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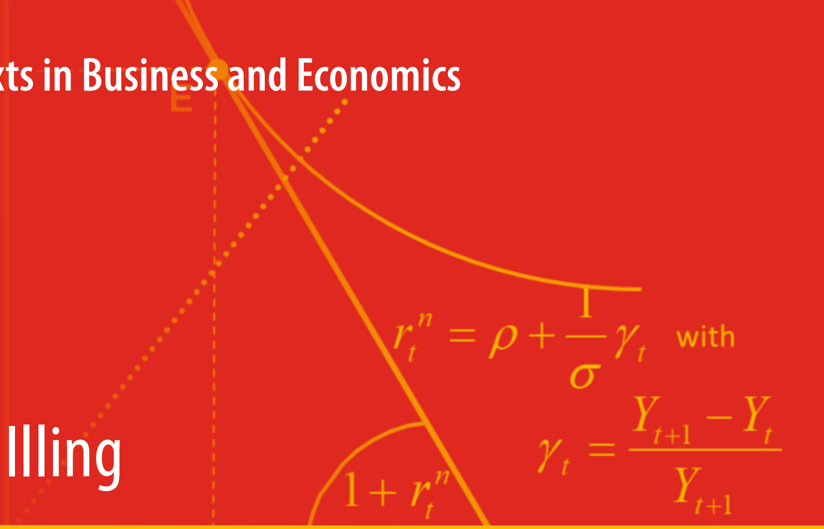

Jin Cao
Gerhard Illing

Instructor's Manual for *Money: Theory and Practice*

Springer

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Instructor's Manual for *Money: Theory and Practice*

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1.1 Exercises

1.1.1 Short Review Questions

(a) Consider in an endowment, cashless economy a representative agent lives for two periods, $t = 1, 2$. She receives a flow of perishable endowment Y_t in each period t. Besides consuming C_t, she has the opportunity to save S_t which pays her a net return r_t in $t + 1$.
   1. Specify her resource constraints in each period.
   2. Her life-time utility from the consumption flow (C_1, C_2) is $U(C_1)+\beta U(C_2)$ in which $U(\cdot)$ is the standard neoclassical utility function with $U'(\cdot) > 0$, $U''(\cdot) < 0$, and $0 < \beta < 1$ is the discount rate. If she is a utility maximizer, compute her optimal consumption flow, denote it as $\left(C_1^*, C_2^*\right)$.
   3. Compute her marginal rate of substitution (MRS). Interpret it.
   4. Under what condition $C_1^* > C_2^*$? Interpret the condition. What is the natural rate of interest?

(b) Following the last exercise, now suppose the government is actually the saving agency. It takes the savings for public expenditure, and repays the agents via taxing the agents.
   1. Specify a representative agent's resource constraints in each period.
   2. What is Ricardian equivalence? Under what condition(s) it holds?

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1.1.2 Dynamic Optimization in Infinite Horizon[1]

Consider an infinitely lived representative consumer receiving an endowment flow Y_t in each period $t = 0, 1, \dots, +\infty$. She may consume C_t in each period and buy or sell bonds B_t at interest rate r. She maximizes her life-time utility

$$u = \sum_{t=0}^{+\infty} \frac{1}{(1+\rho)^t} U(C_t)$$

with $U(\cdot)$ being the neoclassical utility function, subject to the per period budget constraint $B_{t+1} - B_t = Y_t + rB_t - C_t$ with B_0 and $\{Y_t\}_{t=0}^{+\infty}$ given.

(a) Derive the first-order conditions and characterize the optimal consumption path.
(b) Using the No-Ponzi-Game condition, formulate the consumer's intertemporal wealth constraint. Discuss the relation between the No-Ponzi-Game condition and the transversality condition.

1.1.3 Application of Dynamic Optimization in Economic Growth: Ramsey Model

An infinitely lived representative agent has the neoclassical life-time utility function

$$u = \sum_{t=0}^{+\infty} \frac{1}{(1+\rho)^t} U(C_t) \text{ with } U(C_t) = \frac{C_t^{1-\sigma}}{1-\sigma}.$$

The aggregate production function is $Y = K^\alpha N^{1-\alpha}$ $(0 < \alpha < 1)$, in which K is capital input and N is labor input. Across the periods, the growth rate of labor force is n, the rate depreciation of capital is δ. Both rates are constant over time. Economic agents own the capital stock, and work to produce. In each period, a representative agent provides unit labor in production, receives the output from the production. Using the output, she can consume, and change the depreciated capital stock.

(a) What does σ mean for this type of preference? How is it related to the rate of risk aversion (RRA)? Show that $U(C_t) = \ln C_t$ when $\sigma \to 0$. Show that the production function has constant returns to scale and formulate output per capita (suppose that everyone in this economy provides a unit of labor force) as a function of capital intensity (capital per capita).
(b) Derive the transition equation for capital intensity.
(c) Derive first-order conditions of the agents optimization problem.

[1] A brief introduction to solving dynamic optimization problems is provided in the Appendix.

(d) Derive the Euler equation for per capita consumption.
(e) Calculate capital intensity and per capita consumption of the steady state.
(f) Explain the optimal growth path from an arbitrary starting value of capital intensity.
(g) How should the economy respond to a *unforeseeable* change in the growth rate of labor force? To put it clear, suppose the economy is already in the steady state at t_0 with a constant growth rate of labor force n_0, and then for whatever reason from t_1 in the future the growth rate of labor force will be $n_1 > n_0$, $\forall t \in [t_1, +\infty)$. Characterize the response of the economy from t_1 on.
(h) What happens if the shock in question (g) is *foreseeable*, i.e., at t_0 people expect in the future the growth rate of labor force will be $n_1 > n_0$, $\forall t \in [t_1, +\infty)$ due to immigration? Characterize the response of the economy from t_0 on.
(i) The same dynamic optimization problem can be also explored in *continuous* time, such that the life-time utility function becomes $U_0 = \int_0^{+\infty} e^{-\rho t} \, (c(t))^{\beta} \, dt$, $0 < \beta < 1$. Re-do questions (a)–(g). Are results different, compared with those in discrete time?

1.1.4 Dynamic Optimization in Continuous Time

Dynamic optimization problems can be alternatively explored in *continuous* time. Suppose an individual receives a steady stream of income over time $y(t)$. She maximizes her discounted utility from consumption. Her intertemporal utility function is given by

$$\int_0^{+\infty} e^{-\rho t} U(t) dt \quad \text{with} \quad U(t) = \frac{1}{\alpha} c(t)^{\alpha}$$

The consumer has access to a perfect capital market at which she can lend or borrow at an interest rate r.

(a) Give an interpretation of the parameter ρ. Calculate the elasticity of substitution between consumption of two points in time and the rate of relative risk aversion.
(b) What is the transition equation for consumer's wealth?
(c) Formulate the dynamic optimization problem and derive the first-order conditions.
(d) Derive the Euler equation and show how consumption changes over time. Distinguish two cases: a rate of time preference being lower/exceeding the rate of interest.
(e) Let $r = 0.1$ and $\rho = 0.2$. Determine the optimal consumption path, if the present value of the income stream is $y_0 = 100$. Discuss the relation between the transversality condition and the household's intertemporal budget constraint.

1.1.5 Stochastic Optimization: Asset Pricing

Consider a household with expected utility function

$$E[U] = U(c_t) + \frac{1}{1+\rho} \sum_s p_s U(c_{2,s}),$$

where p_s is the probability of state s. Income is y_1 in the first period and $y_{2,s}$ in state s of period 2. There is one asset traded in period 1 that pays an interest rate of r in each state of period 2.

(a) Write down budget constraints and derive the first-order condition. Show that for optimal savings the asset's return equals the expected marginal rate of substitution between present and future consumption.
Assume a constant rate of risk aversion (CRRA) utility function $U(c) = \frac{c^\alpha}{\alpha}$ and assume that the stochastic income in period 2 is such that with optimal savings the MRS has log-normal distribution, i.e. $\ln(MRS) \sim N(\mu, \sigma^2)$.
(b) Show that the difference between interest rate and time preference rate rises with increasing variance.
Consider now an asset with stochastic return $1 + r_s$.
(c) Write down budget constraints and derive the first-order condition. Show how the asset price depends on the covariance between r_s and $y_{2,s}$.

1.1.6 Stochastic Optimization: Permanent Income Hypothesis

Consider a consumer maximizing expected utility in discrete time under uncertainty subject to a budget constraint. The interest rate r is constant, as is the rate of time preference ρ. The consumer has an initial stock of assets A and earns income an Y_t that is uncertain in the future. In each period, the consumer solves the following problem:

$$\max_{\{C_t\}_{t=0}^{+\infty}} E_t \left[\sum_{t=0}^{+\infty} \frac{1}{(1+\rho)^t} U(C_t) + \lambda \left(A - \sum_{t=0}^{+\infty} \frac{1}{(1+r)^t} (C_t - Y_t) \right) \right],$$

in which λ is the Lagrange multiplier.

(a) Using the first-order conditions, show that marginal utility is a random process of the form $X_{t+1} = kX_t + \epsilon_{t+1}$, where X denotes marginal utility, k is a constant, and ϵ is a random term with mean zero.
(b) Show that for quadratic utility, $U(C) = -\frac{(b-C)^2}{2}$, consumption is a stochastic process of the form $C_{t+1} = kC_t + \delta + \epsilon_{t+1}$, where k and δ are constants, and ϵ is a random term with mean zero. What happens to the consumption path if $k = 1$? Does consumption path respond to a temporary income shock that only

lasts for a few periods? Does consumption path respond to a permanent income shock that lasts for life time?

1.1.7 Asset Pricing: The Lucas Tree

Lucas (1978) suppose that the only assets in the economy are some infinitely living trees. Output equals the fruits of the trees (suppose the productivities of the trees are perfectly correlated, i.e. all the trees produce exactly the same amount of fruits in a given period), which is exogenously given positive random variable and cannot be stored—therefore $c_t = y_t$ for each t in which y_t is the exogenously determined per capita output (to make it simple, one can assume that the number of trees is equal to the population, i.e. y_t is also the productivity of the trees in period t) and c_t is the per capita consumption. Assume that in the beginning each one in this economy owns the same number of trees. Since all the agents are assumed to be the same, in equilibrium the behavior of the price of the trees should be such that in each period the representative agent is not willing to either increase or decrease her holdings of the trees.

Let P_t denote the price of a tree in period t, and assume that if the tree is sold the sale occurs after the existing owner receives that period's output. Finally, assume that the representative agent maximizes

$$E_0\left[\sum_{t=0}^{+\infty} \frac{1}{(1+\rho)^t} \frac{c_t^{1-\sigma}}{1-\sigma}\right].$$

(a) Suppose that the representative agent reduces her consumption is period t by an infinitesimal amount, uses the resulting saving to increase her holdings of trees and then sells these additional trees in period $t+1$. Find the condition that c_t and expectations involving y_{t+1}, P_{t+1}, and c_{t+1} must satisfy for this change not to affect expected utility. Solve this condition for P_t in terms of y_t and expectations involving y_{t+1}, P_{t+1}, and c_{t+1}. (Hint: The representative agent's resource constraint can be written as $c_t + P_t e_{t+1} = (y_t + P_t)\, e_t$, in which e_t denotes how many trees she owns in period t.)

(b) Suppose that $\sigma \to 1$ and $\lim_{s\to+\infty} E_t\left[\frac{1}{(1+\rho)^s}\frac{P_{t+s}}{y_{t+s}}\right] = 0$. Iterate the result in (a) forward to solve for P_t.

(c) Give some intuition why in (b) an increase in expectations of future dividends does not affect the price of the asset.

(d) Does consumption follow a random walk in this model?

1.1.8 The Equity Premium Puzzle

Mehra and Prescott (1985) continue with the settings in Exercise 7 (a). Now except the ownership of the tree, we introduce another asset—a riskless asset b_t with price

q_t. We call such riskless asset *bond*, and the risky assets (the ownership of the trees) *equity* or *stock*.

(a) Define the representative agent's optimization problem, and derive the first-order conditions. Note that in addition to the agent's flow budget constraint that you specified in Exercise 7, in each period t the agent now has to decide how much b_{t+1} she has to invest for the next period at current price q_t. (Hint: The representative agent's resource constraint can be written as $c_t + P_t e_{t+1} + q_t b_{t+1} = (y_t + P_t)\, e_t + b_t$.)

(b) Express P_t and q_t in terms of y_t and expectations involving y_{t+1}, P_{t+1}, and c_{t+1}. Show how P_t depends on the covariance between P_{t+1} and c_{t+1}, and define P_t as the sum of the riskless return and the risk premium.

(c) Define the implicit return of the riskless asset, the bond, as

$$R_b = \frac{1}{q_t},$$

and the implicit return of the risky assets, the stock, as

$$R_s = \frac{P_{t+1} + y_{t+1}}{P_t}.$$

Rewrite the expressions in (b) with R_b and R_s.

Now assume that the consumption growth rate is

$$\frac{c_{t+1}}{c_t} = \frac{y_{t+1}}{y_t} = \gamma \exp\left(\epsilon_{yt} - \frac{\sigma_y^2}{2}\right),$$

in which γ is a positive constant and ϵ_{yt} is a normally distributed i.i.d. shock, $\epsilon_{yt} \sim N\left(0, \sigma_y^2\right)$.

And assume that R_s fluctuates around $\overline{R}_s$ as

$$R_s = \overline{R}_s \exp\left(\epsilon_{st} - \frac{\sigma_s^2}{2}\right),$$

in which ϵ_{st} is a normally distributed i.i.d. shock, $\epsilon_{st} \sim N\left(0, \sigma_s^2\right)$.

Find the *equity premium* $\overline{R}_s - R_b$ in terms of σ, ϵ_{yt}, and ϵ_{st}.

(d) Estimated from the US financial market (1890–2003), $R_b = 1.01$, $\overline{R}_s = 1.07$, and $\operatorname{cov}\left(\epsilon_{st}, \epsilon_{yt}\right) = 0.002$. Compute σ using the result of (c). What does σ mean in economics? Why do people call this result a puzzle?

1.1.9 Dixit–Stiglitz Indices for Continuous Commodity Space

Dixit and Stiglitz (1977) Consider a one-person economy. Mr. Rubinson Crusoe is the only agent in this economy, consuming a continuum of commodities $i \in [0, 1]$. Suppose that the consumption index C of him is defined as

$$C = \left[\int_0^1 Z_i^{\frac{1}{\eta}} C_i^{\frac{\eta-1}{\eta}} di\right]^{\frac{\eta}{\eta-1}}$$

in which C_i is the consumption of good i and Z_i is the taste shock for good i. Suppose that Crusoe has an amount of endowment Y to spend on goods with exogenously given price tags. Therefore the budget constraint is

$$\int_0^1 P_i C_i di = Y.$$

(a) Find the first-order condition for the problem of maximizing C subject to the budget constraint. Solve for C_i in terms of Z_i, P_i, and the Lagrange multiplier on the budget constraint.
(b) Use the budget constraint to find C_i in terms of Z_i, P_i, and Y.
(c) Insert the result of (b) into the expression for C and show that $C = \frac{Y}{P}$, in which

$$P = \left(\int_0^1 Z_i P_i^{1-\eta} di\right)^{\frac{1}{1-\eta}}.$$

(d) Use the results in (b) and (c) to show that

$$C_i = Z_i \left(\frac{P_i}{P}\right)^{-\eta} \left(\frac{Y}{P}\right).$$

Interpret this result.

1.1.10 Menu Cost and Nominal Price Rigidity

A representative monopolistically competitive firm sells its output for a nominal price P_i. It faces the demand function

$$Y_i = \left(\frac{P_i}{P}\right)^{-\epsilon} D \text{ with } \epsilon > 1$$

in which P is the general price level of the economy and D is an aggregate demand parameter (both of them are exogenous to the firm). There is only one productive input, labor L_i , which is used according to the production function

$$Y_i = L_i^{\frac{1}{\beta}} \text{ with } \beta > 1.$$

The firm pays workers an exogenous nominal wage w.

(a) Explain the parameter β. What is the economic interpretation of the condition $\beta > 1$?
(b) Draw a diagram with the demand function, the marginal revenue function, and the marginal cost function of the firm. Determine the firm's optimal price and quantity.
(c) Use your diagram to demonstrate the response of the optimal relative price to a fall in D.
(d) Explain that a small *menu cost* of changing prices is able to prevent the firm from adjusting price freely. Which factors determine the degree of such rigidity? You may want to come back to this question after you read Chap. 4

1.2 Solutions for Selected Exercises

1.2.1 Dynamic Optimization in Discrete Time

Consider an infinitely lived representative consumer receiving an endowment flow Y_t in each period $t = 0, 1, \ldots, +\infty$. She may consume C_t in each period and buy or sell bonds B_t at interest rate r. She maximizes her life-time utility

$$u = \sum_{t=0}^{+\infty} U(C_t) \frac{1}{(1+\rho)^t}$$

with $U(\cdot)$ being the neoclassical utility function, subject to the per period budget constraint $B_{t+1} - B_t = Y_t + rB_t - C_t$ with B_0 and $\{Y_t\}_{t=0}^{+\infty}$ given.

(a) Derive the first-order conditions and characterize the optimal consumption paths.

Present value Hamiltonian

$$\mathcal{H}_t = \frac{1}{(1+\rho)^t} U(C_t) + \lambda_t (Y_t + rB_t - C_t),$$

with first-order conditions

$$\frac{\partial \mathcal{H}_t}{\partial C_t} = \frac{1}{(1+\rho)^t} U'(C_t) - \lambda_t = 0, \tag{1.1}$$

$$\frac{\partial \mathcal{H}_t}{\partial B_t} = r\lambda_t = \lambda_{t-1} - \lambda_t, \tag{1.2}$$

as well as transversality condition

$$\lim_{t\to+\infty} \lambda_t(B_{t+1} - \overline{B}_{t+1}) = 0$$

with complementary slackness such that if $B_{t+1} < \overline{B}_{t+1}$ then $\lambda_t = 0$; if $\lambda_t = 0$ then $B_{t+1} = \overline{B}_{t+1}$.

Rearranging (1.1) and (1.2) gives

$$\frac{U'(C_{t-1})}{U'(C_t)} = \frac{1+r}{1+\rho},$$

as well as corresponding optimal consumption paths

$$C_t = (U')^{-1}\left[\left(\frac{1+\rho}{1+r}\right)^t U'(C_0)\right].$$

(b) *Using the No-Ponzi-Game condition, formulate the consumer's intertemporal wealth constraint. Discuss the relation between the No-Ponzi-Game condition and the transversality condition.*

Solve the difference equation given by the flow budget constraint $B_{t+1} - B_t = Y_t + rB_t - C_t$. Rearrange to get

$$B_t = \frac{1}{1+r}B_{t+1} - \frac{1}{1+r}\left(Y_t - C_t\right),$$

$$B_0 = \lim_{T\to+\infty}\frac{1}{(1+r)^T}B_T - \sum_{t=0}^{+\infty}\frac{1}{(1+r)^{t+1}}\left(Y_t - C_t\right).$$

No-Ponzi-Game condition requires that[2]

$$\lim_{T\to+\infty}\frac{1}{(1+r)^T}B_T \geq 0,$$

and the transversality condition requires that

$$\lim_{T\to+\infty}\frac{1}{(1+r)^T}B_T = 0.$$

[2]Please note that throughout the problem we take B as the stock of assets.

No-Ponzi-Game condition rules out the economically non-plausible path, and the transversality condition makes it binding.

1.2.2 Application of Dynamic Optimization in Growth Theory: Ramsey Model

An infinitely lived representative agent has the neoclassical life-time utility function in continuous time

$$U_0 = \int_0^{+\infty} e^{-\rho t} \left(c(t)\right)^{\beta} dt, \quad 0 < \beta < 1.$$

The aggregate production function is $Y = K^{\alpha} N^{1-\alpha}$ $(0 < \alpha < 1)$*, in which* K *is capital input and* N *is labor input. Across the periods, the growth rate of labor force is* n*, the rate depreciation of capital is* δ*. Both rates are constant over time. Economic agents own the capital stock, and work to produce. In each period, a representative agent provides unit labor in production, receives the output from the production. Using the output, she can consume, and change the depreciated capital stock.*

(a) What does σ *mean for this type of preference? How is it related to the rate of risk aversion (RRA)? Show that* $U(C_t) = \ln C_t$ *when* $\sigma \to 0$*. Show that the production function has constant returns to scale and formulate output per capita (suppose that everyone in this economy provides a unit of labor force) as a function of capital intensity (capital per capita).*

Note that optimization problems in continuous time generate the same patterns of balanced growth path as in discrete time, we solve this exercise in continuous time.

It's straightforward that

$$Y(\lambda K, \lambda N) = \lambda Y(K, N).$$

And then

$$y = \frac{Y}{N} = \left(\frac{K}{N}\right)^{\alpha} = k^{\alpha}.$$

(b) Derive the transition equation for capital intensity.

From law of motion

$$\dot{K} = Y - \delta K - cN$$

divide both sides by N and get

$$\frac{\dot{K}}{N} = y - \delta k - c.$$

By definition

$$k = \frac{K}{N}$$

it's directly seen that

$$\frac{\dot{k}}{k} = \frac{\dot{K}}{K} - \frac{\dot{N}}{N},$$

that is

$$\dot{k} = \frac{\dot{K}}{N} - nk.$$

Then

$$\dot{k} = k^{\alpha} - (\delta + n)k - c.$$

(c) Derive first-order conditions of the agents optimization problem.

Present value Hamiltonian

$$\mathcal{H} = e^{-\rho t} c^{\beta} + \mu \left[k^{\alpha} - (\delta + n)k - c \right].$$

First-order conditions:

$$\frac{\partial \mathcal{H}}{\partial c} = e^{-\rho t} \beta c^{\beta - 1} - \mu = 0, \tag{1.3}$$

$$\frac{\partial \mathcal{H}}{\partial k} = \mu[\alpha k^{\alpha - 1} - (\delta + n)] = -\dot{\mu}, \tag{1.4}$$

as well as transversality condition

$$\lim_{t \to +\infty} \mu(t) k(t) = 0,$$

$$\lim_{t \to +\infty} \mu(0) \exp\left[- \int_{s=0}^{t} \left(\alpha k(s)^{\alpha - 1} - (\delta + n) \right) ds \right] k(t) = 0,$$

$$\lim_{t \to +\infty} \alpha k(t)^{\alpha - 1} - (\delta + n) > 0,$$

i.e., in steady state $y'^{*} = \alpha k^{*\alpha - 1} > \delta + n$.

(d) Derive the Euler equation for per capita consumption.

Combine (1.3) and (1.4) and simplify, then

$$\frac{\dot{c}}{c} = \frac{\alpha k^{\alpha-1} - \delta - \rho - n}{1-\beta}.$$

(e) Calculate capital intensity and per capita consumption of the steady state.

Taking $\dot{c}^* = 0$ and $\dot{k}^* = 0$, solve to get

$$k^* = \left(\frac{\delta+\rho+n}{\alpha}\right)^{\frac{1}{\alpha-1}},$$

$$c^* = \left(\frac{\delta+\rho+n}{\alpha}\right)^{\frac{\alpha}{\alpha-1}} - (\delta+n)\left(\frac{\delta+\rho+n}{\alpha}\right)^{\frac{1}{\alpha-1}}.$$

(f) Explain the optimal growth path from an arbitrary starting value of capital intensity.

See the phase diagram Fig. 1.1 (black curves).

(g) How should the economy respond to a foreseeable *change in the growth rate of labor force? To put it clear, suppose the economy is already in the steady state*

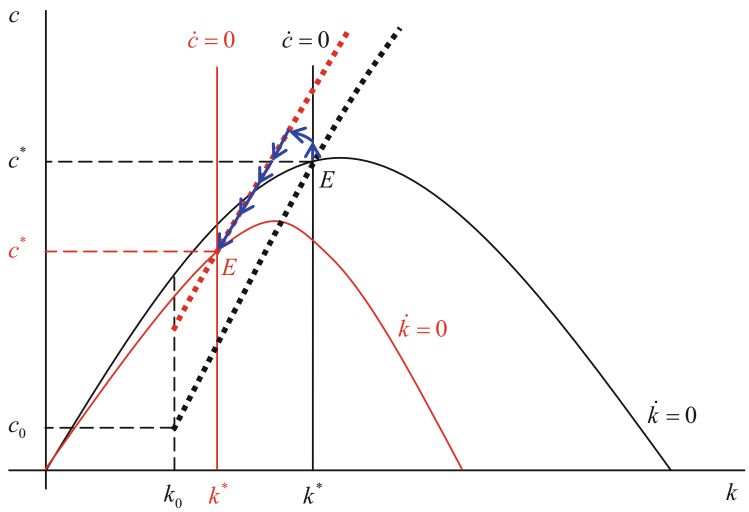

Fig. 1.1 Economic dynamics in phase diagram

at t_0 with a constant growth rate of labor force n_0, and then for whatever reason it becomes public information at t_0 that from t_1 in the future the growth rate of labor force will be $n_1 > n_0$, $\forall t \in [t_1, +\infty)$. Using phase diagram characterize the response of the economy from t_0 on.

See Fig. 1.1 (red and blue curves).

1.2.3 Dynamic Optimization in Continuous Time

An individual receives a steady stream of income over time $y(t)$. She maximizes her discounted utility from consumption. Her intertemporal utility function is given by

$$\int_0^{+\infty} e^{-\rho t} U(t) dt \quad \text{with} \quad U(t) = \frac{1}{\alpha} c(t)^\alpha$$

The consumer has access to a perfect capital market at which she can lend or borrow at an interest rate r.

(a) Give an interpretation of the parameter ρ. Calculate the elasticity of substitution between consumption of two points in time and the rate of relative risk aversion.

ρ is the agent's discount rate; the greater is ρ, the less she values future consumption relative to current consumption.

The elasticity of substitution between consumption of two points in time, t and s, is given by

$$\sigma(c(t), c(s)) := -\frac{\frac{U'(c(s))}{U'(c(t))}}{\frac{c(s)}{c(t)}} \frac{d\left(\frac{c(s)}{c(t)}\right)}{d\left[\frac{U'(c(s))}{U'(c(t))}\right]}.$$

Take t as a reference point and $s \to t$, then the instantaneous elasticity of substitution at t is

$$\sigma(c(t)) = \lim_{s \to t} -\frac{\frac{U'(c(s))}{U'(c(t))}}{\frac{c(s)}{c(t)}} \frac{d\left(\frac{c(s)}{c(t)}\right)}{d\left[\frac{U'(c(s))}{U'(c(t))}\right]} = -\frac{U'(c(t))}{U''(c(t))c(t)}.$$

Apply $U(t) = \frac{1}{\alpha} c(t)^\alpha$ to get

$$\sigma(c(t)) = \frac{1}{1-\alpha}.$$

Arrow–Pratt measure (see Mas-Colell et al. 1995, Chapter 6 for detail) for the rate of relative risk aversion is defined as

$$R := -\frac{U''(c(t))c(t)}{U'(c(t))}.$$

Apply $U(t) = \frac{1}{\alpha}c(t)^\alpha$ to get

$$R = \frac{1}{\sigma} = 1 - \alpha.$$

(b) What is the transition equation for consumer's wealth?

$\dot{b}(t) = y(t) + rb(t) - c(t).$

(c) Formulate the Hamiltonian of this problem and derive first-order conditions.

Present value Hamiltonian:

$$\mathcal{H} = e^{-\rho t}\left[\frac{1}{\alpha}c(t)^\alpha\right] + \mu(t)\left[y(t) + rb(t) - c(t)\right].$$

First-order conditions[3]:

$$\frac{\partial \mathcal{H}}{\partial c} = e^{-\rho t}c^{\alpha-1} - \mu = 0, \tag{1.5}$$

$$\frac{\partial \mathcal{H}}{\partial b} = r\mu = -\dot{\mu}, \tag{1.6}$$

as well as transversality condition

$$\lim_{t\to+\infty} \mu(t)b(t) = 0,$$

$$\lim_{t\to+\infty} \mu(0)e^{-rt}b(t) = 0.$$

(d) Derive the Euler equation and show how consumption changes over time. Distinguish two cases: a rate of time preference being lower/exceeding the rate of interest.

Log-linearize and differentiate (1.5) to get

$$\frac{\dot{\mu}}{\mu} = -\rho + (\alpha - 1)\frac{\dot{c}}{c},$$

[3]Time variable t is sometimes dropped when there is no confusion.

and insert it into (1.6) to get

$$\frac{\dot{c}}{c} = \frac{r - \rho}{1 - \alpha}.$$

Solve for $c(t)$, and this gives

$$c(t) = \exp\left[\frac{r-\rho}{1-\alpha}t + \text{constant}\right] = c(0)\exp\left[\frac{r-\rho}{1-\alpha}t\right].$$

- $r > \rho$, consumption path with a constant positive growth rate;
- $r < \rho$, consumption path with a constant negative growth rate.

In the end, the transversality condition determines optimal growth path.

(e) Let $r = 0.1$ and $\rho = 0.2$. Determine the optimal consumption path, if the present value of the income stream is $y_0 = 100$. Discuss the relation between the transversality condition and the household's intertemporal budget constraint.

Solve the flow budget constraint $\dot{b}(t) = y(t) + rb(t) - c(t)$ and get

$$b(0) = \lim_{T\to+\infty} e^{-rT}b(T) - \int_0^{+\infty} e^{-rt}\left[y(t) - c(t)\right]dt.$$

Because the agent starts from zero assets and the transversality condition makes

$$\lim_{T\to+\infty} e^{-rT}b(T) = 0,$$

therefore one can see that

$$\int_0^{+\infty} e^{-rt}c(t)dt = \int_0^{+\infty} e^{-rt}y(t)dt,$$

$$\int_0^{+\infty} c(0)\exp\left[\left(\frac{r-\rho}{1-\alpha} - r\right)t\right]dt = y_0,$$

$$-\frac{1-\alpha}{\alpha r - \rho}c(0) = y_0.$$

Given that $r = 0.1$, $\rho = 0.2$, and $y_0 = 100$, solve to get

$$c(0) = \frac{20 - 10\alpha}{1 - \alpha}.$$

1.2.4 Stochastic Optimization: Asset Pricing

Consider a household with expected utility function

$$E[U] = U(c_t) + \frac{1}{1+\rho}\sum_s p_s U(c_{2,s}),$$

where p_s is the probability of state s. Income is y_1 in the first period and $y_{2,s}$ in state s of period 2. There is one asset traded in period 1 that pays an interest rate of r in each state of period 2.

(a) Write down budget constraints and derive the first-order condition. Show that for optimal savings the asset's return equals the expected marginal rate of substitution between present and future consumption.

The budget constraint is

$$\begin{aligned} c_1 + d &\leq y_1, \\ c_{2,s} - (1+r)d &\leq y_{2,s} \end{aligned}$$

in which d is period 1 saving.

The first-order condition gives

$$U'(c_1) - \frac{1+r}{1+\rho}E[U'(c_{2,s})] = 0,$$

$$\begin{aligned} 1+r &= (1+\rho)E\left[\frac{U'(c_1)}{U'(c_{2,s})}\right], \\ 1+\rho &= (1+r)E[MRS]. \end{aligned}$$

Assume a constant rate of risk aversion (CRRA) utility function $U(c) = \frac{c^\alpha}{\alpha}$ and assume that the stochastic income in period 2 is such that with optimal savings the MRS has log-normal distribution, i.e. $\ln(MRS) \sim N(\mu, \sigma^2)$.

(b) Show that the difference between interest rate and time preference rate rises with increasing variance.

Since $\ln(MRS) \sim N(\mu, \sigma^2)$, $E[MRS] = \exp\left(\mu + \frac{1}{2}\sigma^2\right)$. Therefore $\rho - r = \mu + \frac{1}{2}\sigma^2$, increasing with variance.

Consider now an asset with stochastic return $1 + r_s$.

(c) Write down budget constraints and derive the first-order condition. Show how the asset price depends on the covariance between r_s and $y_{2,s}$.

The budget constraint is now

$$c_1 + P_1 d \leq y_1,$$
$$c_{2,s} - (1 + r_s)d \leq y_{2,s}.$$

The first-order condition gives

$$U'(c_1) - \frac{1}{1+\rho} E\left[\frac{1+r_s}{P_1} U'(c_{2,s})\right] = 0,$$
$$(1+\rho)E\left[\frac{P_1 U'(c_1)}{U'(c_{2,s})(1+r_s)}\right] = 1.$$

Arrange to get

$$\begin{aligned} P_1 &= \frac{1}{1+\rho} E\left[(1+r_s)\frac{U'(c_{2,s})}{U'(c_1)}\right] \\ &= \frac{1}{1+\rho}\left\{E\left[1+r_s\right] E\left[\frac{U'(c_{2,s})}{U'(c_1)}\right] + \operatorname{cov}\left(1+r_s, \frac{U'(c_{2,s})}{U'(c_1)}\right)\right\}. \end{aligned}$$

1.2.5 Stochastic Optimization: Permanent Income Hypothesis

Consider a consumer maximizing expected utility in discrete time under uncertainty subject to a budget constraint. The interest rate r is constant, as is the rate of time preference ρ. The consumer has an initial stock of assets A and earns income an Y_t that is uncertain in the future. In each period, the consumer solves the following problem:

$$\max_{\{C_t\}_{t=0}^{+\infty}} E_t\left[\sum_{t=0}^{+\infty} \frac{1}{(1+\rho)^t} U(C_t) + \lambda\left(A - \sum_{t=0}^{+\infty} \frac{1}{(1+r)^t}(C_t - Y_t)\right)\right].$$

(a) *Using the first-order conditions, show that marginal utility is a random process of the form $X_{t+1} = kX_t + \epsilon_{t+1}$, where X denotes marginal utility, k is a constant, and ϵ is a random term with mean zero.*

For any t first-order condition gives

$$\frac{1}{(1+\rho)^t} E\left[U'(C_t)\right] - \lambda \frac{1}{(1+r)^t} = 0,$$

and it also holds for one period forward

$$\frac{1}{(1+\rho)^{t+1}} E\left[U'(C_{t+1})\right] - \lambda \frac{1}{(1+r)^{t+1}} = 0.$$

Divide these two equations and get

$$\frac{E\left[U'(C_{t+1})\right]}{E\left[U'(C_t)\right]} = \frac{1+\rho}{1+r}.$$

This is equivalent to the random process $X_{t+1} = kX_t + \epsilon_{t+1}$.

(b) *Show that for quadratic utility, $U(C) = -\frac{(b-C)^2}{2}$, consumption is a stochastic process of the form $C_{t+1} = kC_t + \delta + \epsilon_{t+1}$, where k and δ are constants, and ϵ is a random term with mean zero. What happens to the consumption path if $k = 1$? Does consumption path respond to a temporary income shock that only lasts for a few periods? Does consumption path respond to a permanent income shock that lasts for life time?*

Apply $U(C) = -\frac{(b-C)^2}{2}$ and the result is directly seen.

1.2.6 Asset Pricing: The Lucas Tree

Lucas (1978) suppose that the only assets in the economy are some infinitely living trees. Output equals the fruits of the trees (suppose the productivities of the trees are perfectly correlated, i.e. all the trees produce exactly the same amount of fruits in a given period), which is exogenously given positive random variable and cannot be stored—therefore $c_t = y_t$ for each t in which y_t is the exogenously determined per capita output and c_t is the per capita consumption. Assume that in the beginning each one in this economy owns the same number of trees. Since all the agents are assumed to be the same, in equilibrium the behavior of the price of the trees should be such that in each period the representative agent is not willing to either increase or decrease her holdings of the trees.

Let P_t denote the price of a tree in period t, and assume that if the tree is sold the sale occurs after the existing owner receives that period's output. Finally, assume that the representative agent maximizes

$$E_0\left[\sum_{t=0}^{+\infty} \frac{1}{(1+\rho)^t} \frac{c_t^{1-\sigma}}{1-\sigma}\right].$$

(a) *Suppose that the representative agent reduces her consumption is period t by an infinitesimal amount, uses the resulting saving to increase her holdings of trees and then sells these additional trees in period $t+1$. Find the condition* *that c_t and expectations involving y_{t+1}, P_{t+1}, and c_{t+1} must satisfy for this change not to affect expected utility. Solve this condition for P_t in terms of y_t and expectations involving y_{t+1}, P_{t+1}, and c_{t+1}.*

The representative agent's problem is

$$\max_{\{c_t\}_{t=0}^{+\infty}} \; E_0\left[\sum_{t=0}^{+\infty} \frac{1}{(1+\rho)^t}\frac{c_t^{1-\sigma}}{1-\sigma}\right],$$
$$s.t. \;\; c_t + P_t e_{t+1} = (y_t + P_t)\, e_t.$$

Set up the Bellman equation

$$V(e, y) = \max_{e'} E\left[\frac{c^{1-\sigma}}{1-\sigma} + \frac{1}{1+\rho}V(e', y')\right],$$
$$s.t. \quad c + Pe' = (y + P)\, e.$$

The first-order condition gives

$$-\,c^{-\sigma}P + \frac{1}{1+\rho}\frac{\partial V(e', y')}{\partial e'} = 0, \tag{1.7}$$

and the envelope condition gives

$$\frac{\partial V(e, y)}{\partial e} = c^{-\sigma}(y + P). \tag{1.8}$$

One period update for Eq. (1.8) gives

$$\frac{\partial V(e', y')}{\partial e'} = E\left[c'^{-\sigma}(y' + P')\right]. \tag{1.9}$$

Combine (1.7) and (1.9) and we get

$$P = E\left[\frac{1}{1+\rho}\frac{c^{\sigma}}{c'^{\sigma}}(y' + P')\right].$$

Add the time parameters to the variables,

$$P_t = E_t\left[\frac{1}{1+\rho}\frac{c_t^{\sigma}}{c_{t+1}^{\sigma}}(y_{t+1} + P_{t+1})\right]. \tag{1.10}$$

(b) Suppose that $\sigma \to 1$ and $\lim_{s\to+\infty} E_t\left[\frac{1}{(1+\rho)^s}\frac{P_{t+s}}{y_{t+s}}\right] = 0$. Iterate the result in (a) forward to solve for P_t.

Insert $\sigma = 1$ and $c_t = y_t$ into Eq. (1.10)

$$\begin{aligned} P_t &= E_t\left[\frac{1}{1+\rho}\frac{y_t}{y_{t+1}}(y_{t+1}+P_{t+1})\right] \\ &= \frac{y_t}{1+\rho} + \frac{y_t}{1+\rho}E_t\left[\frac{P_{t+1}}{y_{t+1}}\right]. \end{aligned}$$

Iterate this equation and one can see that

$$P_t = \frac{y_t}{1+\rho} + \frac{y_t}{(1+\rho)^2} + \ldots + \frac{y_t}{(1+\rho)^s} + \frac{y_t}{(1+\rho)^s}E_t\left[\frac{P_{t+s}}{y_{t+s}}\right],$$

therefore

$$\begin{aligned} \lim_{s\to+\infty} P_t &= \lim_{s\to+\infty}\left\{\frac{y_t}{1+\rho} + \frac{y_t}{(1+\rho)^2} + \ldots + \frac{y_t}{(1+\rho)^s} + \frac{y_t}{(1+\rho)^s}E_t\left[\frac{P_{t+s}}{y_{t+s}}\right]\right\} \\ &= \frac{\frac{1}{1+\rho}}{1-\frac{1}{1+\rho}}y_t \\ &= \frac{y_t}{\rho}. \end{aligned}$$

(c) Give some intuition why in (b) an increase in expectations of future dividends does not affect the price of the asset.

An increase in expectations of future dividends increases expectations of future consumption flow, motivating people to buy more trees today—this tends to raise today's price (income effect). However, high level of future consumption flow implies low marginal utility, i.e. people have abundant dividends when their marginal utility is low—this tends to lower today's price (substitution effect). The instantaneous utility becomes logarithmic utility when $\sigma \to 1$, in which these two effects offset each other.

(d) Does consumption follow a random walk in this model?

The consumption follows a random walk only if the output does so, because $c_t = y_t, \forall t$ in this model.

1.2.7 The Equity Premium Puzzle

Mehra and Prescott (1985) continue with the settings in Exercise 7 (a). Now except the ownership of the tree, we introduce another asset—a riskless asset b_t with price

q_t. *We call such riskless asset* bond*, and the risky assets (the ownership of the trees)* equity *or* stock.

(a) Define the representative agent's optimization problem, and derive the first-order conditions. Note that in addition to the agent's flow budget constraint that you specified in Exercise 7, in each period t the agent now has to decide how much b_{t+1} she has to invest for the next period at current price q_t.

The representative agent's problem is

$$\max_{\{c_t\}_{t=0}^{+\infty}} \quad E_0 \left[\sum_{t=0}^{+\infty} \frac{1}{(1+\rho)^t} \frac{c_t^{1-\sigma}}{1-\sigma} \right],$$
$$s.t. \quad c_t + P_t e_{t+1} + q_t b_{t+1} = (y_t + P_t)\, e_t + b_t.$$

Set up the Lagrangian

$$\mathscr{L} = E_0 \left\{ \sum_{t=0}^{+\infty} \frac{1}{(1+\rho)^t} \frac{c_t^{1-\sigma}}{1-\sigma} + \lambda_t \left[(y_t + P_t)\, e_t + b_t - c_t - P_t e_{t+1} - q_t b_{t+1} \right] \right\},$$

$\forall t \in \{0, 1, \ldots, +\infty\}$ the first-order conditions are

$$\frac{\partial \mathscr{L}}{\partial c_t} = \frac{1}{(1+\rho)^t} c_t^{-\sigma} - \lambda_t, \tag{1.11}$$
$$\frac{\partial \mathscr{L}}{\partial e_{t+1}} = -\lambda_t P_t + E_t \left[\lambda_{t+1} (y_{t+1} + P_{t+1}) \right] = 0, \tag{1.12}$$
$$\frac{\partial \mathscr{L}}{\partial b_{t+1}} = -\lambda_t q_t + E_t \left[\lambda_{t+1} \right] = 0. \tag{1.13}$$

Insert (1.11) into (1.12) and (1.13)

$$\frac{1}{(1+\rho)^t} c_t^{-\sigma} P_t = E_t \left[\frac{1}{(1+\rho)^{t+1}} c_{t+1}^{-\sigma} (y_{t+1} + P_{t+1}) \right], \tag{1.14}$$
$$\frac{1}{(1+\rho)^t} c_t^{-\sigma} q_t = E_t \left[\frac{1}{(1+\rho)^{t+1}} c_{t+1}^{-\sigma} \right]. \tag{1.15}$$

Rearrange to get

$$P_t = \frac{1}{1+\rho} E_t \left[\left(\frac{c_t}{c_{t+1}} \right)^{\sigma} (y_{t+1} + P_{t+1}) \right], \tag{1.16}$$
$$q_t = \frac{1}{1+\rho} E_t \left[\left(\frac{c_t}{c_{t+1}} \right)^{\sigma} \right]. \tag{1.17}$$

(b) Express P_t and q_t in terms of y_t and expectations involving y_{t+1}, P_{t+1}, and c_{t+1}. Show how P_t depends on the covariance between P_{t+1} and c_{t+1}, and define P_t as the sum of the riskless return and the risk premium.

Equations (1.16) and (1.17) are the expressions for P_t and q_t. Further manipulation on (1.16)

$$\begin{aligned} P_t &= \frac{1}{1+\rho} E_t\left[\left(\frac{c_t}{c_{t+1}}\right)^{\sigma}\right] E_t\left(y_{t+1}+P_{t+1}\right) + \frac{1}{1+\rho}\text{cov}\left[y_{t+1}+P_{t+1}, \left(\frac{c_t}{c_{t+1}}\right)^{\sigma}\right] \\ &= \underbrace{q_t E_t\left(y_{t+1}+P_{t+1}\right)}_{(A)} + \underbrace{\frac{1}{1+\rho}\text{cov}\left[y_{t+1}+P_{t+1}, \left(\frac{c_t}{c_{t+1}}\right)^{\sigma}\right]}_{(B)}. \end{aligned}$$

Part (A) of the right-hand side captures the riskless return, and part (B) the risk premium.

(c) Define the implicit return of the riskless asset, the bond, as

$$R_b = \frac{1}{q_t},$$

and the implicit return of the risky assets, the stock, as

$$R_s = \frac{P_{t+1}+y_{t+1}}{P_t}.$$

Rewrite the expressions in (b) with R_b and R_s.

Manipulate the Eqs. (1.16) and (1.17)

$$1 = \frac{1}{1+\rho} E_t\left[\left(\frac{c_t}{c_{t+1}}\right)^{\sigma} \frac{y_{t+1}+P_{t+1}}{P_t}\right] = \frac{1}{1+\rho} E_t\left[\left(\frac{c_t}{c_{t+1}}\right)^{\sigma} R_s\right], \tag{1.18}$$

$$1 = \frac{1}{q_t}\frac{1}{1+\rho} E_t\left[\left(\frac{c_t}{c_{t+1}}\right)^{\sigma}\right] = \frac{1}{1+\rho} R_b E_t\left[\left(\frac{c_t}{c_{t+1}}\right)^{\sigma}\right]. \tag{1.19}$$

Now assume that the consumption growth rate is

$$\frac{c_{t+1}}{c_t} = \frac{y_{t+1}}{y_t} = \gamma \exp\left(\epsilon_{yt} - \frac{\sigma_y^2}{2}\right),$$

in which γ is a positive constant and ϵ_{yt} is a normally distributed i.i.d. shock, $\epsilon_{yt} \sim N\left(0, \sigma_y^2\right)$.

And assume that R_s fluctuates around $\overline{R}_s$ as

$$R_s = \overline{R}_s \exp\left(\epsilon_{st} - \frac{\sigma_s^2}{2}\right),$$

in which ϵ_{st} is a normally distributed i.i.d. shock, $\epsilon_{st} \sim N\left(0, \sigma_s^2\right)$.

Find the equity premium $\overline{R}_s - R_b$ *in terms of σ, ϵ_{yt}, and ϵ_{st}.*

Insert the growth rate of consumption, Eq. (1.19) becomes

$$1 = \frac{1}{1+\rho} R_b E_t \left\{\gamma^{-\sigma} \exp\left[-\sigma\left(\epsilon_{yt} - \frac{\sigma_y^2}{2}\right)\right]\right\}, \tag{1.20}$$

and Eq. (1.18) becomes

$$1 = \frac{1}{1+\rho} E_t \left\{\gamma^{-\sigma} \exp\left[-\sigma\left(\epsilon_{yt} - \frac{\sigma_y^2}{2}\right)\right] R_s\right\}. \tag{1.21}$$

Further manipulation on (1.20)

$$\begin{aligned}
1 &= \frac{1}{1+\rho} R_b \gamma^{-\sigma} E_t \left\{\exp\left[-\sigma\epsilon_{yt} + \frac{\sigma\sigma_y^2}{2}\right]\right\} \\
&= \frac{1}{1+\rho} R_b \gamma^{-\sigma} E_t \left\{\exp\left[-\sigma\epsilon_{yt} - \frac{\sigma^2\sigma_y^2}{2} + \frac{\sigma^2\sigma_y^2}{2} + \frac{\sigma\sigma_y^2}{2}\right]\right\} \\
&= \frac{1}{1+\rho} R_b \gamma^{-\sigma} \exp\left[\sigma(1+\sigma)\frac{\sigma_y^2}{2}\right] E_t \left\{\exp\left[-\sigma\epsilon_{yt} - \frac{\sigma^2\sigma_y^2}{2}\right]\right\}.
\end{aligned}$$

Define a new random variable

$$\kappa = \exp\left[-\sigma\epsilon_{yt} - \frac{\sigma^2\sigma_y^2}{2}\right],$$

then take logarithm on κ

$$\ln\kappa = -\sigma\epsilon_{yt} - \frac{\sigma^2\sigma_y^2}{2}.$$

Since $\epsilon_{yt} \sim N\left(0, \sigma_y^2\right)$, it's easy to see that

$$\ln\kappa \sim N\left(-\frac{\sigma^2\sigma_y^2}{2}, \sigma^2\sigma_y^2\right),$$

i.e., κ follows a log-normal distribution. Therefore

$$E_t[\kappa] = \exp\left(-\frac{\sigma^2\sigma_y^2}{2} + \frac{\sigma^2\sigma_y^2}{2}\right) = 1.$$

Then Eq. (1.20) is simplified as

$$1 = \frac{1}{1+\rho} R_b \gamma^{-\sigma} \exp\left[\sigma(1+\sigma)\frac{\sigma_y^2}{2}\right]. \tag{1.22}$$

Further manipulation on (1.21) by inserting the expression of R_s

$$\begin{aligned}
1 &= \frac{1}{1+\rho} E_t \left\{\gamma^{-\sigma} \exp\left[-\sigma\left(\epsilon_{yt} - \frac{\sigma_y^2}{2}\right)\right] \overline{R}_s \exp\left(\epsilon_{st} - \frac{\sigma_s^2}{2}\right)\right\} \\
&= \frac{1}{1+\rho} \overline{R}_s \gamma^{-\sigma} E_t \left\{\exp\left[\epsilon_{st} - \frac{\sigma_s^2}{2} - \sigma\left(\epsilon_{yt} - \frac{\sigma_y^2}{2}\right)\right]\right\}.
\end{aligned}$$

Again define a new random variable

$$\zeta = \exp\left[\epsilon_{st} - \frac{\sigma_s^2}{2} - \sigma\left(\epsilon_{yt} - \frac{\sigma_y^2}{2}\right)\right],$$

then take logarithm on ζ

$$\ln\zeta = \epsilon_{st} - \frac{\sigma_s^2}{2} - \sigma\left(\epsilon_{yt} - \frac{\sigma_y^2}{2}\right).$$

Since $\ln\zeta$ is just a linear combination of two normally distributed variables, it should be also normally distributed, i.e. ζ follows a log-normal distribution. And

$$E_t[\ln\zeta] = E_t\left[\epsilon_{st} - \frac{\sigma_s^2}{2} - \sigma\left(\epsilon_{yt} - \frac{\sigma_y^2}{2}\right)\right] = -\frac{\sigma_s^2}{2} + \frac{\sigma\sigma_y^2}{2},$$

as well as

$$\begin{aligned}
\text{var}[\ln\zeta] &= E_t[(\ln\zeta)^2] - (E_t[\ln\zeta])^2 \\
&= E_t\left[(\epsilon_{st} - \sigma\epsilon_{yt})^2\right] \\
&= \sigma_s^2 - 2\sigma\,\text{cov}\left(\epsilon_{st}, \epsilon_{yt}\right) + \sigma^2\sigma_y^2.
\end{aligned}$$

Then Eq. (1.21) is simplified as

$$
\begin{aligned}
1 &= \frac{1}{1+\rho}\overline{R}_s \gamma^{-\sigma} \exp\left\{E_t[\ln\zeta] + \frac{\text{var}[\ln\zeta]}{2}\right\} \\
&= \frac{1}{1+\rho}\overline{R}_s \gamma^{-\sigma} \exp\left\{-\frac{\sigma_s^2}{2} + \frac{\sigma\sigma_y^2}{2} + \frac{\sigma_s^2 - 2\sigma\text{cov}\left(\epsilon_{st}, \epsilon_{yt}\right) + \sigma^2\sigma_y^2}{2}\right\} \\
&= \frac{1}{1+\rho}\overline{R}_s \gamma^{-\sigma} \exp\left[\sigma(1+\sigma)\frac{\sigma_y^2}{2} - \sigma\text{cov}\left(\epsilon_{st}, \epsilon_{yt}\right)\right].
\end{aligned}
$$

Divide this equation by (1.22)

$$
\frac{\overline{R}_s}{R_b} = \exp\left[\sigma\text{cov}\left(\epsilon_{st}, \epsilon_{yt}\right)\right].
$$

Take logarithm on both sides and get

$$
\ln\overline{R}_s - \ln R_b = \sigma\text{cov}\left(\epsilon_{st}, \epsilon_{yt}\right).
$$

Since $\overline{R}_s$ and R_b are not too different from 1, the equation can be approximately written as

$$
\overline{R}_s - R_b \approx \sigma\text{cov}\left(\epsilon_{st}, \epsilon_{yt}\right). \tag{1.23}
$$

(d) *Estimated from the US financial market,* $R_b = 1.01$, $\overline{R}_s = 1.07$, *and* $\text{cov}\left(\epsilon_{st}, \epsilon_{yt}\right) = 0.002$. *Compute* σ *using the result of (c). What does* σ *mean in economics? Why do people call this result a puzzle?*

Apply the numbers into (1.23) and compute $\sigma = 30$. σ is Arrow–Pratt measure of relative risk aversion in this problem, whose normal value seldom exceeds 6—risk premium is too high!

1.2.8 Dixit–Stiglitz Indices for Continuous Commodity Space

Consider a one-person economy. Mr. Rubinson Crusoe is the only agent in this economy, consuming a continuum of commodities $i \in [0, 1]$. *Suppose that the consumption index* C *of him is defined as*

$$
C = \left[\int_0^1 Z_i^{\frac{1}{\eta}} C_i^{\frac{\eta-1}{\eta}} di\right]^{\frac{\eta}{\eta-1}}
$$

in which C_i is the consumption of good i and Z_i is the taste shock for good i. Suppose that Crusoe has an amount of endowment Y to spend on goods. Therefore the budget constraint is

$$\int_0^1 P_i C_i di = Y.$$

(a) Find the first-order condition for the problem of maximizing C subject to the budget constraint. Solve for C_i in terms of Z_i, P_i, and the Lagrange multiplier on the budget constraint.

The agent's problem is to

$$\max_{C_i} \; C = \left[\int_0^1 Z_i^{\frac{1}{\eta}} C_i^{\frac{\eta-1}{\eta}} di\right]^{\frac{\eta}{\eta-1}},$$
$$s.t. \; \int_0^1 P_i C_i di = Y.$$

Set up the Lagrangian for this problem

$$\mathscr{L} = \left[\int_0^1 Z_i^{\frac{1}{\eta}} C_i^{\frac{\eta-1}{\eta}} di\right]^{\frac{\eta}{\eta-1}} + \lambda \left(Y - \int_0^1 P_i C_i di\right),$$

$\forall i \in [0, 1]$ the first-order condition is

$$\frac{\partial \mathscr{L}}{\partial C_i} = \frac{\eta}{\eta - 1}\left[\int_0^1 Z_i^{\frac{1}{\eta}} C_i^{\frac{\eta-1}{\eta}} di\right]^{\frac{\eta}{\eta-1}-1} \frac{\eta-1}{\eta} Z_i^{\frac{1}{\eta}} C_i^{\frac{\eta-1}{\eta}-1} - \lambda P_i = 0.$$

Rearrange to get

$$C_i^{-\frac{1}{\eta}} = \frac{\lambda P_i}{\left[\int_0^1 Z_i^{\frac{1}{\eta}} C_i^{\frac{\eta-1}{\eta}} di\right]^{\frac{1}{\eta-1}} Z_i^{\frac{1}{\eta}}},$$

$$C_i = \frac{\left[\int_0^1 Z_i^{\frac{1}{\eta}} C_i^{\frac{\eta-1}{\eta}} di\right]^{\frac{\eta}{\eta-1}} Z_i}{(\lambda P_i)^{\eta}}.$$

(b) Use the budget constraint to find C_i in terms of Z_i, P_i and Y.

