Young, 2012):

$$\min\left\{f_t + (rw - c + A)f_w - \lambda(t)f,\ f_A - \bar{a}^o(t)f_w\right\} = 0, \tag{3.36}$$

with the same boundary conditions. Clearly, $\psi_\infty(t, w, A) = 0$ satisfies this variational inequality when $(c-A)\bar{a}^o(t) < w < (c-A)/r$. When $0 < w < (c-A)\bar{a}^o(t)$, $\psi_\infty(t, w, A) = e^{-\int_t^{s_0} \lambda(s)ds}$ satisfies $\big(\partial_t + (rw - c + A)\partial_w - \lambda(t)\big)\psi_\infty = 0$. Therefore, we only need to show that $\big(\partial_A - \bar{a}^o(t)\partial_w\big)\psi_\infty \geq 0$ in this case, which is true from (3.33). We deduce that ψ_∞ equals the minimum probability of lifetime ruin when the individual is allowed to purchase annuity income in lump sums. □

In Milevsky et al. (2006), individuals are allowed to invest in both a risky and a riskless asset. If one were to restrict investment to a riskless asset only in the anything-anytime annuitization model of Milevsky et al. (2006), then the minimum probability of lifetime ruin would equal ψ_∞ in Proposition 3.4.

4. Conclusions

In this paper, we solved an individual's optimal annuitization problem to minimize her probability of lifetime ruin with age-dependent forces of mortality, both for pricing annuities and for the individual's mortality. We considered a deterministic financial model in which the individual purchases annuity income gradually under a bounded, absolutely continuous rate. In our model, we found that the annuity's buy boundary is *identical* to the one in the anything-anytime annuitization model of Milevsky et al. (2006). Specifically, we found that it is optimal for the individual not to purchase additional annuity income when her wealth is less than $(c-A)\bar{a}^o(t)$. There is another barrier $\bar{w}(t, A)$, which we call the *safe level*; when wealth is above the safe level, lifetime ruin is preventable. When wealth lies between $(c-A)\bar{a}^o(t)$ and $\bar{w}(t, A)$, it is optimal for the individual to purchase annuity income at the maximum allowable rate $\bar{q}$ till the wealth decreases to the buy boundary $(c-A)\bar{a}^o(t)$. We provided numerical examples to illustrate our results.

For future research, we plan to incorporate investment factors into our model, that is, we will allow the individual to invest in a financial market consisting of one risky asset and one riskless asset. We will consider the individual's optimal annuitization purchasing strategy and optimal investment strategy in this stochastic environment framework, and compare them with the strategies in the deterministic model in our paper and in the anything-anytime annuitization model. We also plan to consider the problem in this paper when the insurer might default on its annuities.

Declaration of competing interest

There is no competing interest.

Data availability

No data was used for the research described in the article.

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