First one has to eliminate the Lagrange multiplier. Notice that the expression C_i holds for any good $j \in [0, 1]$, $j \neq i$, i.e.

$$C_i = \frac{CZ_i}{(\lambda P_i)^\eta},$$
$$C_j = \frac{CZ_j}{\left(\lambda P_j\right)^\eta}.$$

Divide C_i by C_j to eliminate λ

$$\frac{C_i}{C_j} = \frac{Z_i}{Z_j}\left(\frac{P_j}{P_i}\right)^\eta.$$

Insert it into the budget constraint

$$\int_0^1 P_i C_i di = \int_0^1 P_i \frac{Z_i}{Z_j}\left(\frac{P_j}{P_i}\right)^\eta C_j di = \frac{C_j P_j^\eta}{Z_j}\int_0^1 Z_i P_i^{1-\eta} di = Y,$$

solve to get

$$C_j = \frac{YZ_j}{P_j^\eta \int_0^1 Z_i P_i^{1-\eta} di}.$$

Since i and j are arbitrarily taken, replace j with i and get

$$C_i = \frac{YZ_i}{P_i^\eta \int_0^1 Z_i P_i^{1-\eta} di}.$$

(c) Insert the result of (b) into the expression for C and show that $C = \frac{Y}{P}$, *in which*

$$P = \left(\int_0^1 Z_i P_i^{1-\eta} di\right)^{\frac{1}{1-\eta}}.$$

Insert the result of (b) into the expression for C

$$C = \left[\int_0^1 Z_i^{\frac{1}{\eta}}\left(\frac{YZ_i}{P_i^\eta \int_0^1 Z_i P_i^{1-\eta} di}\right)^{\frac{\eta-1}{\eta}} di\right]^{\frac{\eta}{\eta-1}}$$
$$= \left[\int_0^1 Z_i^{\frac{1}{\eta}} \frac{Y^{\frac{\eta-1}{\eta}} Z_i^{\frac{\eta-1}{\eta}}}{P_i^{\eta-1}\left(\int_0^1 Z_i P_i^{1-\eta} di\right)^{\frac{\eta-1}{\eta}}} di\right]^{\frac{\eta}{\eta-1}}$$

$$
\begin{aligned}
&= \left[\int_0^1 Z_i P_i^{1-\eta} di \frac{Y^{\frac{\eta-1}{\eta}}}{\left(\int_0^1 Z_i P_i^{1-\eta} di \right)^{\frac{\eta-1}{\eta}}} \right]^{\frac{\eta}{\eta-1}} \\
&= Y \left[\int_0^1 Z_i P_i^{1-\eta} di \right]^{\frac{\eta}{\eta-1}-1} \\
&= Y \left[\int_0^1 Z_i P_i^{1-\eta} di \right]^{\frac{1}{\eta-1}} \\
&= \frac{Y}{P}.
\end{aligned}
$$

(d) Use the results in (b) and (c) to show that

$$
C_i = Z_i \left(\frac{P_i}{P} \right)^{-\eta} \left(\frac{Y}{P} \right).
$$

Interpret this result.

Manipulate the result of (b) using the definition of P

$$
C_i = \frac{Y Z_i}{P_i^{\eta} P^{1-\eta}} = Z_i \left(\frac{P_i}{P} \right)^{-\eta} \left(\frac{Y}{P} \right). \tag{1.24}
$$

This gives the demand function of good i in terms of the taste shock, relative price and the agent's income.

(Addendum) The motivation of adding Z_i in designing this exercise is to access some recent research, i.e. Ravn et al. (2006, 2007) on deep habit. Suppose that Z_i follows a random process such that

$$
Z_{i,t} = \theta Z_{i,t-1} + \epsilon_t,
$$

in which $\epsilon_t \sim N\left(\mu, \sigma^2\right)$. Then in a dynamic context at time t Eq. (1.24) can be rewritten as

$$
\begin{aligned}
C_{i,t} &= \left(\theta Z_{i,t-1} + \epsilon_t\right) \left(\frac{P_{i,t}}{P} \right)^{-\eta} \left(\frac{Y}{P} \right) \\
&= \theta \left(Z_{i,t-1} + \epsilon_{t-1}\right) \left(\frac{P_{i,t}}{P} \right)^{-\eta} \left(\frac{Y}{P} \right) + \left(\epsilon_t - \theta\epsilon_{t-1}\right) \left(\frac{P_{i,t}}{P} \right)^{-\eta} \left(\frac{Y}{P} \right) \\
&= \theta C_{i,t} + \left(\epsilon_t - \theta\epsilon_{t-1}\right) \left(\frac{P_{i,t}}{P} \right)^{-\eta} \left(\frac{Y}{P} \right),
\end{aligned}
$$

which shows the *deep habit* in individual goods, rather than the conventional *superficial habit* in an aggregate level, in consumption.

When habits are formed at the level of individual goods, firms take into account that the demand they will face in the future depends on their current sales. This is because higher consumption of a particular good in the current period makes consumers, all other things equal, more willing to buy that good in the future through the force of habit. Thus, when habits are deeply rooted, the optimal pricing problem of the firm becomes dynamic.

To see this, notice that the demand function of $C_{i,t}$ is composed of two terms. The second term displays a price elasticity of η, and the first originates exclusively from habitual consumption of good i. Therefore, the first term is perfectly price inelastic. Therefore the price elasticity of the demand for good i is a weighted average of the elasticities of the two terms just described, namely η and 0. The weight on η is given by the share of the price-elastic term in total demand. When aggregate demand rises, the weight of the price-elastic term in total demand increases, and as a result the price elasticity increases. We refer to this effect as the *price-elasticity effect* of deep habits. Because the mark-up is inversely related to the price elasticity of demand, it follows that under deep habits, an expansion in aggregate demand induces a *decline* in mark-ups.

In addition to the price-elasticity effect, deep habits influence the equilibrium dynamics of mark-ups through an *intertemporal effect*. This effect arises because firms take into account that current price decisions affect future demand conditions via the formation of habits. According to the intertemporal effect, when the present value of future per-unit profits are expected to be high, firms have an incentive to invest in customer base today. They do so by building up the current stock of habit. In turn, this increase in habits is brought about by inducing higher current sales via a decline in current mark-ups.

Seemingly similar as its counterparts in industrial organization, the deep-habit differs from the switching cost/customer-market formulations such that in the deep-habit model there is gradual substitution between differentiated goods rather than discrete switches among suppliers. One advantage of this implication of the deep-habit model, from the point of view of analytical tractability, is that under the deep-habit formulation one does not face an aggregation problem. In equilibrium, buyers can distribute their purchases identically, and still suppliers face a gradual loss of customers if they raise their relative prices. The deep-habit-formation model can therefore be viewed as a natural vehicle for incorporating switching cost / customer-market models into a dynamic general equilibrium framework.

1.2.9 Menu Cost and Nominal Price Rigidity

Instead of solving the exercise, we briefly explain, in a more general set-up, how menu cost with monopolistic competition generates price rigidities.

Consider an economy of monopolistically competitive firms facing downward sloping demand curves and being initially at the equilibrium level such that the

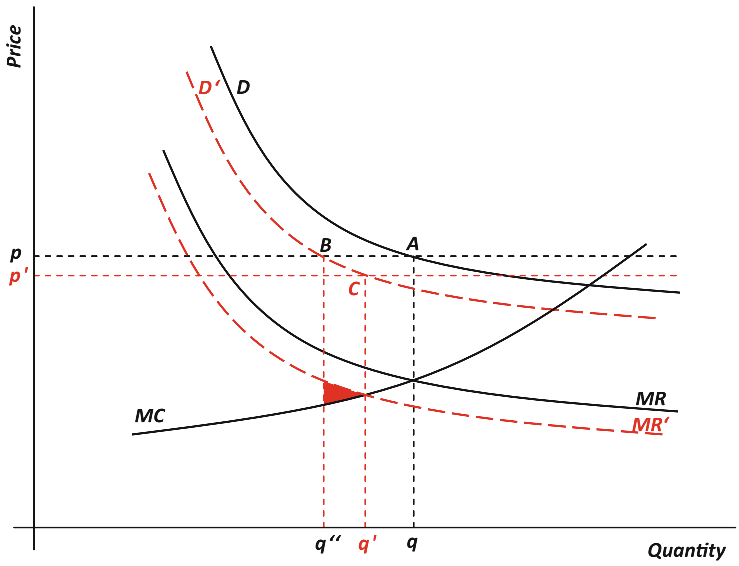

Fig. 1.2 The incentive for price adjustment

marginal revenue equals the marginal cost for each firm. Figure 1.2 shows the optimal price level of a representative firm: Given the demand curve D and the corresponding marginal revenue curve MR the optimal output q is achieved where MR and the marginal cost curve, MC, cross each other, and the optimal price is set by $p = D^{-1}(q)$ as point A shows.

Then suppose now there is an unexpected fall in aggregate demand Y_t; this implies a proportional drop in the representative firm's demand which shifts D curve inward to D' and MR curve to MR', therefore the new optimal strategy for the firm becomes (p', q') as point C in the figure.

Now the question is, how high the incentive it is for the firm to adjust its price level from p to p'? The motivation behind this question is that if the incentive is small enough, even a minor exogenous cost associated with such price adjustment may deter the setting of the new price.

Suppose that firm simply keeps the old price p, then the new output is determined by the new demand curve, as point B shows. Notice that the profit for the firm is just the area between MR and MC curves, the loss from keeping the old price, or the incentive to adjust the price, is the red triangle area (denoted by $\Delta\Pi(z)_1$ for simplicity)—It is indeed very small. And the area would be even smaller, if the demand elasticity becomes higher, i.e., when the firm faces a flatter D curve.

Given that the firm's incentive for price adjustment in response to an aggregate demand shock is small, and smaller when the consumers are more sensitive to the price change (under a flatter D curve), now we assume that there is a *menu cost* c^{MENU} associated with the price change (such cost may be as small as printing a new version of your menu with the new prices). Then if the gain from any price change is no higher than the menu cost, $\Delta\Pi(z)_1 \leq c^{MENU}$, such price adjustment would be deterred.

References

Dixit, A. K., & Stiglitz, J. E. (1977). Monopolistic Competition and optimal product diversity. *American Economic Review, 67*, 297–308.

Lucas, R. E., Jr. (1978). Asset prices in an exchange economy. *Econometrica, 46*, 1429–1445.

Mas-Colell, A., Whinston, M. D., & Green, J. R. (1995). *Microeconomic theory*. New York: Oxford University Press.

Mehra, R., & Prescott, E. C. (1985). The equity premium: A puzzle. *Journal of Monetary Economics, 15*, 145–161.

Ravn, M., Schmitt-Grohé, S., & Uribe, M. (2006). Deep habits. *Review of Economic Studies, 73*, 195–218.

Ravn, M., Schmitt-Grohé, S., & Uribe, M. (2007). Pricing To habits and the law of one price. *American Economic Review, 97*, 232–238.

Money and Long Run Growth

2

2.1 Exercises

2.1.1 Short Review Questions

(a) What are the roles of money in the economy? What are the roles of money in the banking sector? What determine(s) the value of money in a closed economy?
(b) What determine(s) the price level in a closed economy? Under which condition(s) an explosive growth path, or, the path of hyperinflation, cannot be ruled out? What determine(s) the demand for money?
(c) For the long run equilibrium of monetary policy, what are the differences between price level targeting and inflation targeting?
(d) What is seignorage? What does seignorage imply for optimal inflation?
(e) How is the demand for money motivated in different modelling frameworks: (1) money-in-the-utility, (2) cash-in-advance, (3) shopping time? Is optimal monetary policy different across these frameworks?
(f) What is *Friedman Rule*? Why, in reality, is positive inflation desired?

2.1.2 Seigniorage and Inflation

Seignorage, which is the real revenue the government obtains from printing new currency, and exchanging it for real goods, is defined à la Cagan (1956),

$$S = \frac{M_t - M_{t-1}}{P_t}$$

J. Cao, G. Illing, *Instructor's Manual for* Money: Theory and Practice, Springer Texts in Business and Economics,
https://doi.org/10.1007/978-3-030-23618-2_2

where M_t is the money supply at t and P_t the price level at t. Assume that the demand for real money balances is given by

$$\frac{M_t}{P_t} = \left(\frac{P_{t+1}}{P_t}\right)^{-\eta}$$

with $\eta > 0$.

(a) Give some interpretation to the money demand function.
(b) Assume the government controls the growth rate of money supply $\frac{M_t}{M_{t-1}} = 1 + \mu$. Show that, correspondingly, inflation will be constant μ all the time. If the government tries to maximize its revenue, what is the optimal μ? Provide some interpretation to your solution.
(c) Compute the loss in consumer surplus and the deadweight loss arising with this optimal μ.

2.1.3 Money in the Utility: The Steady State

Consider an infinitely lived agent with utility function

$$\int_0^{+\infty} [c(t) + V(m(t))]\, e^{-\rho t} dt,$$

where c is the consumption, m are real money holdings, and V is an increasing and concave function. Money is the only asset, yielding a nominal interest rate flow $i(t)$. Income is exogenously given by $y(t)$.

(a) Formulate the transition equation in real balances (money holdings).
(b) Formulate the Hamiltonian and first-order conditions.
(c) The growth rate of nominal money supply is given by μ. Derive a differential equation describing the optimal real balances.
(d) Discuss potential steady state equilibria and their stability. Characterize conditions that rule out hyperinflationary bubbles.
(e) Discuss the special case of $V(m) = m^\alpha$.

2.1.4 Money in the Utility: The Dual Form

Consider a discrete version of Sidrauski's money in the utility approach: An infinitely lived representative agent maximizes discounted life-time utility

$$\sum_{t=0}^{+\infty} \beta^t U(c_t, m_t)$$

with $\beta \in (0, 1)$ as discount rate, c_t consumption, and $m_t = \frac{M_t}{P_t}$ as real money balances. Each period, the agent is endowed with y_t. y_t can be used for private or government consumption: $y_t = c_t + g_t$. Initially, the agent owns the money stock M_0 and one period nominal government bonds B_0. Period t bonds B_t yield a return i_t. The government finances g_t via taxes τ_t , seigniorage or government bonds. To make it easier, assume that there is no nominal return for holding money, i.e. $i_{m,t} = 0$; also the endowment economy implies that there's no other investment opportunity except holding government bonds.

(a) Formulate the period budget constraint of both the agent and the government and derive the present value budget constraint. To make the computation easier, you may define an auxiliary state variable, *beginning of period t wealth* of either party, as $W_t = (1 + i_{t-1})B_{t-1} + M_{t-1}$.
(b) Characterize the first-order conditions for the agent's optimal path.
(c) Show that with additive separable preferences $U(c_t, m_t) = u(c_t) + v(m_t)$, the real rate of interest depends only on the time path of the real resources available for consumption.
(d) Assume that $U(c_t, m_t) = c_t^\alpha + m_t^\alpha$. Derive the money demand function $m(c_t, i_t)$ and characterize elasticity with respect to c_t and i_t . Show why the price level may not be determinate if the central bank pegs the interest rate to a fixed level $i_t = \bar{i}$.
(e) Assume that both endowment and government spending are constant: $y_t = y$; $g_t = g$. Characterize conditions for steady state. Show that the Friedman rule maximizes per period utility. Discuss reasons why this rule may not be optimal in a more general setting.

2.1.5 Cash-in-Advance Models of Money Demand

Consider a representative household that receives an exogenous income Y_t each period and gains utility only from consumption C_t. Specifically, its utility function is given by

$$U_0 = \sum_{t=0}^{\infty} \beta^t u(C_t).$$

At the beginning of each period t, the household needs to make two decisions about his period income. On the one hand, the household decides how much to consume in period t and how much to save for the future. On the other hand, the household needs to decide on how to allocate its savings across one-period bonds B_t which yield a nominal interest rate of i_t and money balances M_t which do not pay interest. However, money balances are necessary to cover all consumption purchases the household makes in period t at price P_t.

(a) Write down and interpret the household's *cash-in-advance* constraint and the period budget constraint.
(b) Derive the Euler equation and the optimal money demand of the household under the additional assumption that $i_t > 0$. Discuss intuitively the economics underlying the optimal money demand.
(c) Describe the effects of expansionary monetary policy and illustrate them graphically in an $P_t - i_t$ diagram.
(d) How does the analysis change once you take the zero lower bound on nominal interest rates into account, i.e., if $i_t = 0$?

2.1.6 Cost of Inflation and Optimal Monetary Policy

Based on Lucas (2000) the canonical theory of monetary policy assumes that inflation is costly for society. In the following we explore a justification for that assumption, using Sidrauski's money-in-the-utility function.

Consider an economy with a representative infinitely lived consumer whose preference is given by

$$\sum_{t=0}^{+\infty} \beta^t \left[u(c_t) + v(m_t)\right] \tag{2.1}$$

in which $u(\cdot)$ and $v(\cdot)$ are increasing and strictly concave utility functions, c_t is the consumption at date t, m_t is the real money balance at the end of period t, and $\beta \in (0, 1)$ is the discount factor.

Let b_t be real bond holdings at the end of period t that pay a nominal interest rate i_{t+1} at the beginning of the next period, $M_t = P_t m_t$ be nominal money balances, P_t be the price level, and y be the time-invariant and exogenous real income received by the consumer each period. The consumer also receives real net transfers from the government, τ_t. Then, to ease your computation, the nominal budget constraint of the consumer is given by

$$P_{t-1}b_{t-1}(1 + i_t) + P_t y + P_t \tau_t = P_t c_t + M_t - M_{t-1} + P_t b_t.$$

(a) Let r_t be the real interest rate and π_t be the inflation rate, such that

$$1 + i_t = (1 + r_t)(1 + \pi_t). \tag{2.2}$$

Show that consumption can be written as

$$c_t = b_{t-1}(1 + r_t) + y + \tau_t - m_t + \frac{m_{t-1}}{1 + \pi_t} - b_t. \tag{2.3}$$

(b) Using (2.1) and (2.3), show that the following efficiency conditions hold:

$$-u'(c_t) + \beta \frac{u'(c_{t+1})}{1+\pi_{t+1}} + v'(m_t) = 0, \tag{2.4}$$

$$-u'(c_t) + \beta u'(c_{t+1})(1+r_{t+1}) = 0. \tag{2.5}$$

Provide some intuitions for Eqs. (2.4) and (2.5), and show that (2.4) and (2.5) define a money demand function

$$v'(m_t) = \frac{i_{t+1}}{1+i_{t+1}} u'(c_t). \tag{2.6}$$

What is the relationship between money demand and nominal interest rates for a given level of consumption? What is the relationship between money demand and consumption for a given nominal interest rate?

(c) Assume, for the rest of the problem, that there is no government expenditure and no public debt, so that government prints money only to make net transfers to the consumer, i.e.

$$M_t - M_{t-1} = P_t \tau_t.$$

Since there is only one consumer and the government does not issue public debt, equilibrium requires $b_t = 0$. What is c_t in equilibrium? Using your expression for c_t and Eq. (2.5), derive the expression for the real interest rate.

(d) Assume, in addition, that the government follows a constant nominal money growth rule

$$M_t = (1+\mu) M_{t-1}. \tag{2.7}$$

Define a steady state in this model as a situation in which real variables do not change. In particular, in the steady state $m_t = \overline{m}$. Given (2.7) and the fact that $m_t = \frac{M_t}{P_t}$, find the steady-state level of inflation in this model, call it $\overline{\pi}$.

(e) Using Eq. (2.6) evaluated in steady state, find an expression for $\overline{m}$ in terms of $\overline{\pi}$. What is the steady state effect of $\overline{\pi}$ on $\overline{m}$? What is the effect of $\overline{\pi}$ on steady-state consumption? What is the welfare effect of increasing $\overline{\pi}$? What is the optimal level of steady-state inflation?

2.1.7 Overlapping Generations with Money

Samuelson (1958) alternatively, macro economy can be modelled in an infinite horizon with *finitely-lived* agents. Suppose, as in the Diamond (1965) model, that N_t 2-period-lived individuals are born in period t and that generations are growing with rate n. The utility function of a representative individual is $U_t = \ln c_{1,t} + \ln c_{2,t+1}$.

Each individual is born with an endowment of A units of the economy's single good. The good can either be consumed or stored. Each unit stored yields $x > 0$ units next period.

In period 0, there are N_0 young individuals and $\frac{1}{1+n}N_0$ old individuals endowed with some amount Z of the good. Their utility is simply $c_{2,0}$.

(a) Describe the decentralized equilibrium of this economy. (Hint: Will members of any generation trade with members of another generation?)
(b) Consider paths where the fraction of agents' endowment that is stored, s_t, is constant over time. What is per capita consumption (weighted average from young and old) on such a path as a function of s?
(c) If $x < 1 + n$, which value of $s \in [0, 1]$ is maximizing per capita consumption?
(d) Is the decentralized equilibrium Pareto-efficient? If not, how could a social planner raise welfare?

Suppose now that old individuals in period 0 are also endowed with M units of a storable, divisible commodity, which we call money. Money is not a source of utility. Assume $x < 1 + n$.
(e) Suppose the price of the good in units of money in periods t and $t + 1$ is given by P_t and P_{t+1}, respectively. Derive the demand functions of an individual born in t.
(f) Describe the set of equilibria.
(g) Explain why there is an equilibrium with $P_t \to +\infty$. Explain why this must be the case if the economy ends at some date T that is common knowledge among all generations.

2.2 Solutions for Selected Exercises

2.2.1 Seigniorage and Inflation

Seignorage, which is the real revenue the government obtains from printing new currency, and exchanging it for real goods, is defined à la Cagan (1956),

$$S = \frac{M_t - M_{t-1}}{P_t}$$

where M_t is the money supply at t and P_t the price level at t. Assume that the demand for real money balances is given by

$$\frac{M_t}{P_t} = \left(\frac{P_{t+1}}{P_t}\right)^{-\eta}$$

with $\eta > 0$.

(a) Give some interpretation to the money demand function.

The general form of the money demand is given by

$$\frac{M_t}{P_t} = \mathcal{L}\left(i_{t+1}, Y_t\right)$$

where $\mathcal{L}(\cdot)$ stands for Keynes' "Liquidity Preference" theory. This form of the money demand comes both from Baumol–Tobin-type models of inventories and from microfounded dynamic models with money-in-the-utility (such as Sidrauski model).

Log-linearizing the equation above (and denoting the logs by the small letters) yields

$$m_t - p_t = \phi y_t - \eta i_{t+1},$$

where ϕ and η are the income and interest rate elasticities of the money demand, respectively.

Cagan made the following assumptions in his model: y_t and r_t (real interest rate) are given exogenously and constant. It is a normal long-run assumption for the analysis of the monetary policy (i.e., it implies money neutrality). It turns out to be a reasonable assumption for the economies in hyperinflation because in this situation prices are essentially fully flexible. Then we can rewrite the money demand in log form as

$$m_t - p_t = \text{const} - \eta \pi^e_{t+1},$$

since by the Fisher identity $i_{t+1} = r_{t+1} + \pi^e_{t+1}$. Omitting the constant and assuming a deterministic equilibrium path for prices (i.e., $\pi^e_{t+1} = \pi_{t+1}$ in the case of no uncertainty) yields exactly the same money demand function in the problem set.

Intuitively, money demand should depend negatively on the expected inflation which is an important cost of holding money balances.

(b) Assume the government controls the growth rate of money supply $\frac{M_t}{M_{t-1}} = 1 + \mu$. Show that, correspondingly, inflation will be constant μ all the time. If the government tries to maximize its revenue, what is the optimal μ? Provide some interpretation to your solution.

To show that inflation is equal to the growth rate of money we solve forward the following difference equation (money demand):

$$m_t - p_t = -\eta \pi_{t+1} = -\eta \left(p_{t+1} - p_t\right),$$

then solve for p_t

$$p_t = \frac{1}{1+\eta} \sum_{j=0}^{+\infty} \left(\frac{\eta}{1+\eta} \right)^j m_{t+j},$$

therefore

$$\begin{aligned} \pi_{t+1} = p_{t+1} - p_t &= \frac{1}{1+\eta} \sum_{j=0}^{+\infty} \left(\frac{\eta}{1+\eta} \right)^j \Delta m_{t+j+1} \\ &= \frac{1}{1+\eta} \sum_{j=0}^{+\infty} \left(\frac{\eta}{1+\eta} \right)^j \mu \\ &= \mu. \end{aligned}$$

The government seigniorage revenue is given by

$$\begin{aligned} S &= \frac{M_t - M_{t-1}}{P_t} \\ &= \frac{M_t - M_{t-1}}{M_t} \frac{M_t}{P_t} \\ &= \frac{M_{t-1}}{M_t} \frac{M_t - M_{t-1}}{M_{t-1}} \left(\frac{P_{t+1}}{P_t} \right)^{-\eta} \\ &= \mu (1+\mu)^{-\eta-1}. \end{aligned}$$

It is maximized when the first-order condition holds,

$$\begin{aligned} \frac{\partial S}{\partial \mu} &= (1+\mu)^{-\eta-1} + \mu(-\eta-1)(1+\mu)^{-\eta-2} = 0, \\ 1 + \mu + \mu(-\eta-1) &= 0, \\ \mu^* &= \frac{1}{\eta}. \end{aligned}$$

All we have solved was a simple monopoly problem. The government is a monopolist supplier of currency, with a zero marginal cost of currency creation. The "price" of currency is the inflation rate or the rate of money creation. Since it faces a well-defined demand and cares about maximizing revenue, then the solution is a standard monopoly solution: set the price, μ, to be inversely related to the elasticity of demand.

(c) *Compute the loss in consumer surplus and the deadweight loss arising with this optimal* μ.

The loss in consumer surplus is given by the area under the demand for money,

$$\int_0^{\frac{1}{\eta}} \left(\frac{P_{t+1}}{P_t}\right)^{-\eta} d\pi = \int_0^{\frac{1}{\eta}} (1+\pi)^{-\eta}\, d\pi = \frac{\left(\frac{1+\eta}{\eta}\right)^{1-\eta} - 1}{1-\eta}.$$

The deadweight loss is the difference between the loss of consumer surplus and the seigniorage revenue of the government

$$\frac{\left(\frac{1+\eta}{\eta}\right)^{1-\eta} - 1}{1-\eta} - S^* = \frac{\left(\frac{1+\eta}{\eta}\right)^{1-\eta} - 1}{1-\eta} - \frac{1}{\eta}\left(\frac{1+\eta}{\eta}\right)^{-1-\eta} = \left(\frac{1+\eta}{\eta}\right)^{-\eta} > 0.$$

2.2.2 Money in the Utility: The Steady State

Consider an infinitely lived agent with utility function

$$\int_0^{+\infty} [c(t) + V(m(t))]\, e^{-\rho t} dt,$$

where c is the consumption, m are real money holdings, and V is an increasing and concave function. Money is the only asset. Income is exogenously given by $y(t)$.

(a) *Formulate the transition equation in real balances (money holdings).*

The dynamic of the nominal money holding can be expressed as

$$\dot{M}(t) = P(t)y(t) - P(t)c(t) + i(t)M(t).$$

Therefore the transition equation in real balances can be written as

$$\begin{aligned}
\dot{m}(t) &= \frac{\dot{M}(t)}{M(t)}m(t) - \frac{\dot{P}(t)}{P(t)}m(t) \\
&= \frac{\dot{M}(t)}{P(t)} - \pi(t)m(t) \\
&= y(t) - c(t) + (i(t) - \pi(t))m(t).
\end{aligned}$$

in which the change in real money balance $\dot{m}(t)$ is determined by the increment in wealth adjusted by the loss because of inflation.

(b) Formulate the Hamiltonian and first-order conditions.

The present value Hamiltonian can be written as

$$\mathcal{H}(t) = [c(t) + V(m(t))]\, e^{-\rho t} + \lambda(t)\, [y(t) - c(t) + (i(t) - \pi(t))m(t)]\,.$$

The first-order conditions are given by

$$\frac{\partial \mathcal{H}}{\partial c} = e^{-\rho t} - \lambda = 0, \tag{2.8}$$

$$\frac{\partial \mathcal{H}}{\partial m} = e^{-\rho t} V' + (i - \pi)\lambda = -\dot{\lambda} \tag{2.9}$$

as well as the transversality condition

$$\lim_{T \to +\infty} \lambda(T)m(T) = 0. \tag{2.10}$$

Equation (2.8) gives

$$\frac{\dot{\lambda}}{\lambda} = -\rho, \tag{2.11}$$

$$\lambda(t) = \lambda(0)e^{-\rho t} \tag{2.12}$$

and Eq. (2.9) gives

$$\frac{\dot{\lambda}}{\lambda} = -V' - i + \pi. \tag{2.13}$$

Combine (2.11) and (2.13) to get the Euler equation

$$V' = \rho + \pi - i. \tag{2.14}$$

Insert (2.12) into the transversality condition (2.13) and get

$$\lim_{T \to +\infty} \lambda(0)e^{-\rho T} m(T) = 0,$$

and it holds when $\lim_{T \to +\infty} m(T) < +\infty$.

(c) The growth rate of nominal money supply is given by μ. Derive a differential equation describing the optimal real balances.

By the definition of real money balances $m(t) = \frac{M(t)}{P(t)}$, using log-linearization and get

$$\begin{aligned}\frac{\dot{m}(t)}{m(t)} &= \frac{\dot{M}(t)}{M(t)} - \frac{\dot{P}(t)}{P(t)},\\ \dot{m}(t) &= \mu m(t) - \pi(t)m(t)\\ &= (\mu - \pi(t))m(t)\\ &= (\mu + \rho - i - V')m(t)\end{aligned}$$

describing the optimal real balances.

(d) Discuss potential steady state equilibria and their stability. Characterize conditions that rule out hyperinflationary bubbles.

The solution to the problem $\dot{m}(t) = (\mu + \rho - i - V')m(t) = 0$ gives the steady state equilibria. There are several possible cases as following:

- Function $V(m)$ is linear in m. Then
  - If $V' = \mu+\rho-i > 0$, then the steady state equilibria are all the $m \in [0, +\infty)$;
  - If $V' \neq \mu + \rho - i > 0$, then the only steady state equilibrium is $m = 0$;
- Function $V(m)$ is strictly concave in m. Then
  - If $0 < V'(0) \leq \mu + \rho - i$, then the only steady state equilibrium is $m = 0$;
  - If $\mu + \rho - i < V'(0)$ and $\lim_{m\to 0} V'(m)m = 0$, then there are two steady state equilibria, $m_1^* = 0$ and $m_2^* = (V')^{-1}(\mu + \rho - i)$;
  - If $\lim_{m\to 0} V'(m)m \neq 0$, then there is a unique steady state equilibrium $m^* = (V')^{-1}(\mu + \rho - i) > 0$.

Now one can see that the condition to rule out hyperinflationary bubbles is $\lim_{m\to 0} V'(m)m \neq 0$.

(e) Discuss the special case of $V(m) = m^\alpha$.

Because $V(m)$ is increasing and concave in m, then $\alpha \in (0, 1]$. For $\alpha = 1$, one only has to compare $\mu + \rho - i$ with 1:

- If $\mu + \rho - i = 1$, then the steady state equilibria are all the $m \in [0, +\infty)$;
- Otherwise the only steady state equilibrium is $m = 0$.

For $\alpha \in (0, 1)$, $\mu+\rho-i < V'(0) = +\infty$ but $\lim_{m\to 0} V'(m)m = \lim_{m\to 0} \alpha m^\alpha = 0$. Then there are two steady state equilibria, $m_1^* = 0$ and $m_2^* = \left(\frac{\mu+\rho-i}{\alpha}\right)^{\frac{1}{\alpha-1}}$.

2.2.3 Money in the Utility: The Dual Form

Consider a discrete version of Sidrauski's money in the utility approach: An infinitely lived representative agent maximizes discounted life-time utility

$$\sum_{t=0}^{+\infty} \beta^t U(c_t, m_t)$$

with $\beta \in (0,1)$ as discount rate, c_t consumption, and $m_t = \frac{M_t}{P_t}$ as real money balances. Each period, the agent is endowed with y_t. y_t can be used for private or government consumption: $y_t = c_t + g_t$. Initially, the agent owns the money stock M_0 and one period nominal government bonds B_0 . Period t bonds B_t yield a return i_t. The government finances g_t via taxes τ_t, seigniorage or government bonds.

(a) Formulate the period budget constraint of both the agent and the government and derive the present value budget constraint.

For the representative agent, the flow budget constraint can be written as

$$P_t c_t + P_t \tau_t + M_t + B_t = P_t y_t + W_t, \tag{2.15}$$

in which her wealth at t, W_t, consists of

$$W_t = (1 + i_{t-1}) B_{t-1} + M_{t-1}. \tag{2.16}$$

Combine (2.15) and (2.16) to get the reduced form

$$P_t c_t + P_t \tau_t + \frac{i_t}{1+i_t} M_t + \frac{1}{1+i_t} W_{t+1} = P_t y_t + W_t.$$

Rearrange to get the present value constraint

$$\begin{aligned}
W_t &= \frac{1}{1+i_t} W_{t+1} + P_t c_t + P_t \tau_t + \frac{i_t}{1+i_t} M_t - P_t y_t \\
&= \frac{1}{1+i_t} \left(\frac{1}{1+i_{t+1}} W_{t+2} + P_{t+1} c_{t+1} + P_{t+1} \tau_{t+1} + \frac{i_{t+1}}{1+i_{t+1}} M_{t+1} - P_{t+1} y_{t+1} \right) \\
&\quad + P_t c_t + P_t \tau_t + \frac{i_t}{1+i_t} M_t - P_t y_t \\
&= \dots\dots \\
&= \left(\prod_{j=0}^{s} \frac{1}{1+i_{t+j}} \right) W_{t+j+1} \\
&\quad + \sum_{j=0}^{s} \left[\prod_{k=1}^{j} \frac{1}{1+i_{t+k}} \left(P_{t+j} c_{t+j} + P_{t+j} \tau_{t+j} + \frac{i_{t+j}}{1+i_{t+j}} M_{t+j} - P_{t+j} y_{t+j} \right) \right]
\end{aligned}$$

in which one can simplify the expression by defining

$$Q_{t,t+s} = \prod_{j=1}^{s} \frac{1}{1+i_{t+j-1}} \text{ with } Q_{t,t} = 1.$$

Then the expression of W_0 can be written as

$$W_0 = \lim_{T\to+\infty} Q_{0,T} W_T + \sum_{t=0}^{+\infty} Q_{0,t} \left(P_t c_t + P_t \tau_t + \frac{i_t}{1+i_t} M_t - P_t y_t \right).$$

No-Ponzi-Game constraint requires that

$$\lim_{T\to+\infty} Q_{0,T} W_T = 0,$$

so the present value budget constraint for the agent is

$$W_0 = \sum_{t=0}^{+\infty} Q_{0,t} \left(P_t c_t + P_t \tau_t + \frac{i_t}{1+i_t} M_t - P_t y_t \right), \tag{2.17}$$

in which $W_0 = M_0 + B_0$.

For the government, the flow budget constraint can be written as

$$P_t g_t + i_{t-1} B_{t-1} = P_t \tau_t + B_t - B_{t-1} + M_t - M_{t-1}, \tag{2.18}$$

adding the government's wealth at t, W_t^g

$$W_t^g = (1 + i_{t-1}) B_{t-1} + M_{t-1}, \tag{2.19}$$

one can get the reduced form by combining (2.18) and (2.19)

$$P_t g_t + W_t^g = P_t \tau_t + \frac{1}{1+i_t} W_{t+1}^g + \frac{i_t}{1+i_t} M_t.$$

By the similar approach the expression of W_0^g can be written as

$$W_0^g = \lim_{T\to+\infty} Q_{0,T} W_T^g + \sum_{t=0}^{+\infty} Q_{0,t} \left(P_t \tau_t + \frac{i_t}{1+i_t} M_t - P_t g_t \right).$$

No-Ponzi-Game constraint requires that

$$\lim_{T\to+\infty} Q_{0,T} W_T^g = 0,$$

so the present value budget constraint for the government is

$$W_0^g = \sum_{t=0}^{+\infty} Q_{0,t}\left(P_t\tau_t + \frac{i_t}{1+i_t}M_t - P_t g_t\right), \tag{2.20}$$

in which $W_0^g = M_0 + B_0$.

(b) Characterize the first-order conditions for the agent's optimal path.

The agent's problem is an optimization problem with equality constraint

$$\max_{\{c_t,m_t\}_{t=0}^{+\infty}} \sum_{t=0}^{+\infty} \beta^t U(c_t, m_t)$$

$$s.t.\ \ M_0 + B_0 = \sum_{t=0}^{+\infty} Q_{0,t}\left(P_t c_t + P_t\tau_t + \frac{i_t}{1+i_t}M_t - P_t y_t\right).$$

Set up Lagrangian for this problem

$$\begin{aligned}\mathscr{L} &= \sum_{t=0}^{+\infty} \beta^t U(c_t, m_t) + \lambda\left[M_0 + B_0 - \sum_{t=0}^{+\infty} Q_{0,t}\left(P_t c_t + P_t\tau_t + \frac{i_t}{1+i_t}M_t - P_t y_t\right)\right]\\ &= \sum_{t=0}^{+\infty} \beta^t U(c_t, m_t) + \lambda\left[M_0 + B_0 - \sum_{t=0}^{+\infty} Q_{0,t}\left(P_t c_t + P_t\tau_t + \frac{i_t}{1+i_t}P_t m_t - P_t y_t\right)\right]\end{aligned}$$

and derive the first-order conditions

$$\begin{aligned}\frac{\partial \mathscr{L}}{\partial c_t} &= \beta^t \frac{\partial U(c_t, m_t)}{\partial c_t} - \lambda Q_{0,t} P_t = 0,\\ \frac{\partial \mathscr{L}}{\partial m_t} &= \beta^t \frac{\partial U(c_t, m_t)}{\partial m_t} - \lambda Q_{0,t} P_t \frac{i_t}{1+i_t} = 0.\end{aligned}$$

Therefore the marginal rate of substitution for intertemporal consumption is

$$\frac{\frac{\partial U(c_t,m_t)}{\partial c_t}}{\frac{\partial U(c_t,m_t)}{\partial c_{t+1}}} = \beta\frac{Q_{0,t}}{Q_{0,t+1}}\frac{P_t}{P_{t+1}} = \beta\frac{1+i_{t+1}}{1+\pi_{t+1}} = \frac{1+r_{t+1}}{1+\rho}, \tag{2.21}$$

by Fisher equation. The intratemporal marginal rate of substitution between consumption and money holding is

$$\frac{\frac{\partial U(c_t,m_t)}{\partial c_t}}{\frac{\partial U(c_t,m_t)}{\partial m_t}} = \frac{1+i_t}{i_t}, \tag{2.22}$$

and the marginal rate of substitution for intertemporal money holding is

$$\frac{\frac{\partial U(c_t,m_t)}{\partial m_t}}{\frac{\partial U(c_t,m_t)}{\partial m_{t+1}}} = \beta \frac{Q_{0,t}}{Q_{0,t+1}} \frac{P_t}{P_{t+1}} \frac{i_t}{1+i_t} \frac{1+i_{t+1}}{i_{t+1}} = \beta \frac{1+i_{t+1}}{1+\pi_{t+1}} \frac{i_t}{1+i_t} \frac{1+i_{t+1}}{i_{t+1}}. \quad (2.23)$$

The equation above equals to $\frac{1+r_{t+1}}{1+\rho}$ in the steady state.

(c) Show that with additive separable preferences $U(c_t, m_t) = u(c_t) + v(m_t)$, the real rate of interest depends only on the time path of the real resources available for consumption.

From (2.21) apply $U(c_t, m_t) = u(c_t) + v(m_t)$ and one can see that

$$\frac{u'(c_t)}{u'(c_{t+1})} = \frac{1+r_{t+1}}{1+\rho},$$

meaning that the real rate of interest depends only on the time path of the real resources available for consumption.

(d) Assume that $U(c_t, m_t) = c_t^\alpha + m_t^\alpha$. Derive the money demand function $m(c_t, i_t)$ and characterize elasticity with respect to c_t and i_t . Show why the price level may not be determinate if the central bank pegs the interest rate to a fixed level $i_t = \bar{i}$.

From (2.22) apply $U(c_t, m_t) = c_t^\alpha + m_t^\alpha$ and one can see that

$$\frac{\alpha c_t^{\alpha-1}}{\alpha m_t^{\alpha-1}} = \frac{1+i_t}{i_t},$$

$$m_t = \left(\frac{i_t}{1+i_t}\right)^{\frac{1}{\alpha-1}} c_t.$$

Take logarithm on both sides

$$\ln m_t = \frac{1}{\alpha-1} \ln\left(\frac{i_t}{1+i_t}\right) + \ln c_t$$
$$\approx \frac{1}{\alpha-1} \ln i_t + \ln c_t.$$

The elasticity of money demand with respect to i_t is $\frac{1}{\alpha-1}$, and 1 with respect to c_t.

(e) Assume that both endowment and government spending are constant: $y_t = y$; $g_t = g$. Characterize conditions for steady state. Show that the Friedman rule maximizes per period utility. Discuss reasons why this rule may not be optimal in a more general setting.

In steady state the consumption level is constant, i.e. $c_t = c_{t+1} = c^*$. The value of c^* is determined by (2.17) and (2.20), $c^* = y - g =$ constant.

Apply this fact in the first-order conditions, by (2.21) one can see that $\rho = r$. By (2.23) money supply is also constant in the steady state, i.e. $m_t = m_{t+1} = m^*$.

In steady state from the intratemporal marginal rate of substitution (2.22) the marginal utility from holding money is

$$v'(m^*) = u'(c^*)\frac{i}{1+i}.$$

Since c^* is constant, normalize $u'(c^*) = 1$ and get $v'(m^*) = \frac{i}{1+i}$.

On the other hand, since $v(m)$ is strictly concave in m the marginal utility of holding money $v'(m^*)$ is a downward sloping curve. The intersection of these two $v'(m^*)$ curves determines the equilibrium level of money holding m^*. See Fig. 2.1.

However consider a central planner's problem of optimal money supply

$$\max_{c_t, m_t} \mathscr{L} = \sum_{t=0}^{+\infty} \beta^t U(c_t, m_t) + \lambda_t(y - g - c_t - \kappa m_t)$$

in which κ denotes the marginal cost of money supply. Then the first-order condition gives

$$\frac{\partial \mathscr{L}}{\partial c_t} = \beta^t u'(c_t) - \lambda_t = 0,$$
$$\frac{\partial \mathscr{L}}{\partial m_t} = \beta^t v'(m_t) - \lambda_t \kappa = 0.$$

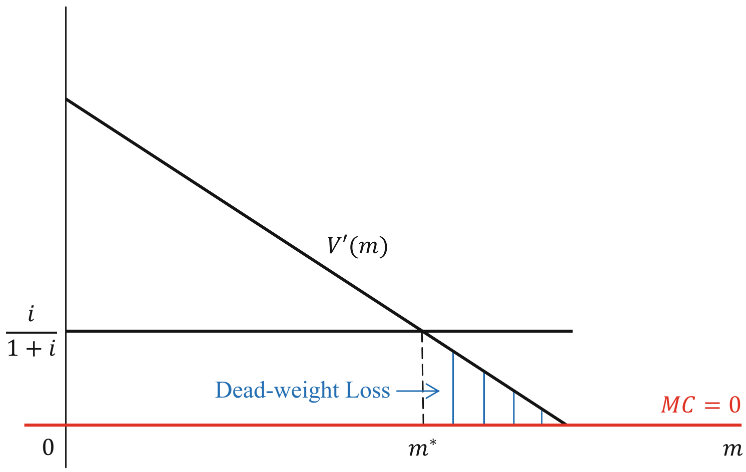

Fig. 2.1 Money holding in equilibrium

Combine these two equations and get $v'(m^*) = \kappa$ by normalizing $u'(c^*) = 1$. Usually the marginal cost of money supply is so low that one can simply set $\kappa = 0$. Therefore for any $\frac{i}{1+i} \neq 0$ the decentralized equilibrium is inefficient and Friedman rule (setting $i = 0$) maximizes per period utility.

However Friedman rule implies that $\pi = i - r = -r < 0$, and this may not be optimal in more general settings. One can think about the following arguments:

- People actually prefer $\pi > 0$, because one is able to adjust the relative prices—e.g., to reduce real wage by setting the increasing rate of the wage lower than π—a strictly positive inflation "greases the wheel";
- Inflation as a tax on money holding;
- Stabilizing economy (think about liquidity traps?);
- ...

2.2.4 Cash-in-Advance Models of Money Demand

Consider a representative household that receives an exogenous income Y_t each period and gains utility only from consumption C_t. Specifically, its utility function is given by

$$U_0 = \sum_{t=0}^{\infty} \beta^t u(C_t).$$

At the beginning of each period t, the household needs to make two decisions about his period income. On the one hand, the household decides how much to consume in period t and how much to save for the future. On the other hand, the household needs to decide on how to allocate its savings across one-period bonds B_t which yield a nominal interest rate of i_t and money balances M_t which do not pay interest. However, money balances are necessary to cover all consumption purchases the household makes in period t at price P_t.

(a) *Write down and interpret the household's* cash-in-advance *constraint and the period budget constraint.*

Timing of events in each period t is as follows:

- Beginning of period t: Household decides on consumption C_t, bonds B_t, and money holdings M_t;
- End of period t: Repayment of bonds with interest $(1 + i_t)B_t$ is part of initial wealth in period $t + 1$;
- End of period t: Money M_t is either spent on consumption (CIA) or used as store of wealth with $i_M = 0$.

Cash-in-advance constraint implies that money is needed to finance consumption purchases, i.e., $M_t \geq P_t C_t$. Therefore, period budget constraint requires that the household consumes and invests his savings in bonds, all financed by income and interest plus principal on bonds, i.e., $P_t C_t + B_t = P_t Y_t + (1 + i_{t-1}) B_{t-1}$.

(b) *Derive the Euler equation and the optimal money demand of the household under the additional assumption that $i_t > 0$. Discuss intuitively the economics underlying the optimal money demand.*

Suppose $i_t > 0$, CIA constraint is binding (because of opportunity cost of money), $M_t = P_t C_t$. This is optimal money demand and reflects transactions view of money demand.

Euler equation is derived from the following optimization problem:

$$\max_{C_t, B_t} \; U_0 = \sum_{t=0}^{\infty} \beta^t u(C_t),$$
$$s.t. \;\; P_t C_t + B_t = P_t Y_t + (1 + i_{t-1}) B_{t-1}.$$

Using Lagrangian,

$$L = \sum_{t=0}^{\infty} \beta^t u(C_t) + \sum_{t=0}^{\infty} \lambda_t \left[P_t Y_t + (1 + i_{t-1}) B_{t-1} - P_t C_t - B_t \right],$$

the first-order conditions give

$$\frac{\partial L}{\partial C_t} = \beta^t u'(C_t) - \lambda_t P_t = 0,$$
$$\frac{\partial L}{\partial C_{t+1}} = \beta^{t+1} u'(C_{t+1}) - \lambda_{t+1} P_{t+1} = 0,$$
$$\frac{\partial L}{\partial B_t} = \lambda_{t+1}(1 + i_t) - \lambda_t = 0.$$

Combining the first two equations and using the third one to substitute for the Lagrange parameter yields the Euler equation

$$\frac{u'(C_t)}{\beta u'(C_{t+1})} = \frac{\lambda_t P_t}{\lambda_{t+1} P_{t+1}} = \frac{1 + i_t}{1 + \pi_{t+1}} = 1 + r_t.$$

In equilibrium (of an endowment economy), $C_t = Y_t$, $\forall t$, the Euler equation becomes $\frac{u'(Y_t)}{u'(Y_{t+1})} = \beta(1 + r_t)$, and money demand is $M_t = P_t Y_t$.

(c) Describe the effects of expansionary monetary policy and illustrate them graphically in an $P_t - i_t$ diagram.

Expansionary monetary policy increases current prices and lowers the interest rate

- M_t increases: P_t increases as Y_t is fixed (CIA constraint);
- r_t decreases for given P_{t+1} (Fisher Equation);
- Households want to consume less in the current period, yet this is not possible because of endowment economy;
- i_t needs to decrease to encourage savings.

(d) How does the analysis change once you take the zero lower bound on nominal interest rates into account, i.e., if $i_t = 0$?

Expansionary monetary policy has no effect on prices or interest rates

- Once $i_t = 0$, CIA constraint ceases to bind;
- Individuals are indifferent between money and bonds since both yield zero interest rate;
- Money holdings are no longer constrained by opportunity costs of money: Excess money holdings.

2.2.5 Cost of Inflation and Optimal Monetary Policy

Based on Lucas (2000) The canonical theory of monetary policy assumes that inflation is costly for society. In the following we explore a justification for that assumption, using Sidrauski's money-in-the-utility function.

Consider an economy with a representative infinitely lived consumer whose preference is given by

$$\sum_{t=0}^{+\infty} \beta^t \left[u(c_t) + v(m_t)\right]$$

in which $u(\cdot)$ and $v(\cdot)$ are increasing and strictly concave utility functions, c_t is the consumption at date t, m_t is the real money balance at the end of period t, and $\beta \in (0, 1)$ is the discount factor.

Let b_t be real bond holdings at the end of period t that pay a nominal interest rate i_{t+1} at the beginning of the next period, $M_t = P_t m_t$ be nominal money balances, P_t be the price level, and y be the time-invariant and exogenous real income received by the consumer each period. The consumer also receives real net transfers from *the government, τ_t. Then, to ease your computation, the nominal budget constraint of the consumer is given by*

$$P_{t-1}b_{t-1}(1+i_t) + P_t y + P_t \tau_t = P_t c_t + M_t - M_{t-1} + P_t b_t.$$

(a) Let r_t be the real interest rate and π_t be the inflation rate, such that

$$1 + i_t = (1 + r_t)(1 + \pi_t).$$

Show that consumption can be written as

$$c_t = b_{t-1}(1 + r_t) + y + \tau_t - m_t + \frac{m_{t-1}}{1 + \pi_t} - b_t.$$

Divide both sides of the nominal budget constraint by P_t and use (2.2), then the result is immediately seen.

(b) Using (2.1) and (2.3), show that the following efficiency conditions hold:

$$-u'(c_t) + \beta \frac{u'(c_{t+1})}{1 + \pi_{t+1}} + v'(m_t) = 0,$$

$$-u'(c_t) + \beta u'(c_{t+1})(1 + r_{t+1}) = 0.$$

Provide some intuitions for Eqs. (2.4) and (2.5), and show that (2.4) and (2.5) define a money demand function

$$v'(m_t) = \frac{i_{t+1}}{1 + i_{t+1}} u'(c_t).$$

What is the relationship between money demand and nominal interest rates for a given level of consumption? What is the relationship between money demand and consumption for a given nominal interest rate?

The first-order conditions of the dynamic optimization problem directly yield (2.4) and (2.5). Then combine them to see (2.6).

Intuition of (2.4): Suppose that we are already in the optimal path with (c^*, m^*). Consider the following scheme: Take one real euro (i.e., P_t paper euro) out of my consumption today and transfer it into consumption tomorrow via money holding. Then my loss today is $u'(c_t)$, and my gain in present value is $\beta \frac{u'(c_{t+1})}{1+\pi_{t+1}} + v'(m_t)$. If I was already in my optimal path, the loss and gain should cancel out.

Intuition of (2.5): Similar as above, take one euro out of my consumption today and transfer it into consumption tomorrow via bonds holding.

Relationship between money demand and nominal interest rates for a given level of consumption: Using implicit function theorem, differentiate (2.6) with respect to i_{t+1}

$$\frac{\partial m_t}{\partial i_{t+1}} = \frac{1}{(1+i_{t+1})^2}\frac{u'(c_t)}{v''(m_t)} < 0.$$

Relationship between money demand and consumption for a given nominal interest rate: Using implicit function theorem, differentiate (2.6) with respect to c_t

$$\frac{\partial m_t}{\partial c_t} = \frac{i_{t+1}}{1+i_{t+1}}\frac{u''(c_t)}{v''(m_t)} > 0.$$

Hence money demand decreases with the nominal interest rate, which represents the opportunity cost of holding money; and it increases with consumption.

(c) *Assume, for the rest of the problem, that there is no government expenditure and no public debt, so that government prints money only to make net transfers to the consumer, i.e.*

$$M_t - M_{t-1} = P_t\tau_t.$$

Since there is only one consumer and the government does not issue public debt, equilibrium requires $b_t = 0$. What is c_t in equilibrium? Using your expression for c_t and Eq. (2.5), derive the expression for the real interest rate.

Apply all these facts in the nominal budget constraint and get $c_t = y$. Insert this into Eq. (2.5) and derive the expression for the real interest rate, $r = \frac{1-\beta}{\beta}$.

(d) *Assume, in addition, that the government follows a constant nominal money growth rule*

$$M_t = (1+\mu)M_{t-1}.$$

Define a steady state in this model as a situation in which real variables do not change. In particular, in the steady state $m_t = \overline{m}$. Given (2.7) and the fact that $m_t = \frac{M_t}{P_t}$, find the steady-state level of inflation in this model, call it $\overline{\pi}$.

In the steady state $\overline{m} = \frac{M_{t-1}}{P_{t-1}} = \frac{M_t}{P_t}$, therefore $\overline{\pi} = \frac{M_t}{M_{t-1}} - 1 = \mu$.

(e) *Using Eq. (2.6) evaluated in steady state, find an expression for $\overline{m}$ in terms of $\overline{\pi}$. What is the steady state effect of $\overline{\pi}$ on $\overline{m}$? What is the effect of $\overline{\pi}$ on steady-state consumption? What is the welfare effect of increasing $\overline{\pi}$? What is the optimal level of steady-state inflation?*

Using Eq. (2.6) evaluated in steady state

$$\begin{aligned} v'(\overline{m}) &= \left(1 - \frac{1}{1+i}\right) u'(y) \\ &= \left(1 - \frac{\beta}{1+\overline{\pi}}\right) u'(y). \end{aligned}$$

The steady state effect of $\overline{\pi}$ on $\overline{m}$ can therefore be explored by the implicit function theorem

$$\frac{\partial \overline{m}}{\partial \overline{\pi}} = \frac{\beta}{(1+\overline{\pi})^2} \frac{u'(y)}{v''(\overline{m})} < 0,$$

which means that higher steady state inflation (money growth) reduces steady-state real money balances. Since steady state consumption is unaffected by nominal variables, and since real money balances enter directly the utility function, this implies that an increase in inflation reduces welfare. This also implies that the optimal level of steady state inflation is the lowest possible given the constraint that nominal interest rates be non-negative, i.e.

$$\begin{aligned} 1+i &= (1+r)(1+\overline{\pi}) \\ &= \frac{1}{\beta}(1+\overline{\pi}) \\ &= 1 \end{aligned}$$

– Friedman Rule shows up again. The optimal level of steady-state inflation is therefore given by $\overline{\pi} = \beta - 1 < 0$, i.e. a deflationary economy as Friedman argues.

2.2.6 Overlapping Generations with Money

Samuelson (1958) alternatively, macro economy can be modelled in an infinite horizon with finitely-lived *agents. Suppose, as in the Diamond (1965) model, that N_t 2-period-lived individuals are born in period t and that generations are growing with rate n. The utility function of a representative individual is $U_t = \ln c_{1,t} + \ln c_{2,t+1}$. Each individual is born with an endowment of A units of the economy's single good. The good can either be consumed or stored. Each unit stored yields $x > 0$ units next period.*

In period 0, there are N_0 young individuals and $\frac{1}{1+n}N_0$ old individuals endowed with some amount Z of the good. Their utility is simply $c_{2,0}$.

(a) Describe the decentralized equilibrium of this economy. (Hint: Will members of any generation trade with members of another generation?)

For any generation $t > 0$ the intertemporal optimization problem is

$$\max_{c_{1,t}, c_{2,t+1}} \quad U_t = \ln c_{1,t} + \ln c_{2,t+1}$$
$$s.t. \quad c_{1,t} + S_t \leq A$$
$$c_{2,t+1} \leq x S_t,$$

noting that members of any generation will not trade with members of another generation because the older generation dies in the next period and cannot pay or get paid.

Solve by first-order condition and get $c^*_{1,t} = \frac{A}{2}$, and $c^*_{2,t} = \frac{xA}{2}$. For generation $t = 0$, $c^*_{1,0} = \frac{Z}{2}$, and $c^*_{2,0} = \frac{xZ}{2}$. For generation $t = -1$, $c^*_{2,-1} = Z$.

Briefly speaking, when young, each individual consumes half of her endowment and stores the other half. This allows her to consume the savings when old. Note that the utility function is logarithmic, the fraction of her endowment that she stores doesn't depend on the return to storage.

(b) *Consider paths where the fraction of agents' endowment that is stored, s_t, is constant over time. What is per capita consumption (weighted average from young and old) on such a path as a function of s?*

In any period t, the consumption for generation t is $(1 - s_t)AN_t$ and the consumption for generation $t - 1$ is xs_tAN_{t-1}. Therefore per capita consumption is

$$c_t = \frac{(1 - s_t)AN_t + xs_tAN_{t-1}}{N_t + N_{t-1}} = \frac{(1 - s_t)A(1 + n) + xs_tA}{2 + n}$$

noting that $N_t = (1 + n)N_{t-1}$.

(c) *If $x < 1 + n$, which value of $s \in [0, 1]$ is maximizing per capita consumption?*

It's directly seen that $s_t = 0$ maximizes c_t for $x < 1 + n$.

(d) *Is the decentralized equilibrium Pareto-efficient? If not, how could a social planner raise welfare?*

No (except when $x = 1 + n$) (why do the welfare theorems break down here?). Redistribute the wealth according to the value of x.

The intrinsic reason is that the decentralized equilibrium, with saving rate being one half, is not Pareto efficient. Since intergenerational trade is not possible, individuals are forced into storage because this is the only way they can save and consume in old age. They must do this even if the return on storage, x, is poor. However, at any point in time, a social planner could take one unit from each of the young generation and give the each of the old $1 + n$ units since there are fewer of them. With $x < 1 + n$, this gives a better return than storage. Therefore, the social planner could raise the social welfare by taking the half of the young generation's endowment that it was going to be stored and instead give it to the old. The planner could do this in each period, which allows the individuals to consume $\frac{A}{2}$ units when young—the same as in the decentralized equilibrium—but now they get to consume $\frac{(1+n)A}{2}$ when old. This is greater than $\frac{xA}{2}$ in the decentralized equilibrium and Pareto improving.

Suppose now that old individuals in period 0 are also endowed with M units of a storable, divisible commodity, which we call money. Money is not a source of utility. Assume $x < 1 + n$.

(e) Suppose the price of the good in units of money in periods t and $t + 1$ is given by P_t and P_{t+1}, respectively. Derive the demand functions of an individual born in t.

For any generation $t > 0$ the intertemporal optimization problem becomes

$$\max_{c_{1,t}, c_{2,t+1}} \quad U_t = \ln c_{1,t} + \ln c_{2,t+1}$$

$$s.t. \quad P_t c_{1,t} + P_t s_t + M_t^d \leq P_t A$$

$$P_{t+1} c_{2,t+1} \leq P_{t+1} x s_t + M_t^d.$$

Now the individual has two decisions to make when she is young. The first is on how much of her endowment to consume and how much to save. The second is by which way to save, i.e. through the storage technology, holding money, or both. With log utility we can separate the two decisions since the rate of return on saving will not affect the fraction of the first period endowment that is saved, i.e. she will still consume half of her endowment in the first period, $c_{1,t} = \frac{A}{2}$.

Next, for her decision on saving technology, it depends on the gross rate of return on storage, x, relative to that on holding money, $\frac{P_t}{P_{t+1}}$. There are three cases:

Case 1 $\frac{P_t}{P_{t+1}} > x$ The storage technology is dominated by holding money. She will consume half of her endowment and then sell the rest for money when she is young,

$$c_{1,t} = \frac{A}{2},$$

$$s_t = 0,$$

$$M_t^d = \frac{P_t A}{2},$$

$$c_{2,t} = \frac{P_t}{P_{t+1}} \frac{A}{2}.$$

Case 2 $\frac{P_t}{P_{t+1}} < x$ The storage technology dominates holding money. Then she will consume half of her endowment and then save the rest instead of holding money when she is young,

$$\begin{aligned}
c_{1,t} &= \frac{A}{2}, \\
s_t &= \frac{A}{2}, \\
M_t^d &= 0, \\
c_{2,t} &= \frac{xA}{2}.
\end{aligned}$$

Case 3 $\frac{P_t}{P_{t+1}} = x$ The storage technology works as good as holding money. Then she will consume half of her endowment and then be indifferent between saving the rest and holding money when she is young. Suppose that a share $\alpha_t \in [0, 1]$ of the rest is invested for money, then

$$\begin{aligned}
c_{1,t} &= \frac{A}{2}, \\
s_t &= \frac{(1-\alpha_t)A}{2}, \\
M_t^d &= \frac{P_t \alpha_t A}{2}, \\
c_{2,t} &= \frac{xA}{2}.
\end{aligned}$$

(f) Describe the set of equilibria.

Equilibrium requires that the aggregate money demand equal the aggregate money supply. And we analyze the equilibrium for each of the cases.

Case 1: $\frac{P_t}{P_{t+1}} > x$ The aggregate money demand is $L_t M_t^d = L_t \frac{P_t A}{2}$; and the aggregate money supply is $M = L_t \frac{P_t A}{2}$. Since this holds for all periods, then we update it one period forward and get $M = L_{t+1} \frac{P_{t+1} A}{2}$, therefore

$$\begin{aligned}
L_t \frac{P_t A}{2} &= L_{t+1} \frac{P_{t+1} A}{2}, \\
L_t \frac{P_t A}{2} &= (1+n) L_t \frac{P_{t+1} A}{2}, \\
\frac{P_t}{P_{t+1}} &= 1 + n.
\end{aligned}$$

This shows that if money is introduced into a dynamically inefficient economy, storage will not be used. The monetary equilibrium will thus result in achieving the golden-rule level of storage.

Case 2: $\frac{P_t}{P_{t+1}} < x$ In this case holding money gives one lower return than the storage technology. Therefore there is no positive demand for money, or, the price level is $P_t = +\infty$ for all the periods.

Case 3: $\frac{P_t}{P_{t+1}} = x$ The aggregate money demand is $L_t M_t^d = L_t \frac{P_t \alpha_t A}{2}$; and the aggregate money supply is $M = L_t \frac{P_t \alpha_t A}{2}$. Since this holds for all periods, then we update it one period forward and get $M = L_{t+1} \frac{P_{t+1}\alpha_{t+1}A}{2}$, therefore

$$\begin{aligned}
L_t \frac{P_t \alpha_t A}{2} &= L_{t+1} \frac{P_{t+1}\alpha_{t+1}A}{2}, \\
L_t \frac{P_t \alpha_t A}{2} &= (1+n) L_t \frac{P_{t+1}\alpha_{t+1}A}{2}, \\
\frac{P_t}{P_{t+1}} &= \frac{\alpha_{t+1}}{\alpha_t}(1+n), \\
x &= \frac{\alpha_{t+1}}{\alpha_t}(1+n).
\end{aligned}$$

(g) *Explain why there is an equilibrium with $P_t \to +\infty$. Explain why this must be the case if the economy ends at some date T that is common knowledge among all generations.*

The equilibrium with $P_t \to +\infty$ is just included in Case 2. Other situations in which the equilibrium is $P_t \to +\infty$ are

- When the young generation at $t = 0$ doesn't believe that money will be valued in the next period and thus that the generation one will also not accept money for goods. In that case, in period 0, the young simply consume half of their endowment and store the rest, and the old have some useless paper to go with their endowment. This is an equilibrium with real money demand $m_t^d = \frac{M_t^d}{P_t}$ equal to 0 at $P_t \to +\infty$ and real money supply equal to 0 as well. If no one believes that the next generation will accept money for goods, this equilibrium continues for all future periods;
- When t is finite and ends at some date T. Then $P_t \to +\infty$ is the unique equilibrium for this economy. The young at T will not sell any of their endowment for money, instead they will maximize their utility of their one-period life by consuming all of their endowment in T. Therefore if the old at T hold any money, they would be stuck with it and it would be useless to them. Knowing that the money will be of no use when old, when they are young in $T-1$ they would not sell any of their endowment for money. Thus if the old at $T-1$ hold

any money they would be stuck with it and it would be useless to them—the old at $T - 1$ will not want any money when they are young and so on. By backward induction, no one would ever want to sell goods for money and money would not be valued at all.

References

Cagan, P. (1956). The monetary dynamics of hyperinflation. In M. Friedman (Ed.) *Studies in the quantity theory of money*. Chicago: University of Chicago Press.

Diamond, P. (1965). National debt in a neoclassical growth model. *American Economic Review, 55*, 1126–1150.

Lucas, R. E. (2000). Inflation and welfare. *Econometrica, 68*, 247–274.

Samuelson, P. A. (1958). An exact consumption-loan model of interest with or without the social contrivance of money. *Journal of Political Economy, 66*, 467–482.

Interaction Between Monetary and Fiscal Policy: Active and Passive Monetary Regimes

3

3.1 Exercises

3.1.1 Short Review Questions

(a) Suppose, in a closed economy, the government funds public expenditure through seignorage. Briefly explain how the seignorage is generated through inflation. Under which condition(s) government's incentive in generating seignorage leads to hyperinflation? How realistic is seignorage-generating inflation today?

(b) Suppose that the government is able to fund its public expenditure through both taxation and seignorage. Discuss government's incentive in taking debts. Under which condition(s) are public debts sustainable?

(c) Explain how price level in an economy is determined by the fiscal regime.

(d) Explain how fiscal policy and monetary policy anchor the macro economy, respectively. How should fiscal policy and monetary policy coordinate in reality?

3.1.2 Public Sector Budget and Seignorage

Assume that the public sector comprises a government and a central bank. The government finances its expenditure through taxes T_t, new issuance of debt $(B_{t+1}^T - B_t^T)$ and profits made by the central bank $\tilde{S}_t$. Government expenditure consists of payments G_t and interest payment on the stock of debt $i_t B_t^T$. The central bank finances its net purchases of government debt $(B_{t+1}^{CB} - B_t^{CB})$ and its profits which are being paid to the government by printing money $\Delta M_{t+1} = M_{t+1} - M_t$ and through its interest income from holding government bonds.

J. Cao, G. Illing, *Instructor's Manual for* Money: Theory and Practice, Springer Texts in Business and Economics,
https://doi.org/10.1007/978-3-030-23618-2_3

(a) Write down the period budget constraint of the government and the central bank and give an interpretation.
(b) Derive the *consolidated* public sector budget constraint. What are the different sources of finance for the public sector?
(c) *Current seignorage income* as a share of GDP is given by $S_{M,t} = \frac{M_{t+1}-M_t}{P_t Y_t}$. Discuss why $S_{M,t}$ is often referred to as *inflation tax*.
(d) Discuss how the concept of *seignorage* $S = ik$ with $k = \frac{M}{PY}$ differs from that of the inflation tax, i.e. from S_M.
(e) Assume demand for real balances depends on the nominal interest rate. Derive that level of the nominal interest rate that maximizes steady-state seignorage revenues S. Discuss your result.

3.1.3 Sustainability of Government Debt in a Monetary Economy

Suppose that the consolidated government debt in an economy evolves as follows:

$$B_t = G_t - T_t - \Delta M_t + (1 + i_t)B_{t-1}$$

in which B_t is nominal government debt held by the private sector, $G_t - T_t$ is the primary deficit, M_t is the money supply, and i_t is the nominal rate of interest.

(a) Derive a difference equation for the debt to GDP ratio.
(b) Solve this equation for the steady-state debt ratio. What does the steady-state debt ratio depend on? Interpret.
(c) Now use a continuous time analogue to the obtained difference equation and solve it. Discuss the four different adjustment mechanisms if a shock was to temporarily increase the current debt level above its sustainable long-run equilibrium level.
(d) The total government deficit—which includes interest payments on government debt—as a share of GDP is now given as $d_t = g_t - \tau_t - \mu k + i_t b_t$. Derive an expression for the steady-state deficit-to-GDP ratio.

3.1.4 Sustainability of Debt in a Small Open Economy

Consider a small open economy with an infinitely lived representative consumer with CES utility, the instantaneous elasticity of substitution being σ and discount rate ρ. The economy's GDP growth rate is $\frac{\dot{Y}}{Y} = y$, the growth rate of population is a constant n, and the interest rate is a constant r. Initial asset positions are zero.

(a) Derive the present value budget constraint. Assume that the representative consumer is a Ramsey consumer, i.e., her optimal consumption path follows that

in Ramsey model, derived from Exercise 3 of Chap. 1. Under which conditions is this optimal consumption path well defined?
(b) Compute the interest rate in autarky.
(c) Suppose that the country is initially running a current account deficit. How are current account and debt/GDP-ratio evolving over time?

3.1.5 Sustainability of Government Debt

Consider an economy with constant GDP growth rate y and interest rate $r > y$. The government finances its expenditure G through tax income T and debt B.

(a) Suppose that the government's primary deficit rate $d_p = \frac{G-T}{Y}$ is given exogenously and constant over time. Show how the debt/GDP ratio (suppose that the initial value of this ratio is b_0) evolves over time. Which restrictions are needed to ensure long-run stability of the debt/GDP ratio? How does your answer change, if $r < y$?
(b) Suppose that the government's total deficit rate $d_t = \frac{G-T+rB}{Y}$ is given exogenously and constant over time. Show how the debt/GDP ratio evolves over time. Which restrictions are needed to ensure long-run stability of the debt/GDP ratio? How does your answer depend on the interest rate?
(c) What are the long-run implications of a constant total deficit rate for the primary deficit?
(d) Suppose now that r is the *nominal* interest rate and y is the growth rate of *nominal* output. How would a rise in the rate of inflation affect the long-run primary deficit if the total deficit rate is held constant?
(e) Discuss the relation of your results with the Maastricht criteria.

3.2 Solutions for Selected Exercises

3.2.1 Public Sector Budget and Seignorage

Assume that the public sector comprises a government and a central bank. The government finances its expenditure through taxes T_t, new issuance of debt ($B_{t+1}^T - B_t^T$) and profits made by the central bank $\tilde{S}_t$. Government expenditure consists of payments G_t and interest payment on the stock of debt $i_t B_t^T$. The central bank finances its net purchases of government debt ($B_{t+1}^{CB} - B_t^{CB}$) and its profits which are being paid to the government by printing money $\Delta M_{t+1} = M_{t+1} - M_t$ and through its interest income from holding government bonds.

(a) Write down the period budget constraint of the government and the central bank and give an interpretation.

Period budget constraints

- Government: $T_t + B_{t+1}^T - B_t^T + \tilde{S}_t = G_t + i_t B_t^T$;
- Central bank: $B_{t+1}^{CB} - B_t^{CB} + \tilde{S}_t = M_{t+1} - M_t + i_t B_t^{CB}$.

Note that total government debt can be decomposed in debt held by the central bank and debt held by the public: $B_t^T = B_t + B_t^{CB}$.

(b) *Derive the* consolidated *public sector budget constraint. What are the different sources of finance for the public sector?*

To derive the consolidated budget constraint of the public sector, eliminate $\tilde{S}_t$ and use the fact that $B_t^T = B_t + B_t^{CB}$

$$G_t + i_t B_t^T = T_t + B_{t+1}^T - B_t^T + M_{t+1} - M_t + i_t B_t^{CB} - B_{t+1}^{CB} + B_t^{CB},$$

$$G_t + i_t(B_t^T - B_t^{CB}) = T_t + (B_{t+1}^T - B_{t+1}^{CB}) - (B_t^T - B_t^{CB}) + M_{t+1} - M_t,$$

$$G_t + i_t B_t = T_t + B_{t+1} - B_t + M_{t+1} - M_t.$$

Therefore, there are three sources of government finance:

- Taxation: T_t;
- Issuance of new debt to households: $B_{t+1} - B_t$;
- Issuance of new money: $M_{t+1} - M_t$.

(c) Current seignorage income *as a share of GDP is given by* $S_{M,t} = \frac{M_{t+1}-M_t}{P_t Y_t}$. *Discuss why* $S_{M,t}$ *is often referred to as* inflation tax.

Current seignorage income, or, real income from money creation as a share of GDP

$$S_{M,t} = \frac{M_{t+1} - M_t}{P_t Y_t} = \frac{M_{t+1} - M_t}{M_t} \cdot \frac{M_t}{P_t Y_t} = \mu_{t+1} \cdot \frac{M_t}{P_t Y_t} = (\pi_t + y_t) \cdot \frac{M_t}{P_t Y_t}.$$

Note: The last equality follows directly from the Quantity Equation assuming that the velocity of money is stable:

$$\mu_t = \pi_t + y_t.$$

Assume that real economic growth is determined by real factors and thus unaffected by monetary policy. For simplicity, assume $y_t = y$. It follows directly that:

$$S_{M,t} = (\pi_{t+1} + y) \cdot \frac{M_t}{P_t Y_t} = \pi_{t+1} \cdot k_t + C.$$

Therefore, current seignorage income resembles a tax with tax rate π and tax base k (apart from some lump-sum revenue C). It is referred to as inflation tax.

(d) *Discuss how the concept of* seignorage $S = ik$ *with* $k = \frac{M}{PY}$ *differs from that of the inflation tax, i.e. from* S_M.

Now look at total seignorage income (in steady state)

$$S = i \cdot \frac{M}{PY} = (r + \pi + y - y) \cdot k = (\pi + y) \cdot k + (r - y) \cdot k = S_M + (r - y) \cdot k$$

where the Fisher Equation ($i_t = r_t + \pi_{t+1}^e$) is used to transform the equation. How to differentiate S_M and S?

- Total seignorage S: All revenues the government obtains due to its monopoly over the provision of money (cf. monopolistic rents);
- Current seignorage income S_M: Real resources the government obtains via the issuance of money.

In more detail: S captures the opportunity costs of issuing money since the government saves interest payments by making private agents hold money instead of interest-rate bearing bonds.

(e) *Assume demand for real balances depends on the nominal interest rate. Derive that level of the nominal interest rate that maximizes steady-state seignorage revenues S. Discuss your result.*

Steady state seignorage is given by $S(i) = i \cdot k(i)$ where $k(i)$ is money demand that depends negatively on the nominal interest rate. Condition for maximum is given by:

$$\frac{\partial S(i)}{\partial i} = k(i) + i \cdot \frac{\partial k(i)}{\partial i} = 0,$$

$$\eta_k = \frac{\partial k(i)}{\partial i} \cdot \frac{i}{k} = -1.$$

3.2.2 Sustainability of Government Debt in a Monetary Economy

Suppose that the consolidated government debt in an economy evolves as follows:

$$B_t = G_t - T_t - \Delta M_t + (1 + i_t) B_{t-1}$$

in which B_t *is nominal government debt held by the private sector,* $G_t - T_t$ *is the primary deficit,* M_t *is the money supply, and* i_t *is the nominal rate of interest.*

(a) Derive a difference equation for the debt to GDP ratio.

Difference equation for debt-to-GDP ratio is given by

$$
\begin{aligned}
B_t &= G_t - T_t - \Delta M_t + (1+i_t)B_{t-1},\\
\frac{B_t}{P_tY_t} &= \frac{G_t}{P_tY_t} - \frac{T_t}{P_tY_t} - \frac{\Delta M_t}{P_tY_t} + (1+i_t)\frac{B_{t-1}}{P_tY_t},\\
b_t &= g_t - \tau_t - \frac{\Delta M_t}{M_t}\cdot\frac{M_t}{P_tY_t} + (1+i_t)\frac{B_{t-1}}{P_{t-1}Y_{t-1}}\cdot\frac{P_{t-1}Y_{t-1}}{P_tY_t}\\
&= g_t - \tau_t - \mu_t\cdot k_t + b_{t-1}\cdot\frac{(1+i_t)}{(1+y_t)(1+\pi_t)}\\
&= g_t - \tau_t - \mu_t\cdot k_t + b_{t-1}\cdot(1+r_t-y_t),\\
b_t - b_{t-1} &= g_t - \tau_t - \mu_t\cdot k_t + (r_t-y_t)\cdot b_{t-1}.
\end{aligned}
$$

(b) Solve this equation for the steady-state debt ratio. What does the steady-state debt ratio depend on? Interpret.

In the steady state, $b_t = b_{t-1} = b$, then

$$
\begin{aligned}
(r-y)\cdot b &= \tau - g + \mu\cdot k,\\
b &= \frac{\tau - g + \mu\cdot k}{r-y}.
\end{aligned}
$$

(c) Now use a continuous time analogue to the obtained difference equation and solve it. Discuss the four different adjustment mechanisms if a shock was to temporarily increase the current debt level above its sustainable long-run equilibrium level.

Starting from the differential equation

$$
\begin{aligned}
\dot{b}(t) &= g(t) - \tau(t) - \mu(t)k(t) + (r(t)-y(t))b(t),\\
\dot{b}(t) - (r(t)-y(t))b(t) &= g(t) - \tau(t) - \mu(t)k(t).
\end{aligned}
$$

Use product rule to find function $\lambda(t)$ such that:

$$
\lambda(t)\dot{b}(t) - \lambda(t)(r(t)-y(t))b(t) = [\lambda(t)\dot{b(t)}].
$$

This function can be recovered from

$$
\begin{aligned}
\dot{\lambda(t)} &= -\lambda(t)(r(t)-y(t)),\\
\frac{\dot{\lambda(t)}}{\lambda(t)} \equiv \dot{\lambda(t)} &= -(r(t)-y(t)),
\end{aligned}
$$

$$\lambda(t) = c_0 \cdot e^{-\int_0^t (r(s)-y(s))ds}.$$

Use this function to solve the differential equation:

$$[\lambda(t)\dot{b}(t)] = \lambda(t)[g(t) - \tau(t) - \mu(t)k(t)],$$

$$\lambda(T)b(T) + c_1 = \int_0^T \lambda(t)[g(t) - \tau(t) - \mu(t)k(t)]dt,$$

$$b(T) = \lambda(T)^{-1}\left(\int_0^T \lambda(t)[g(t) - \tau(t) - \mu(t)k(t)]dt + c_2\right).$$

Using the expression for $\lambda(t)$ derived above

$$\begin{aligned} b(T) &= e^{\int_0^T (r(s)-y(s))ds}\left(\int_0^T e^{-\int_0^t (r(s)-y(s))ds}[g(t) - \tau(t) - \mu(t)k(t)]dt + c_0^{-1}c_2\right) \\ &= Ae^{\int_0^T (r(s)-y(s))ds} + \int_0^T e^{\int_0^T (r(s)-y(s))ds - \int_0^t (r(s)-y(s))ds} \cdot [g(t) - \tau(t) - \mu(t)k(t)]dt \\ &= Ae^{\int_0^T (r(s)-y(s))ds} + \int_0^T e^{\int_t^T (r(s)-y(s))ds}[g(t) - \tau(t) - \mu(t)k(t)]dt. \end{aligned}$$

Solve for A by setting looking at initial (time $T = 0$) debt level $b(0)$

$$b(0) = Ae^{\int_0^0 (r(s)-y(s))ds} + \int_0^T e^{\int_t^0 (r(s)-y(s))ds}[g(t) - \tau(t) - \mu(t)k(t)]dt = A.$$

Therefore, the following expression solves the differential equation for initial debt to GDP ratio $b(0)$

$$b(T) = b(0)e^{\int_0^T (r(s)-y(s))ds} + \int_0^T [g(t) - \tau(t) - \mu(t)k(t)]e^{\int_t^T (r(s)-y(s))ds}dt.$$

Equivalently, the current value of the debt to GDP ratio is given by

$$b(0) = b(T)e^{-\int_0^T (r(s)-y(s))ds} + \int_0^T [\tau(t) - g(t) + \mu(t)k(t)]e^{-\int_0^t (r(s)-y(s))ds}dt.$$

Now, let's look at a simplified version. Suppose that the real interest rate and the real growth rate are constant as well as τ, g and μk

$$\begin{aligned} b(T) &= b(0)e^{\int_0^T (r-y)ds} + \int_0^T [g - \tau - \mu k]e^{\int_t^T (r-y)ds}dt \\ &= b(0)e^{(r-y)T} + [g - \tau - \mu k]\int_0^T e^{(r-y)(T-t)}dt \end{aligned}$$

$$= b(0)e^{(r-y)T} + [g - \tau - \mu k]e^{(r-y)T} \int_0^T e^{-(r-y)t} dt$$

$$= b(0)e^{(r-y)T} + \frac{g - \tau - \mu k}{r - y} e^{(r-y)T} \left(1 - e^{-(r-y)T}\right)$$

$$= \left(b(0) + \frac{g - \tau - \mu k}{r - y}\right) e^{(r-y)T} - \frac{g - \tau - \mu k}{r - y}.$$

Now, back to the general form but keep r and y constant. When talking about debt sustainability, we look at T $\to \infty$ and impose the transversality condition. This yields

$$b(0) \equiv \frac{B(0)}{P(0)Y(0)} = \int_0^\infty [\tau(t) - g(t) + \mu(t)k(t)]e^{-(r-y)t} dt.$$

Idea: The current debt to GDP ratio equals the present value of future primary surpluses and seignorage income discounted by the effective real interest rate $r - y$.

Adjustment in case of shocks:

- Fiscal consolidation: Cut spending and increase taxes $\tau(t) - g(t) \to$ monetary dominance;
- *Printing press*: Increase seignorage revenues $\mu(t)k(t) \to$ fiscal dominance;
- (Partial) default on debt;
- Fiscal theory of the price level: Budget constraint as a valuation equation (cf. stock market). Current real value depends on discounted sum of expected budget surpluses. Adjustment of $P(0)$.

(d) The total government deficit—which includes interest payments on government debt—as a share of GDP is now given as $d_t = g_t - \tau_t - \mu k + i_t b_t$. Derive an expression for the steady-state deficit-to-GDP ratio.

The total government deficit is given by $d_t = g_t - \tau_t - \mu k + i_t b_t$.

Evolution of government debt is given by $b_t - b_{t-1} = g_t - \tau_t - \mu_t k_t + (r_t - y_t) b_{t-1}$.

Taken together and using the Fisher Equation with $r_t = i_t - \pi_t$, this yields $b_t - b_{t-1} = d_t - (\pi_t + y_t) b_{t-1}$.

In steady state, $b^* = \dfrac{d^*}{\pi^* + y^*}$.

3.2.3 Sustainability of Debt in a Small Open Economy

Consider a small open economy with an infinitely lived representative consumer with CES utility and time preference parameter ρ. The economy's growth rate is y. Initial asset positions are zero.

(a) *Derive the present value budget constraint. Assume that the representative consumer is a Ramsey consumer, i.e., her optimal consumption path follows that in Ramsey model, derived from Exercise 3 of Chap. 1. Under which conditions is this optimal consumption path well defined?*

The present value *aggregate* budget constraint is given by

$$\int_0^{+\infty} e^{-rt} C(t) dt = \int_0^{+\infty} e^{-rt} Y(t) dt + B(0).$$

As a result of Ramsey–Cass–Koopmans model, the consumption path is given as

$$\frac{\dot{c}(t)}{c(t)} = \sigma(r - n - \rho),$$

given $c(0)$ it is equivalent to

$$c(t) = c(0) \exp\left[\sigma(r - n - \rho)t\right].$$

The population at time t is $L(t) = L(0)e^{nt}$, therefore in the aggregate level the consumption path can be represented as

$$\begin{aligned} c(t)L(t) &= c(0)L(0) \exp(nt) \exp\left[\sigma(r - n - \rho)t\right], \\ C(t) &= C(0) \exp\left[\left(\sigma(r - n - \rho) + n\right) t\right]. \end{aligned}$$

Insert it into the budget constraint, one can get

$$\begin{aligned} \int_0^{+\infty} C(0) \exp\left[\left(\sigma(r - n - \rho) + n - r\right) t\right] dt &= \int_0^{+\infty} e^{-rt} Y(t) dt + B(0), \\ \frac{1}{r - \sigma(r - n - \rho) - n} C(0) &= \frac{1}{r - y} Y(0) + B(0). \end{aligned}$$

This is only feasible when $r - \sigma(r - n - \rho) - n > 0, r > y$.

(b) *What is the interest rate in autarky?*

In autarky $C(t) = Y(t), \forall t \in [0, +\infty)$. Therefore

$$\begin{aligned} r - \sigma(r - n - \rho) - n &= r - y, \\ r &= n + \rho + \frac{y - n}{\sigma}. \end{aligned}$$

(c) *Suppose that the country is initially running a current account deficit. How are current account and debt/GDP-ratio evolving over time?*

The definition of current account gives

$$CA(t) = \dot{B}(t) = Y(t) - C(t) + rB(t).$$

Solve this linear differential equation with propagator, one can get

$$B(0) = \lim_{T \to +\infty} e^{-rT} B(T) + \int_0^{+\infty} e^{-rt} [Y(t) - C(t)] \, dt.$$

The transversality condition requires that the first term of the right-hand side is equal to 0, and this implies that

$$-\infty < B(0) = \int_0^{+\infty} e^{-rt} [Y(t) - C(t)] \, dt < 0.$$

The boundary condition $B(0) = Y(0) - C(0) < 0$ plus the fact that $Y(t)$ and $C(t)$ are exponential functions imply that

$$\lim_{T \to +\infty} \frac{C(T)}{Y(T)} = 0,$$

otherwise $\int_0^{+\infty} e^{-rt} [Y(t) - C(t)] \, dt = -\infty$.

Debt/GDP-ratio is defined as $b(t) = \frac{B(t)}{Y(t)}$, by log-linearization one can see that

$$\begin{aligned} \frac{\dot{b}(t)}{b(t)} &= \frac{\dot{B}(t)}{B(t)} - \frac{\dot{Y}(t)}{Y(t)}, \\ \dot{b}(t) &= \frac{\dot{B}(t)}{Y(t)} - yb(t) \\ &= 1 - \frac{C(t)}{Y(t)} + (r - y)b(t). \end{aligned}$$

The steady state is obtained by

$$\lim_{t \to +\infty} \dot{b}(t) = \lim_{t \to +\infty} 1 - \frac{C(t)}{Y(t)} + (r - y)b(t) = 0,$$

therefore the steady state level of debt/GDP-ratio is

$$b^* = -\frac{1}{r - y}.$$

3.2.4 Sustainability of Government Debt

Consider an economy with constant growth rate y and interest rate $r > y$.

(a) *Suppose that the government's primary deficit rate $d_p = \frac{G-T}{Y}$ is given exogenously and constant over time. Show how the debt ratio evolves over time. Which restrictions are needed to ensure long-run stability of the debt ratio. How does your answer change, if $r < y$?*

Define B_t as the government's debt at time t, then

$$\begin{aligned}
\dot{B}_t &= G_t - T_t + rB_t, \\
\dot{b}_t &= \frac{\dot{B}_t}{Y_t} - yb_t \\
&= \frac{G_t - T_t}{Y_t} + (r - y)b_t \\
&= d_p + (r - y)b_t.
\end{aligned}$$

Solve the linear ordinary differential equation of b_t with propagator. First solve the differential equation

$$\dot{b}_t = (r - y)b_t$$

and get the solution

$$b_t = C \exp[(r - y)t].$$

Now add the effect of propagator

$$b_t = C(t) \exp[(r - y)t]$$

and take derivation with respect to t

$$\begin{aligned}
\dot{b}_t &= \dot{C}(t) \exp[(r - y)t] + (r - y)C(t) \exp[(r - y)t] \\
&= \dot{C}(t) \exp[(r - y)t] + (r - y)b_t.
\end{aligned}$$

Compare with the original problem

$$\dot{b}_t = d_p + (r - y)b_t$$

one can see that

$$\dot{C}(t) \exp[(r - y)t] = d_p.$$

Solve this equation for $C(t)$

$$C(t) = \int d_p \exp[-(r-y)t]dt + c$$
$$= -\frac{d_p}{r-y}\exp[-(r-y)t] + c,$$

then plug it into the expression of b_t

$$b_t = \left(-\frac{d_p}{r-y}\exp[-(r-y)t] + c\right)\exp[(r-y)t].$$

Now consider the problem for $t \in [0, +\infty)$ with b_0 at $t = 0$

$$b_0 = -\frac{d_p}{r-y} + c,$$
$$c = b_0 + \frac{d_p}{r-y}.$$

Substitute for c and get b_t

$$b_t = \left(b_0 + \frac{d_p}{r-y}\right)\exp[(r-y)t] - \frac{d_p}{r-y}.$$

Given $r > y$ long-run stability of the debt ratio can only be ensured when

$$b_0 = -\frac{d_p}{r-y},$$

because $\lim_{t\to+\infty}\exp[(r-y)t] = +\infty$.

If $r < y$, then the long-run stability of the debt ratio is always ensured because $\lim_{t\to+\infty)}\exp[(r-y)t] = 0$.

(b) *Suppose that the government's total deficit rate* $d_t = \frac{G-T+rB}{Y}$ *is given exogenously and constant over time. Show how the debt ratio evolves over time. Which restrictions are needed to ensure long-run stability of the debt ratio. How does your answer depend on the interest rate?*

By the definitions of d_t and d_p, one can see that

$$d_t = \frac{G-T+rB}{Y} = d_p + rb.$$

Replace d_p by d_t in the ordinary differential equation

$$\begin{aligned}\dot{b}_t &= d_p + (r - y)b_t \\ &= d_t - yb_t.\end{aligned}$$

Solve this linear ordinary differential equation and get

$$b_t = \left(b_0 - \frac{d_t}{y}\right)\exp(-yt) + \frac{d_t}{y}.$$

The long-run stability of the debt ratio is ensured whenever $y > 0$. The answer doesn't depend on the interest rate.

(c) What are the long-run implications of a constant total deficit rate for the primary deficit?

From the result of (b) one can see that

$$\lim_{t\to+\infty} d_p = d_t - rb_t = d_t\left(1 - \frac{r}{y}\right).$$

Then d_p depends on the ratio of r and y:

- If $r > y$, then $d_p < 0$, i.e. the government has surplus;
- If $r < y$, then $d_p > 0$, i.e. the government has deficit.

(d) Suppose that r is the nominal interest rate and y is the growth rate of nominal output. How would a rise in the rate of inflation affect the long-run primary deficit if the total deficit rate is held constant?

Now $r = \pi + r^*$ and $y = \pi + y^*$ in which variables with $*$ are real terms. Then

$$\lim_{t\to+\infty} d_p = d_t\left(1 - \frac{r^* + \pi}{y^* + \pi}\right) = g(\pi).$$

It's easy to see that

$$\frac{dg}{d\pi} = -\frac{y^* - r^*}{(y^* + \pi)^2}.$$

Then the effect of a rise in π on the long-run primary deficit is:

- The long-run primary deficit decreases if $y^* > r^*$;
- The long-run primary deficit increases if $y^* < r^*$;
- The long-run primary deficit is independent of π if $y^* = r^*$.

(e) Discuss the relation of your results with the Maastricht criteria.

Maastricht criteria:

- Debt ratio $b_t \leq 60\%$;
- Total deficit ratio $d_t \leq 3\%$.

The result of (b) implies that

$$\lim_{t \to +\infty} b_t = \lim_{t \to +\infty} \left(b_0 - \frac{d_t}{y} \right) \exp(-yt) + \frac{d_t}{y} = \frac{d_t}{y}.$$

Apply empirical data $y^* = 3\%$ and $\pi = 2\%$, $y = y^* + \pi = 5\%$. Insert into the equation above, one can see that the steady state $b_t = 60\%$.

Part II

Monetary Policy in the Short Run

4 New Keynesian Macroeconomics

4.1 Exercises

4.1.1 Short Review Questions

(a) Explain how monopolistic competition leads to the price mark-up. What does this imply for the natural rate of output in an economy?
(b) Explain, in Blanchard–Kiyotaki model, how aggregate demand externalities prevent the price from being adjusted freely. What does price stickiness imply for the natural rate of output in an economy?
(c) Name a few other sources of price stickiness and explain how they prevent firms from adjusting their prices freely.
(d) What does sticky price imply for aggregate demand and supply in the short-run equilibrium? What are optimal monetary policy under (1) demand shocks, (2) supply shocks, and (3) mark-up shocks?

4.1.2 Sticky Price Models: The Policy Implication

Consider the models with sticky prices, such that some firms don't make immediate responses to the changes in the price level (such as Calvo 1983 and Yun 1996).

(a) Explain the difference between *ex ante* and *ex post* mark-up.
(b) Explain how monetary policy affects the real economy.

J. Cao, G. Illing, *Instructor's Manual for* Money: Theory and Practice, Springer Texts in Business and Economics,
https://doi.org/10.1007/978-3-030-23618-2_4

4.1.3 Staggered Price Setting: The Driving Forces

Consider the models with staggered price setting such as Calvo (1983) and Yun (1996).

(a) Show that increases in output have a positive impact on inflation.
(b) Explain why the resulting aggregate supply curve is forward looking.
(c) Explain how the economy is distorted by monopolistic competition and staggered price setting. Provide some intuitions on how economic policies may restore the efficiency of equilibrium allocations.
(d) Show that stabilizing output and stabilizing inflation are no conflicting goals.

4.1.4 Price Setting with Differentiated Goods

Consider a representative agent with utility function

$$U = \left(\sum_{i=1}^{m} C_i^{\gamma}\right)^{\frac{\alpha}{\gamma}} \left(\frac{M}{P}\right)^{1-\alpha} - N^{\beta}, \text{ with } 0 < \gamma < 1, 0 < \alpha < 1, \beta > 1.$$

Assume that firm's profits are distributed to consumers, but a single consumer's decision has no impact on these profits. Thus, profit income is taken as exogenous by consumers.

(a) Derive the demand functions for commodities C_i and for money M and the supply for labor N. To ease your calculations, use aggregate indices for consumption and prices:

$$C = \left(\sum_{i=1}^{m} C_i^{\gamma}\right)^{\frac{1}{\gamma}}, P = \left(\sum_{i=1}^{m} P_i^{-\frac{\gamma}{1-\gamma}}\right)^{-\frac{1-\gamma}{\gamma}}.$$

(b) Assume that firms produce goods with production function $C_i = \theta N_i$, where N_i is the labor input of firm i. Labor is homogeneous and the labor market is competitive. Firms are setting prices P_i in order to maximize profits. Show that equilibrium prices are above marginal costs (assume that firms are small to the extent that a single firm's decisions has no impact on average income of households).
(c) Show that equilibrium levels of production and employment are below the efficient levels.
(d) Following Blanchard and Kiyotaki (1987), explain how menu costs can prevent price adjustments to monetary expansion and how this influences overall efficiency.

4.1.5 Monopolistic Competition, Catalogue Cost, and Monetary Policy

Consider an economy of yeomen farmers, which has the following features:

- The economy is populated by a continuum of farmers, each of whom is indexed by $i \in [0, 1]$, i.e., the population is normalized to be 1;
- Each farmer, say, farmer i, is the unique producer of a monopolistically competitive good i whose price P_i is set by herself;
- Farmers don't produce their goods by themselves. Instead, each farmer goes to a competitive labor market (which means that each individual employer takes the nominal wage rate, W, as given) where she sells her labor to the other farmers and hires the other farmers as the labor input for her production. In equilibrium, the labor supply of farmer i, L_i, should be equal to the labor input in her production (although she doesn't work for herself).

All the farmers have the same technology which transforms one unit of labor into on unit of product (denoted by Y), i.e., for farmer i

$$Y_i = L_i.$$

The representative farmer i's utility comes from consuming all the varieties of goods, C_j ($\forall j \in [0, 1]$), and displeasure from providing labor,

$$U_i = C - \frac{L_i^\beta}{\beta}, \text{ with } \beta > 1 \tag{4.1}$$

in which C is the Dixit–Stiglitz index for her consumption. By market clearing, C is equal to per capita GDP as well. Further, using Dixit–Stiglitz aggregation implies that, for her consumption bundle,

$$\int_0^1 P_j C_j dj = PC,$$

in which P is the Dixit–Stiglitz index for the price level. Therefore, her budget constraint is

$$PC = (P_i - W)Y_i + WL_i. \tag{4.2}$$

Under monopolistic competition, it is known that

$$Y_i = C\left(\frac{P_i}{P}\right)^{-\eta}, \text{ with } \eta > 1, \tag{4.3}$$

therefore, the budget constraint becomes

$$PC = (P_i - W)C\left(\frac{P_i}{P}\right)^{-\eta} + WL_i. \tag{4.4}$$

The left-hand side is her total expenditure on consumption, and the right-hand side is her total income made by the profit from selling good i plus her wage from selling labor.

Thus farmer i's decision problem is choosing the optimal P_i and L_i to maximize her utility function (4.1), subject to her budget constraint (4.4).

(a) Give some interpretation to Eq. (4.3).
(b) Show that the optimal price P_i is set as

$$\frac{P_i}{P} = \frac{\eta}{\eta - 1}\frac{W}{P}, \tag{4.5}$$

and the optimal labor supply is given by

$$L_i = \left(\frac{W}{P}\right)^{\frac{1}{\beta-1}}. \tag{4.6}$$

Show *analytically* that equilibrium levels of production and employment are below the efficient levels (Hint: Equilibrium implies symmetry in price setting and labor supply).

Suppose that in this economy, all the payments are required to be done with fiat money, i.e., the prices of both goods and labor are paid with money. Regarding the representative consumer's budget constraint, P_i and W on the right-hand side of (4.2) are paid with money. The aggregate money supply is under the control of the central bank. Per capita money supply is M, therefore, Eq. (4.2) can be equivalently written as $PC = M$.

(c) Show *analytically* that in equilibrium money is neutral in this economy, i.e., changing money supply doesn't have any impact on real resource allocation, as long as the prices are flexible.

Now suppose that in this economy, all the prices of the goods are listed on a catalogue. One farmer can only adjust the price of her product if she pays a fixed cost for the publisher in order to publish a new edition of the catalogue. Then the central bank starts an expansionary monetary policy.

(d) Explain *verbally* under which condition(s) the farmers' price adjustments are prevented.
(e) Show *analytically* that in equilibrium money is no longer neutral and the overall efficiency can be improved by the expansionary monetary policy (assume that the expansion in money supply is small enough so that the farmers' price adjustments are totally prevented by the catalogue cost).

4.1.6 Monopolistic Competition, Aggregate Demand Externalities, and Sticky Price

Consider a representative household that lives for two periods with utility function

$$V(C_1, C_2, N_1, N_2) = U(C_1) - V(N_1) + \frac{1}{1+\rho} E_1 \left[U(C_2) - V(N_2) \right]$$

in which N is the labor supply and C is a consumption index given by

$$C_t \equiv \left(\int_0^1 C_t(i)^{\frac{\theta-1}{\theta}} di \right)^{\frac{\theta}{\theta-1}}$$

with $C_t(i)$ representing the quantity of good i consumed in period t at price $P_t(i)$. Assume that the period utility function is given by

$$U(C_t) = \frac{C_t^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}},$$

$$V(N_t) = \frac{N_t^{1+\varphi}}{1+\varphi}.$$

The household saves by buying one-period risk-free bonds B_t at price Q_t. His budget constraint is given by

$$(1+\tau_t^C) \int_0^1 P_t(i) C_t(i) di + Q_t B_t = B_{t-1} + (1-\tau_t^N) W_t N_t + T_t$$

in which T_t are lump-sum components of income including firms' profits, and τ_t^C and τ_t^N represent taxes on consumption and wages. In addition, there is a continuum of producers with measure 1 each of which produces one of these varieties under monopolistic competition with the linear production technology

$$Y_t(i) = A_t N_t(i).$$

Assume that firms do not face any additional frictions in setting their prices so that prices are perfectly flexible.

(a) Derive the optimal allocation of consumption expenditures among different goods for the household and the demand function for good i.
(b) Give an interpretation of θ and plot the demand function for good i.
(c) Derive the optimal labor supply and consumption decision of the household in log-linear form.

(d) Derive the optimal price setting rule for the monopolistic producer of good i under the assumption of perfectly flexible prices. What is the effect of monopolistic competition relative to the competitive case?
(e) What is the *natural* level of output $Y_{t,n}$ in this economy. Discuss its properties by comparing it to the output level that would result under perfect competition.
(f) Assume that agents expect the economy to be at its natural level before the realization of shocks, i.e. $y_2^e = y_1^e = y_n$ and $p_2^e = p^e = p^*$. Derive the AD curve under this set-up.
(g) Now assume that there are price rigidities so that in period 1 only a fraction of firms $1 - \alpha$ can adjust their prices whereas the remaining fraction α is unable to do so. Prices are perfectly flexible in period 2. Derive the optimal price setting rule in this case and the AS curve.

4.2 Solutions for Selected Exercises

4.2.1 Sticky Price Models: The Policy Implication

Consider the models with sticky prices, such that some firms don't make immediate responses to the changes in the price level (such as Calvo 1983 and Yun 1996).

(a) Explain the difference between ex ante *and* ex post *mark-up.*

Ex ante mark-up is the mark-up the firm desires at the time when it sets its price, which could be expressed as

$$E_t\left[P_s(z)\right] = E_t\left[(1+\mu_s)P_s MC_s\right]$$

in which $s > t$. Ex post mark-up is the mark-up when the uncertainties are resolved, which could be expressed as

$$P_t(z) = (1+\mu_s')P_s MC_s$$

in which $s > t$, and $P_t(z)$ is the firm's price at time s which was already fixed at time t. Suppose that there is an unexpected increase in aggregate demand, and therefore an increase in the demand of labor which drives up MC_s. Since the firm's price was already fixed, then its ex post mark-up goes down. So with sticky prices in assumption, we have the counter-cyclical mark-ups.

(b) Explain how monetary policy affects the real economy.

Now money matters because of sticky price settings. For any change in price level, due to monetary policy, the firms' ex post mark-ups differ from each other. Therefore, the firms differ from each other on their *relative* prices, and this leads to a dispersion in the consumption of differentiated goods.

4.2.2 Staggered Price Setting: The Driving Forces

Consider the models with staggered price setting such as Calvo (1983) and Yun (1996).

(a) Show that increases in output have a positive impact on inflation.

The aggregate supply relation in Calvo–Yun model is captured in the new Keynesian Phillips curve, which could be expressed as following:

$$\pi_t = \kappa \hat{mc}_t + \beta E_t \pi_{t+1}. \tag{4.7}$$

The increase in output reflects the excess demand, which in turn causes scarcity of output and running capital at high intensity. This results in higher marginal cost, hence higher prices and greater inflation.

(b) Explain why the resulting aggregate supply curve is forward looking.

Because the price is sticky such that the firms set prices in advance. Since the firms have market power, so if the price is flexible the price that maximizes one firm's profit is a constant mark-up over marginal cost. But when the price is sticky, the firm's mark-up would be affected when there are any changes in the price level, during the periods when the firm is not able to adjust its price. Therefore, whenever the firm has the chance to set its price, it has to take into account both current real marginal cost and expected future real marginal cost—so that the resulting aggregate supply curve is forward looking.

(c) Explain how the economy is distorted by monopolistic competition and staggered price setting. Provide some intuitions on how economic policies may restore the efficiency of equilibrium allocations.

First, we examine the distortion related to *monopolistic competition*.

Monopolistic competition distorts factor prices, hence the agent's intratemporal decisions on consumption and labor,

$$-\frac{\frac{\partial u_t}{\partial N_t}}{\frac{\partial u_t}{\partial C_t}} = \frac{W_t}{P_t} = \frac{1}{1+\mu}\frac{\partial Y(z)}{\partial N(z)}.$$

This suggests that it would be optimal to subsidize the employment cost. Suppose that at the rate τ the employment is subsidized, then

$$-\frac{\frac{\partial u_t}{\partial N_t}}{\frac{\partial u_t}{\partial C_t}} = (1+\tau)\frac{W_t}{P_t} = \frac{1+\tau}{1+\mu}\frac{\partial Y(z)}{\partial N(z)}$$

and the optimality would be restored if the social planner sets $\tau = \mu$;

Second, we examine the distortion related to *staggered price setting*.

It is known that the output level of a monopolistically competitive firm, $Y_t(z)$, is determined by the aggregate output Y_t and the relative price $\frac{P_t(z)}{P_t}$ in a way that

$$Y_t(z) = \left[\frac{P_t(z)}{P_t}\right]^{-\epsilon} Y_t.$$

Apply this expression to calculate the aggregate output Y_t^z

$$\begin{aligned}
Y_t^z &= \int_0^1 Y_t(z)dz \\
&= \int_0^1 \left[\frac{P_t(z)}{P_t}\right]^{-\epsilon} Y_t dz \\
&= Y_t \int_0^1 \left[\frac{P_t(z)}{P_t}\right]^{-\epsilon} dz, \\
A_t N_t^\alpha K_{t-1}^{1-\alpha} &= Y_t \int_0^1 \left[\frac{P_t(z)}{P_t}\right]^{-\epsilon} dz,
\end{aligned}$$

and the last step uses the fact that the production function is constant return to scale. Now let's define a new variable to finish the aggregation of production

$$s_t = \int_0^1 \left[\frac{P_t(z)}{P_t}\right]^{-\epsilon} dz,$$

which measures the gap between aggregate output of intermediate goods and final goods, i.e.

$$Y_t^z = A_t N_t^\alpha K_{t-1}^{1-\alpha} = s_t Y_t.$$

Obviously $s_t = 1$ if there is no price dispersion, which is caused by fluctuations in price level. Define $\zeta_t = \left[\frac{P_t(z)}{P_t}\right]^{1-\epsilon}$, and obviously $s_t = \int_0^1 \zeta_t^{\frac{\epsilon}{\epsilon-1}} dz$. Notice that

$$\begin{aligned}
\left(\int_0^1 \zeta_t dz\right)^{\frac{\epsilon}{\epsilon-1}} &= \left\{\int_0^1 \left[\frac{P_t(z)}{P_t}\right]^{1-\epsilon} dz\right\}^{\frac{\epsilon}{\epsilon-1}} \\
&= P_t^\epsilon \left\{\left[\int_0^1 P_t(z)^{1-\epsilon} dz\right]^{\frac{1}{1-\epsilon}}\right\}^{-\epsilon} \\
&= 1
\end{aligned}$$

using the definition of price index for the last step, one can see that

$$1 = \left(\int_0^1 \zeta_t dz \right)^{\frac{\epsilon}{\epsilon-1}} \leq \int_0^1 \zeta_t^{\frac{\epsilon}{\epsilon-1}} dz = s_t$$

by Jensen's inequality because $\epsilon > 1$ and $\frac{\epsilon}{\epsilon-1} > 1$, and the equality holds only for ζ_t being constant, i.e. $P_t(z) = P_t, \forall z \in [0, 1]$—when there exists no price dispersion.

Since s_t is bounded below by 1, the output level, i.e. the production of the final goods, is distorted by the factor of s_t due to the existence of price dispersion. Since the prices are adjusted in a staggering manner in our economy, the only way to wipe out such inefficient price dispersion is to keep the price level constant, such that the firms don't have to adjust their prices at all. Therefore, it is a desirable policy to stabilize the price level, i.e. to eliminate inflation, in this economy.

(d) Show that stabilizing output and stabilizing inflation are no conflicting goals.

There is no conflict between a policy designed to maintain inflation at zero and a policy designed to keep the output gap equal to zero. Rewrite (4.7) in terms of output gap

$$\pi_t = \kappa \left(\gamma + \gamma_n \right) x_t + \beta E_t \pi_{t+1}, \tag{4.8}$$

then when output is stabilized such that $x_{t+i} = 0$ for all $i > 0$, then $\pi_{t+i} = 0$, and the same argument holds when inflation is maintained at zero. Because firms adjust prices in a staggered manner, inflation generates a costly dispersion of prices; the central bank can eliminate this source of distortion by ensuring price stability. When firms do not need to adjust their prices, the fact that prices are sticky is no longer relevant, and the output is thus stabilized.

4.2.3 Price Setting with Differentiated Goods

Consider a representative agent with utility function

$$U = \left(\sum_{i=1}^{m} C_i^{\gamma} \right)^{\frac{\alpha}{\gamma}} \left(\frac{M}{P} \right)^{1-\alpha} - N^{\beta}, \text{ with } 0 < \gamma < 1, 0 < \alpha < 1, \beta > 1.$$

Assume that firm's profits are distributed to consumers, but a single consumer's decision has no impact on these profits. Thus, profit income is taken as exogenous by consumers.

(a) *Derive the demand functions for commodities C_i and for money M and the supply for labor N. To ease your calculations, use aggregate indices for consumption and prices:*

$$C = \left(\sum_{i=1}^{m} C_i^{\gamma}\right)^{\frac{1}{\gamma}}, P = \left(\sum_{i=1}^{m} P_i^{-\frac{\gamma}{1-\gamma}}\right)^{-\frac{1-\gamma}{\gamma}}.$$

The representative agent's problem is to maximize her utility

$$\max_{C_i, M, N} \; U = \left(\sum_{i=1}^{m} C_i^{\gamma}\right)^{\frac{\alpha}{\gamma}} \left(\frac{M}{P}\right)^{1-\alpha} - N^{\beta}$$

$$s.t. \; \sum_{i=1}^{m} P_i C_i + M \leq wN + Y.$$

Set up the Lagrangian for this problem

$$\mathscr{L} = \left(\sum_{i=1}^{m} C_i^{\gamma}\right)^{\frac{\alpha}{\gamma}} \left(\frac{M}{P}\right)^{1-\alpha} - N^{\beta} - \lambda \left(\sum_{i=1}^{m} P_i C_i + M - wN - Y\right)$$

and derive the first-order conditions

$$\frac{\partial \mathscr{L}}{\partial C_i} = \frac{\alpha}{\gamma} \left(\sum_{i=1}^{m} C_i^{\gamma}\right)^{\frac{\alpha}{\gamma}-1} \left(\frac{M}{P}\right)^{1-\alpha} \gamma C_i^{\gamma-1} - \lambda P_i = \alpha C^{\alpha-\gamma} \left(\frac{M}{P}\right)^{1-\alpha} C_i^{\gamma-1} - \lambda P_i = 0, \tag{4.9}$$

$$\frac{\partial \mathscr{L}}{\partial M} = C^{\alpha}(1-\alpha)\left(\frac{M}{P}\right)^{-\alpha} \frac{1}{P} - \lambda = 0, \tag{4.10}$$

$$\frac{\partial \mathscr{L}}{\partial N} = -\beta N^{\beta-1} + w\lambda = 0. \tag{4.11}$$

Combine (4.9) and (4.10) to get

$$\frac{\alpha}{1-\alpha} C^{-\gamma} M C_i^{\gamma-1} = P_i,$$

$$\frac{\alpha}{1-\alpha} C^{-\gamma} P_i^{-\gamma} M = P_i^{1-\gamma} C_i^{1-\gamma},$$

$$P_i C_i = \left(\frac{\alpha}{1-\alpha} C^{-\gamma} P_i^{-\gamma} M\right)^{\frac{1}{1-\gamma}},$$

$$\sum_{i=1}^{m} P_i C_i = \left(\frac{\alpha}{1-\alpha} C^{-\gamma} M\right)^{\frac{1}{1-\gamma}} \sum_{i=1}^{m} P_i^{-\frac{\gamma}{1-\gamma}},$$
$$\sum_{i=1}^{m} P_i C_i = \left(\frac{\alpha}{1-\alpha} M\right)^{\frac{1}{1-\gamma}} (PC)^{-\frac{\gamma}{1-\gamma}}.$$

And from the first line one can also solve for C_i

$$C_i^{\gamma} = \left(\frac{\alpha}{1-\alpha} C^{-\gamma} \frac{M}{P_i}\right)^{\frac{\gamma}{1-\gamma}},$$
$$\sum_{i=1}^{m} C_i^{\gamma} = \left(\frac{\alpha}{1-\alpha} C^{-\gamma} M\right)^{\frac{\gamma}{1-\gamma}} \sum_{i=1}^{m} P_i^{-\frac{\gamma}{1-\gamma}},$$
$$C^{\gamma} = \left(\frac{\alpha}{1-\alpha} C^{-\gamma} M\right)^{\frac{\gamma}{1-\gamma}} P^{-\frac{\gamma}{1-\gamma}},$$
$$PC = \frac{\alpha}{1-\alpha} M,$$

giving the consumption index in the equilibrium

$$C = \frac{\alpha}{1-\alpha} \frac{M}{P}. \tag{4.12}$$

Insert this result into the expression of $\sum_{i=1}^{m} P_i C_i$, it's easy to see that

$$\sum_{i=1}^{m} P_i C_i = (PC)^{\frac{1}{1-\gamma}} (PC)^{-\frac{\gamma}{1-\gamma}} = PC.$$

Now the budget constraint can be manipulated by the relations between $\sum_{i=1}^{m} P_i C_i$, PC and M

$$\frac{\alpha}{1-\alpha} M + M = wN + Y,$$
$$M = (1-\alpha)(wN + Y)$$

as well as the aggregate consumption index

$$C = \frac{\alpha(wN + Y)}{P}.$$

Apply the result above into the expression for C_i and get the demand function for commodity i

$$\begin{aligned} C_i &= \left(\frac{\alpha}{1-\alpha} C^{-\gamma} \frac{M}{P_i}\right)^{\frac{1}{1-\gamma}} \\ &= \left[\frac{\alpha}{1-\alpha} \frac{(1-\alpha)(wN+Y)}{P_i} \left(\alpha \frac{wN+Y}{P}\right)^{-\gamma}\right]^{\frac{1}{1-\gamma}} \\ &= \alpha(wN+Y)\left(\frac{P^\gamma}{P_i}\right)^{\frac{1}{1-\gamma}}. \end{aligned}$$

Note that if $P_i = P$ the demand function for commodity i reduces to

$$C_i = \frac{\alpha(wN+Y)}{P}.$$

Combine (4.10) and (4.11) to see the supply of labor N

$$\begin{aligned} N^{\beta-1} &= \frac{w\lambda}{\beta} \\ &= \frac{wC^\alpha(1-\alpha)\left(\frac{M}{P}\right)^{-\alpha} \frac{1}{P}}{\beta} \\ &= \frac{wC^\alpha(1-\alpha)\left(\frac{C(1-\alpha)}{\alpha}\right)^{-\alpha} \frac{1}{P}}{\beta} \\ &= \frac{w(1-\alpha)^{1-\alpha}\alpha^\alpha}{\beta P}, \end{aligned}$$

and we find the labor supply decision of the representative agent

$$N = \left[\frac{w(1-\alpha)^{1-\alpha}\alpha^\alpha}{\beta P}\right]^{\frac{1}{\beta-1}}. \tag{4.13}$$

(b) Assume that firms produce goods with production function $C_i = \theta N_i$, where N_i is the labor input of firm i. Labor is homogeneous and the labor market is competitive. Firms are setting prices P_i in order to maximize profits. Show that equilibrium prices are above marginal costs (assume that firms are small to the extent that a single firm's decision has no impact on average income of households).

A firm's problem is to maximize its profit

$$\max_{P_i, N_i} \ \Pi_i = P_i C_i(P_i) - wN_i$$
$$s.t. \ \ C_i(P_i) = \theta N_i.$$

The equality constraint simply captures the firm's demand for labor,

$$N_i = \frac{C_i(P_i)}{\theta}. \tag{4.14}$$

Insert the equation above into the object function and the original problem becomes

$$\max_{P_i, N_i} \ \Pi_i = P_i C_i(P_i) - w\frac{C_i(P_i)}{\theta}.$$

The first-order condition gives

$$\frac{\partial \Pi_i}{\partial P_i} = C_i(P_i) + \left(P_i - \frac{w}{\theta}\right)\frac{\partial C_i(P_i)}{\partial P_i} = 0 \tag{4.15}$$

in which $\frac{\partial C_i(P_i)}{\partial P_i}$ can be derived from the demand function for commodity i

$$C_i = \alpha(wN + Y)\left(\frac{P^\gamma}{P_i}\right)^{\frac{1}{1-\gamma}},$$
$$\frac{\partial C_i}{\partial P_i} = -\frac{1}{1-\gamma}\alpha(wN + Y)\left(\frac{P^\gamma}{P_i}\right)^{\frac{1}{1-\gamma}}\frac{1}{P_i}$$
$$= -\frac{1}{1-\gamma}\frac{C_i}{P_i}.$$

Apply this result into the first-order condition (4.15)

$$C_i = \left(P_i - \frac{w}{\theta}\right)\frac{1}{1-\gamma}\frac{C_i}{P_i}, \tag{4.16}$$
$$P_i = \frac{w}{\gamma\theta}. \tag{4.17}$$

The marginal cost of production is $\frac{w}{\theta}$, therefore the price is above the marginal cost for those $\gamma \in (0, 1)$.

(c) Show that equilibrium levels of production and employment are below the efficient levels.

The efficient level of production achieves at the point where the marginal revenue is equal to the marginal cost, i.e. $P_i^e = \frac{w}{\theta}$ and $\theta = \frac{w}{P_i^e}$. And from the representative's decision on labor supply

$$N = \left[\frac{w(1-\alpha)^{1-\alpha}\alpha^{\alpha}}{\beta P}\right]^{\frac{1}{\beta-1}}$$

one can see that it only depends on the ratio $\frac{w}{P}$. However in equilibrium $\frac{w}{P} = \gamma\theta < \theta$, therefore the employment is below the efficient level.

Also from the demand function for commodity i

$$C_i = \frac{\alpha(wN + Y)}{P}$$

one can see that the equilibrium consumption level is also lower than the efficient level since $N < N^e$, $\frac{w}{P} = \gamma\theta < \theta$, and $P > P^e$. By market clearing condition this simply means that the output level is also lower than the efficient level.

(d) Following Blanchard and Kiyotaki (1987), explain how menu costs can prevent price adjustments to monetary expansion and how this influences overall efficiency.

Suppose that money supply rises a little from M to $\tilde{M}$. Now given the fact that all the firms still keep their prices at the equilibrium price level under M, then if a single firm i deviates to a new price level $\tilde{P}_i$ she may face two possible situations:

- She makes a higher profit than before. But this would make all the other firms deviate and they end up with a new price level under $\tilde{M}$;
- She makes no higher profit than before. Then all the firms would keep the old price level.

To see which one is the true outcome, assume that all the firms still keep their prices at the equilibrium price level under M. Then from (4.13) the labor supply of the representative agent is

$$\tilde{N}^s = \left[\frac{\tilde{w}(1-\alpha)^{1-\alpha}\alpha^{\alpha}}{\beta P}\right]^{\frac{1}{\beta-1}}.$$

Note that $\tilde{w}$ and $\tilde{N}^s$ are achieved under the new money supply $\tilde{M}$ and old price level P. Comparing with N^s under M one can easily see that

$$\tilde{N}^s = \left[\frac{\tilde{w}}{w}\right]^{\frac{1}{\beta-1}} N^s.$$

And from (4.14) the labor demand of the firms is

$$\tilde{N}^d = \frac{\tilde{C}(P)}{\theta} = \frac{\alpha}{1-\alpha}\frac{\tilde{M}}{\theta P}$$

in which that $\tilde{C}(P)$ is achieved under the new money supply $\tilde{M}$ and old price level P, and the second step is derived by using (4.12). Comparing with N^d under M one can easily see that

$$\tilde{N}^d = \frac{\tilde{M}}{M}N^d.$$

Market clearing conditions require that $\tilde{N}^s = \tilde{N}^d$ and $N^s = N^d$, and one can see that

$$\left[\frac{\tilde{w}}{w}\right]^{\frac{1}{\beta-1}} = \frac{\tilde{M}}{M}$$
$$\tilde{w} = w\left(\frac{\tilde{M}}{M}\right)^{\beta-1}.$$

Then if one firm i wants to deviate, her best response $\tilde{P}_i$ is featured by (4.17)

$$\begin{aligned}\tilde{P}_i &= \frac{\tilde{w}}{\gamma\theta} \\ &= \left(\frac{\tilde{M}}{M}\right)^{\beta-1}\frac{w}{\gamma\theta} \\ &= \left(\frac{\tilde{M}}{M}\right)^{\beta-1}P \\ &> P.\end{aligned}$$

Her profit from deviating is

$$\tilde{\Pi}_i = \left(\tilde{P}_i - \frac{\tilde{w}}{\theta}\right)\tilde{C}_i(\tilde{P}_i),$$

in comparison to her profit from not deviating

$$\Pi_i = \left(P_i - \frac{\tilde{w}}{\theta}\right)C_i(P_i)$$

the deviation makes sense only if $\tilde{\Pi}_i > \Pi_i$. Otherwise all the firms keep the old price level.

4.2.4 Monopolistic Competition, Aggregate Demand Externalities, and Sticky Price

Consider a representative household that lives for two periods with utility function

$$V(C_1, C_2, N_1, N_2) = U(C_1) - V(N_1) + \frac{1}{1+\rho} E_1 \left[U(C_2) - V(N_2)\right]$$

in which N is the labor supply and C is a consumption index given by

$$C_t \equiv \left(\int_0^1 C_t(i)^{\frac{\theta-1}{\theta}} di\right)^{\frac{\theta}{\theta-1}}$$

with $C_t(i)$ representing the quantity of good i consumed in period t at price $P_t(i)$. Assume that the period utility function is given by

$$U(C_t) = \frac{C_t^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}},$$

$$V(N_t) = \frac{N_t^{1+\varphi}}{1+\varphi}.$$

The household saves by buying one-period risk-free bonds B_t at price Q_t. His budget constraint is given by

$$(1+\tau_t^C)\int_0^1 P_t(i)C_t(i)di + Q_t B_t = B_{t-1} + (1-\tau_t^N)W_t N_t + T_t$$

in which T_t are lump-sum components of income including firms' profits, and τ_t^C and τ_t^N represent taxes on consumption and wages. In addition, there is a continuum of producers with measure 1 each of which produces one of these varieties under monopolistic competition with the linear production technology

$$Y_t(i) = A_t N_t(i).$$

Assume that firms do not face any additional frictions in setting their prices so that prices are perfectly flexible.

(a) Derive the optimal allocation of consumption expenditures among different goods for the household and the demand function for good i.

Optimal allocation of expenditures: For given expenditures E, the household maximizes its utility from consumption by choosing optimal quantities of each variety

$$\max U(C_t) = U\left(\left(\int_0^1 C_t(i)^{\frac{\theta-1}{\theta}} di\right)^{\frac{\theta}{\theta-1}}\right),$$

$$s.t. \quad \int_0^1 P_t(i)C_t(i)di \leq E.$$

Using Lagrangian

$$L = U\left(\left(\int_0^1 C_t(i)^{\frac{\theta-1}{\theta}} di\right)^{\frac{\theta}{\theta-1}}\right) - \lambda\left(\int_0^1 P_t(i)C_t(i)di - E\right),$$

first-order condition gives

$$\frac{\partial L}{\partial C_t(k)} = \frac{\partial U}{\partial C_t}\frac{\partial C_t}{\partial C_t(k)} - \lambda\frac{\partial \int_0^1 P_t(i)C_t(i)di}{\partial C_t(k)} = \frac{\partial U}{\partial C_t}\frac{\partial C_t}{\partial C_t(k)} - \lambda P_t(k) = 0,$$

arrange to get

$$\frac{\partial U}{\partial C_t}\frac{\theta}{\theta-1}\left(\int_0^1 C_t(i)^{\frac{\theta-1}{\theta}} di\right)^{\frac{\theta}{\theta-1}-1}\frac{\theta-1}{\theta}C_t(k)^{\frac{\theta-1}{\theta}-1} - \lambda P_t(k) = 0,$$

$$\frac{\partial U}{\partial C_t}\left(\int_0^1 C_t(i)^{\frac{\theta-1}{\theta}} di\right)^{\frac{1}{\theta-1}} C_t(k)^{-\frac{1}{\theta}} - \lambda P_t(k) = 0,$$

$$\frac{\partial U}{\partial C_t}\left(\int_0^1 C_t(i)^{\frac{\theta-1}{\theta}} di\right)^{\frac{\theta}{\theta-1}\frac{\theta-1}{\theta}\frac{1}{\theta-1}} C_t(k)^{-\frac{1}{\theta}} - \lambda P_t(k) = 0,$$

$$\frac{\partial U}{\partial C_t}C_t^{\frac{1}{\theta}} C_t(k)^{-\frac{1}{\theta}} - \lambda P_t(k) = 0.$$

How can we interpret the Lagrange parameter? λ measures the shadow price of a marginal increase in nominal expenditures E in terms of utility. One additional unit of E increases real consumption by P_t^{-1}. Therefore

$$\lambda = \frac{\partial U}{\partial C_t}\frac{1}{P_t}.$$

Taken together, this yields

$$\begin{aligned}
\frac{\partial U}{\partial C_t} C_t^{\frac{1}{\theta}} C_t(k)^{-\frac{1}{\theta}} - \frac{\partial U}{\partial C_t} \frac{P_t(k)}{P_t} &= 0, \\
C_t^{\frac{1}{\theta}} C_t(k)^{-\frac{1}{\theta}} - \frac{P_t(k)}{P_t} &= 0. \\
C_t(k) = C_t \left(\frac{P_t}{P_t(k)}\right)^{\theta}.
\end{aligned}$$

Substitute this into the definition of the consumption index

$$\begin{aligned}
C_t &= \left(\int_0^1 C_t(k)^{\frac{\theta-1}{\theta}} dk\right)^{\frac{\theta}{\theta-1}} \\
&= \left(\int_0^1 \left(C_t \left(\frac{P_t}{P_t(k)}\right)^{\theta}\right)^{\frac{\theta-1}{\theta}} dk\right)^{\frac{\theta}{\theta-1}} \\
&= \left(\int_0^1 C_t^{\frac{\theta-1}{\theta}} \left(\frac{P_t}{P_t(k)}\right)^{\theta-1} dk\right)^{\frac{\theta}{\theta-1}} \\
&= C_t P_t^{\theta} \left(\int_0^1 P_t(k)^{1-\theta} dk\right)^{\frac{\theta}{\theta-1}}, \\
P_t &= \left(\int_0^1 P_t(k)^{1-\theta} dk\right)^{\frac{1}{1-\theta}}.
\end{aligned}$$

Now solve for the relative demand for good k and good l. Note that the first-order condition holds for all varieties

$$\begin{aligned}
&\frac{\partial U}{\partial C_t} C_t^{\frac{1}{\theta}} C_t(k)^{-\frac{1}{\theta}} - \lambda P_t(k) = 0, \\
&\frac{\partial U}{\partial C_t} C_t^{\frac{1}{\theta}} C_t(l)^{-\frac{1}{\theta}} - \lambda P_t(l) = 0, \\
&C_t(k)^{-\frac{1}{\theta}} - \lambda P_t(k) = C_t(l)^{-\frac{1}{\theta}} - \lambda P_t(l) = 0, \\
&\frac{C_t(k)}{C_t(l)} = \left(\frac{P_t(l)}{P_t(k)}\right)^{\theta}, \\
&C_t(l) = C_t(k) \left(\frac{P_t(k)}{P_t(l)}\right)^{\theta}.
\end{aligned}$$

Check for consistency. Solve for the demand function for good k by substituting this expression in the definition of the consumption bundle:

$$
\begin{aligned}
C_t &= \left(\int_0^1 C_t(l)^{\frac{\theta-1}{\theta}} dl\right)^{\frac{\theta}{\theta-1}} = \left(\int_0^1 \left(C_t(k)\left(\frac{P_t(k)}{P_t(l)}\right)^{\theta}\right)^{\frac{\theta-1}{\theta}} dl\right)^{\frac{\theta}{\theta-1}} \\
&= C_t(k)\left(\int_0^1 \left(\frac{P_t(k)}{P_t(l)}\right)^{\theta-1} dl\right)^{\frac{\theta}{\theta-1}} \\
&= C_t(k)P_t(k)^{\theta}\left(\int_0^1 P_t(l)^{1-\theta} dl\right)^{\frac{1}{1-\theta}(1-\theta)\frac{\theta}{\theta-1}} \\
&= C_t(k)P_t(k)^{\theta}P_t^{(1-\theta)\frac{\theta}{\theta-1}} = C_t(k)P_t(k)^{\theta}P_t^{-\theta}, \\
C_t(k) &= C_t\left(\frac{P_t}{P_t(k)}\right)^{\theta}.
\end{aligned}
$$

(b) Give an interpretation of θ and plot the demand function for good i.

θ measures the elasticity of substitution between varieties and thus gives a measure of the elasticity of the demand for good k with respect to changes in its price

$$
\theta = -\frac{\partial C_t(k)}{\partial P_t(k)}\frac{P_t(k)}{C_t(k)}.
$$

The higher θ, the stronger the response of the demand for good k to changes in its price. Thus, higher values of θ imply that varieties are better substitutable. In the limit, for $\theta \to \infty$, goods are perfect substitutes which implies that there is perfect competition between firms.

Graphically, the demand function for good k is a decreasing function of the price of variety k and its slope is decreasing in θ.

(c) Derive the optimal labor supply and consumption decision of the household in log-linear form.

Note that we can simplify the budget constraint as follows:

$$
\begin{aligned}
\int_0^1 P_t(i)C_t(i)di &= \int_0^1 C_t\left(\frac{P_t}{P_t(i)}\right)^{\theta} P_t(i)di = C_tP_t^{\theta}\int_0^1 P_t(i)^{\theta-1}di, \\
\int_0^1 P_t(i)C_t(i)di &= C_tP_t^{\theta}\left(\int_0^1 P_t(i)^{1-\theta}di\right)^{\frac{1-\theta}{1-\theta}} = C_tP_t^{\theta}P_t^{1-\theta} = P_tC_t.
\end{aligned}
$$

Therefore, the household's optimization problem reduces to

$$\max \quad V(C_1, C_2, N_1, N_2) = U(C_1) - V(N_1) + \frac{1}{1+\rho} E_1 \left[U(C_2) - V(N_2)\right],$$
$$s.t. \quad (1+\tau_t^C)P_t C_t + Q_t B_t = B_{t-1} + (1-\tau_t^N)W_t N_t + T_t.$$

Using Lagrangian,

$$L = V(C_1, C_2, N_1, N_2) - \sum_{t=1}^{2} \lambda_t \left[(1+\tau_t^C)P_t C_t + Q_t B_t - B_{t-1} - (1-\tau_t^N)W_t N_t - T_t\right],$$

the first-order conditions give

$$\begin{aligned}
\frac{\partial L}{\partial C_1} &= U'(C_1) - \lambda_1(1+\tau_1^C)P_1 = 0,\\
\frac{\partial L}{\partial C_2} &= \frac{1}{1+\rho}U'(C_2) - \lambda_2(1+\tau_2^C)P_2 = 0,\\
\frac{\partial L}{\partial N_1} &= -V'(N_1) + \lambda_1(1-\tau_1^N)W_1 = 0,\\
\frac{\partial L}{\partial N_2} &= -\frac{1}{1+\rho}V'(N_2) + \lambda_2(1-\tau_2^N)W_2 = 0,\\
\frac{\partial L}{\partial B_1} &= \lambda_1 Q_1 - \lambda_2 = 0.
\end{aligned}$$

Also remember that the bond price is related to the interest rate on the bond and use the specific expressions for the utility functions

$$\begin{aligned}
Q_t &= (1+i_t)^{-1},\\
U'(C_t) &= C_t^{-\sigma^{-1}},\\
V'(N_t) &= N_t^{\varphi}.
\end{aligned}$$

This yields the Euler equation

$$\begin{aligned}
\frac{U'(C_1)}{U'(C_2)} &= \frac{1}{1+\rho}\frac{1+\tau_1^C}{1+\tau_2^C}\frac{P_1}{P_2}\frac{1}{Q_1}\\
&= \frac{1}{1+\rho}\frac{1+\tau_1^C}{1+\tau_2^C}\frac{(1+i_1)P_1}{P_2},
\end{aligned}$$

$$\frac{U'(C_1)}{U'(C_2)} = \frac{1+r_1}{1+\rho}\frac{1+\tau_1^C}{1+\tau_2^C},$$

$$\frac{C_2}{C_1} = \left(\frac{1+r_1}{1+\rho}\frac{1+\tau_1^C}{1+\tau_2^C}\right)^{\sigma}.$$

In addition, the optimal labor supply schedule is given by

$$\frac{V'(N_1)}{U'(C_1)} = N_1^{\varphi}C_1^{\sigma^{-1}} = \frac{(1-\tau_1^N)W_1}{(1+\tau_1^C)P_1},$$

$$\frac{V'(N_2)}{U'(C_2)} = N_2^{\varphi}C_2^{\sigma^{-1}} = \frac{(1-\tau_2^N)W_2}{(1+\tau_2^C)P_2}$$

Log-linearize these expressions with smaller case letters denoting natural logarithms,

$$c_2 - c_1 = \sigma(r_1 - \rho + \tau_1^C - \tau_2^C),$$

$$c_1 = c_2 - \sigma(r_1 - \rho) - \sigma(\tau_1^C - \tau_2^C),$$

$$\varphi n_1 + \frac{1}{\sigma}c_1 = w_1 - p_1 - \tau_1^N - \tau_1^C,$$

$$n_1 = \frac{1}{\varphi}(w_1 - p_1 - \tau_1^N - \tau_1^C) - \frac{1}{\sigma\varphi}c_1,$$

$$n_2 = \frac{1}{\varphi}(w_2 - p_2 - \tau_2^N - \tau_2^C) - \frac{1}{\sigma\varphi}c_2.$$

(d) *Derive the optimal price setting rule for the monopolistic producer of good i under the assumption of perfectly flexible prices. What is the effect of monopolistic competition relative to the competitive case?*

The monopolistic producer of good i maximizes his profit taking into account the effect of his price setting on the demand for good i

$$\max \Pi_t(i) = P_t(i)Y_t^d(i) - W_t N_t^d(i) = P_t(i)Y_t^d(i) - \frac{W_t}{A_t}Y_t^d(i)$$

$$= P_t(i)Y_t\left(\frac{P_t}{P_t(i)}\right)^{\theta} - \frac{W_t}{A_t}Y_t\left(\frac{P_t}{P_t(i)}\right)^{\theta}$$

$$= \left(P_t(i)^{1-\theta} - P_t(i)^{-\theta}\frac{W_t}{A_t}\right)Y_t P_t^{\theta}.$$

The first-order condition gives

$$\frac{\partial \Pi_t(i)}{\partial P_t(i)} = (1-\theta)P_t(i)^{-\theta} + \theta P_t(i)^{-\theta-1}\frac{W_t}{A_t} = 0,$$

$$\theta P_t(i)^{-1}\frac{W_t}{A_t} = \theta - 1,$$

$$P_t(i) = \frac{\theta}{\theta-1}\frac{W_t}{A_t} = (1+\mu)\frac{W_t}{A_t} \equiv P_t \rightarrow \frac{W_t}{P_t} = \frac{1}{1+\mu}A_t < A_t,$$

$$p_t = \mu + w_t - a_t.$$

Monopolistic competition results in a constant mark-up μ that depends on the elasticity of substitution. The higher θ, the more elastic is demand in response to a price change and thus the lower the mark-up. With perfect competition, $\mu \rightarrow 0$.

(e) *What is the* natural *level of output $Y_{t,n}$ in this economy. Discuss its properties by comparing it to the output level that would result under perfect competition.*

Natural level of output means the output with flexible prices (but possibly distorted because of monopolistic competition).

Efficient level of output is the first-best level of output with perfect competition and without distortionary taxation.

Goods market equilibrium implies $y_t = c_t + g_t$ and $y_t = a_t + n_t$.

Start from labor market equilibrium and replace the real wage and consumption

$$\varphi n_t + \frac{1}{\sigma}c_t = w_t - p_t - \tau_t^N - \tau_t^C,$$

$$\varphi(y_t - a_t) + \frac{1}{\sigma}(y_t - g_t) = a_t - \mu - \tau_1^N - \tau_1^C,$$

$$\varphi y_t + \frac{1}{\sigma}y_t = (1+\varphi)a_t + \frac{1}{\sigma}g_t - \mu - \tau_1^N - \tau_1^C,$$

$$y_{t,n} = \frac{1}{\varphi + \sigma^{-1}}\left((1+\varphi)a_t + \frac{1}{\sigma}g_t - \mu - \tau_1^N - \tau_1^C\right),$$

$$n_{t,n} = \frac{1}{\varphi + \sigma^{-1}}\left((1-\sigma^{-1})a_t + \frac{1}{\sigma}g_t - \mu - \tau_1^N - \tau_1^C\right).$$

Solve for the natural rate of interest via the Euler equation

$$c_2 - c_1 = \sigma(r_1 - \rho + \tau_1^C - \tau_2^C),$$

$$y_2 - g_2 - y_1 + g_1 = \sigma(r_1 - \rho + \tau_1^C - \tau_2^C),$$

$$r_{1,n} = \rho + \frac{1}{\sigma}\left[(y_{2,n} - y_{1,n}) - (g_2 - g_1)\right] - (\tau_1^C - \tau_2^C).$$

Notice the structural inefficiency

$$y_t^* - y_{t,n} = \frac{1}{\varphi + \sigma^{-1}}\left(\mu + \tau_1^N + \tau_1^C\right).$$

Also note that these natural values are independent of monetary policy shocks.

(f) *Assume that agents expect the economy to be at its natural level before the realization of shocks, i.e. $y_2^e = y_1^e = y_n$ and $p_2^e = p^e = p^*$. Derive the AD curve under this set-up.*

Start from the Euler equation and use the goods market equilibrium to substitute for consumption. Now, take expectations into account

$$\begin{aligned}
c_2^e - c_1 &= \sigma(r_1 - \rho + \tau_1^C - \tau_2^C),\\
c_t &= y_t - g_t,\\
y_2^e - y_1 - g_2^e + g_1 &= \sigma(i_1 - (p_2^e - p_1) - \rho + \tau_1^C - \tau_2^C),\\
y_1 &= y_n - \sigma(i_1 - (p^* - p_1) - \rho + \tau_1^C - \tau_2^C) - (g_2^e - g_1).
\end{aligned}$$

(g) *Now assume that there are price rigidities so that in period 1 only a fraction of firms $1-\alpha$ can adjust their prices whereas the remaining fraction α is unable to do so. Prices are perfectly flexible in period 2. Derive the optimal price setting rule in this case and the AS curve.*

Since prices are perfectly flexible in period 2, firms that can adjust the price in period 1 will simply choose the optimal price in the flexible price set-up above

$$p_1(i) = \mu + w_1 - a_1.$$

Combine this with the labor market equilibrium to substitute for the wage and goods market equilibrium to solve for c_1 and n_1

$$\begin{aligned}
\varphi n_1 + \frac{1}{\sigma}c_1 &= w_1 - p_1 - \tau_1^N - \tau_1^C = p_1(i) - \mu + a_1 - p_1 - \tau_1^N - \tau_1^C,\\
p_1(i) - p_1 &= \mu + \tau_1^N + \tau_1^C - a_1 + \varphi n_1 + \frac{1}{\sigma}c_1\\
&= \mu + \tau_1^N + \tau_1^C - a_1 + \varphi(y_1 - a_1) + \frac{1}{\sigma}(y_1 - g_1)\\
&= \mu + \tau_1^N + \tau_1^C - (1+\varphi)a_1 + \varphi y_1 + \frac{1}{\sigma}y_1 - \frac{1}{\sigma}g_1.
\end{aligned}$$

Note that this price is identical for all firms that can adjust, i.e., $p_1(i) = \tilde{p}_1$. Also recall the definition of the natural level of output

$$y_{1,n} = \frac{1}{\varphi + \sigma^{-1}} \left((1+\varphi)a_1 + \frac{1}{\sigma} g_1 - \mu - \tau_1^N - \tau_1^C \right).$$

Combining these equations yields the price adjustment rule for those firms that are able to adjust prices given the average price level in the economy

$$\tilde{p}_1 - p_1 = \left(\varphi + \frac{1}{\sigma} \right) (y_1 - y_{1,n}).$$

In contrast, those firms that cannot adjust their prices in period 1 will set them equal to the expected price level when the economy is at the expected natural level of production

$$p_1^{sticky}(i) = p_1^{sticky} = p_1^e = E_0(p_1).$$

The aggregate price level in period 1 is a mix of fixed and flexible prices:

$$P_1^{1-\theta} = \alpha \left(P_1^{sticky} \right)^{1-\theta} + (1-\alpha)\tilde{P}_1^{1-\theta}.$$

Use a first-order Taylor approximation around the steady state price level P to linearize this equation

$$P_1^{1-\theta} = P^{1-\theta} + (1-\theta)\alpha P^{-\theta}(P_1^{sticky} - P) + (1-\theta)(1-\alpha)P^{-\theta}(\tilde{P}_1 - P),$$

$$\frac{P_1^{1-\theta} - P^{1-\theta}}{P^{1-\theta}} = (1-\theta)\alpha \frac{P^{-\theta}(P_1^{sticky} - P)}{P^{1-\theta}} + (1-\theta)(1-\alpha)\frac{P^{-\theta}(\tilde{P}_1 - P)}{P^{1-\theta}},$$

$$\left(\frac{P_1 - P}{P} \right)^{1-\theta} = (1-\theta)\alpha \frac{(P_1^{sticky} - P)}{P} + (1-\theta)(1-\alpha)\frac{(\tilde{P}_1 - P)}{P}.$$

Also use the first-order approximation

$$\ln(x_t) - \ln(x) = \ln\left(\frac{x_t}{x}\right) = \ln\left(\frac{x_t}{x} - 1 + 1\right) \approx \frac{x_t}{x} - 1 = \frac{x_t - x}{x}.$$

Let small letters denominate logs. Then, this yields

$$(p_1 - p)^{1-\theta} = (1-\theta)\alpha(p_1^{sticky} - p) + (1-\theta)(1-\alpha)(\tilde{p}_1 - p),$$

$$(1-\theta)(p_1 - p) = (1-\theta)\alpha p_1^{sticky} + (1-\theta)(1-\alpha)\tilde{p}_1 - (1-\theta)p,$$

$$p_1 = \alpha p_1^{sticky} + (1-\alpha)\tilde{p}_1.$$

Now, use the price setting behavior of sticky price firms

$$p_1 = \alpha p_1^{sticky} + (1-\alpha)\tilde{p}_1 = \alpha p_1^e + (1-\alpha) p_1 + (1-\alpha)\left(\varphi + \sigma^{-1}\right)(y_1 - y_{1,n}),$$

$$\alpha p_1 = \alpha p_1^e + (1-\alpha)\left(\varphi + \sigma^{-1}\right)(y_1 - y_{1,n}),$$

$$p_1 - p_1^e = \frac{(1-\alpha)\varphi + \sigma^{-1}}{\alpha}(y_1 - y_{1,n}).$$

Therefore, the AS curve relates the output gap in period 1 to deviations of prices from their expected values

$$p_1 - p_1^e = \kappa(y_1 - y_{1,n}).$$

References

Blanchard, O. J., & Kiyotaki, N. (1987). Monopolistic competition and the effects of aggregate demand. *American Economic Review, 77*, 647–666.

Calvo, G. A. (1983). Staggered prices in a utility-maximizing framework. *Journal of Monetary Economics, 12*, 383–398.

Yun, T. (1996). Nominal price rigidity, money supply endogeneity, and business cycles. *Journal of Monetary Economics, 37*, 345–370.

5 Optimal Monetary Policy

5.1 Exercises

5.1.1 Short Review Questions

(a) Briefly explain how sticky price leads to price dispersion, and why this implies a loss in social welfare. What does this imply for optimal monetary policy?
(b) Briefly explain how conflicts of interests induce central bank to generate surprise inflation, and how inflation sustains in such scenario.
(c) Following question (b), name a few solutions to the inflation problem. Explain how they work.
(d) Explain the pros and cons of the following monetary policy rules: (1) price level targeting, (2) inflation targeting, (3) GDP targeting, and (4) money growth targeting.
(e) Grilli et al. (1991) defined an independence index of a central bank, which is computed from the following factors: (1) Central bank governor not appointed by the government; (2) Central bank governor's tenure longer than 5 years; (3) All the central bank's Board not appointed by the government; (4) No government approval of monetary policy formulation is required; (5) No mandatory participation of government representative in the Board; (6) Legal provisions that strengthen the central bank's position in conflicts with the government are present. Then the authors found that the higher one central bank's index is, the lower the country's inflation. In the framework of Barro–Gordon model, provide some intuitions why central bank independence may help to fight against inflation.

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5.1.2 Barro–Gordon Model

As Barro and Gordon (1983a,b), assume a social loss function depending on employment l and prices p

$$L = (l - l^*)^2 + \beta(p - p^*)^2,$$

where l^* is the efficient employment and p^* is the price level consistent with optimal inflation. All lowercase letters denote logarithmic terms. The short-run Phillips curve is given by

$$l = \bar{l} + c(p - p^e + \theta),$$

where $c > 0$ is a parameter and θ is a random shock.

(a) Assume that the central bank can control the price level and aims at minimizing social losses after observing productivity shock θ. Derive the first-order condition for optimal monetary policy and solve the model for its rational expectations equilibrium described by $p^e = E(p)$ and policy rule $p(\theta)$.
(b) Discuss the impact of exogenous parameters on the inflation bias $p^e - p^*$ and on the policy rule $p(\theta)$ obtained in (b).
(c) Assume now that the central bank commits to stabilize inflation in such a way that $p = p^*$. Compare the resulting variance of employment, the inflation bias and expected welfare loss with your solution from (b).

5.1.3 Solving Time-Inconsistency Problem: Delegation

Rogoff (1985) consider an Economy in which efficient employment and optimal price level are both normalized to 1, $L^* = 1 > \overline{L}$, $P^* = 1$, and $\overline{L}$ is the natural rate of employment. For simplicity, in the following we use log values of variables; therefore, $l^* = \ln L^* = 0$, $p^* = \ln P^* = 0$, and $l = \ln L$, $p = \ln P$ are the percentage deviations from their efficient levels.

Suppose the government wants to maximize the social welfare as given by

$$W = \gamma l - a\frac{p^2}{2},$$

and delegates monetary policy to a central banker who follows an objective function

$$\tilde{W} = c\gamma l - a\frac{p^2}{2},$$

in which γ is a random variable with mean $\overline{\gamma}$ and variance σ_γ^2. Suppose that the short-run Phillips curve is given by

$$l = \bar{l} + b\left(p - p^e\right).$$

Note that the expected price level p^e is determined before γ is observed, and the central banker chooses p after γ is known.

(a) Compute the central banker's optimal solution for p, with p^e, γ, and c being given.
(b) Is the central banker able to resist the temptation to aim at efficient employment, i.e. $l^* = 0$? Compute p^e.
(c) Compute the expected value of W.
(d) Compute c that maximizes W. Provide some intuitions on your result.

5.1.4 Optimal Monetary Policy: The New Keynesian Perspective

Based on Clarida et al. (1999) consider the "new Keynesian perspective" featured by an IS curve

$$x_t = -\phi(i_t - E_t\pi_{t+1}) + E_t x_{t+1} + g_t,$$

where x_t is the output gap, and a Phillips curve

$$\pi_t = \lambda x_t + \beta E_t \pi_{t+1} + u_t,$$

g_t and u_t are shocks that obey $g_t = \mu g_{t-1} + \hat{g}_t$ and $u_t = \rho u_{t-1} + \hat{u}_t$, where $\mu \geq 0$, $\rho \leq 1$ and $\hat{g}_t$, and $\hat{u}_t$ are i.i.d. random variables with zero mean and constant variances.

The policy objective is given by

$$\min E_t \left[\sum_{i=0}^{+\infty} \beta^i \left(\alpha x_{t+i}^2 + \pi_{t+i}^2\right)\right].$$

(a) Explain the "new" IS curve and the forward looking Phillips curve.
(b) Explain the policy objective. What are the differences from a Barro–Gordon type policy objective?
(c) Derive the optimal discretionary policy for rational expectations and show that there is a short-run trade-off between inflation and output variability.
(d) Using discretionary policy: How must the interest rate respond to a rise in expected inflation?

5.1.5 Optimal Monetary Policy in a Small DSGE Model

Start from the same model of the economy as in Chap. 4, Exercise 6 with sticky prices. The model features two periods, the short-run ($t = 1$) and the long-run ($t = 2$), and is characterized by the following equations:

1. AD curve: $y_1 = \bar{y}_n - \sigma(i_1 - (p_2^e - p_1) - \rho) + \eta_1$;
2. AS curve: $p_1 - p_1^e = \kappa(y_1 - y_{1,n})$;
3. Natural output: $y_{1,n} = \bar{y}_n + \epsilon_1 - u_1$;
4. Efficient output: $y_1^* = y_{1,n} + \Delta + u_1 = \bar{y}_n + \Delta + \epsilon_1$.

The three shocks η, ϵ, and u have zero expected mean and their variances are given by σ_η^2, σ_ϵ^2, and σ_u^2.

Assume that the central bank sets the interest rate to minimize the loss function

$$L = \frac{1}{2}E(y_1 - y_1^*)^2 + \frac{\theta}{2\kappa}E(p_1 - p^*)^2.$$

(a) Discuss the effects of the three types of shocks on output and prices as well as the associated transmission mechanisms.

(b) Welfare losses result in parts from price dispersion. Start from the labor market equilibrium condition

$$N_t = \int_{i=0}^{1} N_t(i)di$$

and show via a second-order approximation that the price dispersion index

$$D_t = \int_{i=0}^{1} \left(\frac{P_t(i)}{P_t}\right)^{-\theta} di$$

can be expressed in terms of the cross-sectional variance of prices. Show that this variance is related to surprise inflation.

(c) Assume that the central bank follows an interest rate rule $i_1 = f(\eta_1, \epsilon_1, u_1)$. The structure of the problem follows a three-stage process.

- Stage 1: Central bank announces a policy rule $i_1 = f(\eta_1, \epsilon_1, u_1)$;
- Stage 2: The share α of firms sets (fixed) prices based on rational expectations. Then shocks occur;
- Stage 3: Central bank reacts to the shocks by setting the actual interest rate i_1.

Obtain the optimal policy rule under the assumption that there is no structural inefficiency (i.e. $\Delta = 0$). Discuss how optimal monetary policy responds to the different shocks.

(d) Show that the loss function of the central bank can be derived from the household's period utility function $V(C, N) = U(C) - V(N)$ via a second order approximation.

5.1.6 Time Inconsistency Problem and Inflation Bias

Consider the same set-up as in the previous exercise but now assume that the structural inefficiency is not corrected via paying subsidies to firms, i.e., there is a wedge $\Delta > 0$ between the first-best (efficient) level of output and the natural level.

Assume that the central bank minimizes the loss function

$$L = \frac{1}{2}E(y_1 - y_1^*)^2 + \frac{\theta}{2\kappa}E(p_1 - p^*)^2$$

under the constraints

- AD curve: $y_1 = \bar{y}_n - \sigma(i_1 - (p_2^e - p_1) - \rho) + \eta_1$;
- AS curve: $p_1 - p_1^e = \kappa(y_1 - y_{1,n})$;
- Natural output: $y_{1,n} = \bar{y}_n + \epsilon_1 - u_1$;
- Efficient output: $y_1^* = y_{1,n} + \Delta + u_1 = \bar{y}_n + \Delta + \epsilon_1$

in which the three shocks η, ϵ, and u have zero expected mean and their variances are given by σ_η^2, σ_ϵ^2, and σ_u^2.

Agents have rational expectations. Assume that the central bank follows an interest rate rule $i_1 = f(\eta_1, \epsilon_1, u_1)$. The structure of the problem follows a three-stage process.

- Stage 1: Central bank announces a policy rule $i_1 = f(\eta_1, \epsilon_1, u_1)$;
- Stage 2: The share α of firms sets (fixed) prices based on rational expectations. Then shocks occur;
- Stage 3: Central bank reacts to the shocks by setting the actual rate i_1.

(a) First, assume that the central bank can credibly commit itself to the announced price level in stage 1 so that $p^e = E(p_1)$. Derive the commitment equilibrium under these assumptions.
(b) Discuss why there is an incentive for surprise inflation and derive the equilibrium if surprise inflation is possible, i.e. under the assumption that inflation expectations are fixed at $p^e = p^*$.
(c) Derive the equilibrium under discretion and the welfare loss L associated with discretionary central bank policy.

5.2 Solutions for Selected Exercises

5.2.1 Short Review Questions

(e) Grilli et al. (1991) defined an independence index of a central bank, which is computed from the following factors: (1) Central bank governor not appointed by the government; (2) Central bank governor's tenure longer than 5 years; (3) All the central bank's Board not appointed by the government; (4) No government approval of monetary policy formulation is required; (5) No mandatory participation of government representative in the Board; (6) Legal provisions that strengthen the central bank's position in conflicts with the government are present. Then the authors found that the higher one central bank's index is, the lower the country's inflation. In the framework of Barro–Gordon model, provide some intuitions why central bank independence may help to fight against inflation.

The answer consists of the following points: First, where does inflation in Barro–Gordon model come from? The Nash inflation comes from the fact that the policy maker fails to resist the temptation of improving employment via generating surprising inflations. Second, what does central bank independence mean? Independence enables central banks to concentrate on monetary policy (e.g., factor (1), (3), and (4)) and not to yield upon government's pressure of raising employment (e.g., factor (5) and (6)). Third, why is inflation lower under a higher degree of independence? Therefore in the loss function the central bank is able to put a much higher weight on price stabilization, and the realized inflation becomes lower. Furthermore, Longer tenure (factor (2)) of central bank governors makes the game approximately infinitely repeated—imagine that the Fed adjusts its rates every few weeks, but Alan Greenspan stayed in office for 20 years—which makes cooperative solution more likely to arise.

5.2.2 Barro–Gordon Model

Assume a social loss function depending on employment l and prices p

$$L = (l - l^*)^2 + \beta(p - p^*)^2,$$

where l^* is the efficient employment and p^* is the price level consistent with optimal inflation. All lowercase letters denote logarithmic terms. The short-run Phillips curve is given by

$$l = \bar{l} + c(p - p^e + \theta),$$

where $c > 0$ is a parameter and θ is a random shock.

(a) *Assume that the central bank can control the price level and aims at minimizing social losses after observing productivity shock θ. Derive the first-order condition for optimal monetary policy and solve the model for its rational expectations equilibrium described by $p^e = E(p)$ and policy rule $p(\theta)$.*

The central bank's problem is to

$$\min_p \ L = (l - l^*)^2 + \beta(p - p^*)^2$$
$$s.t. \ \ l = \bar{l} + c(p - p^e + \theta).$$

Insert the equality constraint into the object function and the original problem turns out to be

$$\min_p \ L = \left[\bar{l} + c(p - p^e + \theta) - l^*\right]^2 + \beta(p - p^*)^2.$$

The first-order condition gives

$$\frac{\partial L}{\partial p} = 2\left[\bar{l} + c(p - p^e + \theta) - l^*\right]c + 2\beta(p - p^*) = 0,$$

and rearrange to get the policy rule $p(\theta)$

$$p(\theta) = \frac{c(l^* - \bar{l}) + c^2(p^e - \theta) + \beta p^*}{c^2 + \beta}. \tag{5.1}$$

And in rational expectation equilibrium

$$p^e = E[p] = E\left[\frac{c(l^* - \bar{l}) + c^2(p^e - \theta) + \beta p^*}{c^2 + \beta}\right] = \frac{c(l^* - \bar{l}) + c^2 p^e + \beta p^*}{c^2 + \beta},$$

solve and get p^e

$$p^e = \frac{c(l^* - \bar{l})}{\beta} + p^*. \tag{5.2}$$

(b) *Discuss the impact of exogenous parameters on the inflation bias $p^e - p^*$ and on the policy rule $p(\theta)$ obtained in (a).*

In the expression above the inflation bias is

$$p^e - p^* = \frac{c(l^* - \bar{l})}{\beta}.$$

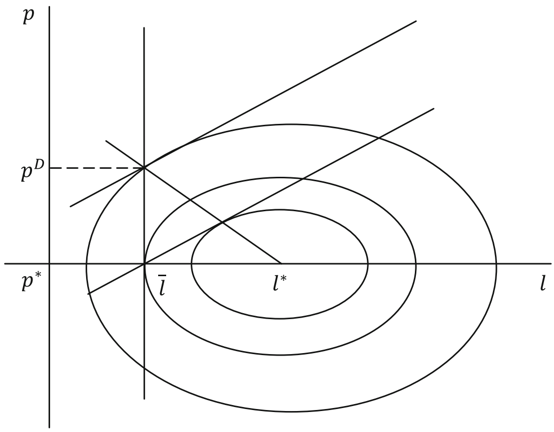

Fig. 5.1 Discretionary monetary policy

β is the weight on inflation in the loss function, defining the shape of the ellipses; and c measures the impact of unexpected inflation on the employment. Explain by Fig. 5.1.

(c) Assume now that the central bank commits to stabilize inflation in such a way that $p = p^*$. Compare the resulting variance of employment, the inflation bias and expected welfare loss with your solution from (b).

Insert (5.2) into (5.1) to eliminate p^*

$$\begin{aligned} p &= \frac{c(l^* - \bar{l}) + c^2(p^e - \theta) + \beta p^e - c(l^* - \bar{l})}{c^2 + \beta} \\ &= p^e - \frac{c^2\theta}{c^2 + \beta}. \end{aligned}$$

Apply it into the short-run Phillips curve and one can get

$$l = \bar{l} + \frac{c\beta}{c^2 + \beta}\theta,$$

as well as

$$\text{var}(l) = \left(\frac{\beta}{c^2 + \beta}\right)^2 c^2\sigma_\theta^2.$$

When the central bank commits to stabilize inflation in such a way that $p = p^*$, then $p^e = p = p^*$. Now the short-run Phillips curve becomes

$$l = \bar{l} + c\theta,$$
$$\mathrm{var}(l) = c^2\sigma_\theta^2.$$

The variance now is larger than the case before.

5.2.3 Solving Time-Inconsistency Problem: Delegation

Rogoff (1985) consider an Economy in which efficient employment and optimal price level are both normalized to 1, $L^* = 1 > \overline{L}$, $P^* = 1$ and $\overline{L}$ is the natural rate of employment. For simplicity, in the following we use log values of variables; therefore, $l^* = \ln L^* = 0$, $p^* = \ln P^* = 0$, and $l = \ln L$, $p = \ln P$ are the percentage deviations from their efficient levels.

Suppose the government wants to maximize the expected welfare as given by

$$W = \gamma l - a\frac{p^2}{2},$$

and delegates monetary policy to a central banker who follows an objective function

$$\tilde{W} = c\gamma l - a\frac{p^2}{2},$$

in which γ is a random variable with mean $\overline{\gamma}$ and variance σ_γ^2. Suppose that the short-run Phillips curve is given by

$$l = \bar{l} + b\left(p - p^e\right).$$

Note that p^e is determined before γ is observed, and the central banker chooses p after γ is known.

(a) Compute the central banker's optimal solution for p, with p^e, γ and c being given.

The central banker's problem is to

$$\max_p \;\; \tilde{W} = c\gamma l - a\frac{p^2}{2},$$
$$s.t. \;\; l = \bar{l} + b\left(p - p^e\right).$$

The first-order condition gives

$$c\gamma b - ap = 0,$$
$$p = \frac{c\gamma b}{a}.$$

(b) Is the central banker able to resist the temptation to aim at efficient employment, i.e. $l^* = 0$? Compute p^e.

No—As soon as $\bar{l} < 0$, the central banker does nothing different from the government in this sense.

The public knows that the central banker would set inflation $p = \frac{c\gamma b}{a}$, therefore rational expectation requires that

$$p^e = E\left[\frac{c\gamma b}{a}\right] = \frac{c\bar{\gamma}b}{a}.$$

(c) Compute the expected value of W.

Take expectation on W, then apply p and p^e

$$\begin{aligned}
E[W] &= E\left[\gamma l - a\frac{p^2}{2}\right] \\
&= E\left\{\gamma\left[\bar{l} + b\left(p - p^e\right)\right] - a\frac{p^2}{2}\right\} \\
&= E\left\{\gamma\left[\bar{l} + b\left(\frac{c\gamma b}{a} - \frac{c\bar{\gamma}b}{a}\right)\right] - a\frac{\left(\frac{c\gamma b}{a}\right)^2}{2}\right\} \\
&= \bar{l}\bar{\gamma} + \frac{cb^2}{a}\sigma_\gamma^2 - \frac{c^2b^2}{2a}\left(\sigma_\gamma^2 + \bar{\gamma}^2\right).
\end{aligned}$$

(d) Compute c that maximizes W. Give some intuitions on your result.

Find the optimal solution for c by the first-order condition

$$\frac{\partial E[L]}{\partial c} = \frac{b^2}{a}\sigma_\gamma^2 - \frac{cb^2}{a}\left(\sigma_\gamma^2 + \bar{\gamma}^2\right) = 0,$$

rearrange to get c

$$c = \frac{\sigma_\gamma^2}{\sigma_\gamma^2 + \bar{\gamma}^2} \leq 1.$$

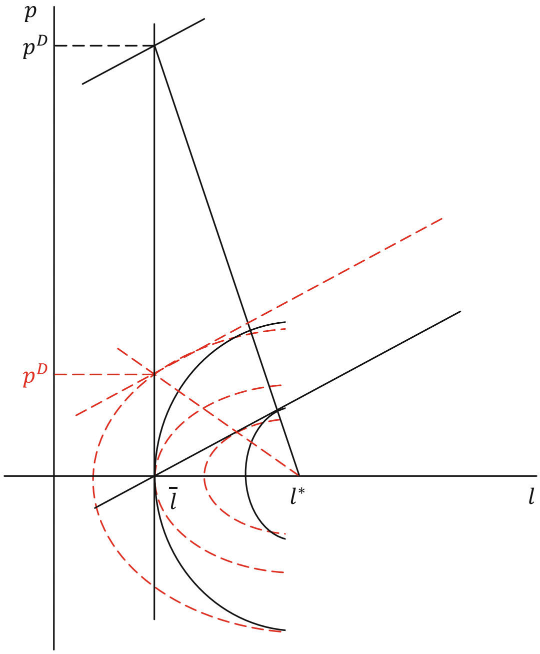

Fig. 5.2 Lower inflation under delegation

The lesson we learn here is that the government should delegate monetary policy to a central banker who is more conservative than the government, i.e. a central banker with a lower weight c on unemployment. By doing so, the average inflation is kept lower since $p = \frac{c\gamma b}{a} \leq \frac{\gamma b}{a} = p^d$, in which p^d is the discretionary solution achieved by the government itself, hence a higher social welfare level. However, such a policy maker would not respond well to the shocks, and c needs to be properly chosen to reflect such trade off (Fig. 5.2).

The optimal value of c is decreasing in $\overline{\gamma}$. If $\overline{\gamma}$ is higher, then the expected inflation would also be higher for a given c, therefore it would be welfare improving to offset this effect by choosing a central banker with a lower c and keep inflation lower.

However, the optimal value of c is increasing in σ_γ^2. Because the central banker acts after γ reveals, she can offset the deviation in γ from its expected value and raise social welfare. Therefore, in order to achieve this, it would be better to have a central banker who cares more about the shock's effect, i.e. a central banker with a higher c.

5.2.4 Optimal Monetary Policy: The New Keynesian Perspective

Based on Clarida et al. (1999) consider the "new Keynesian perspective" featured by an IS curve

$$x_t = -\phi(i_t - E_t\pi_{t+1}) + E_t x_{t+1} + g_t,$$

where x_t is the output gap, and a Phillips curve

$$\pi_t = \lambda x_t + \beta E_t \pi_{t+1} + u_t,$$

g_t and u_t are shocks that obey $g_t = \mu g_{t-1} + \hat{g}_t$ and $u_t = \rho u_{t-1} + \hat{u}_t$, where $\mu \geq 0$, $\rho \leq 1$ and $\hat{g}_t$ and $\hat{u}_t$ are i.i.d. random variables with zero mean and constant variances.

The policy objective is given by

$$\min E_t \left[\sum_{i=0}^{+\infty} \beta^i \left(\alpha x_{t+i}^2 + \pi_{t+i}^2 \right) \right].$$

(a) Explain the "new" IS curve and the forward looking Phillips curve.

These equations constitute the non-policy block of the basic new Keynesian model. That block has a simple recursive structure: the new Keynesian Phillips curve determines inflation given a path for the output gap, whereas the IS curve determines the output gap given a path for the actual real rate. Solve x_t and π_t forward and get

$$\begin{aligned} x_t &= -\phi(i_t - E_t \pi_{t+1}) + E_t x_{t+1} + g_t \\ &= E_t \left[\sum_{i=0}^{+\infty} -\phi(i_{t+i} - \pi_{t+1+i}) + g_{t+i} \right], \end{aligned}$$

– Note that $r_t = i_t - E_t \pi_{t+1}$ is the expected real return on a one period bond (i.e., the real interest rate), it emphasizes the fact that the output gap is proportional to the sum of current and anticipated real interest rate;

$$\begin{aligned} \pi_t &= \lambda x_t + \beta E_t \pi_{t+1} + u_t \\ &= E_t \left[\sum_{i=0}^{+\infty} \beta^i \left(\lambda x_{t+i} + u_{t+i} \right) \right], \end{aligned}$$

– Inflation depends on current and expected output gap, hence marginal cost, as explained in Exercise 2(b).

(b) Explain the policy objective. What are the differences from a Barro–Gordon type policy objective?

The policy objective is to minimize the fluctuations in output and inflation, and α determines the importance of output fluctuations relative to inflation fluctuations.

The policy objective looks a lot like the standard quadratic loss function used in Barro–Gordon model. There are, however, two critical differences. First, the output gap is measured relative to equilibrium output under flexible prices. In the traditional

Barro–Gordon model, the output variable was more commonly interpreted as output relative to trend or output relative to the natural rate of output. The natural rate of output varies with productivity shocks, but it is not the same as the flexible-price equilibrium level of output used to define the gap variable x_t. A second difference is the reason inflation variability enters the loss function. When prices are sticky, inflation results in an inefficient dispersion of output among the individual producers. The representative household's utility depends on its consumption of a composite good; faced with a dispersion of prices for the differentiated goods produced in the economy, the household buys more of the relatively cheaper goods and less of the relatively more expensive goods. Because of diminishing marginal utility, the increase in utility derived from consuming more of some goods is less than the loss in utility due to consuming less of the more expensive goods. Hence, price dispersion reduces utility. Similarly, dispersion on the production side is costly. The increased cost of producing more of some goods is greater than the cost saving from reducing production of other goods. For these reasons, price dispersion reduces utility, and, when each firm does not adjust its price every period, price dispersion is caused by inflation.

However, in Barro–Gordon model, the efficiency distortion that leads to x^* was used to motivate the presence of an overly ambitious output target in the central bank's objective function. As a consequence, the presence of $x^* > \overline{x}$ implies that a central bank acting under discretion to maximize the policy objective would produce an average inflation bias, which is the source of the Barro–Gordon inflation.

(c) Derive the optimal discretionary policy for rational expectations and show that there is a short-run trade-off between inflation and output variability.

The optimal discretionary policy under rational expectations is derived in two stages:

1. Choose x_t and π_t in order to maximize the policy objective, given the new Keynesian Phillips curve;
2. Given these optimal values of inflation and output gap, determine the optimal setting of the interest rate implied by the IS curve, i.e. the interest rate that will support the inflation and output levels determined in the last step.

With discretionary policy the central bank cannot manipulate the beliefs, therefore it takes the expectation of the private sector as given in solving the optimization problem. (Surely, conditional on the central bank's optimal rule, the private sector forms the beliefs rationally.) As a consequence, policy decisions today do not affect future expectations, which implies that the maximization problem can be written as

$$\min \; E_t\left[\sum_{i=0}^{+\infty}\beta^i\left(\alpha x_{t+i}^2+\pi_{t+i}^2\right)\right]=\alpha x_t^2+\pi_t^2+F_t,$$

$$s.t. \;\; \pi_t=\lambda x_t+\beta E_t\pi_{t+1}+u_t=\lambda x_t+f_t,$$

in which

$$F_t = E_t\left[\sum_{i=1}^{+\infty}\beta^i\left(\alpha x_{t+i}^2 + \pi_{t+i}^2\right)\right],$$
$$f_t = \beta E_t \pi_{t+1} + u_t$$

are the terms taken as given in the optimization, capturing the ideas such that

- Future inflation and output are not affected by today's actions;
- The central bank cannot directly manipulate expectations.

By the first-order conditions one can see that

$$x_t = -\frac{\lambda}{\alpha}\pi_t. \tag{5.3}$$

Remember that the inflation target here is zero, so if inflation is positive, the central bank needs to create a recession to reduce inflation, i.e. "leaning against the wind" argument. To achieve this, whenever inflation is above target the central bank contracts demand by raising the interest rate. How aggressive such action is depends on

- The gain in reduced inflation per unit of output loss (positive effect);
- The weight on output fluctuation in preferences (negative effect)

– The short-run trade-off between inflation and output variability.

(d) *Using discretionary policy: How must the interest rate respond to a rise in expected inflation?*

Now solve the nominal interest rate rule by applying the first-order condition in the new Keynesian model. Insert (5.3) into new Keynesian Phillips curve

$$\begin{aligned}\pi_t &= \lambda x_t + \beta E_t \pi_{t+1} + u_t \\ &= \lambda\left(-\frac{\lambda}{\alpha}\pi_t\right) + \beta E_t \pi_{t+1} + u_t, \\ \pi_t &= \frac{\alpha\beta}{\alpha+\lambda^2}E_t\pi_{t+1} + \frac{\alpha}{\alpha+\lambda^2}u_t.\end{aligned}$$

Then in order to pin down $E_t\pi_{t+1}$, we impose the rational expectation condition of the private sector, i.e. the expectations should be consistent with the predictions of the model and the updated version of the equation above should still hold

$$\pi_{t+1} = \frac{\alpha\beta}{\alpha+\lambda^2}E_{t+1}\pi_{t+2} + \frac{\alpha}{\alpha+\lambda^2}u_{t+1}.$$

Take expectations on both sides and get

$$E_t \pi_{t+1} = \frac{\alpha\beta}{\alpha+\lambda^2} E_t \left[E_{t+1} \pi_{t+2} \right] + \frac{\alpha}{\alpha+\lambda^2} E_t u_{t+1}.$$

Recall that the supply shock follows $u_t = \rho u_{t-1} + \hat{u}_t$, then

$$E_t u_{t+1} = \rho u_t + E_t \hat{u}_{t+1} = \rho u_t.$$

Therefore $E_t \pi_{t+1}$ can be computed

$$E_t \pi_{t+1} = \frac{\alpha\beta}{\alpha+\lambda^2} E_t \left[\pi_{t+2} \right] + \frac{\alpha\rho}{\alpha+\lambda^2} u_t,$$

and substitute $E_t \pi_{t+1}$ in the expression for π_t to get

$$\begin{aligned} \pi_t &= \frac{\alpha\beta}{\alpha+\lambda^2} \left(\frac{\alpha\beta}{\alpha+\lambda^2} E_t \left[\pi_{t+2} \right] + \frac{\alpha\rho}{\alpha+\lambda^2} u_t \right) + \frac{\alpha}{\alpha+\lambda^2} u_t \\ &= \left(\frac{\alpha\beta}{\alpha+\lambda^2} \right)^2 E_t \left[\pi_{t+2} \right] + \left(\frac{\alpha}{\alpha+\lambda^2} \right)^2 \beta\rho u_t + \frac{\alpha}{\alpha+\lambda^2} u_t \\ &= \frac{\alpha}{\alpha+\lambda^2} \frac{1}{1 - \frac{\alpha\beta\rho}{\alpha+\lambda^2}} u_t, \end{aligned}$$

in which the last step is ensured by the transversality condition. Update π_t one period forward to get

$$\begin{aligned} E_t \pi_{t+1} &= E_t \left[\frac{\alpha}{\alpha+\lambda^2} \frac{1}{1 - \frac{\alpha\beta\rho}{\alpha+\lambda^2}} u_{t+1} \right] \\ &= \frac{\alpha}{\alpha+\lambda^2} \frac{\rho}{1 - \frac{\alpha\beta\rho}{\alpha+\lambda^2}} u_t. \end{aligned}$$

By Eq. (5.3) one can see that

$$\begin{aligned} x_t &= -\frac{\lambda}{\alpha} \pi_t \\ &= -\frac{\lambda}{\alpha} \frac{\alpha}{\alpha+\lambda^2} \frac{1}{1 - \frac{\alpha\beta\rho}{\alpha+\lambda^2}} u_t \\ &= -\frac{\lambda}{\alpha+\lambda^2} \frac{1}{1 - \frac{\alpha\beta\rho}{\alpha+\lambda^2}} u_t. \end{aligned}$$

Update x_t one period forward to get

$$E_t x_{t+1} = -\frac{\lambda}{\alpha+\lambda^2} \frac{\rho}{1 - \frac{\alpha\beta\rho}{\alpha+\lambda^2}} u_t.$$

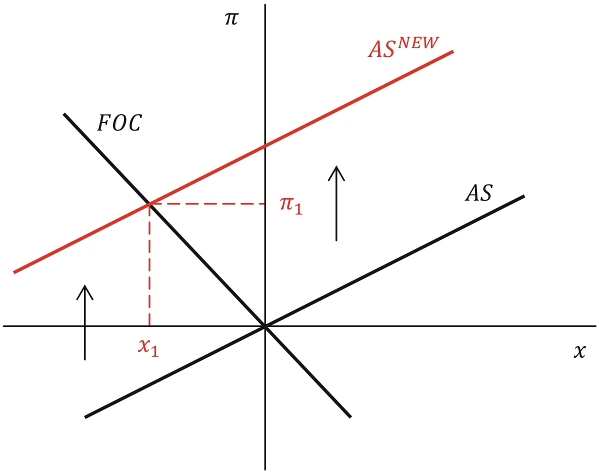

Fig. 5.3 Central bank's response to a rise in expected inflation

Then insert everything into the IS curve

$$
\begin{aligned}
x_t &= -\phi(i_t - E_t\pi_{t+1}) + E_t x_{t+1} + g_t, \\
i_t &= E_t\pi_{t+1} - \frac{1}{\phi}x_t + \frac{1}{\phi}E_t x_{t+1} + \frac{1}{\phi}g_t \\
&= E_t\pi_{t+1} + \frac{1}{\phi}\frac{\lambda}{\alpha\rho}E_t\pi_{t+1} - \frac{1}{\phi}\frac{\lambda}{\alpha}E_t\pi_{t+1} + \frac{1}{\phi}g_t \\
&= E_t\pi_{t+1}\left[1 + \frac{\lambda(1-\rho)}{\alpha\phi\rho}\right] + \frac{1}{\phi}g_t,
\end{aligned}
$$

showing how the interest rate responds to a rise in expected inflation. The result is that the central bank creates a recession to deter the inflation, as shown in Fig. 5.3.

5.2.5 Optimal Monetary Policy in a Small DSGE Model

Start from the same model of the economy as in Chap. 4, Exercise 6 with sticky prices. The model features two periods, the short-run ($t = 1$) and the long-run ($t = 2$), and is characterized by the following equations:

1. AD curve: $y_1 = \bar{y}_n - \sigma(i_1 - (p_2^e - p_1) - \rho) + \eta_1$;
2. AS curve: $p_1 - p_1^e = \kappa(y_1 - y_{1,n})$;
3. Natural output: $y_{1,n} = \bar{y}_n + \epsilon_1 - u_1$;
4. Efficient output: $y_1^* = y_{1,n} + \Delta + u_1 = \bar{y}_n + \Delta + \epsilon_1$.

The three shocks η, ϵ, and u have zero expected mean and their variances are given by σ_η^2, σ_ϵ^2, and σ_u^2.

Assume that the central bank sets the interest rate to minimize the loss function

$$L = \frac{1}{2}E(y_1 - y_1^*)^2 + \frac{\theta}{2\kappa}E(p_1 - p^*)^2.$$

(a) Discuss the effects of the three types of shocks on output and prices as well as the associated transmission mechanisms.

Recall from Chap. 4, Exercise 6

$$y_t^* - y_{t,n} = \frac{1}{\varphi + \sigma^{-1}}\left(\mu + \tau_1^N + \tau_1^C\right).$$

Demand shock η_1 (see Fig. 5.4)

- $\eta_1 > 0$ shifts AD curve upward and drives output above the natural level. This induces firms to increase their prices;
- Optimal response: Increase in interest rate dampens current demand once real interest rate increases. Perfect stabilization possible;
- Vice versa for $\eta_1 < 0$.

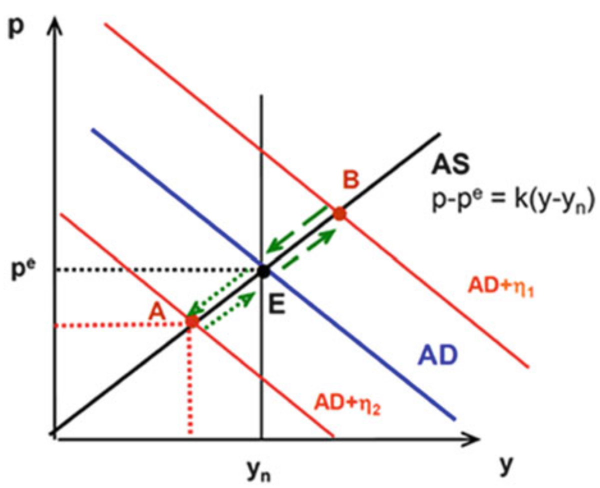

Fig. 5.4 Effects of a demand shock

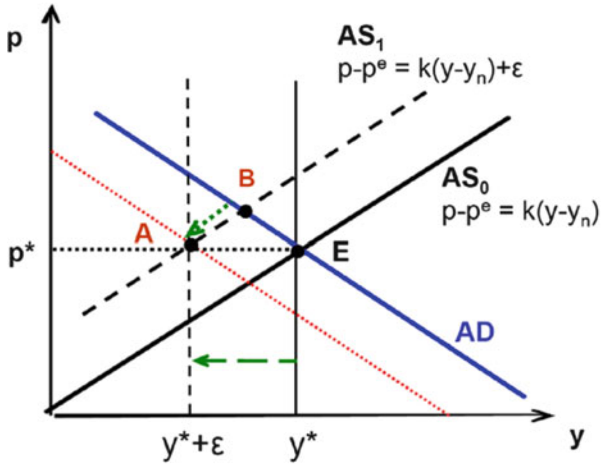

Fig. 5.5 Effects of a supply shock

Supply shock ϵ_1 (based on productivity a_t or willingness to work z_t, see Fig. 5.5)

- Shocks ϵ_1 affect both natural and efficient output but the wedge Δ remains the same (remember that this wedge only depends on the mark-up and distortionary taxation);
- Divine coincidence: Stabilization of price level also stabilizes the welfare-relevant output gap. No trade-off between output and inflation stabilization.

Cost-push shock u_1 (based on mark-up μ or changes in distortionary taxes)

- Shocks u_1 affect the gap between natural and efficient output since they only affect the natural level of output;
- Trade-off between output and inflation stabilization.

(b) *Welfare losses result in parts from price dispersion. Start from the labor market equilibrium condition*

$$N_t = \int_{i=0}^{1} N_t(i)di$$

and show via a second-order approximation that the price dispersion index

$$D_t = \int_{i=0}^{1} \left(\frac{P_t(i)}{P_t}\right)^{-\theta} di$$

can be expressed in terms of the cross-sectional variance of prices. Show that this variance is related to surprise inflation.

Start from the labor market clearing condition (ignoring time subscripts)

$$N = \int_{i=0}^{1} N(i)di = \int_{i=0}^{1} \frac{Y(i)}{A}di = \frac{Y}{A}\int_{i=0}^{1}\left(\frac{P(i)}{P}\right)^{-\theta} di = \frac{Y}{A}D$$

in which D refers to a measure of price dispersion in the economy that creates welfare losses. Now, consider the definition of the price index

$$P = \left(\int_{i=0}^{1} P(i)^{1-\theta}di\right)^{1-\theta} \rightarrow\ 1 = \int_{i=0}^{1}\left(\frac{P(i)}{P}\right)^{1-\theta} di,$$

$$1 = \int_{i=0}^{1} e^{ln\left[\left(\frac{P(i)}{P}\right)^{1-\theta}\right]}di$$

$$= \int_{i=0}^{1} e^{(1-\theta)(p(i)-p)}di$$

$$= \int_{i=0}^{1}\sum_{k=0}^{\infty}\frac{[(1-\theta)(p(i)-p)]^k}{k!}di,$$

$$1 \approx 1 + \int_{i=0}^{1}(1-\theta)(p(i)-p)di + \int_{i=0}^{1}\frac{(1-\theta)^2}{2}(p(i)-p)^2di,$$

$$p = \int_{i=0}^{1} p_i di + \frac{(1-\theta)}{2}\int_{i=0}^{1}(p(i)-p)^2di$$

$$= E[p(i)] + \frac{(1-\theta)}{2}\int_{i=0}^{1}(p(i)-p)^2di,$$

$$\int_{i=0}^{1}(p_i - p)di = -\frac{(1-\theta)}{2}\int_{i=0}^{1}(p(i)-p)^2di$$

in which $E[p(i)] \equiv \int_{i=0}^{1} p_i di$ denotes the cross-sectional mean of (log) prices.

Start from the definition of the price dispersion index and use the above result for substitution

$$D = \int_{i=0}^{1}\left(\frac{P(i)}{P}\right)^{-\theta} di = \int_{i=0}^{1} e^{-\theta(p(i)-p)}di = \int_{i=0}^{1}\sum_{k=0}^{\infty}\frac{[-\theta(p(i)-p)]^k}{k!}$$

$$\approx 1 - \int_{i=0}^{1}\theta(p(i)-p)di + \int_{i=0}^{1}\frac{\theta^2}{2}(p(i)-p)^2di$$

$$\approx 1 + \theta\frac{(1-\theta)}{2}\int_{i=0}^{1}(p(i)-p)^2di + \frac{\theta^2}{2}\int_{i=0}^{1}(p(i)-p)^2di$$

$$= 1 + \frac{\theta}{2}\int_{i=0}^{1}(p(i)-p)^2di.$$

Note that up to second order

$$\int_{i=0}^{1} (p(i) - p)^2 di \approx \int_{i=0}^{1} (p(i) - E[p(i)])^2 di \equiv \text{var}[p(i)].$$

Therefore, the price dispersion term can be written in logs as

$$d = \ln D \approx \frac{\theta}{2}\text{var}[p(i)].$$

What about the term var$[p(i)]$? Recall that a fraction α of firms sets prices according to $P^{sticky} = E_{-1}P$ or $p^{sticky} = p^e$. Taking this into account, the overall price level is given by

$$P^{1-\theta} = \alpha(P^{sticky})^{1-\theta} + (1-\alpha)\bar{P}^{1-\theta}.$$

The first-order approximation of this term has been derived in the last chapter and is given by

$$\begin{aligned}
p &= \alpha p^{sticky} + (1-\alpha)\bar{p}, \\
\text{var}[p(i)] &= E[p(i)^2] - E[p(i)]^2 \\
&= \alpha(p^{sticky})^2 + (1-\alpha)\bar{p}^2 - [\alpha p^{sticky} + (1-\alpha)\bar{p}]^2 \\
&= \alpha(1-\alpha)(p^{sticky})^2 + (1-\alpha)\alpha\bar{p}^2 - 2\alpha(1-\alpha)\bar{p}(p^{sticky})^2 \\
&= \alpha(1-\alpha)[\bar{p} - p^{sticky}]^2 \\
&= \alpha(1-\alpha)[\bar{p} - p^e]^2.
\end{aligned}$$

Note that $\pi = p - p_{-1}$ so that surprise inflation is given by

$$\pi - E_{-1}\pi = p - E_{-1}p = \alpha p^e + (1-\alpha)\bar{p} - p^e = (1-\alpha)(\bar{p} - p^e).$$

Therefore

$$\begin{aligned}
\text{var}[p(i)] &= \frac{\alpha}{1-\alpha}[\pi - E_{-1}\pi]^2, \\
d &\approx \frac{\theta}{2}\frac{\alpha}{1-\alpha}[\pi - E_{-1}\pi]^2.
\end{aligned}$$

(c) Assume that the central bank follows an interest rate rule $i_1 = f(\eta_1, \epsilon_1, u_1)$. The structure of the problem follows a three-stage process.

- *Stage 1: Central bank announces a policy rule $i_1 = f(\eta_1, \epsilon_1, u_1)$;*
- *Stage 2: The share α of firms sets (fixed) prices based on rational expectations. Then shocks occur;*
- *Stage 3: Central bank reacts to the shocks by setting the actual interest rate i_1.*

Obtain the optimal policy rule under the assumption that there is no structural inefficiency (i.e. $\Delta = 0$). Discuss how optimal monetary policy responds to the different shocks.

Following the three-stage game, the central bank needs to solve the following problem

$$\begin{aligned}
\min \quad & L = \frac{1}{2}E(y_1 - y_1^*)^2 + \frac{\theta}{2\kappa}E(p_1 - p^*)^2, \\
s.t. \quad & p_1 - p_1^e = \kappa(y_1 - y_{1,n}) = \kappa(y_1 - y_1^* + u_1), \\
& \text{rational expectations of agents.}
\end{aligned}$$

Start from stage 3 after realizations of shocks (with no uncertainty) so that

$$\Lambda = \frac{1}{2}(y_1 - y_1^*)^2 + \frac{\theta}{2\kappa}(p_1 - p^*)^2 + \lambda(p_1 - p_1^e - \kappa(y_1 - y_1^* + u_1)),$$

the first-order conditions give

$$\begin{aligned}
\frac{\partial \Lambda}{\partial y_1} &= (y_1 - y_1^*) - \lambda\kappa = 0, \\
\frac{\partial \Lambda}{\partial p_1} &= \frac{\theta}{\kappa}(p_1 - p^*) + \lambda = 0, \\
\frac{\partial \Lambda}{\partial \lambda} &= p_1 - p_1^e - \kappa(y_1 - y_1^* + u_1) = 0.
\end{aligned}$$

Solving the second equation for λ and substituting into the first one yields the optimal stabilization curve

$$\begin{aligned}
y_1 - y_1^* &= -\frac{\theta}{\kappa}\kappa(p_1 - p^*), \\
p_1 - p^* &= -\frac{1}{\theta}(y_1 - y_1^*).
\end{aligned}$$

This curve is independent of η_1 and u_1 as neither affects efficient output y^*. However, shocks ϵ_1 shift y^* and thus affect the optimal stabilization curve.

Combining the optimal stabilization curve with the AS curve yields the optimal price level p_1

$$\begin{aligned}
p_1 - p^* = -\frac{1}{\theta}(y_1 - y_1^*) &= -\frac{1}{\theta\kappa}(p_1 - p_1^e) + \frac{1}{\theta}u_1, \\
p_1 + \frac{1}{\theta\kappa}p_1 &= p^* + \frac{1}{\theta\kappa}p_1^e + \frac{1}{\theta}u_1,
\end{aligned}$$

$$\frac{1+\theta\kappa}{\theta\kappa}p_1 = \left(1+\frac{1}{\theta\kappa}\right)p^* + \frac{1}{\theta\kappa}(p_1^e - p^*) + \frac{1}{\theta}u_1,$$
$$p_1 = p^* + \frac{1}{1+\theta\kappa}(p_1^e - p^*) + \frac{\kappa}{1+\theta\kappa}u_1.$$

This is the central bank's reaction function that characterizes the optimal response of the central bank for any expected price level p_1^e after the realization of shocks.

At stage 2, firms set their prices using the best prediction of the price level under anticipation of the central bank's reaction. Their expectation of the price level in stage 3 under rational expectations is given by

$$p_1^e = E(p_1) = p^* + \frac{1}{1+\theta\kappa}(E(p_1) - p^*) + \frac{\kappa}{1+\theta\kappa}E(u_1),$$
$$\left(1-\frac{1}{1+\theta\kappa}\right)E(p_1) = \left(1-\frac{1}{1+\theta\kappa}\right)p^*,$$
$$p_1^e = E(p_1) = p^*.$$

This is combined with the central bank's reaction function to yield:

$$p_1 = p^* + \frac{1}{1+\theta\kappa}(p^* - p^*) + \frac{\kappa}{1+\theta\kappa} + u_1,$$
$$p_1 = p^* + \frac{\kappa}{1+\theta\kappa}u_1,$$
$$p_1 - p^* = \frac{\kappa}{1+\theta\kappa}u_1.$$

Thus, cost-push shocks are the only shocks that result in deviations of prices from their target. All other shocks can be perfectly stabilized in the absence of structural inefficiencies. Combining this results with the AS curve yields

$$p_1 - p_1^e = p_1 - p^* = \frac{\kappa}{1+\theta\kappa}u_1 = \kappa(y_1 - y_{1,n}),$$
$$y_1 - y_{1,n} = \frac{1}{1+\theta\kappa}u_1,$$
$$y_1 - \bar{y}_{1,n} = \epsilon_t - u_1 + \frac{1}{1+\theta\kappa}u_1 = \epsilon_t - \frac{\theta\kappa}{1+\theta\kappa}u_1,$$
$$y_1 - y^* = y_1 - y_{1,n} - u_1 = -\frac{\theta\kappa}{1+\theta\kappa}u_1.$$

Use the AD curve to calculate the optimal interest rate rule

$$
\begin{aligned}
y_1 - \bar{y}_n &= -\sigma(i_1 - (p^* - p_1) - \rho) + \eta_1, \\
\epsilon_t - \frac{\theta\kappa}{1+\theta\kappa}u_1 &= -\sigma(i_1 + \frac{\kappa}{1+\theta\kappa}u_1 - \rho) + \eta_1, \\
i_1 + \frac{\kappa}{1+\theta\kappa}u_1 - \rho &= \frac{1}{\sigma}\left(\frac{\theta\kappa}{1+\theta\kappa}u_1 - \epsilon_t + \eta_1\right), \\
i_1 &= \rho + \frac{1}{\sigma}\left(\frac{\theta\kappa}{1+\theta\kappa}u_1 - \epsilon_t + \eta_1\right) - \frac{\kappa}{1+\theta\kappa}u_1, \\
i_1 &= \rho + \frac{1}{\sigma}\left(\eta_1 - \epsilon_t - \frac{(\sigma-\theta)\kappa}{1+\kappa\theta}u_1\right).
\end{aligned}
$$

(d) *Show that the loss function of the central bank can be derived from the household's period utility function $V(C,N) = U(C) - V(N)$ via a second order approximation.*

Start from second-order Taylor approximation of the utility function around the natural levels of consumption and labor, and note that the cross-derivatives are zero (i.e., $V_{CN} = U_{CN} = 0$)

$$
\begin{aligned}
V(C,N) &= U(C) + V(N) = \frac{C_t^{1-\frac{1}{\sigma}}}{1-\frac{1}{\sigma}} + \frac{N_t^{1+\varphi}}{1+\varphi} \\
&\approx V(C^n, N^n) + U'(C^n)(C - C^n) + V'(N^n)(N - N^n) \\
&\quad + \frac{1}{2}U''(C^n)(C - C^n)^2 + \frac{1}{2}V''(N^n)(N - N^n)^2 \\
&= V(C^n, N^n) + U'(C^n)C^n\frac{(C - C^n)}{C^n} + V'(N^n)N^n\frac{(N - N^n)}{N^n} \\
&\quad + \frac{1}{2}(C^n)^2U''(C^n)\cdot\left(\frac{(C - C^n)}{C^n}\right)^2 + \frac{1}{2}(N^n)^2V''(N^n)\left(\frac{(N - N^n)^2}{N^n}\right)^2.
\end{aligned}
$$

Note that

$$
\begin{aligned}
\frac{1}{\sigma} &= -\frac{U''(C^n)C^n}{U'(C^n)}, \\
\varphi &= \frac{V''(N^n)N^n}{V'(N^n)}.
\end{aligned}
$$

In addition, note that

$$\begin{aligned}\frac{X-X^n}{X^n} &= e^{\ln\left(\frac{X}{X^n}\right)} - 1\\ &= \sum_{k=0}^{\infty}\frac{\left(\ln\left(\frac{X}{X^n}\right)\right)^k}{k!} - 1\\ &= \ln\left(\frac{X}{X^n}\right) + \frac{1}{2}\left(ln\left(\frac{X}{X^n}\right)\right)^2 + \cdots,\end{aligned}$$

therefore, the second-order approximation yields

$$\frac{X-X^n}{X^n} \approx \hat{x} + \frac{1}{2}\hat{x}^2.$$

Use these short-cuts in the derivation of the loss function

$$\begin{aligned}V(C,N) = {} & V(C^n,N^n) + U'(C^n)C^n\frac{(C-C^n)}{C^n}\left[1+\frac{1}{2}\frac{C^nU''(C^n)}{U'(C^n)}\left(\frac{(C-C^n)}{C^n}\right)\right]\\ & +V'(N^n)N^n\frac{(N-N^n)}{N^n}\left[1+\frac{1}{2}\frac{N^nV''(N^n)}{V'(N^n)}\left(\frac{(N-N^n)^2}{N^n}\right)\right]\\ = {} & V(C^n,N^n)+U'(C^n)C^n\left(\hat{c}+\frac{1}{2}\hat{c}^2\right)\left[1-\frac{1}{2\sigma}\left(\hat{c}+\frac{1}{2}\hat{c}^2\right)\right]\\ & +V'(N^n)N^n\cdot\left(\hat{n}+\frac{1}{2}\hat{n}^2\right)\left[1+\frac{\varphi}{2}\left(\hat{n}+\frac{1}{2}\hat{n}^2\right)\right]\\ = {} & V(C^n,N^n)+U'(C^n)C^n\left[\hat{c}+\left(\frac{1}{2}-\frac{1}{2\sigma}\right)\hat{c}^2-\frac{1}{2\sigma}\hat{c}^3-\frac{1}{8\sigma}\hat{c}^4\right]\\ & +V'(N^n)\cdot N^n\left[\hat{n}+\left(\frac{1}{2}+\frac{\varphi}{2}\right)\hat{n}^2+\frac{\varphi}{2}\hat{n}^3+\frac{\varphi}{8}\hat{n}^4\right].\end{aligned}$$

Drop all terms of order 3 and higher in the second order approximation

$$V-V^n = U'(C^n)C^n\left[\hat{c}+\left(\frac{1}{2}-\frac{1}{2\sigma}\right)\hat{c}^2\right]+V'(N^n)N^n\left[\hat{n}+\frac{1+\varphi}{2}\hat{n}^2\right].$$

Now, derive expressions to substitute for $\hat{c}$ and $\hat{n}$. Goods market clearing requires $Y_t = C_t + G_t$ and thus $\hat{y} = \frac{C^n}{Y^n}\hat{c} + \frac{G^n}{Y^n}\hat{g}$. Let's assume that there is no government, so that $\hat{y} = \hat{c}$. In log terms, it holds that $n = y - a + d$ whereas $n^n = y^n - a$. Therefore, $\hat{n} = n - n^n = \hat{y} + d$. Inserting this in our welfare measure yields

$$V-V^n = U'(C^n)C^n\left[\hat{y}+\left(\frac{1}{2}-\frac{1}{2\sigma}\right)\hat{y}^2\right]+V'(N^n)N^n\left[\hat{y}+d+\frac{1+\varphi}{2}(\hat{y}^2+d^2)\right].$$

Use the results from the approximation of the price dispersion index. Since the second-order approximation of d is proportional to the second moment of prices, we can drop any terms of d with order higher than one

$$\begin{aligned} V - V^n &= U'(C^n)C^n \left[\hat{y} + \left(\frac{1}{2} - \frac{1}{2\sigma} \right) \hat{y}^2 \right] + V'(N^n)N^n \left[\hat{y} + d + \frac{1+\varphi}{2} \hat{y}^2 \right] \\ &= U'(C^n)C^n \left[\hat{y} + \left(\frac{1}{2} - \frac{1}{2\sigma} \right) \hat{y}^2 \right] + V'(N^n)N^n \left[\hat{y} + \frac{\theta}{2} \mathrm{var}[p(i)] + \frac{1+\varphi}{2} \hat{y}^2 \right]. \end{aligned}$$

Now, assume there is a subsidy in place such that the natural level of output equals the efficient one. Then

$$\begin{aligned} -\frac{V'(N^n)}{U'(C^n)} &= \frac{W}{P} = A = \frac{Y^n}{N^n}, \\ -V'(N^n)N^n &= U'(C^n)Y^n = U'(C^n)C^n. \end{aligned}$$

Using this in the welfare measure yields

$$\begin{aligned} V - V^n &= U'(C^n)C^n \left[\hat{y} + \left(\frac{1}{2} - \frac{1}{2\sigma} \right) \hat{y}^2 \right] - U'(C^n)C^n \left[\hat{y} + \frac{\theta}{2} \mathrm{var}[p(i)] + \frac{1+\varphi}{2} \hat{y}^2 \right] \\ &= U'(C^n)C^n \left[\left(-\frac{1}{2\sigma} - \frac{\varphi}{2} \right) \hat{y}^2 - \frac{\theta}{2} \mathrm{var}[p(i)] \right] \\ &= -\frac{1}{2} U'(C^n)C^n \left[\left(\frac{1}{\sigma} + \varphi \right) \hat{y}^2 + \theta \mathrm{var}[p(i)] \right]. \end{aligned}$$

Finally, express the marginal utility of consumption as deviation from the steady state

$$\begin{aligned} U'(C^n)C^n &\approx U'(\bar{C})\bar{C} + \left(U''(\bar{C})\bar{C} + U'(\bar{C}) \right) (C^n - \bar{C}) \\ &\approx U'(\bar{C})\bar{C} + \left(-\frac{1}{\sigma} U'(\bar{C}) + U'(\bar{C}) \right) \bar{C} \frac{(C^n - \bar{C})}{\bar{C}} \\ &\approx U'(\bar{C})\bar{C} + \left(1 - \frac{1}{\sigma} \right) U'(\bar{C})\bar{C} \frac{(C^n - \bar{C})}{\bar{C}} \\ &\approx U'(\bar{C})\bar{C} + tip \end{aligned}$$

in which tip refers to terms independent of monetary policy which can be neglected in the following since they do not matter for optimal monetary policy. This yields the welfare measure

$$L \equiv \frac{V - V^n}{U'(\bar{C})\bar{C}} \approx -\frac{1}{2} \left[\left(\frac{1}{\sigma} + \varphi \right) \hat{y}^2 + \theta \mathrm{var}[p(i)] \right].$$

Finally, substitute the expression for var[$p(i)$] derived above

$$L = -\frac{1}{2}\left[\left(\frac{1}{\sigma} + \varphi\right)\hat{y}^2 + \theta\frac{\alpha}{1-\alpha}\left(\pi - E_{-1}\pi\right)^2\right].$$

5.2.6 Time Inconsistency Problem and Inflation Bias

Consider the same set-up as in the previous exercise but now assume that the structural inefficiency is not corrected via paying subsidies to firms, i.e., there is a wedge $\Delta > 0$ between the first-best (efficient) level of output and the natural level.

Assume that the central bank minimizes the loss function

$$L = \frac{1}{2}E(y_1 - y_1^*)^2 + \frac{\theta}{2\kappa}E(p_1 - p^*)^2$$

under the constraints

- *AD curve:* $y_1 = \bar{y}_n - \sigma(i_1 - (p_2^e - p_1) - \rho) + \eta_1$*;*
- *AS curve:* $p_1 - p_1^e = \kappa(y_1 - y_{1,n})$*;*
- *Natural output:* $y_{1,n} = \bar{y}_n + \epsilon_1 - u_1$*;*
- *Efficient output:* $y_1^* = y_{1,n} + \Delta + u_1 = \bar{y}_n + \Delta + \epsilon_1$

in which the three shocks η, ϵ, and u have zero expected mean and their variances are given by σ_η^2, σ_ϵ^2, and *σ_u^2.*

Agents have rational expectations. Assume that the central bank follows an interest rate rule $i_1 = f(\eta_1, \epsilon_1, u_1)$. The structure of the problem follows a three-stage process.

- *Stage 1: Central bank announces a policy rule $i_1 = f(\eta_1, \epsilon_1, u_1)$;*
- *Stage 2: The share α of firms sets (fixed) prices based on rational expectations. Then shocks occur;*
- *Stage 3: Central bank reacts to the shocks by setting the actual rate i_1.*

(a) First, assume that the central bank can credibly commit itself to the announced price level in stage 1 so that $p^e = E(p_1)$. Derive the commitment equilibrium under these assumptions.

First, suppose that the central bank is able to commit to its announcement in period 1 which then shapes expectations of price-setting firms at stage 2. When making the announcement, the central bank takes the effects on expectations into account. Thus, p^e becomes a choice variable (via the announcement) and the optimization problem becomes

$$\begin{aligned}
\min \quad & L = \frac{1}{2}E(y_1 - y_1^*)^2 + \frac{\theta}{2\kappa}E(p_1 - p^*)^2, \\
s.t. \quad & p_1 - p_1^e = \kappa(y_1 - y_{1,n}) = \kappa(y_1 - y_1^* + \Delta + u_1), \\
& E(p_1) = p_1^e, \\
& \Lambda = \frac{1}{2}E\left(\frac{1}{\kappa}(p_1 - p_1^e) - \Delta - u_1\right)^2 + \frac{\theta}{2\kappa}E\left(p_1 - p^*\right)^2 + \lambda(E(p_1) - p_1^e).
\end{aligned}$$

Keep in mind that the announcement is made before shocks are realized so we need to take expectations into account

$$\begin{aligned}
\frac{\partial\Lambda}{\partial p_1^e} &= -\frac{1}{\kappa}E\left(\frac{1}{\kappa}(p_1 - p_1^e) - \Delta - u_1\right) - \lambda = 0, \\
E\left(\frac{1}{\kappa}(p_1 - p_1^e) - \Delta\right) &= -\kappa\lambda, \\
E(p_1 - p_1^e) &= -\kappa^2\lambda + \kappa\Delta, \\
p_1^C &= E(p_1) + \kappa^2\lambda - \kappa\Delta.
\end{aligned}$$

Since $p_1^C = E(p_1)$, it follows that $\lambda = \dfrac{\Delta}{\kappa}$. In contrast, the actual price decision is made after the realization of shocks is observed

$$\begin{aligned}
& \frac{\partial\Lambda}{\partial p_1} = \frac{1}{\kappa}\left(\frac{1}{\kappa}(p_1 - p_1^e) - \Delta - u_1\right) + \frac{\theta}{\kappa}\left(p_1 - p^*\right) + \lambda = 0, \\
& p_1 - p_1^e - \kappa(\Delta + u_1) + \kappa\theta\left(p_1 - p^*\right) + \kappa^2\lambda = 0, \\
& (1+\kappa\theta)p_1 = p_1^e + \kappa(\Delta + u_1) + \kappa\theta p^* - \kappa^2\lambda, \\
& (1+\kappa\theta)p_1 = p_1^e + \kappa(\Delta + u_1) + \kappa\theta p^* - \kappa\Delta, \\
& (1+\kappa\theta)p_1 = p_1^e + \kappa u_1 + \kappa\theta p^*.
\end{aligned}$$

As $p_1^e = E(p_1)$, it follows that

$$(1+\kappa\theta)p_1^e = p_1^e + \kappa\theta p^* = 0 \rightarrow p_1^e = p^*.$$

This yields

$$\begin{aligned}
(1+\kappa\theta)p_1 &= (1+\kappa\theta)p^* + \kappa u_1, \\
p_C &= p^* + \frac{\kappa}{1+\kappa\theta}u_1.
\end{aligned}$$

Combining this with the AS equation yields the output level under commitment

$$p_C - p_C^e = p^* + \frac{\kappa}{1+\kappa\theta}u_1 - p^* = \frac{\kappa}{1+\theta\kappa}u_1 = \kappa(y_D - y_{1,n}),$$
$$y_D = y_{1,n} + \frac{1}{1+\theta\kappa}u_1 = y_1^* - \Delta - u_1 + \frac{1}{1+\theta\kappa}u_1 = y_1^* - \Delta - \frac{\theta\kappa}{1+\theta\kappa}u_1.$$

This results in the welfare loss

$$\begin{aligned}
EL_C &= \frac{1}{2}E\left(\Delta + \frac{\theta\kappa}{1+\theta\kappa}u_1\right)^2 + \frac{\theta}{2\kappa}E\left(\frac{\kappa}{1+\kappa\theta}u_1\right)^2 \\
&= \frac{1}{2}\left[\Delta^2 + \left(\frac{\theta\kappa}{1+\theta\kappa}\right)^2 E(u_1)^2\right] + \frac{\theta}{2\kappa}\left(\frac{\kappa}{1+\kappa\theta}\right)^2 E(u_1)^2 \\
&= \frac{1}{2}\left[\Delta^2 + \left(\frac{\theta\kappa}{1+\theta\kappa}\right)^2 E(u_1)^2 + \frac{\theta\kappa}{(1+\kappa\theta)^2}E(u_1)^2\right] \\
&= \frac{1}{2}\left[\Delta^2 + \frac{\theta\kappa}{1+\theta\kappa}\sigma_u^2\right].
\end{aligned}$$

(b) Discuss why there is an incentive for surprise inflation and derive the equilibrium if surprise inflation is possible, i.e. under the assumption that inflation expectations are fixed at $p^e = p^*$.

Note that the commitment solution is no Nash Equilibrium since the central bank has an incentive on stage 3 to renege on its promise made at stage 1 after contracts are written. Thus, commitment is not dynamically consistent, i.e., there is a time inconsistency problem.

Now, assume expectations of agents are fixed at the price target of the central bank, i.e., $p_1^e = p^*$, and shocks are observed

$$\begin{aligned}
\min\ & L = \frac{1}{2}E(y_1 - y_1^*)^2 + \frac{\theta}{2\kappa}E(p_1 - p^*)^2, \\
s.t.\ & p_1 - p_1^e = \kappa(y_1 - y_1^* + \Delta + u_1), \\
& p^* = p_1^e.
\end{aligned}$$

That is,

$$\min L = \frac{1}{2}E\left(\frac{1}{\kappa}(p_1 - p_1^*) - \Delta - u_1\right)^2 + \frac{\theta}{2\kappa}E\left(p_1 - p^*\right)^2.$$

The first-order condition gives

$$\begin{aligned}
\frac{\partial L}{\partial p_1} &= \frac{1}{\kappa}E\left(\frac{1}{\kappa}(p_1 - p_1^*) - \Delta - u_1\right) + \frac{\theta}{\kappa}E\left(p_1 - p^*\right) = 0,\\
\frac{1}{\kappa}&(p_1 - p_1^*) - \Delta - u_1 + \theta(p_1 - p^*) = 0,\\
(1+\kappa\theta)&p_1 = (1+\kappa\theta)p^* + \kappa(\Delta + u_1),\\
p_1 &= p^* + \frac{\kappa}{1+\kappa\theta}(\Delta + u_1).
\end{aligned}$$

The higher price level helps to stimulate output above the natural level, as

$$\begin{aligned}
p_1 - p^* &= \frac{\kappa}{1+\theta\kappa}(\Delta + u_1) = \kappa(y_1 - y_{1,n}),\\
y_1 &= y_{1,n} + \frac{1}{1+\theta\kappa}(\Delta + u_1)\\
&= y^* - \frac{\theta\kappa}{1+\theta\kappa}(\Delta + u_1).
\end{aligned}$$

The resulting welfare loss is given by

$$\begin{aligned}
EL_S &= \frac{1}{2}E\left(-\frac{\theta\kappa}{1+\theta\kappa}(\Delta + u_1)\right)^2 + \frac{\theta}{2\kappa}E\left(\frac{\kappa}{1+\theta\kappa}(\Delta + u_1)\right)^2\\
&= \frac{1}{2}\left[\frac{(\theta\kappa)^2}{(1+\theta\kappa)^2}(\Delta^2 + \sigma_u^2) + \frac{\theta}{\kappa}\frac{\kappa^2}{(1+\theta\kappa)^2}(\Delta^2 + \sigma_u^2)\right]\\
&= \frac{1}{2}\left[\frac{(\theta\kappa)^2}{(1+\theta\kappa)^2} + \frac{\theta\kappa}{(1+\theta\kappa)^2}\right](\Delta^2 + \sigma_u^2)\\
&= \frac{1}{2}\left[\frac{\theta\kappa}{1+\theta\kappa}\Delta^2 + \frac{\theta\kappa}{1+\theta\kappa}\sigma_u^2\right].
\end{aligned}$$

Note that the welfare loss can be reduced via surprise inflation compared to the commitment solution. Thus, the central bank has an incentive to deviate from its announced price level.

$$EL_S - EL_C = -\frac{1}{2}\frac{1}{1+\theta\kappa}\Delta^2 < 0.$$

(c) Derive the equilibrium under discretion and the welfare loss L associated with discretionary central bank policy.

However, note that individual anticipate the incentive of the central bank to deviate from its announcement and take this into account when forming expectations. This results in the discretionary outcome with inflationary bias.

Specifically, without effective commitment, the central bank solves the following problem:

$$\begin{aligned}
&\min \quad L = \frac{1}{2}E(y_1 - y_1^*)^2 + \frac{\theta}{2\kappa}E(p_1 - p^*)^2, \\
&s.t. \quad p_1 - p_1^e = \kappa(y_1 - y_{1,n}) = \kappa(y_1 - y_1^* + \Delta + u_1).
\end{aligned}$$

Start from stage 3 after realizations of shocks (no uncertainty) so that

$$\Lambda = \frac{1}{2}(y_1 - y_1^*)^2 + \frac{\theta}{2\kappa}(p_1 - p^*)^2 + \lambda(p_1 - p_1^e - \kappa(y_1 - y_1^* + \Delta + u_1)),$$

the first-order condition gives

$$\begin{aligned}
\frac{\partial \Lambda}{\partial y_1} &= (y_1 - y_1^*) - \lambda\kappa = 0, \\
\frac{\partial \Lambda}{\partial p_1} &= \frac{\theta}{\kappa}(p_1 - p^*) + \lambda = 0, \\
\frac{\partial \Lambda}{\partial \lambda} &= p_1 - p_1^e - \kappa(y_1 - y_1^* + \Delta + u_1) = 0.
\end{aligned}$$

Solving the second equation for λ and substituting into the first one yields the optimal stabilization curve

$$\begin{aligned}
y_1 - y_1^* &= -\frac{\theta}{\kappa}\kappa(p_1 - p^*), \\
p_1 - p^* &= -\frac{1}{\theta}(y_1 - y_1^*).
\end{aligned}$$

This curve is independent of η_1 and u_1 since neither affects efficient output y^*. However, shocks ϵ_1 shift y^* and thus affect the optimal stabilization curve.

Combining the optimal stabilization curve with the AS curve yields the optimal price level p_1

$$\begin{aligned}
p_1 - p^* &= -\frac{1}{\theta}(y_1 - y_1^*) = -\frac{1}{\theta\kappa}(p_1 - p_1^e) + \frac{1}{\theta}(\Delta + u_1), \\
p_1 + \frac{1}{\theta\kappa}p_1 &= p^* + \frac{1}{\theta\kappa}p_1^e + \frac{1}{\theta}(\Delta + u_1), \\
\frac{1 + \theta\kappa}{\theta\kappa}p_1 &= \left(1 + \frac{1}{\theta\kappa}\right)p^* + \frac{1}{\theta\kappa}(p_1^e - p^*) + \frac{1}{\theta}(\Delta + u_1), \\
p_1 &= p^* + \frac{1}{1 + \theta\kappa}(p_1^e - p^*) + \frac{\kappa}{1 + \theta\kappa}(\Delta + u_1).
\end{aligned}$$

This is the central bank's reaction function that characterizes the optimal response of the central bank for any expected price level p_1^e after the realization of shocks.

At stage 2, firms set their prices using the best prediction of the price level under anticipation of the central bank's reaction. Their expectation of the price level in stage 3 under rational expectations is given by

$$p_1^e = E(p_1) = p^* + \frac{1}{1+\theta\kappa}(E(p_1) - p^*) + \frac{\kappa}{1+\theta\kappa}\Delta,$$
$$\left(1 - \frac{1}{1+\theta\kappa}\right) E(p_1) = \left(1 - \frac{1}{1+\theta\kappa}\right) p^* + \frac{\kappa}{1+\theta\kappa}\Delta,$$
$$p_1^e = E(p_1) = p^* + \frac{1}{\theta}\Delta > p^*.$$

As a consequence, announcing a policy of price stabilization at p^* at stage 1 is no longer credible since the central bank has strong incentives to deviate from such an announcement at stage 3 once prices are set and shocks have been realized in order to get closer to the output target. Yet, rational agents anticipate this incentive and increase their inflation expectations. In equilibrium, prices are above its target. Yet, there is no surprise inflation which is why output is not stimulated above its natural level (see Fig. 5.6).

Agents form rational expectations of future prices given the central bank's reaction function. As above, combine the reaction function with the AS curve to get

$$p_D^e = p^* + \frac{1}{\theta}\Delta.$$

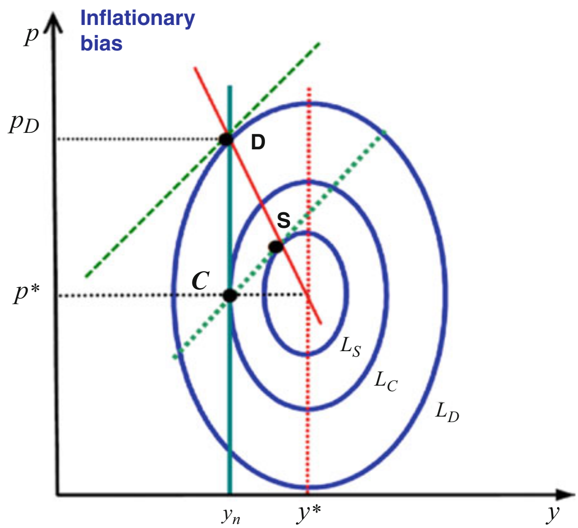

Fig. 5.6 Inflation bias

Given these expectations, the central bank's response at stage 3 is given by

$$\begin{aligned}
p_D &= p^* + \frac{1}{1+\theta\kappa}(p_D^e - p^*) + \frac{\kappa}{1+\theta\kappa}(\Delta + u_1)\\
&= p^* + \frac{1}{1+\theta\kappa}(p^* + \frac{1}{\theta}\Delta - p^*) + \frac{\kappa}{1+\theta\kappa}(\Delta + u_1)\\
&= p^* + \frac{1}{\theta}\Delta + \frac{\kappa}{1+\theta\kappa}u_1.
\end{aligned}$$

Thus, the price level is higher than in the case of $\Delta = 0$. Combining this results with the AS curve yields

$$\begin{aligned}
p_D - p_D^e &= p^* + \frac{1}{\theta}\Delta + \frac{\kappa}{1+\theta\kappa}u_1 - p^* - \frac{1}{\theta}\Delta\\
&= \frac{\kappa}{1+\theta\kappa}u_1\\
&= \kappa(y_D - y_{1,n}),\\
\frac{1}{1+\theta\kappa}u_1 &= y_D - \bar{y}_n - \epsilon_1 + u_1,\\
y_D &= \bar{y}_n + \epsilon_1 - \frac{\theta\kappa}{1+\theta\kappa}u_1,\\
\frac{1}{1+\theta\kappa}u_1 &= y_D - y_1^* + \Delta + u_1,\\
y_D &= y_1^* - \Delta - \frac{\theta\kappa}{1+\theta\kappa}u_1.
\end{aligned}$$

The corresponding policy rule is derived from the AD curve

$$\begin{aligned}
&y_D - \bar{y}_n = -\sigma(i_1 - (p^e - p_D) - \rho) + \eta_1,\\
&\epsilon_1 - \frac{\theta\kappa}{1+\theta\kappa}u_1 = -\sigma\left(i_1 + \frac{\kappa}{1+\theta\kappa}u_1 - \rho\right) + \eta_1,\\
&i_1 + \frac{\kappa}{1+\theta\kappa}u_1 - \rho = \frac{1}{\sigma}\left(\eta_1 - \epsilon_1 + \frac{\theta\kappa}{1+\theta\kappa}u_1\right),\\
&i_1 = \rho + \frac{1}{\sigma}\left(\eta_1 - \epsilon_1 - \frac{(\sigma-\theta)\kappa}{1+\theta\kappa}u_1\right).
\end{aligned}$$

The total welfare loss is given by

$$
\begin{aligned}
E(L_D) &= \frac{1}{2}E(y_1 - y_1^*)^2 + \frac{\theta}{2\kappa}E(p_1 - p^*)^2 \\
&= \frac{1}{2}E\left(-\Delta - \frac{\theta\kappa}{1+\theta\kappa}u_1\right)^2 + \frac{\theta}{2\kappa}E\left(\frac{1}{\theta}\Delta + \frac{\kappa}{1+\theta\kappa}u_1\right)^2 \\
&= \frac{1}{2}\left[\Delta^2 + \left(\frac{\theta\kappa}{1+\theta\kappa}\right)^2 E(u_1)^2\right] \\
&\quad + \frac{\theta}{2\kappa}\left[\left(\frac{1}{\theta}\right)^2\Delta^2 + \left(\frac{\kappa}{1+\theta\kappa}\right)^2 E(u_1)^2\right] \\
&= \frac{1}{2}\left[\Delta^2 + \left(\frac{\theta\kappa}{1+\theta\kappa}\right)^2 E(u_1)^2 + \frac{1}{\theta\kappa}\Delta^2 + \frac{\theta\kappa}{(1+\theta\kappa)^2}E(u_1)^2\right] \\
&= \frac{1}{2}\left[\frac{1+\theta\kappa}{\theta\kappa}\Delta^2 + \frac{\theta\kappa}{1+\theta\kappa}\sigma_u^2\right].
\end{aligned}
$$

The welfare loss results from

- Structural inefficiency;
- Inflationary bias; and
- Variations due to stochastic mark-up shocks that cannot be perfectly stabilized.

Note that the inflationary bias in case of discretion results in a higher loss compared to the commitment solution

$$EL_D - EL_C = \frac{1}{2}\frac{\Delta^2}{\theta\kappa}.$$

References

Barro, R. J., & Gordon, D. B. (1983a). A positive theory of monetary policy in a natural-rate model. *Journal of Political Economy, 91*, 589–610.

Barro, R. J., & Gordon, D. B. (1983b). Rules, discretion, and reputation in a model of monetary policy. *Journal of Monetary Economics, 12*, 101–121.

Clarida, R., Galí, J., & Gertler, M. (1999). The science of monetary policy: A new Keynesian perspective. *Journal of Economic Literature, 37*, 1661–1707.

Grilli, V., Masciandaro, D., & Tabellini, G. (1991). Political and monetary institutions and public financial policies in the industrial countries. *Economic Policy, 13*, 341–392.

Rogoff, K. (1985). The optimal degree of commitment to an intermediate monetary target. *Quarterly Journal of Economics, 100*, 1169–1189.

Monetary Policy Under Uncertainty

6

6.1 Exercises

6.1.1 Short Review Questions

(a) What are the consequences for monetary policy, if central bank has only low quality of data on price and output?
(b) What are the consequences for monetary policy, if central bank has uncertainty on (1) transmission mechanism of monetary policy, (2) the best model of monetary policy, (3) the true source of shock, respectively?
(c) What is *Taylor Rule*? Why does central bank need to react to inflation gap more aggressively than output gap?
(d) Why does more central bank transparency improve the efficiency of monetary policy? Under which condition(s) higher transparency may do harms?

6.1.2 Monetary Policy Under Uncertainty: Reputation

Cukierman and Meltzer (1986) consider that a monetary policy maker has a limited tenure for only two periods. The policy maker is randomly nominated from a pool of candidates, whose object function is as following:

$$W = E\left[b(p_1 - p_1^e) + cp_1 - \frac{ap_1^2}{2} + b(p_2 - p_2^e) + cp_2 - \frac{ap_2^2}{2}\right]$$

in which c is normally distributed over the candidates with mean $\overline{c}$ and variance $\sigma_c^2 > 0$. However, a and b are the same for all candidates.

The policy maker only has a limited control over inflation such that $p_t = \hat{p}_t + \epsilon_t$, $t \in \{1, 2\}$, in which $\hat{p}_t$ is the policy chosen by the policy maker with p_t^e being

J. Cao, G. Illing, *Instructor's Manual for* Money: Theory and Practice,
Springer Texts in Business and Economics,
https://doi.org/10.1007/978-3-030-23618-2_6

given and ϵ_t is a normally distributed random variable with mean zero and variance $\sigma_\epsilon^2 > 0$. The random variables, ϵ_1, ϵ_2, and c are independent on each other. The public cannot observe $\hat{p}_t$ or ϵ_t, but only p_t. The public cannot observe c, either.

The public's expectation on the second-period price level, p_2^e, is formed on the basis of observed first-period price level p_1 in a way such that

$$p_2^e = \alpha + \beta p_1.$$

(a) What is the policy maker's choice on $\hat{p}_2$? Compute the expected value of her second-period objective function in terms of p_2^e.
(b) What is the policy maker's choice on $\hat{p}_1$, with α and β being given and taking account of the impact of p_1 on p_2^e?
(c) Compute the proper value of β. Provide some intuitions on your result.
(d) Provide some intuitions on why the policy maker chooses a lower $\hat{p}$ in the first period than in the second.

6.1.3 Monetary Policy: Limited Control and Incomplete Information

Suppose the central bank wants to minimize a welfare function

$$L = E\left[(\pi - \pi^*)^2\right],$$

where π^* is the optimal inflation rate. The central bank has no direct control over the price level. The inflation rate is given by

$$\pi = \rho Z + \eta,$$

where Z is the instrument to the disposal of the central bank, $\rho > 0$ is some parameter, and η is a random term with standard normal distribution.

(a) What is the optimal reaction of the central bank to shocks η?
(b) Suppose now that the central bank cannot observe η but only some variable $\Psi = \zeta + \eta$, where $\zeta \sim N\left(0, \sigma^2\right)$ and ζ and η are independent. What is the optimal response to observed shocks Ψ in this case?
(c) Suppose now that the central bank can observe η, but not ρ, which has a normal distribution with mean $\overline{\rho}$ and variance τ^2. What is the optimal response of the central bank to observed shocks η?

6.1.4 Monetary Policy: Interest Targeting Versus Monetary Targeting

Suppose the economy is described by linear IS and LM curves that are subject to disturbances

$$y = c - ai + \epsilon, \text{ and } m - p = hy - ki + \eta,$$

where a, h, and k are positive parameters and ϵ and η are independent mean zero shocks with finite variances. The central bank wants to stabilize output, but cannot observe y or the shocks ϵ and η. Other variables are observable. Assume for simplicity that p is fixed.

(a) What is the variance of y if the central bank fixes the interest rate at some level $\bar{i}$?
(b) What is the variance of y if the central bank fixes the money supply rate at some level $\overline{m}$?
(c) Under which conditions does interest targeting lead to a lower variance of output than monetary targeting?
(d) Describe the optimal monetary policy, when there are only IS shocks (the variance of η is zero). Does money or interest rate targeting lead to a lower variance of y?
(e) Describe the optimal monetary policy, when there are only LM shocks (the variance of ϵ is zero). Does money or interest rate targeting lead to a lower variance of y?
(f) Provide some intuitions on your results from (d) and (e).
(g) When there are only IS shocks, is there a policy that produces a variance of y that is lower than either money or interest rate targeting? If so, what policy minimizes the variance of y? If not, why not?

6.2 Solutions for Selected Exercises

6.2.1 Monetary Policy Under Uncertainty: Reputation

Cukierman and Meltzer (1986) consider that a monetary policy maker has a limited tenure for only two periods. The policy maker is randomly nominated from a pool of candidates, whose object function is as following:

$$W = E\left[b(p_1 - p_1^e) + cp_1 - \frac{ap_1^2}{2} + b(p_2 - p_2^e) + cp_2 - \frac{ap_2^2}{2}\right]$$

in which c is normally distributed over the candidates with mean $\overline{c}$ and variance $\sigma_c^2 > 0$. However, a and b are the same for all candidates.

The policy maker only has a limited control over inflation such that $p_t = \hat{p}_t + \epsilon_t$, $t \in \{1, 2\}$, in which $\hat{p}_t$ is the policy chosen by the policy maker with p_t^e being given and ϵ_t is a normally distributed random variable with mean zero and variance $\sigma_\epsilon^2 > 0$. The random variables, ϵ_1, ϵ_2 and c are independent on each other. The public cannot observe $\hat{p}_t$ or ϵ_t, but only p_t. The public cannot observe c, either.

The public's expectation on the second-period price level, p_2^e, is formed on the basis of observed first-period price level p_1 in a way such that

$$p_2^e = \alpha + \beta p_1.$$

(a) What is the policy maker's choice on $\hat{p}_2$? Compute the expected value of her second-period objective function in terms of p_2^e.

The policy maker's problem is to set optimal $\hat{p}_1$ and $\hat{p}_2$ in order to

$$\begin{aligned}
\max_{\hat{p}_1, \hat{p}_2} \quad & W = E\left[b(p_1 - p_1^e) + cp_1 - \frac{ap_1^2}{2} + b(p_2 - p_2^e) + cp_2 - \frac{ap_2^2}{2}\right], \\
s.t. \quad & p_1 = \hat{p}_1 + \epsilon_1, \\
& p_2 = \hat{p}_2 + \epsilon_2, \\
& p_2^e = \alpha + \beta p_1.
\end{aligned}$$

The first-order condition with respect to $\hat{p}_2$ gives

$$\begin{aligned}
\frac{\partial W}{\partial \hat{p}_2} &= E\left[b + c - a\hat{p}_2\right] = 0, \\
\hat{p}_2 &= \frac{b + c}{a}.
\end{aligned}$$

Apply it into the second-period objective function

$$\begin{aligned}
W_2 &= E\left[b(p_2 - p_2^e) + cp_2 - \frac{ap_2^2}{2}\right] \\
&= E\left[b\left(\frac{b+c}{a} + \epsilon_2 - p_2^e\right) + c\left(\frac{b+c}{a} + \epsilon_2\right) - \frac{a\left(\frac{b+c}{a} + \epsilon_2\right)^2}{2}\right] \\
&= \frac{(b+c)^2}{a} - bp_2^e - \frac{a}{2}E\left[\left(\frac{b+c}{a}\right)^2 + 2\frac{b+c}{a}\epsilon_2 + \epsilon_2^2\right] \\
&= \frac{(b+c)^2}{2a} - bp_2^e - \frac{a}{2}\sigma_\epsilon^2.
\end{aligned}$$

(b) What is the policy maker's choice on $\hat{p}_1$, with α and β being given and taking account of the impact of p_1 on p_2^e?

The first-order condition with respect to $\hat{p}_1$ gives

$$\begin{aligned}\frac{\partial W}{\partial \hat{p}_1} &= E\left[b + c - b\beta - a\hat{p}_1\right] = 0,\\ \hat{p}_1 &= \frac{(1-\beta)b + c}{a}.\end{aligned}$$

(c) Compute the proper value of β. Provide some intuitions on your result.

Since p_1 and p_2 are linear functions of normally distributed random variables c and ϵ, therefore p_1 and p_2 should also be normal. Since the public's expectation on the second-period price level, p_2^e, is formed on the basis of observed first-period price level p_1, then it should be that

$$\begin{aligned}p_2^e &= E\left[p_2|P_1\right]\\ &= E\left[p_2\right] + \frac{\text{cov}\left(p_1, p_2\right)}{\text{var}\left(p_1\right)}\left(p_1 - E\left[p_1\right]\right).\end{aligned}$$

Now we compute these first and second moments of p_1 and p_2. The first moments are

$$\begin{aligned}E\left[p_1\right] &= E\left[\hat{p}_1\right] + E\left[\epsilon_1\right]\\ &= \frac{(1-\beta)b + \overline{c}}{a},\end{aligned}$$

and

$$\begin{aligned}E\left[p_2\right] &= E\left[\hat{p}_2\right] + E\left[\epsilon_2\right]\\ &= \frac{b + \overline{c}}{a}.\end{aligned}$$

And the second moments are

$$\begin{aligned}\text{var}\left(p_1\right) &= \text{var}\left(\hat{p}_1 + \epsilon_1\right)\\ &= \text{var}\left[\frac{(1-\beta)b + c}{a} + \epsilon_1\right]\\ &= \frac{\sigma_c^2}{a^2} + \sigma_\epsilon^2,\end{aligned}$$

and

$$\begin{aligned}
\text{cov}\,(p_1, p_2) &= \text{cov}\left[\frac{(1-\beta)b+c}{a}+\epsilon_1, \frac{b+c}{a}+\epsilon_2\right] \\
&= \text{var}\left[\frac{c}{a}\right] \\
&= \frac{\sigma_c^2}{a^2}.
\end{aligned}$$

Therefore the public's expectation on the second-period price level is

$$\begin{aligned}
p_2^e &= E\left[p_2\right] + \frac{\text{cov}\,(p_1, p_2)}{\text{var}\,(p_1)}\left(p_1 - E\left[p_1\right]\right) \\
&= \frac{b+\bar{c}}{a} + \frac{\frac{\sigma_c^2}{a^2}}{\frac{\sigma_c^2}{a^2}+\sigma_\epsilon^2}\left[p_1 - \frac{(1-\beta)b+\bar{c}}{a}\right] \\
&\equiv \alpha + \beta p_1,
\end{aligned}$$

therefore the equivalence in parameters implies that

$$\beta = \frac{\sigma_c^2}{\sigma_c^2 + a^2\sigma_\epsilon^2} \in (0, 1].$$

The public wants to establish its expectation p_2^e, given its observation p_1. In order to do so, the public would like to know for sure what the policy maker's taste, c, is. The problem is that the actual inflation in period 1 does not only depend on true c, but also on the unobservable ϵ_1. Now if the public sees a p_1 greater than its expectation, $\frac{(1-\beta)b+\bar{c}}{a}$, it knows that this could be likely due to a policy maker with a higher-than-expected c—and if this is the case, the public should revise upward its estimate p_2^e from its unconditional mean $\frac{b+\bar{c}}{a}$. However, the reason why p_1 is greater than expected could also be due to a positive shock ϵ_1—and if this is the case, the public shouldn't revise its estimate. The value β just reflects such trade off: If σ_c^2 is very large relative to σ_ϵ^2, β would be close to 1—meaning that the public would attribute most of the above-average realization of p_1 to a policy maker whose c is higher than average, and raise its expectation p_2^e accordingly.

(d) Provide some intuitions on why the policy maker chooses a lower $\hat{p}$ in the first period than in the second.

The policy maker knows that her choice of $\hat{p}_1$ will affect the public's expectation on second-period inflation, p_2^e. When p_1 turns out to be high, the public attributes some of this to a policy maker with a high c and accordingly raises p_2^e, which reduces the policy maker's welfare W_2. Therefore the policy maker chooses a lower

$\hat{p}_1$ and tries to establish a good reputation as someone with a low c in order to keep p_2^e down. In the second period, however, there is no future any more. Therefore there is no need to worry about the effects that current inflation has on the future expected inflation.

6.2.2 Monetary Policy: Limited Control and Incomplete Information

Suppose the central bank wants to minimize a welfare function

$$L = E\left[(\pi - \pi^*)^2\right],$$

where π^* is the optimal inflation rate. The central bank has no direct control over the price level. The inflation rate is given by

$$\pi = \rho Z + \eta,$$

where Z is the instrument to the disposal of the central bank, $\rho > 0$ is some parameter, and η is a random term with standard normal distribution.

(a) What is the optimal reaction of the central bank to shocks η?

The optimal reaction of the central bank to shocks η is to solve the optimization problem

$$\begin{aligned} \min_Z \ & L = E\left[(\pi - \pi^*)^2\right] \\ s.t. \ & \pi = \rho Z + \eta. \end{aligned}$$

Since η is known as the shock comes, then the problem above turns out to be

$$\min_Z \ L = (\rho Z + \eta - \pi^*)^2.$$

The first-order condition gives

$$\begin{aligned} \frac{\partial L}{\partial Z} &= 2\rho(\rho Z + \eta - \pi^*) = 0, \\ Z &= \frac{\pi^* - \eta}{\rho}. \end{aligned}$$

(b) Suppose now that the central bank cannot observe η but only some variable $\Psi = \zeta + \eta$, where $\zeta \sim N\left(0, \sigma^2\right)$ and ζ and η are independent. What is the optimal response to observed shocks Ψ in this case?

Now the observable signal Ψ depends on two normally distributed random variables

$$\begin{bmatrix} \zeta \\ \eta \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma^2 & 0 \\ 0 & 1 \end{bmatrix}\right),$$

and by $\Psi = \zeta + \eta$ it is easy to see that $\Psi \sim (0, 1 + \sigma^2)$.

Then the optimal reaction of the central bank to the observed signal Ψ is to solve the optimization problem

$$\begin{aligned}\min_Z L &= E\left[(\rho Z + \eta - \pi^*)^2 \middle| \Psi\right] \\ &= \rho^2 Z^2 + E\left[(\eta|\Psi)^2\right] + \pi^{*2} + 2\rho Z E\left[\eta|\Psi\right] - 2\rho Z \pi^* - 2\pi^* E\left[\eta|\Psi\right].\end{aligned}$$

The first-order condition gives

$$\begin{aligned}\frac{\partial L}{\partial Z} &= 2\rho^2 Z + 2\rho E\left[\eta|\Psi\right] - 2\rho\pi^* = 0, \\ Z &= \frac{\pi^* - E\left[\eta|\Psi\right]}{\rho}.\end{aligned}$$

Since $\Psi = \zeta + \eta$, then $\operatorname{cov}(\Psi, \zeta) = \operatorname{var}(\zeta) = \sigma^2$, and $\operatorname{cov}(\Psi, \eta) = \operatorname{var}(\eta) = 1$. Therefore

$$\begin{bmatrix} \zeta \\ \eta \\ \Psi \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma^2 & 0 & \sigma^2 \\ 0 & 1 & 1 \\ \sigma^2 & 1 & 1 + \sigma^2 \end{bmatrix}\right),$$

the condition expectations are given by

$$E\left[\begin{bmatrix} \zeta \\ \eta \end{bmatrix} \middle| \Psi\right] = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \sigma^2 \\ 1 \end{bmatrix} \frac{1}{1 + \sigma^2} \Psi.$$

Insert it into the expression for Z and get

$$Z = \frac{\pi^* - \frac{1}{1+\sigma^2}\Psi}{\rho}.$$

(c) Suppose now that the central bank can observe η, but not ρ, which has a normal distribution with mean $\overline{\rho}$ and variance τ^2. What is the optimal response of the central bank to observed shocks η?

Now the optimal reaction of the central bank to the observed shock η is to solve the optimization problem

$$\begin{aligned}\min_Z L &= E\left[(\rho Z + \eta - \pi^*)^2\right] \\ &= Z^2 E\left[\rho^2\right] + \eta^2 + \pi^{*2} + 2Z\eta E[\rho] - 2Z\pi^* E[\rho] - 2\eta\pi^* \\ &= Z^2\left(\tau^2 + \overline{\rho}^2\right) + \eta^2 + \pi^{*2} + 2Z\eta\overline{\rho} - 2Z\pi^*\overline{\rho} - 2\eta\pi^*.\end{aligned}$$

The first-order condition gives

$$\begin{aligned}\frac{\partial L}{\partial Z} &= 2Z\left(\tau^2 + \overline{\rho}^2\right) + 2\eta\overline{\rho} - 2\pi^*\overline{\rho} = 0, \\ Z &= \frac{\pi^*\overline{\rho} - \eta\overline{\rho}}{\tau^2 + \overline{\rho}^2}.\end{aligned}$$

6.2.3 Monetary Policy: Interest Targeting Versus Monetary Targeting

Suppose the economy is described by linear IS and LM curves that are subject to disturbances

$$y = c - ai + \epsilon, \text{ and } m - p = hy - ki + \eta,$$

where a, h, and k are positive parameters and ϵ and η are independent mean zero shocks with finite variances. The central bank wants to stabilize output, but cannot observe y or the shocks ϵ and η. Other variables are observable. Assume for simplicity that p is fixed.

(a) What is the variance of y if the central bank fixes the interest rate at some level $\overline{i}$?

If the central bank fixes the interest rate at some level $\overline{i}$, then the LM curve is irrelevant. Equilibrium output is determined by the IS curve under the fixed nominal interest rate, therefore

$$\text{var}[y] = \text{var}[c - a\overline{i} + \epsilon] = \text{var}[\epsilon] = \sigma_\epsilon^2. \tag{6.1}$$

(b) What is the variance of y if the central bank fixes the money supply rate at some level $\overline{m}$?

When the policy maker fixes m, the equilibrium level of output is determined by the intersection of the two curves. Plug IS curve $y = c - ai + \epsilon$ into LM curve

$m - p = hy - ki + \eta$ to eliminate i, one can get

$$\overline{m} - p = hy - k\frac{c + \epsilon - y}{a} + \eta,$$

$$y = \frac{\overline{m} - p + k\frac{c+\epsilon}{a} - \eta}{h + \frac{k}{a}},$$

the variance can be calculated as

$$\text{var}[y] = \left(\frac{\frac{k}{a}}{h + \frac{k}{a}}\right)^2 \text{var}[\epsilon] + \left(\frac{1}{h + \frac{k}{a}}\right)^2 \text{var}[\eta] = \frac{k^2\sigma_\epsilon^2 + a^2\sigma_\eta^2}{(ha + k)^2}. \quad (6.2)$$

(c) Under which conditions does interest targeting lead to a lower variance of output than monetary targeting?

Compare var[y] obtained in (b) and (c). Interest targeting leads to a lower variance of output than monetary targeting if

$$\sigma_\epsilon^2 < \frac{k^2\sigma_\epsilon^2 + a^2\sigma_\eta^2}{(ha + k)^2},$$

$$\left(h^2a^2 + 2hka + k^2\right)\sigma_\epsilon^2 < k^2\sigma_\epsilon^2 + a^2\sigma_\eta^2,$$

$$\left(h^2a^2 + 2hka\right)\sigma_\epsilon^2 < a^2\sigma_\eta^2,$$

$$\frac{\sigma_\epsilon^2}{\sigma_\eta^2} < \frac{a}{h^2a + 2hk}.$$

(d) Describe the optimal monetary policy, when there are only IS shocks (the variance of η is zero). Does money or interest rate targeting lead to a lower variance of y?

When there are only IS shocks (the variance of η is zero), by Eq. (6.1) interest rate targeting leads to

$$\text{var}[y]|_{i=\bar{i}} = \sigma_\epsilon^2,$$

and by Eq. (6.2) money targeting leads to

$$\text{var}[y]|_{m=\overline{m}} = \frac{k^2\sigma_\epsilon^2}{(ha + k)^2} < \text{var}[y]|_{i=\bar{i}},$$

money targeting leads to a lower variance of y.

(e) Describe the optimal monetary policy, when there are only LM shocks (the variance of ϵ is zero). Does money or interest rate targeting lead to a lower variance of y?

When there are only LM shocks (the variance of ϵ is zero), by Eq. (6.1) interest rate targeting leads to

$$\text{var}[y]|_{i=\bar{i}} = 0,$$

and by Eq. (6.2) money targeting leads to

$$\text{var}[y]|_{m=\overline{m}} = \frac{a^2\sigma_\eta^2}{(ha+k)^2} > \text{var}[y]|_{i=\bar{i}},$$

interest rate targeting leads to a lower variance of y.

(f) Provide some intuitions on your results from (d) and (e).

Consider the situation in (d) in which there are only IS shocks. If the policy maker targets the nominal interest rate, the equilibrium output changes by the full extent of the shift in the IS curve caused by a shock to it. Now consider the case in which the policy maker targets the money supply. A positive IS shock shifts the IS curve to the right. With m fixed, as y rises to equate planned and actual expenditure, i rises as well in order for the money market to remain in equilibrium. This rise in i reduced planned expenditure and thus mitigates some of the positive shock. Therefore y does not end rising as much. The same idea is true in the opposite direction. A negative IS shock shifts IS to the left. If the policy maker targets m, i will fall along with y in order to keep the money market in equilibrium. This fall in i raises planned expenditure and helps to offset the original negative shock to planned expenditure. Thus y does not fall as much as if the policy maker had kept i constant.

Consider the situation in (e) in which there are only LM shocks. If the policy maker targets the nominal money supply, the LM shocks causes the LM curve to shift around and the equilibrium output in the economy is determined by the intersection of that shifting LM curve with the stable IS curve. If the policy maker targets the nominal interest rate, it ensures that i remains constant in the face of any LM shock. Since i is not allowed to change, planned expenditure does not change and thus the level of output that equates planned and actual expenditure does not change in the face of an LM shock.

(g) When there are only IS shocks, is there a policy that produces a variance of y that is lower than either money or interest rate targeting? If so, what policy minimizes the variance of y? If not, why not?

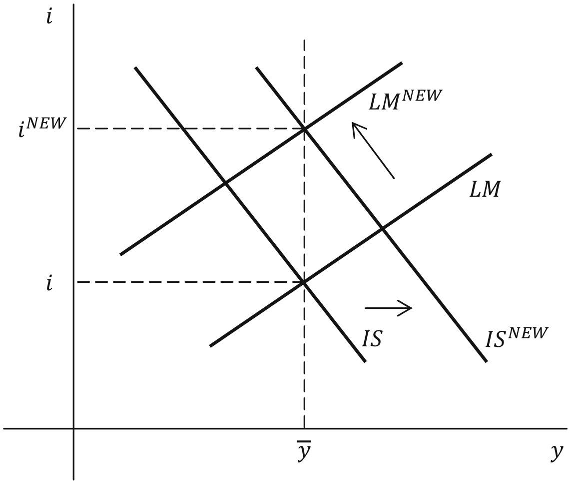

Fig. 6.1 Output stabilization under IS shocks

If there are only IS shocks, it is possible to keep y constant at some target level $\overline{y}$. By rearranging the LM curve with y set to $\overline{y}$, the nominal money supply must be such that

$$m = p + h\overline{y} - ki. \tag{6.3}$$

The policy maker knows that the fixed p has picked $\overline{y}$ and is able to observe i. Thus when i changes—and since there are only IS shocks, we know this must be due to a shift of the IS curve—the policy maker must change m accordingly. As i rises, for example, the policy maker must reduce m.

In Fig. 6.1, as i rises due to the rightward shift of IS curve, the policy maker reduces m which shifts up the LM curve and increases i more. The policy maker can stop reducing m when m and i are such that Eq. (6.3) is satisfied. At this point, the new LM curve would intersect the new IS curve right at the target level of $\overline{y}$.

Reference

Cukierman, A., & Meltzer, A. H. (1986). A theory of ambiguity, credibility and inflation under discretion and asymmetric information. *Econometrica, 54*, 1099–1128.

Liquidity Trap: Limits for Monetary Policy at the Effective Lower Bound

7

7.1 Exercises

7.1.1 Short Review Questions

(a) Why is there a zero lower bound (ZLB) for monetary policy? Why may *effective* zero lower bound be different from zero? What characterizes effective lower bound? Provide reasons why effective zero lower bound may be (1) positive, (2) negative.
(b) Conventional wisdom insists that depositors would hoard cash instead of increasing spending once interest rate falls below zero. However, this did not happen for those countries implementing negative interest rate in the 2010s. What prevent depositors from hoarding cash?
(c) What is liquidity trap? Why is there liquidity trap at the ZLB under Taylor rules?
(d) What is forward guidance? Under which condition(s) is forward guidance policy able to increase inflation expectation? Why is there a commitment problem for forward guidance policy?

7.2 Solutions for Selected Exercises

No solution provided.

J. Cao, G. Illing, *Instructor's Manual for* Money: Theory and Practice, Springer Texts in Business and Economics,
https://doi.org/10.1007/978-3-030-23618-2_7

Part III

Unconventional Monetary Policy, Financial Frictions and Crises

8 Monetary Policy in Practice

8.1 Exercises

8.1.1 Short Review Questions

(a) Why is it that a decrease in the discount rate does not normally lead to an increase in borrowed reserves? Use the supply and demand analysis of the market for reserves to explain.

(b) Suppose that a central bank has just lowered the discount rate. Does this signal that the central bank is moving to a more expansionary monetary policy? Why or why not?

(c) Using the supply and demand analysis of the market for reserves, indicate what happens to the federal funds rate, borrowed reserves, and non-borrowed reserves if

1. The economy is unexpectedly strong, leading to an increase in the amount of bank deposits;
2. Banks expect an unusually large increase in withdrawals from deposit accounts in the future;
3. The Fed raises the target federal funds rate;
4. The Fed raises the interest rate on reserves above the current equilibrium federal funds rate;
5. The Fed reduces reserve requirements;
6. The Fed reduces reserve requirements, and then conducts an open market sale of securities.

8.2 Solutions for Selected Exercises

No solution provided.

J. Cao, G. Illing, *Instructor's Manual for* Money: Theory and Practice,
Springer Texts in Business and Economics,
https://doi.org/10.1007/978-3-030-23618-2_8

Financial Frictions and Monetary Policy

9

9.1 Exercises

9.1.1 Short Review Questions

(a) What is Value-at-Risk (VaR)? How is banks' leverage ratio related to VaR? What happens to banks' leverage ratio immediately after a rise in return to assets (say, as a result of boom in macroeconomy)? What's banks' response to the change in leverage under VaR constraint? What's the impact on new equilibrium asset price? Why is leverage cycle "procyclical"?
(b) How does "financial accelerator" emerge as a result of financial frictions? Explain in words how financial accelerator amplifies shocks from macroeconomy.
(c) How do principal-agent problems affect efficiency in banking? Give a few examples of the social costs due to principal-agent problems in banking and their impacts on macro economy.

9.1.2 Financial Intermediation, Bank Capital, and Credit Supply

Based on Holmström and Tirole (1997) consider an economy in which there are many risky projects to be financed. Each project needs 1 initial input and yields verifiable gross return y if it's successful, 0 if it's unsuccessful. There are two types of projects

- Good projects (type G) with probability of success being p_G;
- Bad projects (type B) with probability of success being $p_B < P_G$, but a bad project gives private benefit $B > 0$ to the entrepreneur. Assume that $p_G y > R >$

J. Cao, G. Illing, *Instructor's Manual for* Money: Theory and Practice,
Springer Texts in Business and Economics,
https://doi.org/10.1007/978-3-030-23618-2_9

$p_B + B$ ($R > 1$ is the risk-free rate, defined by the gross return of government bonds), i.e., bad projects are not socially desirable.

There are many risk-neutral entrepreneurs in the economy, each owns wealth $0 < A < 1$ which is publicly observable. Each A is a random variable, uniformly distributed over (0, 1). Entrepreneurs are the only agents in the economy who have the expertise to run either type of the projects, but their choices of projects are not publicly observable.

There are many risk-neutral investors in the economy who are endowed with money. They can invest the money on government bonds which yield safe gross return $R > 1$. They can also lend to entrepreneurs or banks.

There are intermediaries in this economy called banks, who have a special monitoring technology: after spending a non-observable amount of resource C, entrepreneurs' private benefit falls to $b < B$ if they operate bad projects. Banks start with initial wealth L_B^B, called bank capital which is owned by shareholders. They borrow from investors and lend to entrepreneurs. Shareholders of banks demand gross return-on-equity at least as high as $\beta >> R$.

(a) Separation of market and the role of financial intermediation
  1. If entrepreneurs borrow directly from investors and investors have the market power to charge highest lending rate as possible, what is the highest lending rate investors can actually charge? What is the highest lending rate investors can actually charge, should entrepreneurs' choices on projects be observable? Interpret the difference between these two rates;
  2. Show that there exists a threshold $\overline{A}$ such that all entrepreneurs whose initial wealth $A > \overline{A}$ are able to borrow directly from investors;
  3. If entrepreneurs borrow from banks, given that they are monitored by banks, what is the highest lending rate banks can actually charge? To make sure that banks do exert the effort to monitor, how much profit is needed to be retained by the banks? How much capital do banks need to hold? Explain, in words, why do banks need to hold capital;
  4. Show that there exists a threshold $\underline{A}$ such that all entrepreneurs whose initial wealth $\underline{A} \leq A \leq \overline{A}$ are able to borrow from banks, and entrepreneurs whose initial wealth $A \leq \underline{A}$ are not able to get any funding.

(b) Credit supply
  1. Suppose banks' shareholders are willing to accept a lower return on equity. What is the impact on banks' aggregate credit supply to entrepreneurs?
  2. Suppose the good projects' probability of success p_G falls, and the other assumptions remain unchanged. What is the impact on banks' aggregate credit supply to entrepreneurs?
  3. Suppose banks become more efficient in monitoring: monitoring cost C falls, and entrepreneurs' private benefit b—if they operate bad projects and get monitored—also falls. What is the impact on banks' aggregate credit supply to entrepreneurs?

4. It is known that central bank is able to shift the risk-free rate, or government bond rate R through monetary policy implementation. If central bank decides to loosen monetary policy and cut R, what is the impact on aggregate funding (funding through both direct borrowing and bank lending) in the economy?

9.1.3 Value-at-Risk and Leverage Cycle

Consider an economy that extends to 2 periods: investors invest in risky projects at $t = 0$, and will get paid at $t = 1$. All information is available to public.

There are a fixed number S of ex ante identical risky projects. Each needs 1 unit of initial investment to start at $t = 0$, and at $t = 1$ generates a random gross payoff R that is uniformly distributed over $\left[\overline{R} - z, \overline{R} + z\right]$ with $\overline{R} > 1$ and $z > 0$.

Entrepreneurs who run the projects issue securities to raise funding. Securities are sold at $t = 0$ to investors at price P which is determined by the market. Suppose that funding is scarce so that investors get all the rents, should a project be successful.

There are many risk averse investors, call them passive investors, each gets e endowment at $t = 0$. To spend their endowments, they may buy y_P securities and lend the rest to active investors at gross interest rate equal to 1. A passive investor gets utility from her consumption c at $t = 1$, which contains repaid deposit and return from securities. At $t = 0$ her expected utility is $u(c) = E[c] - \frac{1}{2\tau}\text{var}[c]$ in which $\tau > 0$ is a constant and var$[c]$ is the variance in consumption.

(a) Passive investor's demand for security
   1. Write down passive investors' decision problem at $t = 0$ and derive passive investor's demand for security;
   2. Delineate passive investor's demand for security in $P - y$ space. How does such demand change with τ? Interpret.

There are many risk neutral investors, call them active investors or banks, each gets e endowment at $t = 0$. They may buy y_A securities, using their endowments and borrowing from passive investors at gross interest rate equal to 1. Active investors are subject to Value-at-Risk (VaR) constraint, such that e should be sufficient to cover the largest possible loss.

(b) Active investor's demand for security
   1. Specify active investor's VaR constraint;
   2. Write down active investor's decision problem at $t = 0$ and derive active investor's demand for security;
   3. Delineate active investor's demand for security in the same $P - y$ space, and show how equilibrium security price P is determined.

(c) Asset price and leverage in the bust

Suppose there is a shock to security return at the intermediate date, call it $t = 0.5$, so that both types of investors have the chance to exchange in the security market and adjust their balance sheets: It turns out that the distribution of security return is $\left[\overline{R'} - z, \overline{R'} + z\right]$ with $\overline{R'} < \overline{R}$.

1. Using $P - y$ curves, show the impact on both types' investors demand for securities and the new equilibrium security price;
2. How does the shock to security return affect active investors' balance sheet? How do they adjust the balance sheet to meet VaR constraint? What's the consequence to equilibrium asset price? Why is leverage cycle "procyclical"?

9.2 Solutions for Selected Exercises

9.2.1 Financial Intermediation, Bank Capital, and Credit Supply

Based on Holmström and Tirole (1997) consider an economy in which there are many risky projects to be financed. Each project needs 1 initial input and yields verifiable gross return y if it's successful, 0 if it's unsuccessful. There are two types of projects

- *Good projects (type G) with probability of success being p_G;*
- *Bad projects (type B) with probability of success being $p_B < P_G$, but a bad project gives private benefit $B > 0$ to the entrepreneur. Assume that $p_G y > R > p_B + B$ ($R > 1$ is the risk-free rate, defined by the gross return of government bonds), i.e., bad projects are not socially desirable.*

There are many risk-neutral entrepreneurs in the economy, each owns wealth $0 < A < 1$ which is publicly observable. Each A is a random variable, uniformly distributed over $(0, 1)$. Entrepreneurs are the only agents in the economy who have the expertise to run either type of the projects, but their choices of projects are not publicly observable.

There are many risk-neutral investors in the economy who are endowed with money. They can invest the money on government bonds which yield safe gross return $R > 1$. They can also lend to entrepreneurs or banks.

There are intermediaries in this economy called banks, who have a special monitoring technology: after spending a non-observable amount of resource C, entrepreneurs' private benefit falls to $b < B$ if they operate bad projects. Banks start with initial wealth L_B^B, called bank capital which is owned by shareholders. They borrow from investors and lend to entrepreneurs. Shareholders of banks demand gross return-on-equity at least as high as $\beta >> R$.

(a) *Separation of market and the role of financial intermediation*

1. *If entrepreneurs borrow directly from investors and investors have the market power to charge highest lending rate as possible, what is the highest lending rate investors can actually charge? What is the highest lending rate investors can actually charge, should entrepreneurs' choices on projects be observable? Interpret the difference between these two rates;*

   The first best solution takes place if investors know the entrepreneurs' choices and only lend to good projects, getting the full rent $r_D^{I,FB} = y$. However, when entrepreneurs' choices are not observable, they always have the incentive to choose bad projects and pocket private benefits. The moral hazard problem forces investors to leave some information rents to entrepreneurs, inducing them to choose good projects. With the projects' payoff structure (Fig. 9.1), the incentive compatibility constraint $(IC - D)$ requires

$$p_G \left(y - r_D^I\right) \geq p_B \left(y - r_D^I\right) + B,$$

   that is,

$$r_D^I \leq y - \frac{B}{p_G - p_B} = y - \frac{B}{\Delta p} < r_D^{I,FB}.$$

   At the same time, r_D^I needs to meet investors' participation constraint $(PC - I)$, or, an investor lending L_D^I should be better off than investing on safe assets,

$$p_G r_D^I \geq R L_D^I, \text{ or, } L_D^I \leq \frac{p_G r_D^I}{R}.$$

   The highest lending rate investors can charge here is $r_D^I = y - \frac{B}{\Delta p}$, the difference from $r_D^{I,FB}$ is the information rent component $\frac{B}{\Delta p}$, call it r_D^E.

2. *Show that there exists a threshold $\overline{A}$ such that all entrepreneurs whose initial wealth $A > \overline{A}$ are able to borrow directly from investors;*

   Investors' participation constraint $(PC - I_D)$ implies the maximal lending they can offer is

$$L_D^I \leq \frac{p_G r_D^I}{R} = \frac{p_G}{R}\left(y - \frac{B}{\Delta p}\right).$$

   To start a project, an entrepreneur needs to hold at least A such that

$$A + L_D^I \geq I, \text{ or, } A \geq I - \frac{p_G}{R}\left(y - \frac{B}{\Delta p}\right) \equiv \overline{A}(R).$$

   Or, only well-capitalized entrepreneurs obtain direct funding.

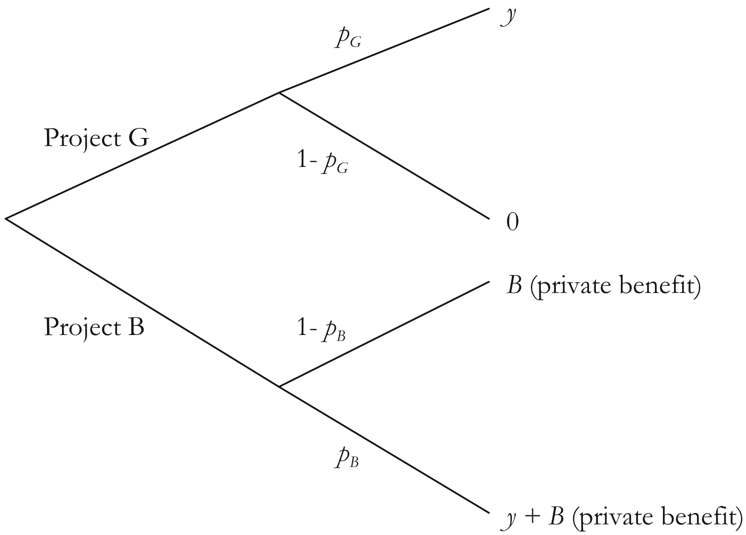

Fig. 9.1 Projects' payoff structure

Assets	Liabilities
Loans to the entrepreneurs L_B *Return y if project is successful*	Bank capital L_B^B *Get the share* r_B^B *if loan performs*
	Deposits L_B^I *Get the share* r_B^I *if loan performs*

Fig. 9.2 Banks' balance sheet

3. *If entrepreneurs borrow from banks, given that they are monitored by banks, what is the highest lending rate banks can actually charge? To make sure that banks do exert the effort to monitor, how much profit is needed to be retained by the banks? How much capital do banks need to hold? Explain, in words, why do banks need to hold capital;*

   Banks, as financial intermediaries, are running the balance sheet shown as Fig. 9.2.

   With banks ensuring entrepreneurs to choose good projects, projects' return will be split among banks (r_B^B), depositing investors (r_B^I) and entrepreneurs (r_B^E), $y = r_B^B + r_B^I + r_B^E$. Entrepreneurs' incentive compatibility constraint $(IC - E)$ requires that

$$p_G r_B^E \geq p_B r_B^E + b, \text{ or, } r_B^B + r_B^I \leq y - \frac{b}{\Delta p}.$$

Banks are subject to moral hazard, too. To ensure banks to monitor with cost C, incentive compatibility constraint $(IC - B)$ requires that

$$p_G r_B^B - C \geq p_B r_B^B, \text{ or, } r_B^B \geq \frac{C}{\Delta p}.$$

Participation constraints must hold for both banks $(PC - B)$ and investors $(PC - I_B)$, i.e., banks' ROE must be high enough to maintain shareholders, and investors should be better off than investing in safe assets

$$p_G r_B^B \geq \beta L_B^B, \text{ or, } L_B^B(\beta) \leq \frac{p_G r_B^B}{\beta} = \frac{p_G C}{\beta \Delta p},$$

and

$$p_G r_B^I \geq R L_B^I, \text{ or, } L_B^I(R) \leq \frac{p_G r_B^I}{R}.$$

Again, banks need to skin-in-the-game, making shirking costlier and reducing moral hazard in monitoring.

4. *Show that there exists a threshold $\underline{A}$ such that all entrepreneurs whose initial wealth $\underline{A} \leq A \leq \overline{A}$ are able to borrow from banks, and entrepreneurs whose initial wealth $A \leq \underline{A}$ are not able to get any funding.*
Combining $(IC - E)$, $(IC - B)$, and $(PC - I_B)$ to get

$$L_B^I \leq \frac{p_G}{R}\left(y - \frac{b + C}{\Delta p}\right).$$

An entrepreneur can borrow from a bank only if she holds at least A such that

$$A + L_B^I + L_B^B \geq I, \text{ or, } A \geq I - L_B^B(\beta) - \frac{p_G}{R}\left(y - \frac{b + C}{\Delta p}\right) \equiv \underline{A}(\beta, R) < \overline{A}(R)$$

if C is small enough. Worst capitalized entrepreneurs with $A \in \left(0, \underline{A}\right)$ will not get any funding. The market is segmented as shown in Figs. 9.3 and 9.4. Note that entrepreneurs with access to direct finance won't borrow from banks, as $r_B^E = y - r_B^B - r_B^I = \frac{b}{\Delta p} < \frac{B}{\Delta p} = r_D^E$.

(b) *Credit supply*

1. *Suppose banks' shareholders are willing to accept a lower return on equity. What is the impact on banks' aggregate credit supply to entrepreneurs?*
Lower β affects $\underline{A}(\beta, R)$. As $\frac{\partial \underline{A}(\beta,R)}{\partial \beta} = -\frac{\partial L_B^B(\beta)}{\partial \beta} = \frac{p_G C}{\beta^2 \Delta p} > 0$, $\underline{A}(\beta, R)$ falls, allowing more under-capitalized entrepreneurs to borrow from banks.

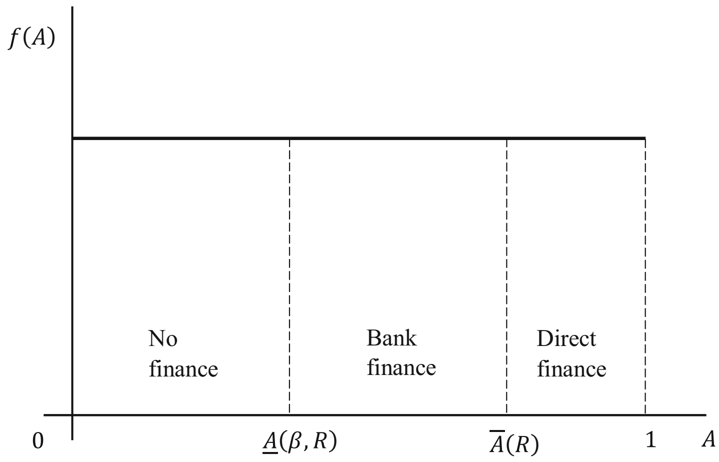

Fig. 9.3 Segmentation of borrowers

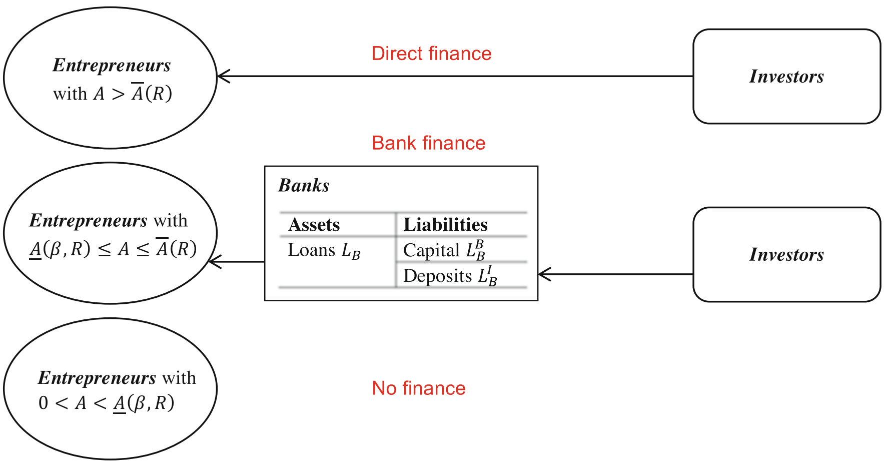

Fig. 9.4 Market segmentation

2. *Suppose the good projects' probability of success p_G falls, and the other assumptions remain unchanged. What is the impact on banks' aggregate credit supply to entrepreneurs?*
Lower p_G affects both $\underline{A}(\beta, R)$ and $\overline{A}(R)$. As $\frac{\partial \underline{A}(\beta,R)}{\partial p_G} = -\frac{C}{\beta\Delta p} - \frac{1}{R}\left(y - \frac{b+C}{\Delta p}\right) < 0$, $\underline{A}(\beta, R)$ rises, allowing less under-capitalized entrepreneurs to borrow from banks. At the same time, $\frac{\partial \overline{A}(R)}{\partial p_G} = -\frac{1}{R}\left(y - \frac{B}{\Delta p}\right) < 0$, $\overline{A}(\beta, R)$ rises, allowing less well-capitalized entrepreneurs to borrow directly. Overall, aggregate credit supply falls.

3. *Suppose banks become more efficient in monitoring: monitoring cost C falls, and entrepreneurs' private benefit b—if they operate bad projects and get monitored—also falls. What is the impact on banks' aggregate credit supply to entrepreneurs?*
   Lower C and b affects $\underline{A}(\beta, R)$. As $\frac{\partial \underline{A}(\beta,R)}{\partial b} = \frac{p_G}{R\Delta p} > 0$, $\frac{\partial \underline{A}(\beta,R)}{\partial C} = -\frac{p_G}{\beta\Delta p} + \frac{p_G}{R\Delta p} > 0$ as $\beta >> R$, $\underline{A}(\beta, R)$ falls, allowing more under-capitalized entrepreneurs to borrow from banks and increasing aggregate credit supply.
4. *It is known that central bank is able to shift the risk-free rate, or government bond rate R through monetary policy implementation. If central bank decides to loosen monetary policy and cut R, what is the impact on aggregate funding (funding through both direct borrowing and bank lending) in the economy?*
   As $\frac{\partial \underline{A}(\beta,R)}{\partial R} = \frac{p_G}{R^2}\left(y - \frac{b+C}{\Delta p}\right) > 0$ and $\frac{\partial \overline{A}(R)}{\partial p_G} = \frac{p_G}{R^2}\left(y - \frac{B}{\Delta p}\right) > 0$, both $\underline{A}(\beta, R)$ and $\overline{A}(R)$ fall with R, increasing aggregate credit supply.

9.2.2 Value-at-Risk and Leverage Cycle

Consider an economy that extends to 2 periods: investors invest in risky projects at $t = 0$, and will get paid at $t = 1$. All information is available to public.

There are a fixed number S of ex ante identical risky projects. Each needs 1 unit of initial investment to start at $t = 0$, and at $t = 1$ generates a random gross payoff R that is uniformly distributed over $\left[\overline{R} - z, \overline{R} + z\right]$ with $\overline{R} > 1$ and $z > 0$.

Entrepreneurs who run the projects issue securities to raise funding. Securities are sold at $t = 0$ to investors at price P which is determined by the market. Suppose that funding is scarce so that investors get all the rents, should a project be successful.

There are many risk averse investors, call them passive investors, each gets e endowment at $t = 0$. To spend their endowments, they may buy y_P securities and lend the rest to active investors at gross interest rate equal to 1. A passive investor gets utility from her consumption c at $t = 1$, which contains repaid deposit and return from securities. At $t = 0$ her expected utility is $u(c) = E[c] - \frac{1}{2\tau}\text{var}[c]$ in which $\tau > 0$ is a constant and var$[c]$ *is the variance in consumption.*

(a) *Passive investor's demand for security*
1. *Write down passive investors' decision problem at $t = 0$ and derive passive investor's demand for security;*
   At $t = 0$ a passive investor maximizes her expected utility by

$$\max_{y_P} \ u(c) = E[c] - \frac{1}{2\tau}\text{var}[c],$$
$$s.t. \ \ c = Ry_P + e - Py_P.$$

Applying the budget constraint in the utility function, $u(c) = E[Ry_P + e - Py_P] - \frac{1}{2\tau}\text{var}[Ry_P + e - Py_P] = \overline{R}y_P + e - Py_P - \frac{1}{2\tau}\frac{z^2}{3}y_P^2$, $\frac{\partial u}{\partial y_P} = \overline{R} - P - \frac{1}{\tau}\frac{z^2}{3}y_P = 0$, or,

$$y_P(P) = \begin{cases} \frac{3\tau(\overline{R}-P)}{z^2} & \text{if } \overline{R} \geq P \\ 0 & \text{otherwise} \end{cases}.$$

2. *Delineate passive investor's demand for security in $P - y$ space. How does such demand change with τ? Interpret.*
Parameter τ affects passive investors' risk aversion. The higher τ, the lower impact of income volatility on their utility, the lower incentive for insurance (saving).

There are many risk neutral investors, call them active investors or banks, each gets e endowment at $t = 0$. They may buy y_A securities, using their endowments and borrowing from passive investors at gross interest rate equal to 1. Active investors are subject to Value-at-Risk (VaR) constraint, such that e should be sufficient to cover the largest possible loss.

(b) *Active investor's demand for security*

1. *Specify active investor's VaR constraint;*
Active investor's VaR constraint is that they should be sufficiently capitalized even in the worst case, i.e., $e \geq VaR$, or $e \geq Py_A - \left(\overline{R} - z\right) y_A = VaR$.
2. *Write down active investor's decision problem at $t = 0$ and derive active investor's demand for security;*
At $t = 0$ an active investor maximizes her expected utility by

$$\max_{y_A} \; E\left[Ry_A - (Py_A - e)\right]$$
$$s.t. \; e \geq Py_A - \left(\overline{R} - z\right) y_A.$$

The active investor maximizes her y_A until VaR constraint is binding, i.e., $y_A(P) = \frac{e}{P - \overline{R} + z}$.
3. *Delineate active investor's demand for security in the same $P - y$ space, and show how equilibrium security price P is determined.*
See Fig. 9.5.

(c) *Asset price and leverage in the bust*

Suppose there is a shock to security return at the intermediate date, call it $t = 0.5$, so that both types of investors have the chance to exchange in the security market and adjust their balance sheets: It turns out that the distribution of security return is $\left[\overline{R'} - z, \overline{R'} + z\right]$ with $\overline{R'} < \overline{R}$.

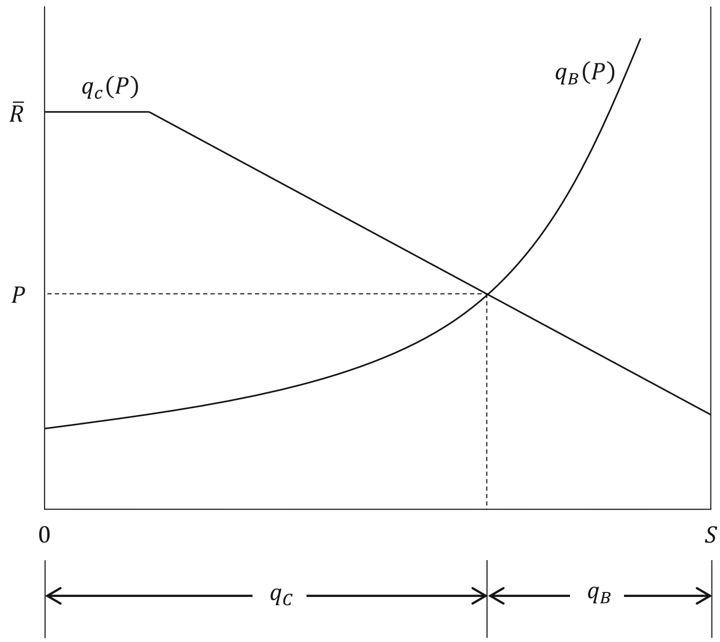

Fig. 9.5 Market equilibrium

1. *Using $P - y$ curves, show the impact on both types' investors demand for securities and the new equilibrium security price;*
   See Fig. 9.6.
2. *How does the shock to security return affect active investors' balance sheet? How do they adjust the balance sheet to meet VaR constraint? What's the consequence to equilibrium asset price? Why is leverage cycle "procyclical"?*
   It's a reverse case of the boom cycle in the textbook.

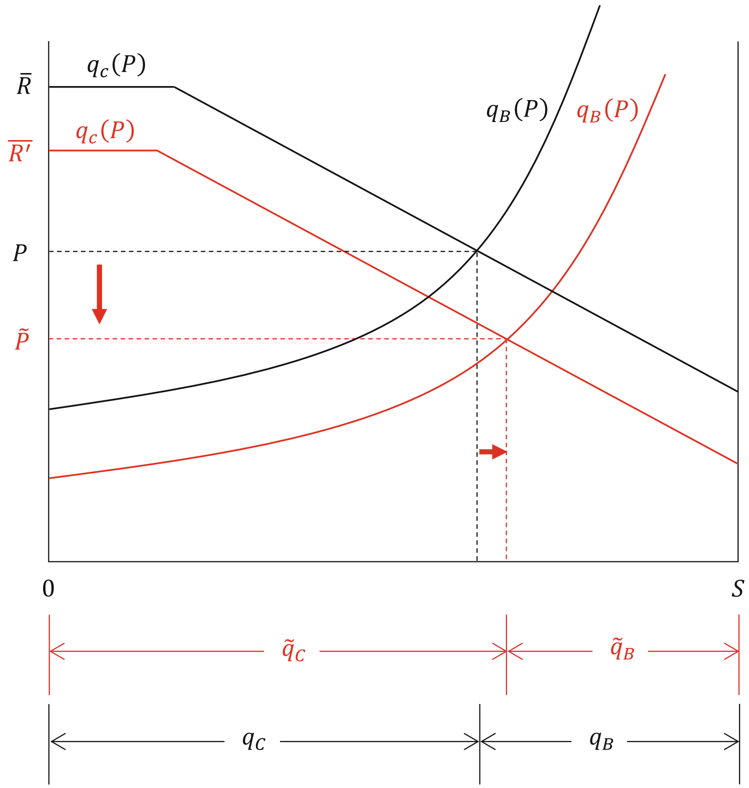

Fig. 9.6 Market equilibrium after the shock

Reference

Holmström, B., & Tirole, J. (1997). Financial intermediation, loanable funds and the real sector. *Quarterly Journal of Economics, 112*, 663–691.

Monetary Policy and Financial Stability 10

10.1 Exercises

10.1.1 Short Review Questions

(a) Explain, in words, how banks achieve optimal risk sharing through maturity transformation, when there is uncertainty in liquidity demand.
(b) Why is fragility in banking desirable as a disciplinary device for banks?

10.1.2 Risk Sharing and Financial Intermediation

Consider a one-good, three-date economy: There are infinitely many ex ante identical consumers, each endowed with one unit of resource at $t = 0$. Consumption takes place either at $t = 1$ or $t = 2$, while the timing preference only gets revealed at $t = 1$: With probability π a consumer is an impatient one (type 1 consumer), who only values consumption at $t = 1$, while with probability $1 - \pi$ a consumer (type 2 consumer) is a patient one, who only values consumption at $t = 2$. A consumer's type is private information.

Let c_i denote the consumption of a type $i = 1, 2$ consumer, and ex post, the utility from consumption is $u(c_i) = \frac{1}{1-\gamma} c_i^{1-\gamma}$ with $\gamma > 1$.

The economy has two technologies of transferring resources between periods: storage technology with gross return equal to 1, and a long-term investment technology with a constant gross return $R > 1$ at $t = 2$ for every per unit invested at $t = 0$. If necessary, an on-going long-term project can be liquidated or stopped prematurely at $t = 1$, with a return $0 < \delta < 1$.

J. Cao, G. Illing, *Instructor's Manual for* Money: Theory and Practice, Springer Texts in Business and Economics,
https://doi.org/10.1007/978-3-030-23618-2_10

(a) Specify the social planner's problem, who wants to maximize a consumer's expected utility at $t = 0$ by allocating her endowments between two technologies.
   1. Compute the optimal allocation, and consumption for each type's consumer—denote the solution as $\left(c_1^*, c_2^*\right)$;
   2. Why aren't consumption levels for two types' consumers identical? Will there be liquidation at $t = 1$?
   3. What will happen to the consumers' optimal consumption when $\gamma \to +\infty$?

(b) Suppose that the economy is in autarky such that every consumer has to allocate her endowments between two technologies by herself at $t = 0$. Show that the consumer's ex post consumption is inferior to the solution in (a) 1.

(c) Suppose there is a bond market available at $t = 1$. At $t = 1$ competitive bond issuers purchase long assets from impatient consumers, issue bonds against these long assets, and sell bonds to the patient consumers (who can pay with the proceeds from their short assets). Each unit of bond bought at $t = 1$ will deliver one unit of consumption good to the bond holder at $t = 2$.
   1. Compute the equilibrium bond price;
   2. Show that the consumer's ex post consumption is inferior to the solution in (a) 1.

(d) Suppose there is a competitive banking sector in the economy, in which banks take consumers' endowments as deposits at $t = 0$ and allocate between the two technologies. Consumers withdraw c_i at $t = i$ according to their type i.
   1. Show that banks can replicate the optimal solution achieved in (a) 1.
   2. Comparing with the result in (b) , how can banks improve social welfare in the economy?

10.1.3 Bank Run and Financial Fragility

Consider the equilibrium with intermediation, as in Exercise 2 (d) in which banks offer consumers the deposit contracts $\left(c_1^*, c_2^*\right)$ at $t = 0$.

(a) Explain why there exist two (Nash) equilibria which are consistent with rational behavior for all agents: one in which only the early consumers withdraw at $t = 1$, and another one in which everyone withdraws at $t = 1$—no matter what type he or she is. What is the individual consumption level in the latter equilibrium? Does the existence of multiple equilibria depend on the value of δ?

(b) Propose a mechanism that can eliminate the bank run equilibrium. Explain how it works.

(c) Suppose that it is known in the economy that a small group of consumers always panic at $t = 1$, i.e., they want to withdraw with certainty at $t = 1$ no matter what type they actually are. Will there still exist two Nash equilibria as in (a)?

10.1.4 Financial Intermediation, Fragility, and Unconventional Monetary Policy

Consider a one-good, three-date economy: There are infinitely many ex ante identical consumers (whose population is normalized to 1), each endowed with one unit of resource at $t = 0$. Consumption may take place either at $t = 1$ or $t = 2$, while each consumer's timing preference of consumption only gets revealed at $t = 1$: With probability p ($0 < p < 1$) a consumer is an impatient one (type 1 consumer), who only values consumption at $t = 1$, while with probability $1 - p$ a consumer is a patient one (type 2 consumer), who only values consumption at $t = 2$. A consumer's type is private information and only known to herself.

Let c_i denote the consumption of a type $i = 1, 2$ consumer. At $t = 0$, without knowing her type, a consumer's expected utility from consumption is $u = p\sqrt{c_1} + (1 - p)\sqrt{c_2}$.

The economy has two technologies of transferring resources between periods: storage technology with gross return equal to 1, and a long-term investment technology with a constant gross return $R > 1$ at $t = 2$ for every unit invested at $t = 0$. If necessary, an on-going long-term project can be liquidated or stopped prematurely at $t = 1$, with a return $0 \leq \delta < 1$.

(a) Suppose at $t = 0$ a social planner allocates all resources in this economy to maximize each consumer's expected utility: At $t = 0$ the planner collects $0 \leq \alpha \leq 1$ from each consumer and invests on the storage technology, and the rest—$(1 - \alpha)$ from each consumer—will be invested on the long-term technology. At $t = 1$ the total proceeds from the storage technology will be evenly distributed among impatient consumers (whose population is p), and at $t = 2$ the total proceeds from the long-term technology will be evenly distributed among patient consumers (whose population is $1 - p$). Show that the optimal solution to type $i = 1, 2$ consumer's consumption is $c_1^* = \frac{1}{p+(1-p)R}$, $c_2^* = \frac{R^2}{p+(1-p)R}$, and the resource invested on storage technology is $\alpha^* = \frac{p}{p+(1-p)R}$.

(b) Suppose that the economy is in autarky such that every consumer has to allocate her endowments between two technologies by herself at $t = 0$. Show that consumers cannot achieve the optimal solution defined in (a) .

(c) Suppose there is a competitive banking sector in the economy, in which banks take consumers' endowments as deposits at $t = 0$ and allocate between the two technologies. Consumers withdraw c_i at $t = i$ according to their type i. Show that banks can implement the optimal solution achieved in (a) in the following way: (1) Banks invest α^* of deposits on storage technology, $1 - \alpha^*$ of deposits on long-term technology; (2) consumers who withdraw at $t = 1$ get c_1^* each, and consumers who withdraw at $t = 2$ get c_2^* each; (3) impatient consumers all withdraw at $t = 1$ and patient consumers all withdraw at $t = 2$.

(d) Banking sector in this economy is fragile: Patient consumers may demand their deposits at $t = 1$, which leads to bank run. However, whether this happens or not crucially depends on the value of δ. Show that as long as $\delta > \frac{1}{R}$ and banks do the same as described in (c) , there will never be bank runs.

(e) During recent crisis, several central banks purchased huge volume of securities, hoping to prevent price of long assets from falling too much. Using your finding in (d), explain why such unconventional policy helps eliminate panics in banking sector.

10.1.5 Monetary Policy, Financial Stability, and Banking Regulation

One way for the banks to cushion adverse shocks is to hold capital buffer so that losses can be absorbed before banks have to go bankruptcy. However, there are reasons that banks prefer being undercapitalized, and such incentive is partially affected by monetary policy. Therefore, monetary policy has strong implication for financial stability, and properly designed regulatory rules is complementary to eliminate instabilities. This exercise explores these issues through a simple model.

Consider an economy where banks could invest either in a safe project that yields G with probability P_G and 0 otherwise, or in a risky project that yields B with probability P_B and 0 otherwise. The projects have constant returns to scale and satisfy $G < B$ and $P_G G > P_B B > 1$. Developing project G requires additional effort with a fixed cost c.

Banks are financed by short-term, unsecured deposits with a return r_D per unit of deposit. Depositors are risk-neutral and will require an expected return equal to the risk-free rate in the economy, which is equal to $R > 0$. We assume the participation constraint for banks is satisfied.

Capital is costly for banks because equity holders require a substantial expected return on equity (RoE) $r > 0$, otherwise banks would not be able to attract investors in the stock market.

(a) Describe the competitive equilibrium in the absence of bank capital, and determine under what conditions the safe project, G, or the risky one, B, will be implemented by banks.

(b) Assume that in the absence of bank capital the only equilibrium obtained is characterized by implementing the risky project. Determine the minimum level of capital a regulator needs in order to restore the possibility of an equilibrium where the safe project is preferred by banks.

(c) Suppose that the risk-free rate R is perfectly under the control of central bank. What's the impact of monetary policy on banks' choice of projects?

(d) Assume depositors are able to observe banks' capital. What will be the amount of capital a profit-maximizing bank will choose?

10.1.6 Countercyclical Capital Buffer Requirement

(a) Read the website of Basel III countercyclical capital buffer (CCyB) from the Bank for International Settlements (https://www.bis.org/bcbs/ccyb/), explain how CCyB works.
(b) Using your findings in the Exercise 3 (c), Chap. 9, explain how CCyB contains leverage cycles.

10.2 Solutions for Selected Exercises

10.2.1 Risk Sharing and Financial Intermediation

Consider a one-good, three-date economy: There are infinitely many ex ante identical consumers, each endowed with one unit of resource at $t = 0$. Consumption takes place either at $t = 1$ or $t = 2$, while the timing preference only gets revealed at $t = 1$: With probability π a consumer is an impatient one (type 1 consumer), who only values consumption at $t = 1$, while with probability $1 - \pi$ a consumer (type 2 consumer) is a patient one, who only values consumption at $t = 2$. A consumer's type is private information.

Let c_i denote the consumption of a type $i = 1, 2$ consumer, and ex post, the utility from consumption is $u(c_i) = \frac{1}{1-\gamma} c_i^{1-\gamma}$ with $\gamma > 1$.

The economy has two technologies of transferring resources between periods: storage technology with gross return equal to 1, and a long-term investment technology with a constant gross return $R > 1$ at $t = 2$ for every per unit invested at $t = 0$. If necessary, an on-going long-term project can be liquidated or stopped prematurely at $t = 1$, with a return $0 < \delta < 1$.

(a) Specify the social planner's problem, who wants to maximize a consumer's expected utility at $t = 0$ by allocating her endowments between two technologies.
   1. *Compute the optimal allocation, and consumption for each type's consumer—denote the solution as $\left(c_1^*, c_2^*\right)$;*
      Suppose the social planner invests s on the short assets, then she solves

$$\begin{aligned} \max_s \quad & \pi u\left(c_1\right) + (1-\pi) u\left(c_2\right), \\ s.t. \quad & \pi c_1 = s, \\ & (1-\pi) c_2 = (1-s) R. \end{aligned}$$

Using Lagrangian for first-order conditions

$$\begin{aligned} \mathcal{L} &= \pi u\left(c_1\right) + (1-\pi) u\left(c_2\right) - \lambda\left[(1-\pi) c_2 - \left(1 - \pi c_1\right) R\right], \\ \frac{\partial \mathcal{L}}{\partial c_1} &= \pi u'\left(c_1\right) - \lambda \pi R = 0, \end{aligned}$$

$$\frac{\partial \mathcal{L}}{\partial c_2} = (1-\pi)u'(c_2) - \lambda(1-\pi) = 0,$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = (1-\pi)c_2 - (1-\pi c_1)R = 0.$$

Rearranging to get

$$u'(c_1) = Ru'(c_2),$$
$$c_1^{-\gamma} = Rc_2^{-\gamma},$$
$$\frac{c_2}{c_1} = R^{\frac{1}{\gamma}}.$$

Together with $(1-\pi)c_2 = (1-\pi c_1)R$, solve to get

$$c_1^* = \frac{R}{(1-\pi)R^{\frac{1}{\gamma}} + \pi R},$$
$$c_2^* = \frac{R^{1+\frac{1}{\gamma}}}{(1-\pi)R^{\frac{1}{\gamma}} + \pi R}.$$

2. *Why aren't consumption levels for two types' consumers identical? Will there be liquidation at $t = 1$?*
First of all, we explore some properties of $\left(c_1^*, c_2^*\right)$. From $u'\left(c_1^*\right) = Ru'\left(c_2^*\right)$, $R > 1$ and $u''(\cdot) < 0$, one can see that $u'\left(c_1^*\right) > u'\left(c_2^*\right)$ and $c_1^* < c_2^*$. We further claim that $1 < c_1^* < c_2^* < R$.

Proof It's known that $\gamma > 1$, or, $-\frac{cu''(c)}{u'(c)} > 1$, this is equivalent to $\frac{\partial[cu'(c)]}{\partial c} < 0$. Together with $c_1^* < c_2^*$ and $R > 1$, it implies that $1 \cdot u'(1) > Ru'(R)$. Further, $\left(c_1^*, c_2^*\right)$ satisfies resource constraint $c_2^* = \frac{(1-\pi c_1^*)R}{1-\pi}$ and optimality condition $u'\left(c_1^*\right) = Ru'\left(c_2^*\right)$. Resource constraint implies either (a) $c_1^* > 1$ and $c_2^* < R$, or (b) $c_1^* \leq 1$ and $c_2^* \geq R$.

If (a) is true, $1 \cdot u'(1) > c_1^* u'\left(c_1^*\right) > c_2^* u'\left(c_2^*\right) > Ru'(R)$, it goes through;
If (b) is true, $u'\left(c_1^*\right) = Ru'\left(c_2^*\right) \leq Ru'(R) < 1 \cdot u'(1)$, which implies $c_1^* > 1$. A contradictions.

■

The result achieves constrained efficiency: should there be no heterogeneous liquidity preference, all resources should be invested on the long assets which give highest yields R. With liquidity preference shock, equilibrium consumption c_1^* and c_2^* are strictly lower than R. The loss in consumption is a liquidity premium.

For social planner, she faces a trade-off between insurance and yield. Risk aversion implies c_1^* closer to c_2^*, while high yield opportunity suggests more investment on long assets, hence c_2^* closer to R. The equilibrium reflects such trade-off.

There is no liquidation at $t = 1$ in the social planner's solution. To see this, suppose that $(\tilde{c}_1, \tilde{c}_2)$ solves social planner's problem, with c long assets liquidated at $t = 1$. Budget constraints become

$$\pi \tilde{c}_1 = s + c\delta,$$
$$(1-\pi)\tilde{c}_2 = (1-s-c)\, R.$$

However, the social planner could make at least type 1 consumers better off (and type 2 consumers indifferent) by investing c instead on short assets at $t = 0$, making $\hat{c}_1 = \frac{s+c}{\pi} > \tilde{c}_1 = \frac{s+c\delta}{\pi}$, given that $0 < \delta < 1$. This contradicts the assumption that $(\tilde{c}_1, \tilde{c}_2)$ solves social planner's problem.

3. *What will happen to the consumers' optimal consumption when $\gamma \to +\infty$?*
   When $\gamma \to +\infty$

$$\lim_{\gamma \to +\infty} \frac{c_2^*}{c_1^*} = R^{\frac{1}{\gamma}} = 1.$$

   When risk aversion dominates, consumers get full insurance.

(b) *Suppose that the economy is in autarky such that every consumer has to allocate her endowments between two technologies by herself at $t = 0$. Show that the consumer's ex post consumption is inferior to the solution in (a) 1.*

Suppose, at $t = 0$, without knowing her type, one consumer invests $0 \le \alpha \le 1$ on short assets and the rest on long assets. At $t = 1$, her type gets revealed.

1. If she is type 1, she gets the storage and liquidates the rest at $t = 1$, $c_1^a = \alpha + (1-\alpha)\delta \le 1$;
2. If she is type 2, she waits and gets returns from both assets at $t = 2$, $c_2^a = \alpha + (1-\alpha)R \le R$.

The allocation $\left(c_1^a, c_2^a\right) \neq \left(c_1^*, c_2^*\right)$ is inferior to the planner's solution.

(c) *Suppose there is a bond market available at $t = 1$. At $t = 1$ competitive bond issuers purchase long assets from impatient consumers, issue bonds against these long assets, and sell bonds to the patient consumers (who can pay with the proceeds from their short assets). Each unit of bond bought at $t = 1$ will deliver one unit of consumption good to the bond holder at $t = 2$.*

1. *Compute the equilibrium bond price;*
   A type 1 consumer sells long assets to type 2 consumers at price b for short assets, then she consumes $c_1^b = \alpha + (1-\alpha)bR$; a type 2 consumer sells short assets to type 1 consumers at price b for long assets, then she consumes $c_2^b = \frac{\alpha}{b} + (1-\alpha)R$. If $b > \frac{1}{R}$, everyone wants to set $\alpha = 0$, implying no

equilibrium; if $b < \frac{1}{R}$, everyone wants to set $\alpha = 1$, implying no equilibrium either. Therefore, in equilibrium it must be $b = \frac{1}{R}$.

2. *Show that the consumer's ex post consumption is inferior to the solution in (a) 1.*
   The allocation then $\left(c_1^b, c_2^b\right) = (1, R) \neq \left(c_1^*, c_2^*\right)$ is inferior to the planner's solution.

(d) *Suppose there is a competitive banking sector in the economy, in which banks take consumers' endowments as deposits at $t = 0$ and allocate between the two technologies. Consumers withdraw c_i at $t = i$ according to their type i.*

1. *Show that banks can replicate the optimal solution achieved in (a) 1.*
   Banks can replicate the optimal solution as follows:
   (a) Collect deposits from consumers at $t = 0$, offering them deposit contracts $d_0 = \left(c_1^*, c_2^*\right)$;
   (b) A type 1 consumer withdraws c_1^* at $t = 1$, and a type 2 consumer withdraws c_2^* at $t = 2$.
   The allocation is indeed an equilibrium outcome, as it fulfills
   (a) Utility maximization;
   (b) Feasible, or, meets budget constraints;
   (c) Incentive compatible, given that each consumer's true type is not observed by banks. Indeed, as $c_1^* < c_2^*$, a truly patient type 2 consumer will wait till $t = 2$ to withdraw; she will not mimic the impatient one of type 1.
2. *Comparing with the result in (b), how can banks improve social welfare in the economy?*
   Banks provide liquidity insurance to consumers, allowing them to withdraw when liquidity shock hits while partially enjoying the high return that long assets generate.

10.2.2 Bank Run and Financial Fragility

Consider the equilibrium with intermediation, as in Exercise 2 (d) in which banks offer consumers the deposit contracts $\left(c_1^*, c_2^*\right)$ at $t = 0$.

(a) *Explain why there exist two (Nash) equilibria which are consistent with rational behavior for all agents: one in which only the early consumers withdraw at $t = 1$, and another one in which everyone withdraws at $t = 1$—no matter what type he or she is. What is the individual consumption level in the latter equilibrium? Does the existence of multiple equilibria depend on the value of δ?*

It's easier to show that no-run strategic profile $\left(c_1^*, c_2^*\right)$ is an equilibrium outcome: it isn't profitable for anyone to mimic the other type as $c_1^* < c_2^*$.
To see that all-run strategic profile $\left(c_1^r, c_2^r\right)$ is an equilibrium outcome, it is sufficient to show that early withdrawers exhaust all resources so that any deviator (waiting instead of running) is worse off, i.e., $c_2^r = 0$. If everyone

demands repayment at $t = 1$, total demand for withdrawal is c_1^*, while the maximum supply of liquidity is $s + (1-s)\delta = \pi c_1^* + (1 - \pi c_1^*)\delta < 1 < c_1^*$ given that $\delta < 1$. Therefore, all resources must be exhausted to meet the demand and nothing is left, i.e., $c_2^r = 0$. The actual payoff for the consumers is $c_1^r = \pi c_1^* + (1 - \pi c_1^*)\delta < c_1^*$. The run equilibrium is thus an inefficient equilibrium.

The existence of run equilibrium hinges on the assumption that $\delta < 1$. If this assumption is dropped, is there a δ that makes bank run impossible, or, deviating from the all-run strategic profile $\left(c_1^r, c_2^r\right)$ is profitable? If yes, there must be resources left after the run, or,

$$\pi c_1^* + (1 - \pi c_1^*)\delta > c_1^*,$$
$$\delta > \frac{(1-\pi)c_1^*}{1 - \pi c_1^*}.$$

We can explore further for a weaker condition. Notice that $c_1^* > 1$,

$$\delta > \frac{(1-\pi)c_1^*}{1 - \pi c_1^*} > \frac{(1-\pi)c_1^*}{c_1^* - \pi c_1^*} = 1.$$

Therefore, eligible $\overline{\delta}$ must be strictly higher than 1, while lower than R.

(b) *Propose a mechanism that can eliminate the bank run equilibrium. Explain how it works.*

One proposal is deposit insurance, that banks pay an insurance premium at $t = 0$ to an insurance company, and the insurance company provides guarantee for d_0. Knowing that higher consumption is guaranteed at $t = 2$, type 2 consumers will never have the incentive to run. However, in reality moral hazard induces banks to abuse insurance scheme, and this makes the scheme collapse. Therefore, deposit insurance seldom provides full guarantees in the real world.

(c) *Suppose that it is known in the economy that a small group of consumers always panic at $t = 1$, i.e., they want to withdraw with certainty at $t = 1$ no matter what type they actually are. Will there still exist two Nash equilibria as in (a) ?*

Depending on the size of the panicking group. There still exist two Nash equilibria only if the group is small enough.

10.2.3 Financial Intermediation, Fragility, and Unconventional Monetary Policy

Consider a one-good, three-date economy: There are infinitely many ex ante identical consumers (whose population is normalized to 1), each endowed with one unit of resource at $t = 0$. Consumption may take place either at $t = 1$ or $t = 2$, while each consumer's timing preference of consumption only gets revealed at $t = 1$: With probability p $(0 < p < 1)$ a consumer is an impatient one (type 1 *consumer), who only values consumption at* $t = 1$*, while with probability* $1 - p$ *a consumer is a patient one (type 2 consumer), who only values consumption at* $t = 2$*. A consumer's type is private information and only known to herself.*

Let c_i *denote the consumption of a type* $i = 1, 2$ *consumer. At* $t = 0$*, without knowing her type, a consumer's expected utility from consumption is* $u = p\sqrt{c_1} + (1 - p)\sqrt{c_2}$*.*

The economy has two technologies of transferring resources between periods: storage technology with gross return equal to 1, and a long-term investment technology with a constant gross return $R > 1$ *at* $t = 2$ *for every unit invested at* $t = 0$*. If necessary, an on-going long-term project can be liquidated or stopped prematurely at* $t = 1$*, with a return* $0 \leq \delta < 1$*.*

(a) *Suppose at* $t = 0$ *a social planner allocates all resources in this economy to maximize each consumer's expected utility: At* $t = 0$ *the planner collects* $0 \leq \alpha \leq 1$ *from each consumer and invests on the storage technology, and the rest—*$(1 - \alpha)$ *from each consumer—will be invested on the long-term technology. At* $t = 1$ *the total proceeds from the storage technology will be evenly distributed among impatient consumers (whose population is* p*), and at* $t = 2$ *the total proceeds from the long-term technology will be evenly distributed among patient consumers (whose population is* $1 - p$*). Show that the optimal solution to type* $i = 1, 2$ *consumer's consumption is* $c_1^* = \frac{1}{p+(1-p)R}$*,* $c_2^* = \frac{R^2}{p+(1-p)R}$*, and the resource invested on storage technology is* $\alpha^* = \frac{p}{p+(1-p)R}$*.*

The social planner solves

$$\begin{aligned} \max_{\alpha} \quad & p\sqrt{c_1} + (1 - p)\sqrt{c_2}, \\ s.t. \quad & pc_1 = \alpha, \\ & (1 - p)c_2 = (1 - \alpha) R. \end{aligned}$$

Solve to get $c_1^* = \frac{1}{p+(1-p)R}$, $c_2^* = \frac{R^2}{p+(1-p)R}$, and $\alpha^* = \frac{p}{p+(1-p)R}$.

(b) *Suppose that the economy is in autarky such that every consumer has to allocate her endowments between two technologies by herself at* $t = 0$*. Show that consumers cannot achieve the optimal solution defined in (a).*

Suppose at $t = 0$, without knowing her type, one consumer invests $0 \leq \alpha \leq 1$ on short assets and the rest on long assets. At $t = 1$, her type gets revealed.

1. If she is type 1, she gets the storage and liquidates the rest at $t = 1$, $c_1^a = \alpha + (1 - \alpha)\delta \leq 1$;
2. If she is type 2, she waits and gets returns from both assets at $t = 2$, $c_2^a = \alpha + (1 - \alpha)R \leq R$.

Given that $c_2^a \leq R < c_2^*$, certainly the allocation $\left(c_1^a, c_2^a\right) \neq \left(c_1^*, c_2^*\right)$ is inferior to the planner's solution. Note: Here the relative rate of risk aversion is smaller than 1, which is different from what is assumed in the standard Diamond–Dybvig model. As a result $c_1^* < 1$ and $c_2^* > R$, therefore, it is crucial here to show that $c_2^* > R$ so that it never falls in the range of c_2^a and $\left(c_1^a, c_2^a\right) \neq \left(c_1^*, c_2^*\right)$ for sure.

(c) *Suppose there is a competitive banking sector in the economy, in which banks take consumers' endowments as deposits at $t = 0$ and allocate between the two technologies. Consumers withdraw c_i at $t = i$ according to their type i. Show that banks can implement the optimal solution achieved in (a) in the following way: (1) Banks invest α^* of deposits on storage technology, $1 - \alpha^*$ of deposits on long-term technology; (2) consumers who withdraw at $t = 1$ get c_1^* each, and consumers who withdraw at $t = 2$ get c_2^* each; (3) impatient consumers all withdraw at $t = 1$ and patient consumers all withdraw at $t = 2$.*

The social planner's solution can be decentralized in the banking economy because the banking allocation $\left(c_1^*, c_2^*\right)$ is (1) utility maximizing as it solves the social planner's problem; (2) feasible as it fulfills the resource constraints as specified in the planner's problem; and (3) implementable: Impatient consumers won't mimic the patient ones because they cannot wait till $t = 2$ and the patient ones won't mimic the impatient ones because they are worse off ($c_1^* < c_2^*$) by mimicking, so that the deposit contract is incentive compatible and consumers self-select the proper outcomes.

(d) *Banking sector in this economy is fragile: Patient consumers may demand their deposits at $t = 1$, which leads to bank run. However, whether this happens or not crucially depends on the value of δ. Show that as long as $\delta > \frac{1}{R}$ and banks do the same as described in (c), there will never be bank runs.*

The bank run outcome is not equilibrium only if it is profitable for a patient consumer to deviate unilaterally given that all other consumers run on the bank; this is equivalent to saying that there is resource left even after all consumers run on the bank, so that it is profitable for a patient consumer to unilaterally deviate, wait till $t = 2$ and get better off: $pc_1^* + \left(1 - pc_1^*\right)\delta > c_1^*$, or, $\delta > \frac{(1-p)c_1^*}{1-pc_1^*} = \frac{1}{R} = \underline{\delta}$ (using c_1^* from (a). However, it is sufficient to reach $\delta > \frac{(1-p)c_1^*}{1-pc_1^*} = \underline{\delta}$).

(e) *During recent crisis, several central banks purchased huge volume of securities, hoping to prevent price of long assets from falling too much. Using your finding in (d), explain why such unconventional policy helps eliminate panics in banking sector.*

As long as the asset purchasing program can maintain $\delta > \underline{\delta}$, bank runs are fully eliminated as explained in (d).

Dynamic Optimization Using Lagrangian and Hamiltonian Methods

A

A.1 The Deterministic Finite Horizon Optimization Problem

We start with a simplest case of the deterministic finite horizon optimization problem, i.e., there is a terminal point in the decision process.

A.1.1 Basic Tools

Problems with Equality Constraints: The General Case

Readers may have already practiced the static optimization problems with equality constraints many times before; the problems won't change much if we simply introduce a finite time dimension, i.e., some constraints must hold for each of the periods $t \in \{0, \ldots, T\}$—In a static problem people do maximization with respect to n variables $(x_1, \ldots, x_n)$, and in a dynamic context with finite periods $t \in \{0, \ldots, T\}$ we just solve basically the same problem with $n(T+1)$ variables $(x_1, \ldots, x_{n(T+1)})$. As we know the Theorem of Lagrange, as Theorem A.1 states, provides a powerful characterization of local optima of equality constrained optimization problems in terms of the behavior of the objective function and the constraint functions at these points. Generally such problems have the form as following:

$$\begin{aligned} \max \quad & f(\mathbf{x}) \\ s.t. \quad & \mathbf{x} \in \mathcal{D} = U \cap \{\mathbf{x} | \mathbf{g}(\mathbf{x}) = 0\}, \end{aligned}$$

in which object function $f : \mathbb{R}^{n(T+1)} \to \mathbb{R}$ and constraints $g_i : \mathbb{R}^{n(T+1)} \to \mathbb{R}^{k(T+1)}, \forall i \in \{1, \ldots, k(T+1)\}$ be continuously differentiable functions, and $U \subseteq \mathbb{R}^{n(T+1)}$ is open. To solve it we set up a function called *Lagrangian* $\mathscr{L}$:

J. Cao, G. Illing, *Instructor's Manual for* Money: Theory and Practice, Springer Texts in Business and Economics,
https://doi.org/10.1007/978-3-030-23618-2

$\mathcal{D} \times \mathbb{R}^{k(T+1)} \to \mathbb{R}$

$$\mathscr{L}(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}) + \sum_{i=1}^{k(T+1)} \lambda_i g_i(\mathbf{x})$$

in which the vector $\boldsymbol{\lambda} = \left(\lambda_1, \ldots, \lambda_{k(T+1)}\right) \in \mathbb{R}^{k(T+1)}$ is called *Lagrange multiplier*.

Theorem A.1 *Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g_i : \mathbb{R}^n \to \mathbb{R}^k$ be continuously differentiable functions, $\forall i \in \{1, \ldots, k\}$. Suppose that $\boldsymbol{x}^*$ is a local maximum or minimum of f on the set*

$$\mathcal{D} = U \cap \{\boldsymbol{x} | g_i(\boldsymbol{x}) = 0, \forall i \in \{1, \ldots, k\}\},$$

in which $U \subseteq \mathbb{R}^n$ is open. Suppose also that rank $(D\boldsymbol{g}(\boldsymbol{x}^*)) = k$. *Then, there exists a vector $\boldsymbol{\lambda}^* = \left(\lambda_1^*, \ldots, \lambda_k^*\right) \in \mathbb{R}^k$ such that*

$$Df\left(\boldsymbol{x}^*\right) + \sum_{i=1}^{k} \lambda_i^* Dg_i(\boldsymbol{x}^*) = 0.$$

Then by Theorem A.1[1] we find the set of all critical points of $\mathscr{L}(\mathbf{x}, \boldsymbol{\lambda})$ for $\mathbf{x} \in U$, i.e., the *first-order conditions*

$$\frac{\partial \mathscr{L}}{\partial x_j} = 0, \quad \forall j \in \{1, \ldots, n(T+1)\},$$

$$\frac{\partial \mathscr{L}}{\partial \lambda_i} = 0, \quad \forall i \in \{1, \ldots, k(T+1)\},$$

which simply say that these conditions should hold for each x and λ in every period.

Now we continue to explore the interpretation for the Lagrange multiplier λ. We relax the equality constraints by adding a sufficiently small constant to each of them, i.e.,

$$\mathbf{g}(\mathbf{x}, \mathbf{c}) = \mathbf{g}(\mathbf{x}) + \mathbf{c}$$

in which $\mathbf{c} = (c_1, \ldots, c_k)$ is a vector of constants. Now the set of constraints becomes

$$\mathcal{D} = U \cap \{\mathbf{x} | \mathbf{g}(\mathbf{x}, \mathbf{c}) = 0\}.$$

[1]Please note that as a tradition people denote the derivative of a multi-variate function $f(\mathbf{x})$: $\mathbb{R}^n \to \mathbb{R}$ by $Df(\mathbf{x})$, which is an n dimensional vector $Df(\mathbf{x}) := \left[\frac{\partial f(x_1,\ldots,x_n)}{\partial x_1}, \ldots, \frac{\partial f(x_1,\ldots,x_n)}{\partial x_n}\right]$.

Then by Theorem A.1 at the optimum $\mathbf{x}^*(\mathbf{c})$ there exists $\lambda^*(\mathbf{c}) \in \mathbb{R}^{k(T+1)}$ such that

$$Df\left(\mathbf{x}^*(\mathbf{c})\right) + \sum_{i=1}^{k(T+1)} \lambda_i^*(\mathbf{c}) Dg_i(\mathbf{x}^*(\mathbf{c})) = 0. \tag{A.1}$$

Define a new function of $\mathbf{c}$, $F(\mathbf{c}) = f(\mathbf{x}^*(\mathbf{c}))$. Then by chain rule,

$$DF(\mathbf{c}) = Df\left(\mathbf{x}^*(\mathbf{c})\right) D\mathbf{x}^*(\mathbf{c}).$$

Insert (A.1) into the equation above, one can get

$$DF(\mathbf{c}) = -\left(\sum_{i=1}^{k(T+1)} \lambda_i^*(\mathbf{c}) Dg_i(\mathbf{x}^*(\mathbf{c}))\right) D\mathbf{x}^*(\mathbf{c}),$$

and this is equivalent to

$$DF(\mathbf{c}) = -\sum_{i=1}^{k(T+1)} \lambda_i^*(\mathbf{c}) Dg_i(\mathbf{x}^*(\mathbf{c})) D\mathbf{x}^*(\mathbf{c}). \tag{A.2}$$

Define another new function of $\mathbf{c}$, $G_i(\mathbf{c}) = g_i(\mathbf{x}^*(\mathbf{c}))$. Then again by chain rule,

$$DG_i(\mathbf{c}) = Dg_i\left(\mathbf{x}^*(\mathbf{c})\right) D\mathbf{x}^*(\mathbf{c}).$$

Insert this into (A.2), and one can get

$$DF(\mathbf{c}) = -\sum_{i=1}^{k(T+1)} \lambda_i^*(\mathbf{c}) DG_i(\mathbf{c}). \tag{A.3}$$

By the equality constraint $\mathbf{g}(\mathbf{x}) + \mathbf{c} = 0$ one can easily see that

$$DG_i(\mathbf{c}) = -\mathbf{e}_i$$

in which $\mathbf{e}_i$ is the i-th unit vector in $\mathbb{R}^{k(T+1)}$, i.e., the vector that has a 1 in the i-th place and zeros elsewhere. Therefore (A.3) turns out to be

$$\begin{aligned} DF(\mathbf{c}) &= -\sum_{i=1}^{k(T+1)} \lambda_i^*(\mathbf{c}) \left(-\mathbf{e}_i\right) \\ &= \lambda^*(\mathbf{c}). \end{aligned}$$

From the equation above one can clearly see that the Lagrange multiplier λ_i measures the sensitivity of the value of the objective function at its maxima $\mathbf{x}^*$ to a small relaxation of the constraint g_i. Therefore λ_i has a very straightforward economic interpretation, that λ_i represents the maximum amount the decision maker would be willing to pay for a marginal relaxation of constraint i—this is sometimes called the *shadow price* of constraint i at the optima.

Problems with Equality Constraints: A Simplified Version

The general case may be a little bit messy to go through, now we deal with the same problem in a much simplified version, i.e., the univariate case which we are quite familiar with. Suppose that an agent maximizes her neoclassical utility function with respect to a single good x, and x must follow an equality constraint,

$$\max_x \quad u(x),$$
$$s.t. \quad g(x) = 0.$$

Then the problem can be easily solved by setting up Lagrangian

$$\mathscr{L} = u(x) + \lambda g(x),$$

and the optimal x, denoted by x^*, can be derived from the first-order conditions

$$\frac{\partial \mathscr{L}}{\partial x} = 0, \frac{\partial \mathscr{L}}{\partial \lambda} = 0.$$

Now relax the constraint a little bit by ϵ around x^*, and rewrite the optimization problem at x^* as

$$\max_\epsilon \quad u(x^*, \epsilon),$$
$$s.t. \quad g(x^*) = \epsilon.$$

By Theorem A.1 the optimal value of ϵ can be solved from the first-order conditions of the Lagrangian

$$\mathscr{L}' = u(x^*, \epsilon) + \lambda \left[g(x^*) - \epsilon \right].$$

However, since we already know that x^* is the optimal solution of the original problem, and the optimal value of ϵ must be achieved when $\epsilon \to 0$, i.e.,

$$\left. \frac{\partial \mathscr{L}'}{\partial \epsilon} \right|_{\epsilon \to 0} = \frac{\partial u(x^*, \epsilon)}{\partial \epsilon} - \lambda = 0,$$

$$\lambda = \frac{\partial u(x^*, \epsilon)}{\partial \epsilon}.$$

The last step clearly shows that the Lagrange multiplier λ measures how much the utility changes when the constraint is relaxed a little bit at the optimum, i.e., the shadow price at the optimum.

Problems with Inequality Constraints: The General Case

The solution of problems with inequality constraints is characterized by the following theorem:

Theorem A.2 *Let f be a concave, continuously differentiable function mapping U into $\mathbb{R}$, where $U \subseteq \mathbb{R}^n$ is open and convex. For $i = 1, \ldots, l$, let $h_i : U \to \mathbb{R}$ be concave, continuously differentiable functions. Suppose there is some $\overline{\boldsymbol{x}} \in U$ such that*

$$h_i(\overline{\boldsymbol{x}}) > 0, i = 1, \ldots, l.$$

Then $\boldsymbol{x}^*$ maximizes f over

$$\mathcal{D} = \{\boldsymbol{x} \in U | h_i(\boldsymbol{x}) \geq 0, i = 1, \ldots, l\}$$

if and only if there is $\boldsymbol{\lambda}^* \in \mathbb{R}^l$ such that the Kuhn–Tucker first-order conditions hold:

$$\begin{aligned}\frac{\partial f(\boldsymbol{x}^*)}{\partial x_j} + \sum_{i=1}^{j} \lambda_i^* \frac{\partial h_i(\boldsymbol{x}^*)}{\partial x_j} &= 0, \quad j = 1, \ldots, n,\\ \lambda_i^* &\geq 0, \quad i = 1, \ldots, l,\\ \lambda_i^* h_i(\boldsymbol{x}^*) &= 0, \quad i = 1, \ldots, l.\end{aligned}$$

For problems with inequality constraints, the solution procedure is pretty similar. The only differences are the following: First, of course, the prototype problem is different in the constraints, which are now

$$\mathbf{x} \in \mathcal{D} = U \cap \{\mathbf{x} | \mathbf{h}(\mathbf{x}) \geq 0\}.$$

Second, besides the first-order conditions, there is an additional *complementary slackness* condition saying that at optimum

$$\begin{aligned}\boldsymbol{\lambda}^* &\geq \mathbf{0},\\ \boldsymbol{\lambda}^* \mathbf{h}^* &= \mathbf{0}.\end{aligned}$$

The economic intuition behind the condition is pretty clear: If any resource i has a positive value at the optima, i.e., $\lambda_i^* > 0$, then it must be exhausted to maximize the object function, i.e., $h_i^* = 0$; and if any resource j is left at a positive value at the optima, i.e., $h_j^* > 0$, then it must be worthless at all, i.e., $\lambda_j^* = 0$. To see how one can arrive at such results, an example is exposed in the next section.

Problems with Inequality Constraints: An Example

Consider the following two-period Ramsey–Cass–Koopmans problem of a farmer. Suppose that

- Time is divided into two intervals of unit length indexed by $t = 0, 1$;
- K_t and N_t denote the amounts of seeds and labor available in period t;
- Seeds and labor input produce an amount Y_t of corn according to the neoclassical production function $Y_t = F(K_t, L_t)$;
- For each period t the farmer must decide
  - how much corn to produce,
  - how much corn to eat, and
  - how much corn to put aside for future production;
- Next period's seed is next period's stock of capital K_{t+1};
- Choice of consumption C_t and investment
  - is constrained by current production

$$C_t + K_{t+1} \leq Y_t,$$

  - aims at maximizing the utility function (assume that $U(\cdot)$ satisfies Inada condition)

$$U(C_0, C_1) = u(C_0) + \beta u(C_1);$$

  - Leisure does not appear in the utility function; assume that the farmer works a given number of hours N each period.

Then the maximization problem turns out to be

$$\begin{aligned}
\max_{C_0, C_1} \quad & U(C_0, C_1) = u(C_0) + \beta u(C_1), \\
s.t. \quad & C_0 + K_1 \leq F(K_0), \\
& C_1 + K_2 \leq F(K_1), \\
& 0 \leq C_0, \\
& 0 \leq C_1, \\
& 0 \leq K_1, \\
& 0 \leq K_2.
\end{aligned}$$

Comparing with the prototype problem presented in Theorem A.2 we may define that

$$\begin{aligned}\mathbf{x} &= (C_0, C_1, K_1, K_2),\\ f(C_0, C_1, K_1, K_2) &= U(C_0, C_1),\\ n &= 4\end{aligned}$$

as well as the constraints

$$\begin{aligned}h_1 &= F(K_0) - C_0 - K_1 \geq 0,\\ h_2 &= F(K_1) - C_1 - K_2 \geq 0,\\ h_3 &= C_0 \geq 0,\\ h_4 &= C_1 \geq 0,\\ h_5 &= K_1 \geq 0,\\ h_6 &= K_2 \geq 0.\end{aligned}$$

By Theorem A.2 the first-order conditions are

$$0 = \frac{\partial U}{\partial C_0} + \lambda_1 \frac{\partial h_1}{\partial C_0} + \ldots + \lambda_6 \frac{\partial h_6}{\partial C_0} = \frac{\partial U}{\partial C_0} - \lambda_1 + \lambda_3, \tag{A.4}$$

$$0 = \frac{\partial U}{\partial C_1} + \lambda_1 \frac{\partial h_1}{\partial C_1} + \ldots + \lambda_6 \frac{\partial h_6}{\partial C_1} = \frac{\partial U}{\partial C_1} - \lambda_2 + \lambda_4, \tag{A.5}$$

$$0 = \frac{\partial U}{\partial K_1} + \lambda_1 \frac{\partial h_1}{\partial K_1} + \ldots + \lambda_6 \frac{\partial h_6}{\partial K_1} = -\lambda_1 + \lambda_2 F'(K_1) + \lambda_5, \tag{A.6}$$

$$0 = \frac{\partial U}{\partial K_2} + \lambda_1 \frac{\partial h_1}{\partial K_2} + \ldots + \lambda_6 \frac{\partial h_6}{\partial K_2} = -\lambda_2 + \lambda_6, \tag{A.7}$$

as well as $\lambda_i \geq 0$, $\forall i \in \{1, \ldots, 6\}$. And complementary slackness gives $\lambda_i h_i = 0$, $\forall i \in \{1, \ldots, 6\}$.

Now let's try to simplify all the statements above. Knowing by Inada condition that

$$\lim_{c_i \to 0} \frac{\partial U}{\partial C_i} = +\infty$$

we infer that $C_0 > 0$ and $C_1 > 0$. From complementary slackness one can directly see that $\lambda_3 = \lambda_4 = 0$. Then by the strict concavity of $U(\cdot)$, $\frac{\partial U}{\partial C_i} > 0$. Therefore (A.4) and (A.5) simply imply that $\lambda_1 = \frac{\partial U}{\partial C_0} > 0$ and $\lambda_2 = \frac{\partial U}{\partial C_1} > 0$, as well as $\lambda_6 > 0$ from (A.7)—this further implies that $K_2 = 0$ by complementary slackness. And from h_2 one can see that $F(K_1) \geq C_1 > 0$, implying that $K_1 > 0$ as well as $\lambda_5 = 0$.

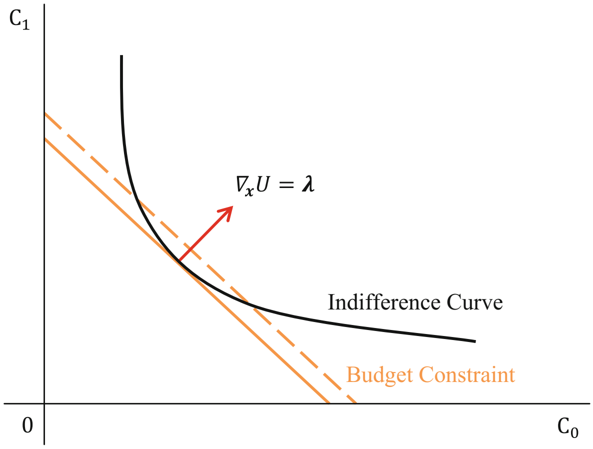

Fig. A.1 Shadow price

From (A.6) one can see that

$$F'(K_1) = \frac{\lambda_1}{\lambda_2} = \frac{\frac{\partial U}{\partial C_0}}{\frac{\partial U}{\partial C_1}}.$$

This is just the Euler condition. With two other conditions $h_1 = 0$ and $h_2 = 0$ one can easily solve for (C_0, C_1, K_1).

Figure A.1 gives a graphical interpretation to this inequality constrained optimization problem. The agent maximizes her life-time utility by choosing the consumption level for each of the two periods, on the basis of her intertemporal budget constraints. The optimum is achieved where the indifferent curve is exactly tangent to the frontier of the budget constraint. Suppose that we relax the budget constraint by adding a little bit to it, the vector $\boldsymbol{\lambda}$ just describes by how much the indifferent curve responds to the relaxation—in mathematical term, exactly the gradient $\nabla_{\mathbf{x}} U$ as the graph shows.

A.1.2 The General Deterministic Finite Horizon Optimization Problems: From Lagrangian to Hamiltonian

Let's take a closer look at the structure of the problem in the example. What makes it interesting is that the variables from the different periods are linked through the constraints (otherwise we can solve the problem by simply repeating dealing with the insulated $T + 1$ static problems), therefore one variable's change in one period may have pervasive effects into the other periods. So one may wonder whether there exists a solution method by exploiting such linkage—this is just the widely applied *optimal control* method.

As a general exposure, the prototype problem can be described as following. Think about the simplest case with only two variables k_t, c_t in each period $t \in \{0, 1, \ldots, T\}$, $T < +\infty$. The problem is to maximize the object function $U : \mathbb{R}^{2(T+1)} \to \mathbb{R}$ which is the summation of the function $u : \mathbb{R}^2 \to \mathbb{R}$ for each period, constrained by the intertemporal relations of k and c as well as the boundary values

$$\begin{aligned}
\max_{\{c_t\}} \;\; & U = \sum_{t=0}^{T} \frac{1}{(1+\rho)^t} u\left(k_t, c_t, t\right), \\
s.t. \;\; & k_{t+1} - k_t = g\left(k_t, c_t\right), \\
& k_{t=0} = k_0, \\
& k_{T+1} \geq \overline{k}_{T+1}.
\end{aligned}$$

k and c represent two kinds of variables. Variable k_t is the one with which each period starts and on which the decision is based, therefore it's usually called *state variable*. And variable c_t is the one the decision maker can change in each period and what is left over is fed back into the next period state variable, therefore it's usually called *control variable*. The constraint linking these variables across periods is called the *law of motion*.

If we express everything in continuous time, we only need to rewrite the summation by integration and the intertemporal change by the derivative with respect to time. However solving the continuous time problems with the Lagrangian would be a bit tricky. And in order to give the readers more exposures to the continuous time models, in the section that follows we start with building up the foundations of finite horizon optimization problems in continuous time. Readers may extend the same idea into the discrete time problems as an exercise.

Continuous Time

Suppose that time is continuous such that $t \in [0, T]$, $T \leq +\infty$. A typical deterministic continuous time optimization problem can be written as (often people simply set $\overline{k}(T)$ to be zero)

$$\begin{aligned}
\max_{\{c(t)\}} \;\; & U = \int_0^T e^{-\rho t} u\left(k(t), c(t), t\right) dt, \\
s.t. \;\; & \dot{k}(t) = g\left(k(t), c(t), t\right), \\
& k(0) = k_0, \\
& k(T) \geq \overline{k}(T).
\end{aligned}$$

Set up Lagrangian for this problem

$$\mathscr{L} = \int_0^T e^{-\rho t} u\left(k(t), c(t), t\right) dt + \int_0^T \mu(t) \left(g\left(k(t), c(t), t\right) - \dot{k}(t)\right) dt + \nu \left[k(T) - \overline{k}(T)\right], \qquad \text{(A.8)}$$

and we are supposed to find the first-order conditions with respect to $k(t)$ and $c(t)$. However the second term in $\mathscr{L}$ involves $\dot{k}(t)$, and this makes it difficult to derive it with respect to $k(t)$. Therefore we rewrite this term with integration by parts

$$\begin{aligned}\int_0^T \mu(t)\dot{k}(t)dt &= \mu(t)k(t)|_0^T - \int_0^T k(t)\dot{\mu}(t)dt \\ &= \mu(T)k(T) - \mu(0)k_0 - \int_0^T k(t)\dot{\mu}(t)dt.\end{aligned}$$

Insert it back into Lagrangian, we get

$$\begin{aligned}\mathscr{L} = &\int_0^T \left[e^{-\rho t} u\left(k(t), c(t), t\right) + \mu(t) g\left(k(t), c(t), t\right)\right] dt \\ &- \left(\mu(T)k(T) - \mu(0)k_0 - \int_0^T k(t)\dot{\mu}(t)dt\right) + \nu\left[k(T) - \overline{k}(T)\right].\end{aligned}$$

Define Hamiltonian function as

$$\mathcal{H}(k, c, \mu, t) = e^{-\rho t} u\left(k(t), c(t), t\right) + \mu(t) g\left(k(t), c(t), t\right), \tag{A.9}$$

then Lagrangian turns out to be

$$\mathscr{L} = \int_0^T \left[\mathcal{H}(k, c, \mu, t) + k(t)\dot{\mu}(t)\right] dt - \mu(T)k(T) + \mu(0)k_0 + \nu\left[k(T) - \overline{k}(T)\right].$$

Now let $k^*(t), c^*(t)$ be the optimal path for state and control variable. Define $p_1(t)$ as an arbitrary perturbation for $c^*(t)$, then a neighboring path around $c^*(t)$ can be defined as

$$c(t) = c^*(t) + \epsilon p_1(t).$$

Similarly define $p_2(t)$ as an arbitrary perturbation for $k^*(t)$, then a neighboring path around $k^*(t)$ can be defined as

$$k(t) = k^*(t) + \epsilon p_2(t)$$

as well as the end-period state variable

$$k(T) = k^*(T) + \epsilon dk(T).$$

Rewrite $\mathscr{L}$ in terms of ϵ

$$\mathscr{L}^*(\cdot, \epsilon) = \int_0^T \left[\mathcal{H}(k(t, \epsilon), c(t, \epsilon), t) + k(t, \epsilon)\dot{\mu}(t)\right] dt - \mu(T)k(T, \epsilon) + \mu(0)k_0 + \nu\left[k(T, \epsilon) - \overline{k}(T)\right]$$

and the first-order condition must hold

$$\begin{aligned}\left.\frac{\partial \mathscr{L}^*(\cdot,\epsilon)}{\partial \epsilon}\right|_{\epsilon\to 0} &= 0\\ &= \int_0^T \left[\frac{\partial \mathcal{H}}{\partial \epsilon} + \dot{\mu}(t)\frac{\partial k}{\partial \epsilon}\right] dt + (\nu - \mu(T))\,\frac{\partial k(T)}{\partial \epsilon}.\end{aligned}$$

By the chain rule

$$\begin{aligned}\frac{\partial \mathcal{H}}{\partial \epsilon} &= \frac{\partial \mathcal{H}}{\partial k}\frac{\partial k}{\partial \epsilon} + \frac{\partial \mathcal{H}}{\partial c}\frac{\partial c}{\partial \epsilon}\\ &= \frac{\partial \mathcal{H}}{\partial k}p_2(t) + \frac{\partial \mathcal{H}}{\partial c}p_1(t),\end{aligned}$$

and insert it into the first-order condition

$$\begin{aligned}\left.\frac{\partial \mathscr{L}^*(\cdot,\epsilon)}{\partial \epsilon}\right|_{\epsilon\to 0} &= \int_0^T \left[\frac{\partial \mathcal{H}}{\partial k}p_2(t) + \frac{\partial \mathcal{H}}{\partial c}p_1(t) + \dot{\mu}(t)p_2(t)\right] dt + (\nu - \mu(T))\,dk(T)\\ &= \int_0^T \left[\left(\frac{\partial \mathcal{H}}{\partial k} + \dot{\mu}(t)\right)p_2(t) + \frac{\partial \mathcal{H}}{\partial c}p_1(t)\right] dt + (\nu - \mu(T))\,dk(T)\\ &= 0.\end{aligned}$$

Therefore the first-order condition is equivalent to the following equations:

$$\frac{\partial \mathcal{H}}{\partial c} = 0, \tag{A.10}$$

$$\frac{\partial \mathcal{H}}{\partial k} = -\dot{\mu}(t), \tag{A.11}$$

$$\mu(T) = \nu. \tag{A.12}$$

Since we assume that $k^*(t), c^*(t)$ be the optimal path, then these conditions must hold. Condition (A.10) is called the *Euler equation*, and condition (A.11) is the *Maximum Principle*. Condition (A.12) requires that the terminal date costate variable, $\mu(T)$, equal the terminal date static Lagrange multiplier ν.

There is still something missing—Go back to the Lagrangian (A.8), we also have to address the concern on complementary slackness regarding the terminal time capital constraint, i.e.,

$$\nu\left[k(T) - \overline{k}(T)\right] = 0 \text{ with } \nu \geq 0.$$

Combining with condition (A.12) the complementary slackness is simply equivalent to

$$\mu(T)\left[k(T)-\overline{k}(T)\right]=0, \tag{A.13}$$

which is often called *transversality condition*. The intuition behind it is pretty clear: If there is strictly positive amount of more capital is left at the end date T than required, i.e., $k(T)-\overline{k}(T)>0$, then its price must be zero, i.e., $\mu(T)=0$, because it is worthless at all. On the other hand, if the capital stock at the end date has a strictly positive value, i.e., $\mu(T)>0$, then the agent must leave no excessive capital at all, i.e., $k(T)-\overline{k}(T)=0$.

Now the lengthy procedure which we went through simply tells us that one can actually start from the Hamiltonian and directly arrive at the first-order conditions. As a summary, to solve the deterministic multi-period optimization problem the whole procedure can be simplified into the following steps:

- Formulate the optimization problem as we did in the beginning of this section, and write down its Hamiltonian as (A.9);
- Derive the first-order conditions regarding control and state variables, respectively, such as (A.10) and (A.11);
- Add the transversality condition such as (A.13);
- Make further treatments on these equations to get whatever you are interested in.

In addition, please note that the menu also works for the problems with more than one state and/or control variables. The first-order conditions are in the same forms as Eqs. (A.10) and (A.11), for control and state variables, respectively.

Discrete Time

Since discrete time problems have the same nature as the ones for continuous time, therefore here we simply present the results without going into the details of proofs.

A typical deterministic discrete time optimization problem can be written as

$$\begin{aligned}
\max_{\{c_t\}} \ U &= \sum_{t=0}^{T}\frac{1}{(1+\rho)^t}u\left(k_t,c_t,t\right),\\
s.t.\ \ k_{t+1}-k_t &= g\left(k_t,c_t\right),\\
k_{t=0} &= k_0,\\
k_{T+1} &\geq \overline{k}_{T+1}.
\end{aligned}$$

Construct the present value Hamiltonian $\mathcal{H}_t = u(k_t, c_t, t) + \lambda_t g(k_t, c_t)$, and the first-order conditions are $\forall t \in \{0, 1, \ldots, T\}$

$$\frac{\partial \mathcal{H}_t}{\partial c_t} = 0,$$
$$\frac{\partial \mathcal{H}_t}{\partial k_t} = -(\lambda_t - \lambda_{t-1}),$$
$$\frac{\partial \mathcal{H}_t}{\partial \lambda_t} = k_{t+1} - k_t,$$

as well as the complementary slackness such that $\lambda_T \geq 0$ and $\lambda_T \left(k_{T+1} - \overline{k}_{T+1}\right) = 0$.

Present Versus Current Value Hamiltonian

Often what we consider in economics is the optimization problem regarding a discounted object function (in contrast to the prototype model by Ramsey), such as

$$\begin{aligned}
\max_{\{c(t)\}} \ & U = \int_0^T e^{-\rho t} u(k(t), c(t), t)\, dt, \\
s.t. \ & \dot{k}(t) = g(k(t), c(t), t), \\
& k(0) = k_0, \\
& k(T) - \overline{k}(T) \geq 0
\end{aligned}$$

in which ρ is the discount rate. As we did in Sect. A.1.2 the *present value* Hamiltonian can be expressed as

$$\mathcal{H} = e^{-\rho t} u(k(t), c(t), t) + \mu(t) g(k(t), c(t), t)$$

– notice that $\mu(t)$ is the present value shadow price, for it correspondents to the discounted object function. Same as before, the first-order conditions can be derived as Eqs. (A.10) and (A.11), plus the transversality condition (A.13).

Sometimes it's convenient to study a problem in the current time terms, and people set up the *current value* Hamiltonian as

$$\hat{\mathcal{H}} = u(k(t), c(t), t) + q(t) g(k(t), c(t), t)$$

in which $q(t) = \mu(t)e^{\rho t}$ is the current value shadow price, for it correspondents to the non-discounted object function. Now the first-order conditions are slightly different in $\frac{\partial \hat{\mathcal{H}}}{\partial k}$

$$\frac{\partial \hat{\mathcal{H}}}{\partial c} = 0, \tag{A.14}$$

$$\frac{\partial \hat{\mathcal{H}}}{\partial k} = \rho q(t) - \dot{q}(t), \tag{A.15}$$

as well as the transversality condition

$$q(T)e^{-\rho t}\left[k(T) - \overline{k}(T)\right] = 0. \tag{A.16}$$

Although Eq. (A.15) is a little more complicated, it is very intuitive. Notice that $\frac{\partial \hat{\mathcal{H}}}{\partial k}$ is just the marginal contribution of the capital to utility, i.e., the dividend received by the agent, the equation reflects the idea of asset pricing: given that $\dot{q}(t)$ is the capital gain (the change in the price of the asset), and ρ is the rate of return on an alternative asset, i.e., consumption, Eq. (A.15) says that at the optimum the agent is indifferent between the two types of the investment, for the overall rate of return to the capital,

$$\frac{\frac{\partial \hat{\mathcal{H}}}{\partial k} + \dot{q}(t)}{q(t)},$$

equals the return to consumption, ρ. For this reason, Eq. (A.15) is also called *non-arbitrage condition*.

A.2 Going Infinite

What will happen when we extend the results of finite horizon optimization problems into the ones with infinite horizon?

The optimization itself is only a little different—$T = +\infty$ in the object function

$$U = \int_0^{+\infty} e^{-\rho t} u\left(k(t), c(t), t\right) dt,$$

and there will be no terminal time condition any more, because the time doesn't terminate at all. But this makes a big change of the problem: Now the optimal time path looks like a kite—we hold the thread at hand, but we don't know where it ends.

Note that the principles behind the finite time optimization problem are that following the optimal time path nothing valuable is left over in the end of the world (such that $\mu(T)\left[k(T) - \overline{k}(T)\right]$ is non-positive) and the agent doesn't exit the world with debt (such that $\mu(T)\left[k(T) - \overline{k}(T)\right]$ is non-negative), which are captured in the transversality condition. To maintain the same principles in the infinite time horizon, we may assume that there is an end of the world, but after a nearly infinitely long time. Therefore we may impose a similar transversality condition for the problems of infinite time horizon

$$\lim_{T \to +\infty} \mu(T)\left[k(T) - \overline{k}(T)\right] = 0,$$

i.e., the value of the state variable must be *asymptotically* zero: If the quantity of $k(T)$ remains different from the constraint asymptotically, then its price, $\mu(T)$, must approach 0 asymptotically; if $k(T) - \overline{k}(T)$ grows forever at a positive rate, then the price $\mu(T)$ must approach 0 at a faster rate so that the product, $\mu(T)\left[k(T) - \overline{k}(T)\right]$, goes to 0.

Dynamic Programming

B

Dynamic programming is another powerful tool to solve dynamic optimization problems.

B.1 Dynamic Programming: The Theoretical Foundation

The theoretical foundation of dynamic programming is *contraction mapping*. Let's start with some formal definitions.

Definition B.1 A *metric space* (S, ρ) is a non-empty set S and a *metric*, or *distance* $\rho : S \times S \to \mathbb{R}$, which is defined as a mapping, $\forall x, y, v$, with

1. $\rho(x, y) = 0 \Leftrightarrow x = y$,
2. $\rho(x, y) = \rho(x, y)$, and
3. $\rho(x, y) \leq \rho(x, v) + \rho(v, y)$.

For example, a plane $\left(\mathbb{R}^2, d_2\right)$ is a metric space, in which the metric $d_2 : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ is defined as

$$d_2(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}||_2 = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}, \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^2,$$

i.e., $d_2(\cdot)$, or $|| \cdot ||_2$, is just the Euclidean distance.

Definition B.2 A *norm* is a mapping $\mathbb{R}^n \ni x \mapsto ||x|| \in \mathbb{R}$ on $\mathbb{R}^n$, with

1. $\forall x \in \mathbb{R}^n, ||x|| = 0 \Leftrightarrow x = 0$,
2. $\forall x \in \mathbb{R}^n, \forall \alpha \in \mathbb{R}, ||\alpha x|| = |\alpha| ||x||$ and
3. $\forall x, y \in \mathbb{R}^n, ||x + y|| \leq ||x|| + ||y||$.

J. Cao, G. Illing, *Instructor's Manual for* Money: Theory and Practice,
Springer Texts in Business and Economics,
https://doi.org/10.1007/978-3-030-23618-2

In the definition of metric space, the set S is just arbitrary. It can be a subset of n dimensional space, i.e., $S \subseteq \mathbb{R}^n$, but it can also be a *function space* $B(X)$—a set containing all (normally, bounded) functions mapping a set X to $\mathbb{R}$, $B : X \to \mathbb{R}$. Then we define a *supremum norm* on such function space

$$d_\infty = ||f - g||_\infty = \sup_{x \in X} |f(x) - g(x)|, \forall f, g \in B(X),$$

and this metric space of bounded functions on X with supremum norm is denoted by $(B(X), d_\infty)$.

Having defined all the necessary terms, we continue with a special mapping.

Definition B.3 Suppose a metric space (S, ρ) and a function $T : S \to S$ mapping S to itself. T is a *contraction mapping* with *modulus* β, if $\exists \beta \in (0, 1)$, $\rho(Tx, Ty) \leq \beta\rho(x, y)$, $\forall x, y \in S$.

An example in Fig. B.1 shows a contraction mapping $T : (0, 1) \to (0, 1)$. The distance between images Tx and Ty is less than $|y - x|$. One may notice that under a contraction mapping like this, a *fixed point* $v \in S = (0, 1)$ exists such that $Tv = v$. Indeed, the following theorem tells us that this is a general phenomenon.

Theorem B.1 (Contraction Mapping Theorem) *If* (S, ρ) *is a complete metric space and* $T : S \to S$ *is a contraction mapping with modulus* β*, then*

1. T *has a unique fixed point* $v \in S$*, and*
2. $\forall v_0 \in S$, $\rho\left(T^n v_0, v\right) \leq \beta^n \rho\left(v_0, v\right)$, $n \in N$.

T^n means that the mapping is applied for n times. But what does the Theorem imply for our questions on dynamic programming? Well, look at the prototype

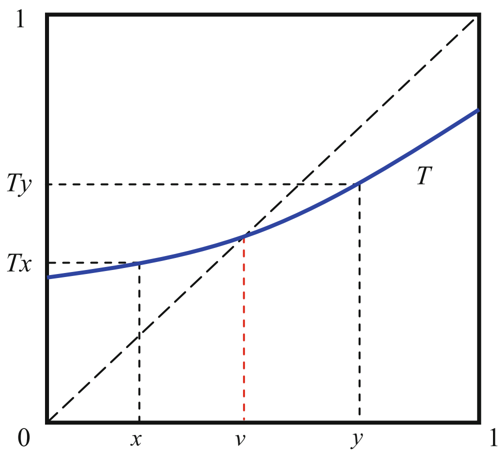

Fig. B.1 Contraction mapping

problem

$$V(k_t) = \max_{c_t, k_{t+1}} \{u(c_t) + \beta V(k_{t+1})\}. \tag{B.1}$$

The right-hand side is just a mapping of function $V(\cdot)$, mapping the function space to itself. And the equilibrium solution making $V = TV$ is simply a fixed point of the mapping! Now the Contraction Mapping Theorem tells us that a unique fixed point exists if the mapping is a contraction mapping, therefore, if we want to say that there is a unique solution for the prototype problem, we have to make sure that the mapping in (B.1) is a contraction mapping.

However, showing a mapping to be a contraction one directly by definition is usually tricky. Fortunately, the following theorem makes the task more tractable.

Theorem B.2 (Blackwell's Sufficient Conditions for a Contraction) *Suppose $X \subseteq \mathbb{R}^n$ and $B(X)$ is the function space for all bounded functions $f : X \to \mathbb{R}$ with supremum norm $||\cdot||_\infty$. If a mapping $T : B(X) \to B(X)$ satisfies*

1. *(Monotonicity condition) $\forall f, g \in B(X)$ and $\forall x \in X$ with $f(x) \leq g(x)$ implies $(Tf)(x) \leq (Tg)(x)$, $\forall x \in X$;*
2. *(Discounting condition) $\exists \beta \in (0, 1)$ such that*

$$[T(f+a)](x) = f(x) + a \leq (Tf)(x) + \beta a, \forall f \in B(X), a \geq 0, x \in X,$$

then T is a contraction mapping with modulus β.

Now we can show that our prototype problem of dynamic programming satisfies Blackwell's sufficient conditions for a contraction, therefore there exists a unique fixed point for the mapping. Suppose that we are going to solve the following dynamic optimization problem with exact utility and production functions,

$$V(k) = \max_{k'} \left\{ \frac{c^{1-\theta}}{1-\theta} + \beta V(k') \right\} = \max_{k'} \left\{ \frac{\left[Ak^\alpha + (1-\delta)k - k'\right]^{1-\theta}}{1-\theta} + \beta V(k') \right\}$$

$$s.t. \quad c + k' = Ak^\alpha + (1-\delta)k,$$

in which we write k and k' instead of k_t and k_{t+1} for simplicity, and the right-hand side defines the mapping T. Since k takes its maximum value when $c = 0$, k is thus bounded above by $\overline{k}$ such that

$$0 + \overline{k} = A\overline{k}^\alpha + (1-\delta)\overline{k},$$

$$\overline{k} = \left(\frac{A}{\delta}\right)^{\frac{1}{1-\alpha}}.$$

Therefore define the state space $X \subseteq \left[0, \overline{k}\right]$ as a complete subspace of $\mathbb{R}$, and $B(X)$ the function space of all bounded continuous functions on X with supremum norm. Then we need to show that the mapping $T : B(X) \to B(X)$ in the complete (why?) metric space $(B(X), d_\infty)$ satisfies Blackwell's sufficient conditions for a contraction.

Check the monotonicity condition. Let $f(x) \leq g(x)$, $\forall x \in X$, then

$$
\begin{aligned}
Tf(k) &= \max_{k'} \left\{ \frac{\left[Ak^\alpha + (1-\delta)k - k'\right]^{1-\theta}}{1-\theta} + \beta f(k') \right\} \\
&\leq \max_{k'} \left\{ \frac{\left[Ak^\alpha + (1-\delta)k - k'\right]^{1-\theta}}{1-\theta} + \beta \left[f(k') + g(k') - f(k') \right] \right\} \\
&= \max_{k'} \left\{ \frac{\left[Ak^\alpha + (1-\delta)k - k'\right]^{1-\theta}}{1-\theta} + \beta g(k') \right\} \\
&= Tg(k).
\end{aligned}
$$

Check the discounting condition.

$$
\begin{aligned}
\left[T(f+a)\right](k) &= \max_{k'} \left\{ \frac{\left[Ak^\alpha + (1-\delta)k - k'\right]^{1-\theta}}{1-\theta} + \beta f(k') + \beta a \right\} \\
&= Tf(k) + \beta a.
\end{aligned}
$$

Both conditions hold. Therefore the dynamic optimization problem has a unique equilibrium solution.

B.2 Defining a Dynamic Programming Problem

Consider a general discrete-time optimization problem

$$
\begin{aligned}
&\max_{\{c_t, k_{t+1}\}_{t=0}^{+\infty}} \sum_{t=0}^{+\infty} \beta^t u(c_t) \\
&s.t. \;\; k_{t+1} = f(c_t, k_t).
\end{aligned}
$$

You may interpret this problem in an economic context. Given any capital stock level k_t in period t, a representative agent maximizes her life-long utility by choosing her period t consumption level c_t (such variables whose value is directly chosen by individuals are called *control variables*; in contrast, those not directly chosen by individuals are called *state variables* such as k_t). So essentially the optimization problem is to seek a *policy function* $c_t = h(k_t)$ which maps the state k_t

into the control c_t. As soon as c_t is chosen, the *transition function* $k_{t+1} = f(c_t, k_t)$ determines next period state k_{t+1} and the same procedure repeats. Such procedure is *recursive*.

The basic idea of dynamic programming is to collapse a multi-periods problem into a sequence of two-periods problem at any t using the recursive nature of the problem

$$\begin{aligned}
V(k_t) &= \max_{c_t, k_{t+1}} \sum_{i=0}^{+\infty} \beta^i u(k_{t+i}) \\
&= \max_{c_t, k_{t+1}} \left\{ u(c_t) + \beta \sum_{i=0}^{+\infty} \beta^i u(c_{t+i+1}) \right\} \\
&= \max_{c_t, k_{t+1}} \{u(c_t) + \beta V(k_{t+1})\} \\
s.t. \quad & k_{t+1} = f(c_t, k_t).
\end{aligned} \tag{B.2}$$

Equation $V(k_t) = \max_{c_t, k_{t+1}} \{u(c_t) + \beta V(k_{t+1})\}$ is known as *Bellman equation*. The *value function* $V(\cdot)$ is only a function of state variable k_t because the optimal value of c_t is just a function of k_t. Then the original problem can be solved by the methods we learned for two-periods problems plus some tricks.

B.3 Getting the Euler Equation

The key step now is to find the proper first-order conditions. There are several possible approaches, and readers may pick up one of them with which he or she feels comfortable.

B.3.1 Using Lagrangian

Since the problem looks pretty similar to a maximization problem with equality constraint, one may suggest Lagrangian—Let's try.

Rewrite $V(k_t)$ as

$$V(k_t) = \max_{c_t, k_{t+1}} \underbrace{\{u(c_t) + \beta V(k_{t+1}) + \lambda_t [f(c_t, k_t) - k_{t+1}]\}}_{\mathscr{L}_t}.$$

Step 1 Since $V(k_t)$ is maximized value for Lagrangian, the first-order conditions with respect to c_t and k_{t+1} must hold,

$$u'(c_t) + \lambda_t \frac{\partial f(c_t, k_t)}{\partial c_t} = 0, \tag{B.3}$$

$$\beta V'(k_{t+1}) - \lambda_t = 0. \tag{B.4}$$

Step 2 Since $V(k_t)$ is optimized at k_t, then

$$
\begin{aligned}
V'(k_t) &= u'(c_t)\frac{dc_t}{dk_t} + \beta V'(k_{t+1})\frac{dk_{t+1}}{dk_t} + \frac{d\lambda_t}{dk_t}\left[f(c_t,k_t) - k_{t+1}\right] \\
&\quad + \lambda_t\left[\frac{\partial f(c_t,k_t)}{\partial k_t} + \frac{\partial f(c_t,k_t)}{\partial c_t}\frac{dc_t}{dk_t} - \frac{dk_{t+1}}{dk_t}\right] \\
&= \underbrace{\left[u'(c_t) + \lambda_t\frac{\partial f(c_t,k_t)}{\partial c_t}\right]}_{(A)}\frac{dc_t}{dk_t} + \underbrace{\left[\beta V'(k_{t+1}) - \lambda_t\right]}_{(B)}\frac{dk_{t+1}}{dk_t} \\
&\quad + \frac{d\lambda_t}{dk_t}\underbrace{\left[f(c_t,k_t) - k_{t+1}\right]}_{(C)} + \lambda_t\frac{\partial f(c_t,k_t)}{\partial k_t}.
\end{aligned}
$$

$(A) = 0$ by (B.3), $(B) = 0$ by (B.4), and $(C) = 0$ by first-order condition of Lagrangian. Therefore

$$
V'(k_t) = \lambda_t\frac{\partial f(c_t,k_t)}{\partial k_t}. \tag{B.5}
$$

Step 3 By (B.3) and (B.4) eliminate λ_t

$$
u'(c_t) + \beta V'(k_{t+1})\frac{\partial f(c_t,k_t)}{\partial c_t} = 0.
$$

And since t is arbitrarily taken, this equation must hold if we take one period backward

$$
u'(c_{t-1}) + \beta V'(k_t)\frac{\partial f(c_{t-1},k_{t-1})}{\partial c_{t-1}} = 0. \tag{B.6}
$$

Next insert (B.3) into (B.5) to eliminate λ and (B.6) into (B.5) to eliminate $V'(k_t)$, then Euler equation is obtained.

B.3.2 Tracing Dynamics of Costate Variable

The other way of thinking is to trace the dynamics of costate variable $V(k_t)$.

Step 1 Since $V(k_t)$ is maximized value of $u(c_t) + \beta V(k_{t+1})$, then the first-order condition with respect to c_t gives

$$
u'(c_t) + \beta V'(k_{t+1})\frac{\partial k_{t+1}}{\partial c_t} = u'(c_t) + \beta V'(k_{t+1})\frac{\partial f(c_t,k_t)}{\partial c_t} = 0. \tag{B.7}
$$

Step 2 Since $V(k_t)$ is optimized at k_t, then

$$\begin{aligned} V'(k_t) &= u'(c_t)\frac{dc_t}{dk_t} + \beta V'(k_{t+1})\left[\frac{\partial f(c_t,k_t)}{\partial k_t} + \frac{\partial f(c_t,k_t)}{\partial c_t}\frac{dc_t}{dk_t}\right] \\ &= \left[u'(c_t) + \beta V'(k_{t+1})\frac{\partial f(c_t,k_t)}{\partial c_t}\right]\frac{dc_t}{dk_t} + \beta V'(k_{t+1})\frac{\partial f(c_t,k_t)}{\partial k_t}. \end{aligned}$$

Apply (B.7) and get

$$V'(k_t) = \beta V'(k_{t+1})\frac{\partial f(c_t,k_t)}{\partial k_t}. \tag{B.8}$$

Since t is arbitrarily taken, (B.7) also holds for one period backward, i.e.,

$$V'(k_t) = -\frac{u'(c_{t-1})}{\beta \frac{\partial f(c_{t-1},k_{t-1})}{\partial c_{t-1}}}. \tag{B.9}$$

Step 3 Apply (B.7) and (B.9) into (B.8) and obtain Euler equation.

B.3.3 Using Envelope Theorem

We may also use the Envelope Theorem to find the first-order condition.

Theorem B.3 *Suppose that value function $m(a)$ is defined as following:*

$$m(a) = \max_x f(x(a), a).$$

Then the total derivative of $m(a)$ with respect to a equals the partial derivative of $f(x(a), a)$ with respect to a, if $f(x(a), a)$ is evaluated at $x = x(a)$ that maximizes $f(x(a), a)$, i.e.,

$$\frac{dm(a)}{da} = \left.\frac{\partial f(x(a), a)}{\partial a}\right|_{x=x(a)}.$$

Proof Since $m(a)$ is maximized value of $f(x(a), a)$ at $x = x(a)$, then

$$\frac{\partial f(x(a), a)}{\partial x} = 0$$

by the first-order condition. Therefore the total derivative of $m(a)$ with respect to a is

$$\begin{aligned}\frac{dm(a)}{da} &= \frac{\partial f(x(a), a)}{\partial x}\frac{dx(a)}{da} + \frac{\partial f(x(a), a)}{\partial a} \\ &= \frac{\partial f(x(a), a)}{\partial a}\end{aligned}$$

since the first term is equal to 0. ■

Solve the budget constraint for c_t and get $c_t = g(k_t, k_{t+1})$. Apply it to $V(k_t)$ and get a univariate optimization problem

$$V(k_t) = \max_{k_{t+1}} \{u(g(k_t, k_{t+1})) + \beta V(k_{t+1})\}.$$

Step 1 Similar as before, since $V(k_t)$ is the maximized value of $u(g(k_t, k_{t+1})) + \beta V(k_{t+1})$, then the first-order condition with respect to k_{t+1} gives

$$u'(g(k_t, k_{t+1}))\frac{\partial g(k_t, k_{t+1})}{\partial k_{t+1}} + \beta V'(k_{t+1}) = 0. \tag{B.10}$$

Step 2 Since $V(k_t)$ is already optimized at k_t, differentiating $V(k_t)$ with respect to k_t gives

$$\frac{dV(k_t)}{dk_t} = \underbrace{\frac{\partial V(k_t)}{\partial k_t}}_{(A)} + \underbrace{\frac{\partial V(k_t)}{\partial k_{t+1}}\frac{\partial k_{t+1}}{\partial k_t}}_{(B)}. \tag{B.11}$$

This is pretty intuitive: k_t may generate a direct effect on $V(k_t)$ as part (A) shows; however, k_t may also generate an indirect effect on $V(k_t)$ through k_{t+1} (remember the dynamic budget constraint). And since $V(k_t)$ is optimized by k_{t+1}, the first-order condition implies that $\frac{\partial V(k_t)}{\partial k_{t+1}} = 0$. Therefore Eq. (B.11) becomes

$$V'(k_t) = \frac{\partial V(k_t)}{\partial k_t} = u'(g(k_t, k_{t+1}))\frac{dg(k_t, k_{t+1})}{dk_t} \tag{B.12}$$

which is also called *Benveniste–Scheinkman condition*.

Step 3 Similar as before, take one period forward for (B.12) and apply it into (B.10) then obtain Euler equation.

B.3.4 Example

Consider a discrete time Ramsey problem for a decentralized economy

$$\max_{\{c_t, b_t\}_{t=0}^{+\infty}} \sum_{t=0}^{+\infty} \beta^t u(c_t)$$
$$s.t. \quad b_{t+1} - b_t = w_t + r_t b_t - c_t - n b_t.$$

Collapse the infinite horizon problem into a sequence of two-periods problem

$$\begin{aligned}
V(b_t) &= \max_{c_t, b_{t+1}} \sum_{i=0}^{+\infty} \beta^i u(c_{t+i}) \\
&= \max_{c_t, b_{t+1}} \left\{ u(c_t) + \beta \sum_{i=0}^{+\infty} \beta^i u(c_{t+i+1}) \right\} \\
&= \max_{c_t, b_{t+1}} \{u(c_t) + \beta V(b_{t+1})\} \\
s.t. \quad & b_{t+1} = w_t + (1 + r_t) b_t - c_t - n b_t.
\end{aligned}$$

Now we solve the problem with all three approaches.

Using Lagrangian

Rewrite Bellman equation in Lagrangian form

$$V(b_t) = \max_{c_t, b_{t+1}} \{u(c_t) + \beta V(b_{t+1}) + \lambda_t [w_t + (1 + r_t) b_t - c_t - n b_t - b_{t+1}]\}.$$

Step 1 The first-order conditions of Lagrangian are

$$u'(c_t) - \lambda_t = 0, \tag{B.13}$$
$$\beta V'(b_{t+1}) - \lambda_t = 0, \tag{B.14}$$

and eliminate λ_t to get

$$u'(c_t) = \beta V'(b_{t+1}). \tag{B.15}$$

Step 2 Now differentiate $V(b_t)$ at b_t

$$\begin{aligned}
V'(b_t) = u'(c_t) \frac{dc_t}{db_t} + \beta V'(b_{t+1})(1 + r_t - n) \\
+ \frac{d\lambda_t}{dk_t} \underbrace{[w_t + (1 + r_t) b_t - c_t - n b_t - b_{t+1}]}_{=0}
\end{aligned}$$

$$
+\lambda_t \left[(1+r_t-n) - \frac{dc_t}{db_t} - (1+r_t-n) \right]
$$

$$
= \underbrace{\left[u'(c_t) - \lambda_t\right]}_{=0} \frac{dc_t}{db_t} + \underbrace{\left[\beta V'(b_{t+1}) - \lambda_t\right]}_{=0} (1+r_t-n) + \lambda_t(1+r_t-n).
$$

That is,

$$
V'(b_t) = \lambda_t(1+r_t-n). \tag{B.16}
$$

Step 3 Insert (B.13) and (B.15) into (B.16) and get the desired result.

Tracing Dynamics of Costate Variable

Now solve the same problem by tracing the dynamics of the costate variable.

Step 1 Since $V(b_t)$ is the maximized value of $u(c_t)+\beta V(b_{t+1})$, then the first-order condition with respect to b_{t+1} gives

$$
-u'(c_t) + \beta V'(b_{t+1}) = 0. \tag{B.17}
$$

Step 2 Now differentiate $V(b_t)$ at b_t

$$
V'(b_t) = u'(c_t)\frac{\partial c_t}{\partial k_t} + \beta V'(b_{t+1}) \left[(1+r_t-n) - \frac{\partial c_t}{\partial k_t} \right].
$$

That is just

$$
V'(b_t) = \beta(1+r_t-n)V'(b_{t+1}). \tag{B.18}
$$

Step 3 Insert (B.17) twice into (B.18) and get the desired result.

Using Envelope Theorem

Now solve the same problem with Envelope Theorem.

Step 1 Since $V(b_t)$ is maximized value of $u(c_t) + \beta V(b_{t+1})$, then the first-order condition with respect to b_{t+1} gives

$$
-u'(c_t) + \beta V'(b_{t+1}) = 0. \tag{B.19}
$$

Step 2 Now the problem is to find $V'(b_{t+1})$. Differentiate $V(b_t)$ at b_t

$$
V'(b_t) = \frac{\partial V(b_t)}{\partial b_t} = (1+r_t-n)u'(c_t). \tag{B.20}
$$

Step 3 Take one period backward for (B.19) and insert into (B.20) to obtain the Euler equation.

B.4 Solving for the Policy Function

As seen in previous sections policy function $c_t = h(k_t)$ captures the optimal solution for each period given the corresponding state variable, therefore one may desire to get the solution of the policy function. Dynamic programming method has a special advantage for this purpose, and we will see several approaches in the following.

Now consider the problem of Brock and Mirman (1972). Suppose utility function takes the form $u_t = \ln c_t$ and the production function follows Cobb–Douglas technology. No depreciation and population growth.

$$\max_{\{c_t,k_t\}_{t=0}^{+\infty}} \sum_{t=0}^{*\infty} \beta^t \ln c_t$$
$$s.t. \;\; k_{t+1} = k_t^\alpha - c_t.$$

B.4.1 Solution by Iterative Substitution the Euler Equation

Recall that the recursive structure of dynamic programming method implies that the problem that the optimizer faces in each period is the same as that she faces last period or next period, so the solution to such a problem should be time-invariant. Thus one can start from deriving the solution under some circumstances and iterate it on an infinite time horizon until it is time invariant. However this approach only works when the problem is simple.

Forward Induction

Set up the Bellman equation and solve for Euler equation. This gives

$$\begin{aligned} \frac{1}{c_t} &= \alpha\beta \frac{k_{t+1}^{\alpha-1}}{c_{t+1}} \\ \frac{k_{t+1}}{c_t} &= \alpha\beta \frac{k_{t+1}^{\alpha}}{c_{t+1}} \\ \frac{k_t^\alpha - c_t}{c_t} &= \alpha\beta \frac{k_{t+1}^{\alpha}}{c_{t+1}} \\ \frac{k_t^\alpha}{c_t} &= \alpha\beta \frac{k_{t+1}^{\alpha}}{c_{t+1}} + 1. \end{aligned}$$

Apply this condition to itself and get a geometric serial

$$\begin{aligned}\frac{k_t^\alpha}{c_t} &= 1+\alpha\beta\left(1+\alpha\beta\frac{k_{t+2}^\alpha}{c_{t+2}}\right)\\ &= 1+\alpha\beta+\alpha^2\beta^2+\alpha^3\beta^3+\ldots\\ &= \frac{1}{1-\alpha\beta},\end{aligned}$$

(why?) and this gives

$$c_t = (1-\alpha\beta)k_t^\alpha.$$

Another way to see this is exploring saving rate dynamics. Express c_t by k_t and k_{t+1}

$$\frac{1}{k_t^\alpha - k_{t+1}} = \beta\frac{1}{k_{t+1}^\alpha - k_{t+2}}\alpha k_{t+1}^{\alpha-1}. \tag{B.21}$$

Define saving rate at time t as

$$s_t = \frac{k_{t+1}}{k_t^\alpha},$$

Rearranging (B.21) gives

$$\begin{aligned}\frac{1}{k_t^\alpha}\frac{1}{1-s_t} &= \beta\frac{1}{k_{t+1}^\alpha}\frac{1}{1-s_{t+1}}\alpha k_{t+1}^{\alpha-1}\\ \frac{k_{t+1}}{k_t^\alpha}\frac{1}{1-s_t} &= \frac{\alpha\beta}{1-s_{t+1}}\\ \frac{s_t}{1-s_t} &= \frac{\alpha\beta}{1-s_{t+1}},\end{aligned}$$

and this is

$$s_{t+1} = 1+\alpha\beta-\frac{\alpha\beta}{s_t}. \tag{B.22}$$

Plot s_{t+1} as a function of s_t as Fig. B.2, and this gives two solutions, $\alpha\beta < 1$ and 1 respectively. Only the former is plausible. Then

$$c_t = (1-s_t)k_t^\alpha = (1-\alpha\beta)k_t^\alpha.$$

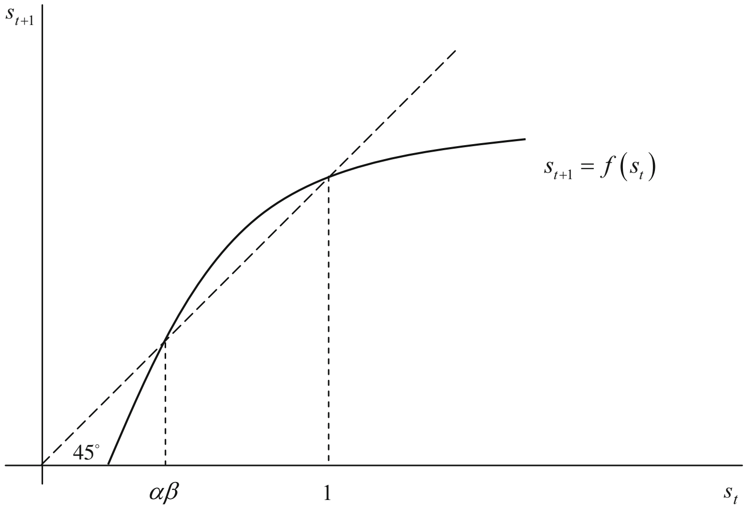

Fig. B.2 Solution for s_t

Backward Induction

Suppose that the world ends after some finite period T. Then surely for the last period

$$s_T = 0.$$

Apply this to (B.22)

$$s_T = 0 = 1 + \alpha\beta - \frac{\alpha\beta}{s_{T-1}},$$

solve to get

$$s_{T-1} = \frac{\alpha\beta}{1 + \alpha\beta}.$$

Continue this process,

$$s_{T-1} = \frac{\alpha\beta}{1 + \alpha\beta} = 1 + \alpha\beta - \frac{\alpha\beta}{s_{T-2}},$$

and this yields

$$s_{T-2} = \frac{\alpha\beta + \alpha^2\beta^2}{1 + \alpha\beta + \alpha^2\beta^2}.$$

We find that for any t between 0 and T

$$s_t = \frac{\sum_{i=1}^{T-t} \alpha^i \beta^i}{1 + \sum_{i=1}^{T-t} \alpha^i \beta^i} = \frac{\frac{\alpha\beta(1-\alpha^{T-t}\beta^{T-t})}{1-\alpha\beta}}{1 + \frac{\alpha\beta(1-\alpha^{T-t}\beta^{T-t})}{1-\alpha\beta}} = \frac{\alpha\beta(1-\alpha^{T-t}\beta^{T-t})}{1-\alpha\beta+\alpha\beta(1-\alpha^{T-t}\beta^{T-t})}.$$

And in the limit

$$\lim_{T-t\to+\infty} s_t = \alpha\beta$$

implying that

$$c_t = (1-\alpha\beta)k_t^{\alpha}.$$

B.4.2 Solution by Value-Function Iteration

Another solution method is based on iteration of the value function. The value function actually will be different in each period, just as we earlier found the function $g(k_t)$ was different depending on how close we were to the terminal period. But it can be shown (but we do not show this here) that as we iterate through time, the value function converges, just as $g(k_t)$ converged in our earlier example as we iterated back further away from the terminal period. This suggests that if we iterate on an initial guess for the value function, even a guess we know is incorrect, the iterations eventually will converge to the true function.

Guess and Verify

One may guess the form of solution and try to verify whether it's true. We guess that

$$V(k_t) = A + B \ln k_t.$$

Then the problem becomes

$$\begin{aligned} V(k_t) &= \max_{c_t, k_{t+1}} \{\ln c_t + \beta V(k_{t+1})\} \\ &= \max_{c_t, k_{t+1}} \{\ln c_t + \beta(A + B \ln k_{t+1})\} \\ s.t. \quad & k_{t+1} = k_t^{\alpha} - c_t. \end{aligned}$$

The first-order condition with respect to k_{t+1} yields

$$\begin{aligned} -\frac{1}{c_t} + \frac{\beta B}{k_{t+1}} &= 0, \\ k_{t+1} &= \beta B(k_t^{\alpha} - k_{t+1}), \end{aligned}$$

$$
\begin{aligned}
k_{t+1} &= \frac{\beta B}{1+\beta B} k_t^{\alpha}, \\
c_t &= \frac{1}{1+\beta B} k_t^{\alpha}.
\end{aligned}
$$

Then apply the results to the Bellman equation, and the following must hold if our conjecture is right

$$
\begin{aligned}
V(k_t) &= \ln\left(\frac{\beta B}{1+\beta B} k_t^{\alpha}\right) + \beta\left[A + B\ln\left(\frac{1}{1+\beta B} k_t^{\alpha}\right)\right] \\
&= \underbrace{\ln \beta B + \beta A - (1+\beta B)\ln(1+\beta B)}_{A} + \underbrace{\alpha(1+\beta B)}_{B} \ln k_t \\
&= A + B \ln k_t.
\end{aligned}
$$

Solve to get

$$
\begin{aligned}
B &= \frac{\alpha}{1-\alpha\beta}, \\
A &= \frac{1}{1-\beta}\left[\ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\ln \alpha\beta\right], \\
c_t &= \frac{1}{1+\beta B} k_t^{\alpha} = (1-\alpha\beta) k_t^{\alpha}.
\end{aligned}
$$

and therefore

$$
\begin{aligned}
k_{t+1} &= \frac{\beta B}{1+\beta B} k_t^{\alpha} = \alpha\beta k_t^{\alpha}, \\
c_t &= \frac{1}{1+\beta B} k_t^{\alpha} = (1-\alpha\beta) k_t^{\alpha}.
\end{aligned}
$$

Value-Function Iteration

Unfortunately few problems can be solved by simple conjectures. As a last resort one needs onerous effort on value functions. Suppose that the world ends after some finite period T. Then surely

$$
V(k_{T+1}) = 0,
$$

as well as

$$
c_T = k_T^{\alpha}, \text{ and } k_{T+1} = 0.
$$

Apply these in Bellman equation,

$$V(k_T) = \ln k_T^\alpha + \beta V(k_{T+1}) = \ln k_T^\alpha.$$

For one period backward,

$$\begin{aligned} V(k_{T-1}) &= \max_{c_{T-1},k_T} \{\ln(c_{T-1}) + \beta V(k_T)\} \\ &= \max_{c_{T-1},k_T} \left\{\ln(c_{T-1}) + \beta \ln k_T^\alpha\right\} \\ s.t. \quad k_T &= k_{T-1}^\alpha - c_{T-1}. \end{aligned}$$

This is simply a two-period intertemporal optimization with an equality constraint. Using Lagrangian

$$\mathscr{L} = \ln(c_{T-1}) + \beta \ln k_T^\alpha + \lambda \left[k_{T-1}^\alpha - c_{T-1} - k_T\right],$$

first-order conditions give

$$\begin{aligned} \frac{\partial \mathscr{L}}{\partial c_{T-1}} &= \frac{1}{c_{T-1}} - \lambda = 0, \\ \frac{\partial \mathscr{L}}{\partial k_T} &= \alpha\beta \frac{k_T^{\alpha-1}}{k_T^\alpha} - \lambda = \alpha\beta \frac{1}{k_T} - \lambda = 0, \\ \frac{\partial \mathscr{L}}{\partial \lambda} &= k_{T-1}^\alpha - c_{T-1} - k_T = 0. \end{aligned}$$

Solve to get

$$\begin{aligned} c_{T-1} &= \frac{1}{1+\alpha\beta} k_{T-1}^\alpha, \\ k_T &= \frac{\alpha\beta}{1+\alpha\beta} k_{T-1}^\alpha. \end{aligned}$$

Then $V(k_{T-1})$ can be expressed as

$$\begin{aligned} V(k_{T-1}) &= \ln\left(\frac{1}{1+\alpha\beta} k_{T-1}^\alpha\right) + \beta \ln\left(\frac{\alpha\beta}{1+\alpha\beta} k_{T-1}^\alpha\right)^\alpha \\ &= \alpha\beta \ln(\alpha\beta) - (1+\alpha\beta)\ln(1+\alpha\beta) + (1+\alpha\beta)\ln k_{T-1}^\alpha. \end{aligned}$$

Again take one period backward,

$$\begin{aligned} V(k_{T-2}) = \max_{c_{T-2},k_{T-1}} & \{\ln(c_{T-2}) + \beta V(k_{T-1})\} \\ s.t.\ \ & k_{T-1} = k_{T-2}^\alpha - c_{T-2}, \end{aligned}$$

and the same procedure applies. After several rounds you may find that for time t long before T the value function converges to

$$\begin{aligned} V(k_t) = \max_{c_t, k_{t+1}} & \left\{ \ln c_t + \beta \left[\frac{1}{1-\beta} \left(\ln(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta} \ln \alpha\beta \right) \right. \right. \\ & \left. \left. + \frac{\alpha}{1-\alpha\beta} \ln k_{t+1} \right] \right\} \\ s.t. \quad & k_{t+1} = k_t^{\alpha} - c_t. \end{aligned}$$

As before since $V(k_t)$ is the maximized value the first-order condition with respect to k_{t+1} still holds

$$-\frac{1}{c_t} + \frac{\alpha\beta}{1-\alpha\beta} \frac{1}{k_{t+1}} = 0,$$

this yields

$$\begin{aligned} c_t &= (1-\alpha\beta) k_t^{\alpha} \\ k_{t+1} &= \alpha\beta k_t^{\alpha}. \end{aligned}$$

Although this solution method is very cumbersome and computationally demanding since it rests on brutal-force iteration of the value function, it has the advantage that it always works if the solution exists. Actually, the convergence of the value function is not incidental. The *Contraction Mapping Theorem* tells us that the convergence result is always achieved as long as the value function contains a *contraction mapping* (which works for most of dynamic optimization problems under the neoclassical assumptions). Such result is crucial for both theory and application. In theory, it ensures that a unique equilibrium solution (the *fixed point*) exists so that we can say what happens in the long run; in application, it implies that even we start iteration from an arbitrary value function, the value function will finally converge to the *true* one. Therefore, in practice when the value function is hardly solvable in an analytical way, people usually set up the computer program to perform the iteration and get a numerical solution.

B.5 Extensions

B.5.1 Extension 1: Dynamic Programming Under Uncertainty

Consider a general discrete-time optimization problem

$$\begin{aligned} \max_{\{c_t, k_{t+1}\}_{t=0}^{+\infty}} \quad & E_0 \left[\sum_{t=0}^{+\infty} \beta^t u(c_t) \right] \\ s.t. \quad & k_{t+1} = z_t f(k_t) - c_t, \end{aligned}$$

in which the production $f(k_t)$ is affected by an i.i.d. process $\{z_t\}_{t=0}^{+\infty}$ (technology shock, which realizes at the beginning of each period t) meaning that such shock varies over time, but its deviations in different periods are uncorrelated (think about the weather for the farmers). Now the agent has to maximize the expected utility over time because future consumption is uncertain.

Since technology shocks realize at the beginning of each period, the value of total output is known when consumption takes place and when the end-of-period capital k_{t+1} is accumulated. The state variables are now k_t and z_t. The control variables are c_t. Similar as before, set up the Bellman equation as

$$V(k_t, z_t) = \max_{c_t, k_{t+1}} \{u(c_t) + \beta E_t [V(k_{t+1}, z_{t+1})]\}$$
$$s.t. \quad k_{t+1} = z_t f(k_t) - c_t.$$

Let's apply the solution strategies introduced before and see whether they work.

Step 1 The first-order condition with respect to k_{t+1} gives

$$-u'(c_t) + \beta E_t \left[\frac{dV(k_{t+1}, z_{t+1})}{dk_{t+1}} \right] = 0. \tag{B.23}$$

Think why it is legal to take derivative within expectation operator.

Step 2 By Envelope Theorem differentiating $V(k_t, z_t)$ with respect to k_t gives

$$\frac{dV(k_t, z_t)}{dk_t} = u'(c_t) z_t f'(k_t). \tag{B.24}$$

Step 3 Take one step forward for (B.24) and apply it into (B.23), then we get

$$u'(c_t) = \beta E_t \left[u'(c_{t+1}) z_{t+1} f'(k_{t+1}) \right].$$

Suppose that

$$f(k_t) = k_t^{\alpha},$$
$$u(c_t) = \ln c_t.$$

Then Euler equation becomes

$$\frac{1}{c_t} = \beta E_t \left[\frac{1}{c_{t+1}} z_{t+1} \alpha k_{t+1}^{\alpha - 1} \right].$$

In deterministic case our solutions were

$$c_t = (1-\alpha\beta)k_t^\alpha,$$
$$k_{t+1} = \alpha\beta k_t^\alpha.$$

Now let's guess that under uncertainty the solution is of similar form such that

$$c_t = (1-A)z_t k_t^\alpha,$$
$$k_{t+1} = A z_t k_t^\alpha,$$

and check whether it's true or false. The Euler equation becomes

$$\begin{aligned}\frac{1}{(1-A)z_t k_t^\alpha} &= \beta E_t\left[\frac{1}{(1-A)z_{t+1}k_{t+1}^\alpha} z_{t+1}\alpha k_{t+1}^{\alpha-1}\right]\\ &= \alpha\beta\left[\frac{1}{(1-A)k_{t+1}}\right]\\ &= \alpha\beta\left[\frac{1}{(1-A)Az_t k_t^\alpha}\right].\end{aligned}$$

Therefore it's easily seen that

$$A = \alpha\beta,$$

which seems quite similar as before.

However since the consumption and capital stock are random variables, it's necessary to explore their properties by characterizing corresponding distributions.

Assume that $\ln z_t \sim N\left(\mu, \sigma^2\right)$. Take log of the solution above,

$$\ln k_{t+1} = \ln\alpha\beta + \ln z_t + \alpha\ln k_t.$$

Apply this result recursively,

$$\begin{aligned}\ln k_t &= \ln\alpha\beta + \ln z_{t-1} + \alpha\ln k_{t-1}\\ &= \ln\alpha\beta + \ln z_{t-1} + \alpha(\ln\alpha\beta + \ln z_{t-2} + \alpha\ln k_{t-2})\\ &\quad \ldots\\ &= \left(1+\alpha+\alpha^2+\ldots+\alpha^{t-1}\right)\ln\alpha\beta\\ &\quad + \left(\ln z_{t-1} + \alpha\ln z_{t-2} + \ldots + \alpha^{t-1}\ln z_0\right)\\ &\quad + \alpha^t\ln k_0.\end{aligned}$$

In the limit the mean of $\ln k_t$ converges to

$$\lim_{t \to +\infty} E_0 [\ln k_t] = \frac{\ln \alpha\beta + \mu}{1 - \alpha}.$$

The variance of $\ln k_t$ is defined as

$$\begin{aligned}
\text{var}[\ln k_t] &= E\left\{(\ln k_t - E[\ln k_t])^2\right\} \\
&= E\left\{\left(\left(1 + \alpha + \alpha^2 + \ldots + \alpha^{t-1}\right)\ln \alpha\beta + (\ln z_{t-1} + \alpha \ln z_{t-2}\right.\right. \\
&\quad \left. + \ldots + \alpha^{t-1} \ln z_0\right) + \alpha^t \ln k_0 - \left[\left(1 + \alpha + \alpha^2 + \ldots + \alpha^{t-1}\right) \ln \alpha\beta \right. \\
&\quad \left.\left. + \left(1 + \alpha + \ldots + \alpha^{t-1}\right)\mu + \alpha^t \ln k_0\right]\right)^2 \Big\} \\
&= E\left\{\left[(\ln z_{t-1} - \mu) + \alpha(\ln z_{t-2} - \mu) + \alpha^2(\ln z_{t-2} - \mu) + \ldots \right.\right. \\
&\quad \left.\left. + \alpha^{t-1}(\ln z_0 - \mu)\right]^2\right\} \\
&= E\left[\left(\sum_{i=1}^{t} \alpha^{i-1}(\ln z_{t-i} - \mu)\right)^2\right] \\
&= \sum_{i=1}^{t} \alpha^{2i-2} E\left[(\ln z_{t-i} - \mu)^2\right] \\
&\quad + \sum_{\forall i,j \in \{1,\ldots,t\} i \neq j} \alpha^{i-1}\alpha^{j-1} E\left[(\ln z_{t-i} - \mu)(\ln z_{t-j} - \mu)\right] \\
&= \frac{1 - \alpha^{2t}}{1 - \alpha^2} \text{var}[\ln z_t] \\
&= \frac{1 - \alpha^{2t}}{1 - \alpha^2} \sigma^2,
\end{aligned}$$

or simply pass the variance operator through the sum and get

$$\begin{aligned}
\text{var}[\ln k_t] &= \text{var}[\ln z_{t-1}] + \alpha^2 \text{var}[\ln z_{t-2}] + \ldots + \alpha^{2t-2} \text{var}[\ln z_0] \\
&= \left(1 + \alpha^2 + \ldots + \alpha^{2t-2}\right)\sigma^2 \\
&= \frac{1 - \alpha^{2t}}{1 - \alpha^2}\sigma^2.
\end{aligned}$$

In the limit the variance of $\ln k_t$ converges to

$$\lim_{t \to +\infty} \operatorname{var}(\ln k_t) = \frac{\sigma^2}{1-\alpha^2}.$$

As a conclusion one can say that in the limit $\ln k_t$ converges to a distribution with mean $\frac{\ln \alpha\beta + \mu}{1-\alpha}$ and variance $\frac{\sigma^2}{1-\alpha^2}$.

B.5.2 Extension 2: Dynamic Programming in Continuous Time

Till now one may get the illusion that dynamic programming only fits discrete time. Now with slight modification we'll see that it works for continuous time problems as well. Consider a general continuous time optimization problem

$$\begin{aligned} \max_{c_t, k_t} \int_{t=0}^{+\infty} & e^{-\rho t} u(c_t) dt \\ s.t. \ \ \dot{k}_t &= \phi(c_t, k_t) = f(k_t) - c_t \end{aligned}$$

in which we assume that $\phi(c_t, k_t)$ is quasi-linear in c_t only for simplicity.

Following Bellman's idea, for arbitrary $t \in [0, +\infty)$ define

$$V(k_t) = \max_{c_t, k_t} \int_t^{+\infty} e^{-\rho(\tau - t)} u(c_\tau) d\tau.$$

Now suppose that time goes from t to $t + \Delta t$, in which Δt is very small. Let's imagine what happened from t on. First $u(c_t)$ accumulates during Δt. Since Δt is so small that it's reasonable to think that $u(c_t)$ is nearly constant from t to $t + \Delta t$, and the accumulation of utility can be expressed as $u(c_t)\Delta t$. Second, from $t + \Delta t$ onwards the total utility accumulation is just $V(k_{t+\Delta t})$. Therefore $V(k_t)$ is just the sum of utility accumulation during Δt, and discounted value of $V(k_{t+\Delta t})$, i.e.,

$$V(k_t) = \max_{c_t, k_t} \left\{ u(c_t)\Delta t + \frac{1}{1 + \rho \Delta t} V(k_{t+\Delta t}) \right\}$$

(Why $V(k_{t+\Delta t})$ is discounted by $\frac{1}{1+\rho\Delta t}$?). Rearrange both sides

$$\begin{aligned} (1 + \rho\Delta t)V(k_t) &= \max_{c_t, k_t} \{u(c_t)(1 + \rho\Delta t)\Delta t + V(k_{t+\Delta t})\} \\ \rho\Delta t V(k_t) &= \max_{c_t, k_t} \{u(c_t)(1 + \rho\Delta t)\Delta t + V(k_{t+\Delta t}) - V(k_t)\} \\ \rho V(k_t) &= \max_{c_t, k_t} \left\{ u(c_t)(1 + \rho\Delta t) + \frac{V(k_{t+\Delta t}) - V(k_t)}{\Delta t} \right\}, \end{aligned}$$

and take limit

$$\rho V(k_t) = \lim_{\Delta t \to 0} \max_{c_t, k_t} \left\{ u(c_t)(1 + \rho \Delta t) + \frac{V(k_{t+\Delta t}) - V(k_t)}{\Delta t} \right\}.$$

Finally this gives

$$\begin{aligned} \rho V(k_t) &= \max_{c_t, k_t} \left\{ u(c_t) + V'(k_t)\dot{k}_t \right\} \\ &= \max_{c_t, k_t} \left\{ u(c_t) + V'(k_t)\phi(c_t, k_t) \right\}. \end{aligned}$$

Then you are able to solve it by any of those three approaches. Here we only try one of them.

Step 1 First-order condition for the maximization problem gives

$$u'(c_t) + V'(k_t)\frac{\partial \phi(c_t, k_t)}{\partial c_t} = u'(c_t) - V'(k_t) = 0. \tag{B.25}$$

Step 2 Differentiating $V(k_t)$ gives

$$\rho V'(k_t) = V''(k_t)\phi(c_t, k_t) + V'(k_t)\frac{\partial \phi(c_t, k_t)}{\partial k_t},$$

that is,

$$\left[\rho - \frac{\partial \phi(c_t, k_t)}{\partial k_t}\right] V'(k_t) = V''(k_t)\phi(c_t, k_t) = V''(k_t)\dot{k}_t.$$

Take derivative of $V'(k_t)$ with respect to t

$$\frac{dV'(k_t)}{dt} = \dot{V}'(k_t) = V''(k_t)\dot{k}_t = \left[\rho - \frac{\partial \phi(c_t, k_t)}{\partial k_t}\right] V'(k_t),$$

and get

$$\frac{\dot{V}'(k_t)}{V'(k_t)} = \rho - \frac{\partial \phi(c_t, k_t)}{\partial k_t} = \rho - f'(k_t). \tag{B.26}$$

Step 3 Take derivative of (B.25) with respect to t and get

$$\frac{\dot{u}'(c_t)}{u'(c_t)} = \frac{u''(c_t)}{u'(c_t)}\dot{c}_t = \frac{\dot{V}'(k_t)}{V'(k_t)} = \rho - f'(k_t)$$

by (B.26), and further arrangement gives

$$-\frac{u''(c_t)c_t}{u'(c_t)}\frac{\dot{c}_t}{c_t} = f'(k_t) - \rho$$
$$\frac{\dot{c}_t}{c_t} = \sigma\left[f'(k_t) - \rho\right].$$

Note that this is exactly the same solution as we got by the optimal control method.

Reference

Brock, W. A., & Mirman, L. J. (1972). Optimal economic growth and uncertainty: The discounted case. *Journal of Economic Theory, 4*, 479–513.